Doodles and Blobs on a Ruled Page: Convex Quasi-envelops of Traversing Flows on Surfaces

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Abstract

Let A denote the cylinder $\mathbb{R} \times S^1$ or the band $\mathbb{R} \times I$, where I stands for the closed interval. We consider 2-moderate immersions of closed curves ("doodles") and compact surfaces ("blobs") in A, up to cobordisms that also are 2-moderate immersions in $A \times [0, 1]$ of surfaces and solids. By definition, the 2-moderate immersions of curves and surfaces do not have tangencies of order ≥ 3 to the fibers of the obvious projections $A \to S^1$, $A \times [0, 1] \to S^1 \times [0, 1]$ or $A \to I$, $A \times [0, 1] \to I \times [0, 1]$. These bordisms come in different flavors: in particular, we consider one flavor based on regular embeddings of doodles and blobs in A. We compute the bordisms of regular embeddings and construct many invariants that distinguish between the bordisms of immersions and embeddings. In the case of oriented doodles on $A = \mathbb{R} \times I$, our computations of 2-moderate immersion bordisms $\mathbf{OC}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A)$ are near complete: we show that they can be described by an exact sequence of abelian groups

$$0 \to \mathbf{K} \to \mathbf{OC}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A) \big/ \mathbf{OC}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \xrightarrow{\mathcal{I}\rho} \mathbb{Z} \times \mathbb{Z} \to 0,$$

where $\mathbf{OC}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \approx \mathbb{Z} \times \mathbb{Z}$, the epimorphism $\mathcal{I}\rho$ counts different types of crossings of immersed doodles, and the kernel $\mathbf{K}$ contains the group $(\mathbb{Z})^\infty$ whose generators are described explicitly.

Keywords Curves · surfaces · immersions · cobordisms · crossing invariants

1 Introduction

This paper illustrates the richness of traversing vector flows (see Definition 2.3) on surfaces with boundary. It also provides tools for constructing such flows (see Fig. 2).

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Some multi-dimensional versions of these ideas and constructions can be found in [14, 15], and [16]. However, the case of, so-called, 2-moderate one- and two-dimensional embeddings and immersions against the background of a fixed 1-dimensional foliation on a target surface A has its unique and pleasing features. One of which is the drastic simplification of the combinatorial considerations that characterize our treatment in [14, 15], and [17] of similar multi-dimensional immersions.

Propositions 1.1 and 1.2 below illustrate the flavor of some of our results.

Consider the vector space $\mathbb{R}^3_{xyz}$ with coordinates x, y, z and the obvious projections $P_z : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$ and $p_z : \mathbb{R}^2_{xz} \to \mathbb{R}^1_x$. Let $\mathcal{C} \subset \mathbb{R}^2_{xz}$ be a smooth simple curve, the boundary of a compact domain $\mathcal{D} \subset \mathbb{R}^2_{xz}$, such that $p_z : \mathcal{C} \to \mathbb{R}^1_x$ has only singularities that are quadratic (folds). We say that $\mathcal{C}$ is positively (negatively) concave if the function $x : \mathcal{D} \to \mathbb{R}$ has more local maxima (minima) than minima (maxima) (see Fig. 5(b)).

Proposition 1.1 *Let $\mathcal{C} \subset \mathbb{R}^2_{xz}$ be a simple smooth curve such that the projection $p_z : \mathcal{C} \to \mathbb{R}^1_x$ has only quadratic folding singularities and $\mathcal{C}$ is positively (negatively) concave.*

Then, any smooth compact surface $\mathcal{S} \subset \mathbb{R}^3_{xyz} \cap \{y \geq 0\}$ that bounds $\mathcal{C}$ must have at least cubic singularities (cusps) under the map $P_z : \mathcal{S} \to \mathbb{R}^2_{xy}$. ◇

Proposition 1.1 is implied directly by Theorem 2.2. The key feature here is the interplay between the curve $\mathcal{C}$, surface $\mathcal{S}$, and the simple 1-dimensional foliation in $\mathbb{R}^3_{xyz}$ whose leaves are the fibers of the projection P_z.

A different, but related, phenomenon is exemplified by the next proposition, which follows from the proof of Theorem 3.1.

Proposition 1.2 *Consider the immersed curve $\mathcal{C}$ in $\mathbb{R}^2_{xz}$ shown in Fig. 9, (a).*

Then, any compact immersed surface $\mathcal{S} \subset \mathbb{R}^3_{xyz} \cap \{y \geq 0\}$, that bounds $\mathcal{C}$ must have two triple-intersection points at least. ◇

Although doodles on surfaces have been a well-traveled destination [4, 5], doodles against the background of a given foliation on a compact surface are not. The same can be said about submersions $\alpha : X \to A$ of compact surfaces X on the cylinders or strips A, equipped with a product foliation $\mathcal{F}(\hat{v})$. This simple product foliation is responsible for the term "ruled" in the title of the paper.

Our general inspiration comes from the pioneering works of V. I. Arnold [1–3], and V. A. Vassiliev [19, 20].

The main problem we are dealing here is to classify such submersions $\alpha : X \to A$ (up to a kind of cobordism), while *controlling the tangency patterns* of the boundary ∂X, or rather of the curves $\alpha(\partial X)$ to the foliation $\mathcal{F}(\hat{v})$. As a byproduct, we getting some computable bordism-like relation among traversing vector fields on compact surfaces with boundary.

Thus, we consider two target spaces, the cylinder $\mathbb{R} \times S^1$ and and the strip $\mathbb{R} \times [0, 1]$. Some of the constructions will work for the cylinder, and some for the strip; for both, we use the same notation "A". The space A is equipped with a traversing (see Definition 2.3 and [9, 10]) vector field $\hat{v}$ and the 1-dimensional oriented foliation $\mathcal{F}(\hat{v})$ it generates.
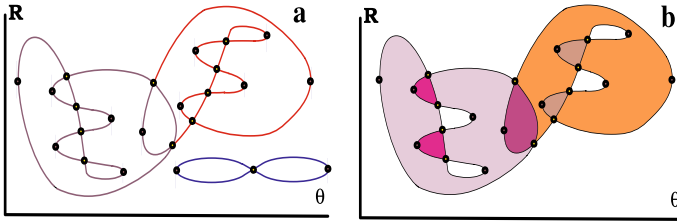
Fig. 1 Diagram (**a**) shows doodles—an immersion $\beta : \mathcal{C} \to A$ of 3 loops $\mathcal{C}$ in the surface $A = \mathbb{R} \times [0, 1]$. Diagram (**b**) shows blobs—an immersion $\alpha : X \to A$ of two disks X in A. The self-intersections of the curves $\beta(\mathcal{C})$ and of $\alpha(\partial X)$ and the points of tangency of $\beta(\mathcal{C})$ and of $\alpha(\partial X)$ to the vertical foliation $\mathcal{F}(\hat{v})$ on A are marked. Thanks to the presence of figure "∞", β does not extend to an immersion α of any compact surface X into A

Its leaves are of the form $\{\mathbb{R} \times \theta\}_\theta$, where θ belongs either to the circle S^1 or to the interval $I = [0, 1]$.

We draw some "doodles" (finite collections of loops) on A (as in Fig. 1, diagram (a)) and pay close attention to the way they intersect the leaves of the foliation $\mathcal{F}(\hat{v})$, especially to the way they are tangent to the leaves. These interactions of doodles with the leaves are combinatorial in nature. We will impose some a priori restrictions (called "2-moderate") on these combinatorial patterns and will classify the doodles that respect the restrictions.

In the paper, we will also consider doodles that are the images of boundaries of compact surfaces X, *immersed* in A (as shown in Fig. 1(b)). The images of X in A form overlapping "blobs".

Our main results about doodles and blobs on a ruled page $(A, \mathcal{F}(\hat{v}))$ are contained in Theorems 2.1–2.2 and Theorems 3.1–3.4. Perhaps, some new invariants of doodles and blobs that lead to these theorems will have an independent life.

Let us set the stage for these results in a more formal way. Let X be a compact surface with boundary and v a traversing vector field (see Definition 2.3) on X. As a function of a point $x \in X$, the v-trajectory $\gamma_x \subset X$ through x exhibits a discontinuous behavior in the vicinity of any "concave" point (see Definition 2.2) of the boundary [9]. In order to get around this fundamental difficulty, we "envelop" the pair (X, v) into a pair $(\hat{X}, \hat{v})$, where an ambient compact surface $\hat{X} \supset X$ with a traversing vector field $\hat{v}$, is such that:

(1) $\partial \hat{X}$ is convex (see Definition 2.2) with respect to the $\hat{v}$-flow on $\hat{X}$,

(2) $\hat{v}|_X = v$.

Without lost of generality, the reader may think of $\hat{X}$ as residing in the cylinder $\mathbb{R} \times S^1$ or in the strip $\mathbb{R} \times [0, 1]$ and the vector field $\hat{v}$ as being the unit vector field, tangent to the leaves of the product foliation $\mathcal{F}(\hat{v})$.

Not any pair (X, v) admits such convex envelop (see Lemma 2.1). However, when available, the convex envelop $(\hat{X}, \hat{v})$ simplifies the analysis of (X, v) greatly.

In this context, our goal is to study convex envelops (and their generalizations, the so called, quasi-envelops, shown in Fig. 2) together with the doodles or blobs they contain, up to some natural equivalence relations that we call in [14, 15] "quasitopies", a crossover between bordisms and pseudoisotopies of immersions. When $A = \mathbb{R} \times [0, 1]$, the quasitopy (bordism) equivalence classes can be organized into

groups. In [14], we compute these algebraic structures for an a priori prescribed set of combinatorial tangency patterns of "*n*-dimensional doodles" to product-like 1-dimensional foliations. For $(n + 1)$-dimensional traversing convexly enveloped flows, this goal is achieved in full generality in [15]. However, the case of quasi-enveloped traversing flows is far from being settled. Although in two dimensions these structures are relatively primitive, as this paper illustrates, they are not completely trivial ether.

Recall that in the study of manifolds and fibrations the universal classifying spaces like Grassmanians play a pivotal role. In the category of convex envelops, the role of universal objects (of "the new Grassmanians") is played by various spaces of smooth functions $f : \mathbb{R} \to \mathbb{R}$ whose zeros (considered with their multiplicities) are modeled after the real divisors of real polynomials. The topology of these functional spaces with constrained zero divisors is interesting on its own right (see [16, 17], where it is described in detail). One particular kind of these functional spaces, called spaces of smooth functions/polynomials with k-moderate singularities, has been introduced and studied in depth by V. I. Arnold [1, 2] and V. A. Vassiliev [19, 20]. In [16, 17], we compute the homology of similar functional spaces, based on real polynomials in one variable, in terms of appropriate *universal* combinatorics. This is reminiscent to the role played by the Schubert calculus in depicting the characteristic classes of classical Grassmanians.

Recall that, for boundary generic 2-dimensional traversing flows (see Definition 2.1), no tangencies of orders ≥ 3 to the boundary may occur [10]. In light of what has been outlined above, we should anticipate a link between boundary generic flows on surfaces and the spaces of smooth functions $f : \mathbb{R} \to \mathbb{R}$ (or even real polynomials) that have no zeros of multiplicities ≥ 3. This connection and its generalizations are validated in [14, 15].

Let $\mathcal{F}$ denote the space of smooths functions $f : \mathbb{R} \to \mathbb{R}$ that are identically 1 outside of a compact interval (the interval may depend on a particular function). The space $\mathcal{F}$ is considered in the C^∞-topology. Let $\mathcal{F}^{<k}$ be its subspace, formed by functions that have zeros only of the multiplicities less than k. Arnold calls such functions "functions with k-moderate singularities". The property of a function to have k-moderate zeros is an *open property* in the C^∞-topology; that is, the space $\mathcal{F}^{<k}$ is open in $\mathcal{F}$.

For $k = 2$, the combinatorial patterns ω of zero divisors of such functions are finite sequences of natural numbers, build of 1's and 2's only, as well as the empty sequence.

A fundamental theorem of Vassiliev ([19], Corollary on page 81 and The First and Second Main Theorems on pages 78–79) identifies the weak homotopy types of the spaces $\mathcal{F}^{<k}$ (for all $k \geq 4$) and their integral homology types (for all $k \geq 3$) as ΩS^{k-1}, the space of loops on a $(k - 1)$-sphere! In particular, the integral homology of the space $\mathcal{F}^{\leq 2} =_{\mathsf{def}} \mathcal{F}^{<3}$ is isomorphic to the homology of the loop space ΩS^2.

Arnold proved that the fundamental group $\pi_1(\mathcal{F}^{\leq 2}) \approx \mathbb{Z}$ ([1]), the result that we will use on a number of occasions. Thus, $H^1(\mathcal{F}^{\leq 2}; \mathbb{Z}) \approx \mathbb{Z}$ as well.

Therefore, for a 2-dimensional convex quasi-envelop $\alpha : (X, v) \subset (A, \hat{v})$ with no tangencies of $\alpha(\partial X)$ to the oriented foliation $\mathcal{F}(\hat{v})$ of order ≥ 3, this fact allows to define a characteristic class $J_\alpha^* \in H^1(A, \partial_\star A; \mathbb{Z}) \approx \mathbb{Z}$ of α. Here, $\partial_\star A = \emptyset$ for $A = \mathbb{R} \times S^1$, and $\partial_\star A = \mathbb{R} \times \partial([0, 1])$ for $A = \mathbb{R} \times [0, 1]$.

Let d be an even positive integer. We will employ the subspaces $\mathcal{F}_d^{\leq 2} \subset \mathcal{F}^{\leq 2}$, formed by functions whose "degree"—the sum of multiplicities of all its zeros—is even and does not exceed d.

2 Convex Quasi-envelops of Traversing Flows on Surfaces and Spaces of Smooth Functions with 2-Moderate Singularities

Let us start with a number of definitions that have their origins in [9–13] and introduce different kinds of vector fields. Unfortunately, the list of definitions is long.

Let v be a smooth vector field on a compact connected surface X with boundary, such that $v \neq 0$ along the boundary ∂X. Following [18], we consider the closed locus $\partial_1^+ X(v) \subset \partial X$, where the field points inside of X or is tangent to ∂X, and the closed locus $\partial_1^- X(v) \subset \partial X$, where v points outside of X or is tangent to ∂X. The intersection

$$\partial_2 X(v) =_{\mathsf{def}} \partial_1^+ X(v) \cap \partial_1^+ X(v)$$

is the locus where v is *tangent* to the boundary ∂X.

Points $z \in \partial_2 X(v)$ come in two flavors: by definition, $z \in \partial_2^+ X(v)$ when $v(z)$ points inside of the locus $\partial_1^+ X(v)$, otherwise $z \in \partial_2^- X(v)$.

Definition 2.1 A vector field v on a compact surface X is boundary generic if:

- $v|_{\partial X} \neq 0$,
- $v|_{\partial X}$, viewed as a section of the normal 1-dimensional (quotient) bundle $n_1 =_{\mathsf{def}} T(X)|_{\partial X}/T(\partial X)$, is transversal to its zero section at the points of the locus $\partial_2 X(v)$. $\diamond$

In particular, for a boundary generic v, the loci $\partial_1^{\pm} X(v)$ are finite unions of closed intervals and circles, residing in ∂X; the loci $\partial_2^{\pm} X(v)$ are finite unions of points, residing in $\partial_1 X$.

Each trajectory γ of a traversing vector field v must reach the boundary both in positive and negative times: otherwise γ is not homeomorphic to a closed interval.

Definition 2.2 A boundary generic vector field v is boundary convex if $\partial_2^+ X(v) = \emptyset$. A boundary generic v is boundary concave if $\partial_2^- X(v) = \emptyset$. $\diamond$

Definition 2.3 A non-vanishing vector field v on a compact surface X is traversing if all its trajectories are closed segments or singletons.[1]

Equivalently, v is traversing if there exists a Lyapunov function $f : X \to \mathbb{R}$ such that $df(v) > 0$ [9]. $\diamond$

Definition 2.4 A traversing vector field v on a compact surface X is called traversally generic, if the following properties hold:

(1) if a trajectory γ is tangent to the boundary ∂X, then the tangency is simple,
(2) no v-trajectory γ contains more then one simple point of tangency to ∂X. $\diamond$

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[1] It easy to see that the ends of these segments, as well as the singletons, must reside in the boundary ∂X.

Fig. 2 A convex quasi-envelop
$\alpha : X \to A$ of a 2-moderately
generic (actually, even
traversally generic) vector field
$\alpha^\dagger(\partial_u)$ on a compact surface X,
the torus from which an open
smooth disk is removed (the top
diagram), and on a compact
surface X, the closed surface of
genus 2 from which an open
smooth disk is removed (the
bottom diagram). In both
examples, the cardinality of the
fibers of the map
$\theta \circ \alpha : \partial X \to \mathcal{T}(\hat{v})$ does not
exceed 6
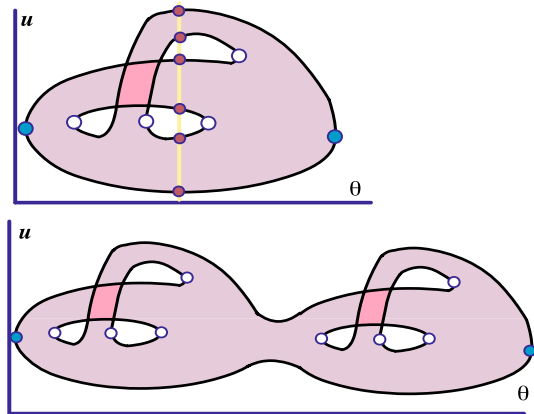
Definition 2.5 Let $\hat{v}$ be the standard traversing vector field on the surface A (a strip or a cylinder), tangent to the fibers of the projections $\mathbb{R} \times [0, 1] \to [0, 1]$ or $\mathbb{R} \times S^1 \to S^1$. Let $\mathcal{C}$ be a closed 1-dimensional manifold (a finite collection of several circles). Consider an *immersion* $\beta : \mathcal{C} \to A$.

We say that such β is 2-moderately generic relatively to $\hat{v}$ if no $\hat{v}$-trajectory $\hat{\gamma}$ has order of tangency ≥ 3 to $\beta(\mathcal{C})$. Here the order of tangency is understood as the sum of tangency orders of local branches of $\beta(\mathcal{C})$ that pass through a given point of $\hat{\gamma}$. ◇

By standard techniques of the singularity theory, we can perturb any given immersion $\beta : \mathcal{C} \to A$ so that $\beta(\mathcal{C})$ will have only transversal self-intersections and will become boundary generic, and thus, 2-moderately generic.

In fact, any orientable connected surface X with boundary admits an immersion $\alpha : X \to A$ (or in the plane $\mathbb{R}^2$) (see Fig. 2). We will use this fact to pull-back to X the standard traversing vector field $\hat{v}$ on A.

Definition 2.6 Consider an *immersion* $\alpha : X \to A$ of a given compact (orientable) surface X into the surface A, equipped with the standard vector field $\hat{v}$ and the foliation $\mathcal{F}(\hat{v})$ it generates.

- Given a transversing vector field v on X, we call an immersion α a convex quasi-envelop of (X, v) if $v = \alpha^\dagger(\hat{v})$, the pull-back of $\hat{v}$ under α.
- Such α is called 2-moderately generic relative to $\hat{v}$, if the restriction $\alpha|_{\partial X}$ is 2-moderately generic with respect to $\hat{v}$ in the sense of Definition 2.5.

◇

The next definition is a special case of Definition 2.6.

Definition 2.7 Let $\alpha : X \to A$ be a *regular embedding* of a given compact surface X into the surface A, carrying the standard vector field $\hat{v}$. We denote by v the pull-back $\alpha^\dagger(\hat{v})$ of $\hat{v}$ under α.

We say that the pair $(A, \hat{v})$ is a convex envelop of (X, v). If $\alpha|_{\partial X}$ is 2-moderately generic, we call α a 2-moderately generic convex envelop. ◇

As the next lemma testifies, the existence of a convex envelop puts significant restrictions on the topology of orientable surfaces X: such X are disks with holes or unions of such.

Lemma 2.1 *If a compact connected surface X with boundary has a pair of loops whose transversal intersection is a singleton, then no traversal flow on X admits a convex envelop. In other words, if a connected surface X with boundary has a handle, then no traversal flow on X can be convexly enveloped.*

Proof By [12], Theorem 1.2, the space $\hat{X}$ of a convex envelop is either a disk or an annulus, both surfaces residing in the plane. No two loops in the plane intersect transversally at a singleton. Thus, for surfaces with a handle, no convex envelops exist.
□

To incorporate surfaces with handles into our constructions, in Definition 2.6, we introduce the less restrictive notion of a convex *quasi*-envelop.

Now, we would like to explore closely a nice connection between:

(1) immersions $\alpha : (X, v) \to (A, \hat{v})$ of a compact surface X in the surface A, such that $v = \alpha^\dagger(\hat{v})$ and $\alpha(\partial X)$ is 2-moderately generic with respect to $\hat{v}$, and

(2) loops in the functional spaces $\mathcal{F}^{\leq 2}$.

Let $\alpha(\partial X)^\times$ denote the finite set of self-intersections of the curves forming the image $\alpha(\partial X)$. Let $\alpha(\partial X)^\circ$ denote the set $\alpha(\partial X) \setminus \alpha(\partial X)^\times$.

With the pattern $\alpha(\partial X) \subset A$, we associate an auxiliary smooth function $z_\alpha : A \to \mathbb{R}$, defined by the following properties:

$$z_\alpha^{-1}(0) = \alpha(\partial X), \tag{2.1}$$

- 0 is the regular value of z_α at the points of $\alpha(\partial X)^\circ$,
- in the vicinity of each transversal crossing point $a \in \alpha(\partial X)^\times$, there exist locally defined smooth functions $x_1, x_2 : A \to \mathbb{R}$ such that: 0 is their regular value, $\{x_1^{-1}(0)\}$ and $\{x_2^{-1}(0)\}$ define the two local intersecting branches of $\alpha(\partial X)$, and $z_\alpha = x_1 \cdot x_2$ locally.
- z_α approaches 1 at infinity in $A = \mathbb{R} \times S^1$ or in $A = \mathbb{R} \times [0, 1]$.

The sign of z_α changes to the opposite, as a path in A crosses an arc from $\alpha(\partial X)^\circ$ transversally, thus providing a "checker board" coloring of the domains in $A \setminus \alpha(\partial X)$. The properties in (2.1) do not determine a unique function z_α, but any such function will serve equally well in the forthcoming constructions.

Given a smooth traversing vector field v on a compact surface X, we denote by $\mathcal{T}(v)$ the space of v-trajectories. For a boundary generic and traversing v, the space $\mathcal{T}(v)$ is a finite graph; for a traversally generic v, the space $\mathcal{T}(v)$ is a finite graph whose vertexes are only of valency 3 and 1 [12]. For the standard vector field $\hat{v}$ on A, the trajectory space $\mathcal{T}(\hat{v})$ is either a circle S^1, or an interval $[0, 1]$.

Let $u : A \to \mathbb{R}$ be the obvious projection on the first ("vertical") factor. In particular, $du(\hat{v}) > 0$ and $u(\hat{\gamma}) = \mathbb{R}$ for all $\hat{v}$–trajectories $\hat{\gamma}$ in A. Then, with the help of z_α and u, we get a map $J_{z_\alpha} : \mathcal{T}(\hat{v}) \to \mathcal{F}^{\leq 2}$ whose source, the trajectory space $\mathcal{T}(\hat{v})$, is either a circle or a closed interval. The image of J_{z_α} belongs to the space of smooth functions

$f : \mathbb{R} \to \mathbb{R}$ such that the set $\{f \leq 0\}$ is compact in $\mathbb{R}$, f has no zeros of multiplicity ≥ 3, and $\lim_{u \to \pm\infty} f(u) = 1$. We define the map J_{z_α} by the formula

$$J_{z_\alpha}([\hat{\gamma}]) = (z_\alpha|_{\hat{\gamma}}) \circ (u|_{\hat{\gamma}})^{-1}, \tag{2.2}$$

where $\hat{\gamma}$ stands for a $\hat{v}$-trajectory in A, and $[\hat{\gamma}]$ for the corresponding point in the trajectory space $\mathcal{T}(\hat{v})$. If A is the strip, $\mathcal{T}(\hat{v}) = [0, 1]$. The two ends of the interval are mapped by J_{z_α} to the convex subspace $\mathcal{F}^+$ of $\mathcal{F}$ that is formed by *strictly positive* smooth functions. Therefore, we get a map of pairs

$$J_{z_\alpha} : ([0, 1], \ \partial[0, 1]) \to (\mathcal{F}^{\leq 2}, \ \mathcal{F}^+).$$

The constant point-function $\mathbf{1}$ is a deformation retract of $\mathcal{F}^+$. Thus, homotopically, J_{z_α} can be regarded as a based loop

$$J_{z_\alpha} : (S^1, pt) \to (\mathcal{F}^{\leq 2}, \ \mathcal{F}^+) \sim (\mathcal{F}^{\leq 2}, \ \mathbf{1}).$$

For a fixed α, the homotopy class $[J_\alpha]$ of the map J_{z_α} does not depend on the choice of the auxiliary function z_α, subject to the four properties in (2.1). Indeed, consider the α_*-image $\alpha_*(\nu)$ of the outer normal vector field ν to the boundary ∂X in X. At the points $a \in \alpha(\partial X)^\times$, we get two vectors $v_1(a), v_2(a) \in \alpha_*(\nu)$, one for each branch of $\alpha(X)$.

Let $\mathcal{L}_w$ denote the directional derivative in the direction of a vector w. If z_α and z'_α are any two functions that satisfy all the properties from the list (2.1), we get $\mathcal{L}_{\hat{v}} z_\alpha > 0$ and $\mathcal{L}_{\hat{v}} z'_\alpha > 0$ at each point of $\alpha(\partial X)^\circ$. Therefore, we get $\mathcal{L}_{\hat{v}}\{\tau z_\alpha + (1 - \tau)z'_\alpha\} > 0$, where $\tau \in [0, 1]$, at each point of $\alpha(\partial X)^\circ$. At the same time, for $i = 1, 2$, at each transversal crossing $a \in \alpha(\partial X)^\times$, we have $\mathcal{L}_{\hat{v}} x_i > 0$, and $\mathcal{L}_{\hat{v}} x'_i > 0$. Hence, $\mathcal{L}_{\hat{v}}\{\tau x_i + (1 - \tau)x'_i\} > 0$. The rest of properties from (2.1) are obviously "convex" .

Therefore, the space of functions z_α that satisfy (2.1) is convex and, thus, contractible, which implies that the homotopy class $[J_\alpha]$ does not depend on the choice of z_α.

As a result, for $A = \mathbb{R} \times S^1$, any moderately generic convex quasi-envelop $\alpha : X \to A$, produces a homotopy class $[J_\alpha] \in [S^1, \mathcal{F}^{\leq 2}]$. For $A = \mathbb{R} \times [0, 1]$, any moderately generic convex quasi-envelop $\alpha : X \to A$, produces a homotopy class $[J_\alpha] \in [(S^1, pt), (\mathcal{F}^{\leq 2}, \mathbf{1})]$.

Note that these homotopy classes do not depend on the orientation of the surface X.

We pick a generator $\kappa \in \pi_1(\mathcal{F}^{\leq 2}, \mathbf{1}) \approx \mathbb{Z}$ (see [2] and Fig. 5(a) or (b), for realistic portraits of κ). For $A = \mathbb{R} \times [0, 1]$, we define an integer J^α by the formula $\mathsf{J}^\alpha \cdot \kappa = [J_\alpha]$.

The isomorphism $\pi_1(\mathcal{F}^{\leq 2}, \mathbf{1}) \approx \mathbb{Z}$ follows from [2]. A slight modification of Arnold's arguments leads to Theorem 2.1 below. The main difference between our constructions and the ones from [2] is that Arnold is concerned with immersions of 1-dimensional oriented "doodles" in A, while we deal also with immersions of "blobs" (compact orientable surfaces) in A (compare diagrams (a) and (b) in Fig. 1).

We denote by $\hat{v}^{\bullet}$ the vector field $(\hat{v}, 0)$ on the solid $A \times [0, 1]$ and by $\pi : A \times [0, 1] \to [0, 1]$ the obvious projection.

We have seen how a 2-moderately generic immersion $\alpha : X \to A$ produces a homotopy class in $[S^1, \mathcal{F}^{\leq 2}]$ or in $[(D^1, \partial D^1), (\mathcal{F}^{\leq 2}, \mathbf{1})]$.

On the other hand, generic (oriented) loops in $\beta : S^1 \to \mathcal{F}^{\leq 2}$ have an interpretation as finite collections $\mathcal{C}$ of smooth *embedded* closed curves (embedded "doodles") in the surface A. These curves have no tangency to the $\hat{v}$-trajectories $\hat{\gamma}$ of order that exceeds two. In particular, any *inflections* of the curves with respect to the $\hat{\gamma}$-trajectories are forbidden. Furthermore, a generic homotopy between the maps $\beta : S^1 \to \mathcal{F}^{\leq 2}$ corresponds to some cobordism like relation between the corresponding plane curves $\mathcal{C}$. The cobordism also avoids the forbidden tangencies of orders ≥ 3 to the standard foliation $\mathcal{F}(\hat{v}^{\bullet})$ on $A \times [0, 1]$.

To define this cobordism, let us spell out more accurately the 2-moderate genericity requirements on the collections of doodles in A:

$$\beta : \mathcal{C} \to A \tag{2.3}$$

- is a smooth immersion of a finite collection of (oriented) circles,
- no intersections of $\beta(\mathcal{C})$ of multiplicities ≥ 3 are permitted,
- the order of tangency between each branch of $\beta(\mathcal{C})$ and each $\hat{v}$-trajectory $\hat{\gamma}$ does not exceed 2.
- if a branch of $\beta(\mathcal{C})$ is quadratically tangent to a $\hat{v}$-trajectory at a point x, then no other branch contains x. ◇

Definition 2.8 Given two immersions $\beta_0 : \mathcal{C}_0 \to A$ and $\beta_1 : \mathcal{C}_1 \to A$ as in (2.3), we say that they are (orientably) 2-moderately cobordant, if there is a compact (oriented) surface S and a smooth immersion $B : S \to A \times [0, 1]$ such that:

(1) $\partial S = \mathcal{C}_0 \coprod -\mathcal{C}_1$,
(2) $B|_{\mathcal{C}_0 \coprod \mathcal{C}_1} = \beta_0 \coprod \beta_1$,
(3) the immersion $B| : S \to A \times [0, 1]$ is 2-to-1 at most,
(4) if two local branches of $B(S)$ intersect at a point $z = a \times \{t\}$, then each of them is transversal to the trajectory $\hat{\gamma} \times \{t\}$ through z,
(5) every $\hat{v}^{\bullet}$-trajectory $\hat{\gamma} \times \{t\} \subset A \times [0, 1]$ is tangent to each local branch of the surface $B(S)$ with the order of tangency that does not exceed 2,[2]
(6) the map $S \xrightarrow{B|} A \times [0, 1] \to [0, 1]$ is a Morse function with the regular values $\{0\}$ and $\{1\}$, ◇

We introduce the bordism set $\mathbf{C}^{\text{imm}}_{\text{moderate} \leq 2}(A)$ of immersions $\beta : \mathcal{C} \to A$ with 2-moderate tangencies to the foliation $\mathcal{F}(\hat{v})$. It is possible to verify that this cobordism between immersions is an *equivalence relation* (see [14, 15]).

Replacing "immersions" with "regular embeddings" in Definition 2.8, we get a modified notion of bordisms of (oriented) doodles. We denote the set of such bordisms by $\mathbf{C}^{\text{emb}}_{\text{moderate} \leq 2}(A)$. Depending on whether we consider the loops, forming $\mathcal{C}$,

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[2] In particular, the cubic tangencies are forbidden.

oriented or not, we get oriented versions $\mathbf{OC}^{\text{imm}}_{\text{moderate}\leq 2}(A)$ and $\mathbf{OC}^{\text{emb}}_{\text{moderate}\leq 2}(A)$ of the bordisms $\mathbf{C}^{\text{imm}}_{\text{moderate}\leq 2}(A)$ and $\mathbf{C}^{\text{emb}}_{\text{moderate}\leq 2}(A)$.

In the next definition, all surfaces and 3-folds that admit submersions in A or in $A \times [0, 1]$ automatically must be orientable, but not necessarily oriented.

Definition 2.9 Given two (oriented) submersions $\alpha_0 : X_0 \to A$ and $\alpha_1 : X_1 \to A$ with $\alpha_0|_{\partial X_0}$ and $\alpha_1|_{\partial X_1}$ as in (2.3), we say that they are 2-moderately cobordant, if there is a compact (oriented) 3-fold W with corners $\partial X_0 \coprod \partial X_1$ and a submersion $B : W \to A \times [0, 1]$ such that:

(1) $\partial W = X_0 \cup -X_1 \cup \delta W$, where $\delta W =_{\text{def}} \partial W \backslash \text{int}(X_0 \coprod X_1)$,

(2) $B|_{X_0 \coprod X_1} = \alpha_0 \coprod \alpha_1$,

(3) $B| : \delta W \to A \times [0, 1]$ has at most double self-intersections,

(4) if two local branches of $B(\delta W)$ intersect at a point $z = a \times \{t\} \in A \times [0, 1]$, then each of them is transversal to the trajectory $\hat{\gamma} \times \{t\}$ through z,

(5) every $\hat{v}^{\bullet}$-trajectory $\hat{\gamma} \times \{t\} \subset A \times [0, 1]$ is tangent to each local branch of the surface $B(\delta W)$ with the order of tangency ≤ 2,

(6) the map $\delta W \xrightarrow{B|} A \times [0, 1] \to [0, 1]$ is a Morse function with the regular values $\{0\}$ and $\{1\}$. $\diamond$

Thus, we can talk about the bordism set $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A)$ or $\mathbf{OB}^{\text{imm}}_{\text{moderate}\leq 2}(A)$ of immersions/embeddings $\alpha : X \to A$ with 2-moderate tangencies to the foliation $\mathcal{F}(\hat{v})$. It is possible to verify that this cobordism between immersions is an *equivalence relation* (see [15]).

If in Definition 2.9 we replace all the "immersions" with the "regular embeddings" and drop vacuous constraints (2) and (3) of the definition, we will get a similar version of (oriented) bordisms of regular embeddings. We denote them by $\mathbf{B}^{\text{emb}}_{\text{moderate}\leq 2}(A)$ and $\mathbf{OB}^{\text{emb}}_{\text{moderate}\leq 2}(A)$.

Note that, if in Definition 2.9 we drop all the constraints related to how the boundary of embedded blobs and of the solid embedded cobordisms interact with the $\hat{v}$-trajectories in A and with the $\hat{v}^{\bullet}$-trajectories in $A \times [0, 1]$, then the corresponding bordism groups $\mathbf{B}^{\text{emb}}(A)$ and $\mathbf{OB}^{\text{emb}}(A)$ of A are trivial. Indeed, by pushing (oriented) blobs $\alpha : X \hookrightarrow A$ inside $A \times [0, 1]$, while keeping their boundaries $\alpha(\partial X)$ fixed in $A \times \{0\}$, we create a (oriented) solid that delivers the cobordism between $\alpha(X)$ and the empty blob.

The following obvious maps are available:

$$\mathcal{A} : \mathbf{B}^{\text{emb}}_{\text{moderate}\leq 2}(A) \longrightarrow \mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A),$$

$$\mathcal{A} : \mathbf{OB}^{\text{emb}}_{\text{moderate}\leq 2}(A) \longrightarrow \mathbf{OB}^{\text{imm}}_{\text{moderate}\leq 2}(A), \quad (2.4)$$

$$\mathcal{A} : \mathbf{C}^{\text{emb}}_{\text{moderate}\leq 2}(A) \longrightarrow \mathbf{C}^{\text{imm}}_{\text{moderate}\leq 2}(A),$$

$$\mathcal{A} : \mathbf{OC}^{\text{emb}}_{\text{moderate}\leq 2}(A) \longrightarrow \mathbf{OC}^{\text{imm}}_{\text{moderate}\leq 2}(A). \quad (2.5)$$

Also, taking the boundaries of (oriented) blobs, we get the obvious maps:

$$\mathcal{B}^{\partial} : \mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A) \longrightarrow \mathbf{C}^{\text{imm}}_{\text{moderate}\leq 2}(A),$$

$$\mathcal{B}^{\partial} : \mathbf{OB}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A) \longrightarrow \mathbf{OC}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A), \qquad (2.6)$$

$$\mathcal{B}^{\partial} : \mathbf{B}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \longrightarrow \mathbf{C}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A),$$

$$\mathcal{B}^{\partial} : \mathbf{OB}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \longrightarrow \mathbf{OC}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A). \qquad (2.7)$$

The maps $\mathcal{B}^{\partial}$ in (2.7) are *bijections* (see Theorem 2.1 and [14, 15]).

Let us add one mild requirement to the list of the six properties in Definition 2.9:

- All the double intersections of $B| : \delta W \to A \times [0, 1]$ are transversal. (2.8)

Combining the six properties from Definition 2.9 with the property in (2.8), we could, at the first glance, get a more rigid notion of cobordisms of immersions. Let us denote them temporarily by $\mathbf{B}^{\times \mathrm{imm}}_{\mathrm{moderate} \leq 2}(A)$.

Thus, we have the obvious map

$$\mathcal{A}^{\times} : \mathbf{B}^{\times \mathrm{imm}}_{\mathrm{moderate} \leq 2}(A) \to \mathbf{B}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A).$$

$$\mathcal{A}^{\times} : \mathbf{OB}^{\times \mathrm{imm}}_{\mathrm{moderate} \leq 2}(A) \to \mathbf{OB}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A).$$

By the general position argument, we may isotope an immersion B via immersions so that all its self-intersections become transversal. Therefore, the maps $\mathcal{A}^{\times}$ are actually bijections [14].

In fact, the sets $\mathbf{B}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A)$, $\mathbf{B}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A)$ and $\mathbf{OB}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A)$, $\mathbf{OB}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A)$ have a group structure. The group operation $\star$ takes a pair of 2-moderate submersions, $\alpha_1 : X_1 \to A$ and $\alpha_2 : X_2 \to A$, to a new 2-moderate submersion $\alpha_1 \star \alpha_2 : X_1 \coprod X_2 \to A$ by placing the image of α_2 in $\mathbb{R} \times S^1$ or in $\mathbb{R} \times [0, 1]$ above the image of α_1 (see Fig. 5(c) and (d)). At the first glance, the operation $\star$ seems to be non-commutative. At least for $A = \mathbb{R} \times [0, 1]$, this first impression is wrong: by an appropriate isotopy, one can switch the vertical order of $\alpha_1(X_1)$ and $\alpha_2(X_2)$ without violating the requirements that all the immersions are 2-moderate.

Note that the cardinality of the fibers of the map $\theta \circ (\alpha_1 \star \alpha_2)$ is less than or equal to the *sum* of cardinalities of the fibers of $\theta \circ \alpha_1$ and of $\theta \circ \alpha_2$, where $\theta : A \to \mathcal{T}(\hat{v})$ is the obvious map.

In the case of $A = \mathbb{R} \times [0, 1]$, another *commutative* group structure is available in the sets $\mathbf{B}^{\mathrm{imm/emb}}_{\mathrm{moderate} \leq 2}(A)$ and $\mathbf{OB}^{\mathrm{imm/emb}}_{\mathrm{moderate} \leq 2}(A)$ (note that, for more general than the 2-moderate combinatorial constraints on the tangency patterns, similar groups may not be commutative [15]). It is induced by the operation

$$\uplus : \mathbf{B}^{\mathrm{imm/emb}}_{\mathrm{moderate} \leq 2}(A) \times \mathbf{B}^{\mathrm{imm/emb}}_{\mathrm{moderate} \leq 2}(A) \to \mathbf{B}^{\mathrm{imm/emb}}_{\mathrm{moderate} \leq 2}(A)$$

$$\uplus : \mathbf{OB}^{\mathrm{imm/emb}}_{\mathrm{moderate} \leq 2}(A) \times \mathbf{OB}^{\mathrm{imm/emb}}_{\mathrm{moderate} \leq 2}(A) \to \mathbf{OB}^{\mathrm{imm/emb}}_{\mathrm{moderate} \leq 2}(A)$$

that places the image of α_2 to the right of the image of α_1 along the segment $[0, 1]$; that is, $\alpha_1(X_1)$ is placed in the strip $\mathbb{R} \times (0, 0.5)$, while $\alpha_2(X_2)$ is placed in the strip $\mathbb{R} \times (0.5, 1)$ (see Fig. 2, lower diagram).

The zero element in $\mathbf{B}^{\text{imm/emb}}_{\text{moderate}\leq 2}(A)$ or in $\mathbf{OB}^{\text{imm/emb}}_{\text{moderate}\leq 2}(A)$ is represented by the empty surface X, and the minus of an immersion $\alpha : X \to A$ by a composition of α with a flip of A with respect to the trajectory $\mathbb{R} \times \{0.5\}$.

In contrast with the operation $\star$, the cardinality of the fibers of the map $\theta \circ (\alpha_1 \uplus \alpha_2)$ is less or equal to the *maximum* of cardinalities of the fibers of $\theta \circ \alpha_1$ and of $\theta \circ \alpha_2$. Due of this good feature of the operation $\uplus$, we choose to study its generalizations in [14, 15].

Lemma 2.2 *For* $A = \mathbb{R} \times [0, 1]$, *the operation* $\uplus$ *in* $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A)$ *or in* $\mathbf{OB}^{\text{imm}}_{\text{moderate}\leq 2}(A)$ *and in* $\mathbf{B}^{\text{emb}}_{\text{moderate}\leq 2}(A)$ *or in* $\mathbf{OB}^{\text{emb}}_{\text{moderate}\leq 2}(A)$ *is commutative.*

Proof By a parallel transport in A of the images $\alpha_1(X_1) \subset \mathbb{R} \times (0, 0.5)$ and $\alpha_2(X_2) \subset \mathbb{R} \times (0.5, 1)$, we can switch their order along $[0, 1]$. This can be done by sliding up $\alpha_2(X_2)$ so that it will reside in $(q, +\infty) \times (0.5, 1)$, then by sliding down $\alpha_1(X_1)$ so that it will reside in $(-\infty, -q) \times (0.5, 1)$. Here q is a positive number that exceeds the vertical size of both images. Then we slide horizontally the new $\alpha_2(X_2)$ and place it $(q, +\infty) \times (0, 0.5)$. Similarly, we slide horizontally the new $\alpha_1(X_1)$ and place it $(q, +\infty) \times (0.5, 1)$. Finally, we slide down $\alpha_1(X_1)$ and slide up $\alpha_2(X_2)$, thus completing the exchange.

Thanks to the nature of all these slides (parallel shifts), the maximal order of tangency of the $\hat{v}$-trajectories to the moving images $\alpha_1(\partial X_1)$ and $\alpha_2(\partial X_2)$ does not exceed 2.

Note that the maximal cardinality of the fibers of the projection $\alpha_1(\partial X_1) \coprod \alpha_2(\partial X_2) \to [0, 1]$ did increase in the exchange process. □

Let $\mathcal{F}^{=2} \subset \mathcal{F}^{\leq 2}$ denote the **discriminant hypersurface** in the space $\mathcal{F}^{\leq 2}$, formed by the functions from $\mathcal{F}^{\leq 2}$ with at least one zero of multiplicity 2.

The following proposition is similar to Theorem from [2]; however, our notion of cobordism of embedded blobs is different from the Arnold's more combinatorial notion of the cobordism of embedded doodles with no vertical inflection points.

Theorem 2.1 • *For* $A = \mathbb{R} \times [0, 1]$, *the construction* $\{\alpha \rightsquigarrow J_{z_\alpha}\}$ *in* (2.2), *where the immersion* $\alpha| : \partial X \to A$ *has only* 2-*moderate tangencies to the foliation* $\mathcal{F}(\hat{v})$, *delivers a group homomorphism*

$$J^{\text{imm}} : \mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A) \to \pi_1(\mathcal{F}^{\leq 2}, \mathbf{1}) \approx \mathbb{Z},$$

where the group addition in $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A)$ *is the operation* $\uplus$.
• *The same construction produces a group* isomorphism

$$J^{\text{emb}} : \mathbf{B}^{\text{emb}}_{\text{moderate}\leq 2}(A) \approx \pi_1(\mathcal{F}^{\leq 2}, \mathbf{1}) \approx \mathbb{Z}.$$

The inverse map $(J^{\text{emb}})^{-1}$ *is delivered by the correspondence*

$$K : \big\{\tau : ([0, 1], \partial[0, 1]) \to (\mathcal{F}^{\leq 2}, \mathbf{1})\big\} \implies \bigcup_{\theta \in [0,1]} \big(\tau(\theta)^{-1}\big((-\infty, 0])\big),\ \theta\big) \subset \mathbb{R}^1 \times [0, 1],$$

where τ is a continuous path in $\mathcal{F}^{\leq 2}$, transversal to the discriminant hypersurface $\mathcal{F}^{=2}$.

• *For* $A = \mathbb{R} \times S^1$, *the construction* $\{\alpha \rightsquigarrow J_{z_\alpha}\}$, *where an embedding* $\alpha : X \to A$ *has only* 2-*moderate tangencies, delivers a* 1-*to*-1 *map*

$$J^{\mathrm{emb}} : \mathbf{B}^{\mathrm{emb}}_{\mathrm{moderate}\leq 2}(A) \approx [S^1, \mathcal{F}^{\leq 2}].$$

The inverse map $(J^{\mathrm{emb}})^{-1}$ *is induced by the correspondence*

$$K : \left\{ \tau : S^1 \to \mathcal{F}^{\leq 2} \right\} \; \Rightarrow \; \bigcup_{\theta \in S^1} \left(\tau(\theta)^{-1}\big((-\infty, 0]\big) \right), \, \theta) \subset \mathbb{R}^1 \times S^1.$$

Proof The validation of this theorem is inspired by the *graphic calculus* in [2] that converts homotopies of loops in the functional space $\mathcal{F}_{\leq 2}$ into cobordisms (surgeries) of doodles in A with 2-moderate tangencies to the foliation $\mathcal{F}(\hat{v})$. With the help of this calculus, the isomorphism $\pi_1(\mathcal{F}^{\leq 2}, \mathbf{1}) \approx \mathbb{Z}$ is established [2]. In the present case, the loops are images of boundaries of oriented compact surfaces under their immersions/embeddings in A. Fig. 6 shows our modification of this graphic calculus in action. A generator of $\pi_1(\mathcal{F}_{\leq 2}, \mathbf{1}) \approx \mathbb{Z}$ is depicted in Fig. 5(b).

If two 2-moderate immersions, $\alpha_0 : X_0 \to A$ and $\alpha_1 : X_1 \to A$, are cobordant with the help of a 2-moderate immersion $B : W \to A \times [0, 1]$ as in Definition 2.9, extending the function $z_{\alpha_0} \coprod z_{\alpha_1} : A \times (\{0\} \coprod \{1\}) \to \mathbb{R}$ to a smooth function $Z : A \times [0, 1] \to \mathbb{R}$ with similar properties delivers a homotopy between the loops $J_{z_{\alpha_0}}$ and $J_{z_{\alpha_1}}$ (see (2.2)) in $\mathcal{F}_{\leq 2}$.

On the other hand, if a path $\tau : ([0, 1], \partial[0, 1]) \to (\mathcal{F}^{\leq 2}, \mathbf{1})$ is transversal to the discriminant hypersurface $\mathcal{F}^{=2} \subset \mathcal{F}^{\leq 2}$, then the locus

$$K(\tau) =_{\mathrm{def}} \bigcup_{\theta \in [0,1]} \left(\tau(\theta)^{-1}(0), \, \theta \right) \subset \mathbb{R}^1 \times [0, 1]$$

is a smooth embedded curve in A (see [16], Lemma 3.4, for validation of this claim and its generalizations). Since the set $f^{-1}((-\infty, 0])$ is compact for any $f \in \mathcal{F}_{\leq 2}$, the curve $K(\tau)$ is the boundary of a compact surface

$$X(\tau) =_{\mathrm{def}} \bigcup_{\theta \in [0,1]} \left(\tau(\theta)^{-1}\big((-\infty, 0]\big) \right), \, \theta) \subset A.$$

Similar arguments hold for a homotopy

$$B : \big([0, 1], \, \partial[0, 1]\big) \times [0, 1] \to (\mathcal{F}^{\leq 2}, \mathbf{1})$$

between two such maps τ_0 and τ_1, where B is transversal to $\mathcal{F}^{=2}$. The 3-fold

$$W(B) =_{\mathrm{def}} \bigcup_{\theta \in [0,1], \, t \in [0,1]} \left(B(\theta)^{-1}\big((-\infty, 0]\big) \right), \, \theta, \, t) \subset A \times [0, 1]$$

delivers the cobordism (as in Definition 2.9) between $X(\tau_0)$ and $X(\tau_1)$. Therefore,

$$J^{\mathrm{emb}} : \mathbf{B}^{\mathrm{emb}}_{\mathrm{moderate}\leq 2}(A) \approx \pi_1(\mathcal{F}^{\leq 2}, \mathbf{1})$$

is a bijection (actually, a group isomorphism with respect to the operation $\uplus$).

This fact has a curious implication for the map $\mathcal{A}$ from (2.6): since the map $J^{\mathrm{imm}} : \mathbf{B}^{\mathrm{imm}}_{\mathrm{moderate}\leq 2}(A) \to \pi_1(\mathcal{F}^{\leq 2}, \mathbf{1})$ is available, we can compose it with the inverse of the bijection J^{emb} to get a surjective map

$$\mathcal{R}_J : \mathbf{B}^{\mathrm{imm}}_{\mathrm{moderate}\leq 2}(A) \to \mathbf{B}^{\mathrm{emb}}_{\mathrm{moderate}\leq 2}(A) \approx \mathbb{Z}, \tag{2.9}$$

which serves as the right inverse of the map (homomorphism) $\mathcal{A}$; that is, $\mathcal{R}_J \circ \mathcal{A} = \mathrm{id}$. In fact, for $A = \mathbb{R} \times [0, 1]$, $\mathcal{R}_J$ is a group epimorphism.

The case $A = \mathbb{R} \times S^1$ is similar, but the operation $\uplus$ is not available. Instead, the operation $\star$ is available, but its commutativity is in question. □

We orient the surface A so that the the θ-coordinate, corresponding to the trajectory space $\mathcal{T}(\hat{v}) = [0, 1]$ or S^1, is the first, and the u-coordinate, corresponding to the multiplier $\mathbb{R}$, is the second. With this counterclockwise orientation of A being fixed, any immersion $\alpha : X \to A$ induces an orientation of the surface X, thus choosing orientations of each component of ∂X. However, this induced orientation, may differ from the original orientation of X, an ingredient in the definition of $\mathbf{OB}^{\mathrm{imm}}_{\mathrm{moderate}\leq 2}(A)$!

Given an immersion $\alpha : (X, v) \subset (A, \hat{v})$ such that $\alpha(\partial X)$ has the properties as in (2.3), we attach a new α-dependent polarity to each "concave" point $a \in \partial_2^+ X(v)$: by definition, the polarity of a is "$\oplus$" if $\alpha_*(v_a)$, where v_a is the inner normal to ∂X at a, points in the direction of the coordinate θ. Otherwise, the new polarity of a is defined to be "$\ominus$" (see Fig. 5). Equivalently, a is of polarity "$\oplus$" if crossing the critical value $\theta(a)$ in the positive direction increases the cardinality of the fiber of the map $\theta : \alpha(\partial X) \to \mathcal{T}(\hat{v})$ in the vicinity of a.

As a result, the loci $\{\partial_j^+ X(v)\}_j$ and $\{\partial_j^- X(v)\}_j$ acquire the additional polarities $\oplus$ and $\ominus$; all together, four flavors for the tangencies of $\alpha(\partial X)$ to $\mathcal{F}(\hat{v})$ are available: $\{(+, \oplus), (+, \ominus), (-, \oplus), (-, \ominus)\}$. These flavors are independent of the orientations of X.

When dealing with *oriented* 2-moderate blobs and doodles, it will be useful to introduce still another polarity (orientation) of tangency points, which will be denoted "$\uparrow$" and "$\downarrow$". Each point $a \in \partial_2^+ X(v)$ has polarity "$\uparrow$" if the vector field $\hat{v}(a)$ is consistent with the orientation of $\alpha(\partial X)$ at a. Otherwise, the polarity of a is "$\downarrow$". Similar definition is applied to the points of the locus $\partial_2^- X(v)$. As a result, each point of the locus $\partial_2 X(v)$, with the help of α, acquires the following eight flavors:

$$\{(+, \oplus, \uparrow), (+, \ominus, \uparrow), (-, \oplus, \uparrow), (-, \ominus, \uparrow), (+, \oplus, \downarrow), (+, \ominus, \downarrow), (-, \oplus, \downarrow), (-, \ominus, \downarrow)\}.$$

Four of these eight flavors will play an essential role:

$$\Big\{ \underbrace{(+, \oplus, \uparrow), (+, \ominus, \downarrow)}_{clockwise}, \underbrace{(+, \oplus, \downarrow), (+, \ominus, \uparrow)}_{counterclockwise} \Big\}, \tag{2.10}$$

the first pair occurring in the blobs oriented clockwise, the second pair in the blobs oriented counterclockwise.

Thus, with any embedded *oriented* surface $\alpha : X \hookrightarrow A$, we associate two integers

$$\mathsf{K}^\alpha =_{\text{def}} \#\{\partial_2^{+,\,\oplus,\,\downarrow} X(\hat{v})\} - \#\{\partial_2^{+,\,\ominus,\,\uparrow} X(\hat{v})\},$$
$$\mathsf{L}^\alpha =_{\text{def}} \#\{\partial_2^{+,\,\oplus,\,\uparrow} X(\hat{v})\} - \#\{\partial_2^{+,\,\ominus,\,\downarrow} X(\hat{v})\} \qquad (2.11)$$

that correspond to the blobs that are oriented counterclockwise and clockwise, respectively.

We introduce also an integer

$$\mathsf{J}^\alpha =_{\text{def}} \#\{\partial_2^{+,\,\oplus} X(\hat{v})\} - \#\{\partial_2^{+,\,\ominus} X(\hat{v})\}$$
$$\text{(in the oriented case, } \mathsf{J}^\alpha = \mathsf{K}^\alpha + \mathsf{L}^\alpha). \qquad (2.12)$$

As the next theorem testifies, this number plays a role as a bordism invariant of immersions of non-oriented blobs and as a tool for characterizing elements of the fundamental group $\pi_1(\mathcal{F}^{\leq 2}, \mathbf{1})$.

Theorem 2.2 *Let $A = \mathbb{R} \times [0, 1]$.*

• *Any regular embedding $\alpha : (X, v) \to (A, \hat{v})$ with only 2-moderate tangencies of $\alpha(\partial X)$ to $\mathcal{F}(\hat{v})$ produces a map J_{z_α} as in Theorem 2.1. Its homotopy class $[J_{z_\alpha}] = \mathsf{J}^\alpha \cdot \kappa$, where κ denotes a generator of $\pi_1(\mathcal{F}^{\leq 2}, \mathbf{1}) \approx \mathbb{Z}$ (shown in Fig. 5(b)), and $\mathsf{J}^\alpha \in \mathbb{Z}$.*

The integer J^α can be computed by the formula (2.12).

• *Any regular embedding $\alpha : (X, v) \to (A, \hat{v})$ of an* oriented *surface X with only 2-moderate tangencies produces two maps*

$$K_{z_\alpha} : ([0, 1], \partial[0, 1]) \to (\mathcal{F}^{\leq 2}, \mathbf{1}) \text{ and } L_{z_\alpha} : ([0, 1], \partial[0, 1]) \to (\mathcal{F}^{\leq 2}, \mathbf{1})$$

as in Theorem 2.1; the first map is generated by the counterclockwise oriented blobs, and the second one by the clockwise oriented blobs. The homotopy classes of these maps are: $[K_{z_\alpha}] = \mathsf{K}^\alpha \cdot \kappa$ and $[L_{z_\alpha}] = \mathsf{L}^\alpha \cdot \kappa$.

The integers K^α and L^α can be computed by the formula (2.11).

• *Moreover, K^α and L^α deliver an isomorphism*

$$\mathsf{K} \times \mathsf{L} : \mathbf{OB}_{\text{moderate}\leq 2}^{\text{emb}}(A) \xrightarrow{\approx} \mathbb{Z} \times \mathbb{Z}. \qquad (2.13)$$

Proof Let $d =_{\text{def}} \max_{\hat{\gamma}} \#\{\hat{\gamma} \cap \alpha(\partial X)\}$ be the maximal cardinality of the intersections of the $\hat{v}$-trajectories $\hat{\gamma}$ with the loop pattern $\alpha(\partial X)$. Since X bounds ∂X, d must be even.

We start with the case of non-oriented blobs.

For any 2-moderate immersion $\alpha : X \to A$, we pick an auxiliary function $z_\alpha : A \to \mathbb{R}$, adjusted to α as in (2.1). By the previous arguments, this choice of z_α produces the relative loop $J_{z_\alpha} : ([0, 1], \partial[0, 1]) \to (F_d^{\leq 2}, \mathbf{1})$.

Consider the restriction $B^\delta =_{\text{def}} B| : \delta W \to A \times [0, 1]$, followed by the obvious projection $\phi : A \times [0, 1] \to [0, 1]$. By an arbitrary C^∞-small perturbation of B, we
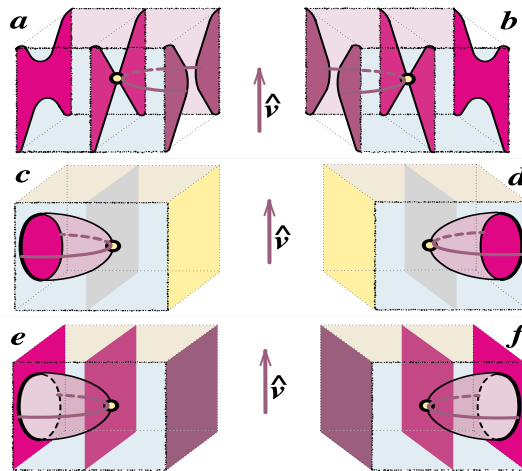
Fig. 3 Changing topology of slices $B^{-1}(A \times \{t\})$, as t crosses a critical value $t_\star$ of the Morse function $f : \delta W \to [0, 1]$. Different shades correspond to different slices; each box is shown with 3 slices. In (**a**) and (**b**), the portion of $B(W)$ over a small interval $[t_\star - \epsilon, t_\star + \epsilon]$ is a pair of solid pants. In (**c**), (**d**), this portion is a solid half-ball. In (**e**) and (**f**), it is the complements to such half-balls in the solid cube. The figure does not show the complements to solid pants, depicted in (**a**) and (**b**). Note the "parabolic locus" (an arc of which is dashed) in $B(\delta W)$, where the vector field $\hat{v}$ is quadratically tangent to the surface $B(\delta W)$

may assume that the composition $f =_{\text{def}} \phi \circ B^\delta$ is a Morse function. Since the space $\mathcal{F}^{\leq 2}$ is open in $\mathcal{F}$, compactly supported perturbations of smooth 2-moderate functions or of their families remain 2-moderate. Therefore, we may assume that the cobordism B is such that $\phi \circ B^\delta$ is a Morse function.

Since, by property (6) from Definition 2.9, $B : W \to A \times [0, 1]$ is a submersion and $f : \delta W \xrightarrow{B^\delta} A \times [0, 1] \to [0, 1]$ is a Morse function, as t ranges in $[0, 1]$, the topology of a regular slice $B^{-1}(A \times \{t\}) \subset W$ may change via a relative, elementary surgery only when t crosses a critical value $t_\star$ of f. The topology of the slice $B^{-1}(A \times \{t\}) \cap \delta W$ changes via an elementary surgery inside δW as t crosses $t_\star$. Thus, there are four types of local surgery, shown in Fig. 3: two types of solid pants that correspond to f-critical points $(x_\star, t_\star) \in B(\delta W)$ of the Morse index 1 and two types of "indented" solids that correspond to f-critical points $(x_\star, t_\star) \in B(\delta W)$ of indexes 0 and 2.

It is crucial for our arguments that the geometry of the elementary surgery blocks $W \subset A \times [0, 1]$ is such that the surface $\delta W \subset \partial W$ has no tangency to the $\hat{v}^\bullet$-trajectories of order 3 or higher (see the parabolic tangency curves with dashed arcs from Fig. 3).

Let us now describe an algorithm for the elementary moves (surgery) (see Fig. 6) that reduces a given pattern $X_0 = J_{z_\alpha}^{-1}((-\infty, 0]) \subset A$ to a pattern from *the canonical set of patterns* $\{n \cdot K\}_{n \in \mathbb{Z}}$ (as in Fig. 5) by a cobordism $B : W \to A \times [0, 1]$ as in Definition 2.9, a cobordism which is a regular embedding.

By a small perturbation of α_0, we may assume that no $\hat{v}$-trajectory has a combinatorial tangency pattern ω with two or more 2's. Then, we select the $\hat{v}$-trajectories $\hat{\gamma}_1, \ldots \hat{\gamma}_s$ that contain the tangency patterns $\omega = (1, \ldots, 1, 2, 1, \ldots, 1)$, where the number of 1's that precede 2 is *odd* (which implies that the unique quadratic tangency

Fig. 4 Eliminating a pair of dark-shaded "kidneys" with their horns facing each other by a surgery: the first surgery transforms the kidneys in the first slice into a ring in the second slice, and then the second surgery transforms the ring into an empty slice
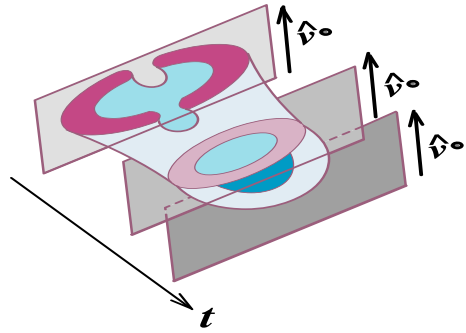
point on $\hat{\gamma}_k$ resides in the *interior* of the set $\hat{\gamma}_k \cap \alpha_0(X_0)$). These trajectories correspond exactly to the "concave" points of $\partial_2^+(\alpha_0(X), \hat{v})$. The trajectories are listed by the order of their images $[\hat{\gamma}_1], \ldots [\hat{\gamma}_s]$ in the oriented trajectory space $\mathcal{T}(\hat{v})$. We notice that such trajectories $\hat{\gamma}_k$ come in two flavors: "⊕" and "⊖", depending on the polarity of the tangency of $\hat{\gamma}_k$ to $\alpha_0(\partial X)$.

Then, for each adjacent pair $\hat{\gamma}_k, \hat{\gamma}_{k+1}$, we choose a $\hat{v}$-trajectory $\hat{\gamma}_k^\star$ in-between $\hat{\gamma}_k$ and $\hat{\gamma}_{k+1}$. Such $\hat{\gamma}_k^\star$ is traversal to $\alpha_0(X_0)$ and its combinatorial type $\omega(\hat{\gamma}_k^\star)$ is a sequence of 1's of a length $2q$. The intersection $\alpha_0(X_0) \cap \hat{\gamma}_k^\star$ is a disjoint union of closed intervals $I_{k,1}, \ldots, I_{k,q}$. In the interior of each interval $I_{k,j}$ we pick a point $p_{k,j}^\star$ and will use it as the critical point for a surgery in $A \times [0, 1]$ on the solid $W_0 = \alpha_0(X_0) \times [0, 0.5)$ (see Fig. 3(a) and (b)). If the polarity of $\hat{\gamma}_k$ is ⊕, we perform a surgery on W_0 as in Fig. 3(a), if the polarity of $\hat{\gamma}_k$ is ⊖, we perform a surgery on W_0 as in Fig. 3(b), by attaching solid pants Z_0 to W_0. We denote by W_1 the resulting solid $W_0 \cup Z_0$ in $A \times [0, 1]$ and by $F_1 : W_1 \to [0, 1]$ the obvious projection.

It is essential that, since F_1 is a Morse function, the attached pants Z_0 do not violate property (5) from Definition 2.9. Note that each elementary surgery of type (a) introduces a pair of new $\hat{v}^\bullet$-tangent points to the slice $\partial(F_1^{-1}(0.5))$; fortunately, none of new tangency pair is of the type $\omega = (1, \ldots, 1, 2, 1, \ldots, 1)$, where the number of 1's that precede 2 is odd (i.e., the pair belongs to $\partial_2^-(F_1^{-1}(0.5))$) .

Now, the preimage $F_1^{-1}(0.5)$ consists of several connected components $U_1, \ldots U_s$, each of which is a domain in $A \times \{0.5\}$ which contains at most a single trajectory of the combinatorial type $(\ldots, 1, 2, 1, \ldots)$, where the number of 1's that precede 2 is odd and whose polarity is either ⊕ or ⊖.

If a connected component U is a ball with several holes, then applying the previous 1-surgery to a segment of trajectory in U that connects different connected components of ∂U, we will convert U into a topological 2-ball without introducing new trajectories of the combinatorial type $(\ldots, 1, 2, 1, \ldots)$ with the number of 1s that precede 2 being odd. Therefore, we may assume that all the connected components U are 2-balls.

Thus, we divide the balls U in three types: (i) U intersects with a single trajectory of the polarity ⊕, (ii) U intersects with a single trajectory of the polarity ⊖, and (iii) U does not intersect with trajectories of the combinatorial type $(\ldots, 1, 2, 1, \ldots)$, where the number of 1's that precede 2 is odd.

Next, we eliminate the type (iii) $\hat{v}^\bullet$-convex 2-balls by a 3-surgery on W_1 that amounts to attaching a few 3-balls to the slice $F_1^{-1}(0.5)$ as in Fig. 3(c). We denote by W_2 the

resulting solid in $A \times [0, 1]$, and by $F_2 : W_2 \to [0, 1]$ the obvious projection. We may assume that the F_2-image of W_2 in $[0, 1]$ resides in the interval $[0, 0.7)$. Because the balls of type (iii) are $\hat{v}^{\bullet}$-convex, we may assume that $\delta W_2 \backslash \delta W_1$ avoids $\hat{v}^{\bullet}$-tangencies of order ≥ 3. In particular, property (5) from Definition 2.9 is respected by these surgeries.

We are left now with balls U in $F_2^{-1}(0.7)$ of types (i) and (ii) (so called, Arnold's "kidneys"). Each pair of 2-balls of the distinct types (i) and (ii) can be eliminated by an isotopy in $A \times \{0.7\}$ (which involves scaling and parallel shifts), followed by a surgery as shown in Fig. 4. Indeed, if the horns of a pair of kidneys U and U' of different polarities $\oplus$ and $\ominus$ are facing each other, then by attaching two solid pants we may surgery their union into a ring in a new slice. Then by attaching a relative handle $(D_+^2 \times S^1, D^1 \times S^1)$ to $A \times \{0\}$ in $A \times [0, 1]$, we eliminate the ring. If the horns of U and U' of different polarities are facing in opposite directions, then by an isotopy in A (as in Lemma 2.2) we may switch their order along the $\mathcal{T}(\hat{v})$-direction. The switching will reduce the situation to the case when the horns face each other. Again, this surgery will not introduce new points of the combinatorial types $(\dots 121 \dots)$, where the number of 1's that precede 2 is odd. Thus, we will be left with a disjoint union of kidneys of the *same* polarity $\oplus$ or $\ominus$.

Let $\alpha : D^2 \to A$ be the regular imbedding realizing a $\oplus$-polarized kidney. By [1], under the map J^{α}, each kidney is mapped to a non-contractible loop Φ_{α} in $\mathcal{F}^{\leq 2}$, a generator of $\pi_1(\mathcal{F}^{\leq 2}, \mathbf{1}) \approx \mathbb{Z}$. Therefore, no further simplifications among the remaining kidneys are possible.

Thus, the difference between the numbers of kidneys of types (i) and (ii) is an invariant of the bordism class, introduced in Definition 2.9. This difference equals to the original difference $\#\{\partial_2^{+,\oplus} X(v)\} - \#\{\partial_2^{+,\ominus} X(v)\}$ between the numbers of $\hat{v}$-trajectories with polarities $\oplus$ and $\ominus$ and was preserved under all the surgeries that led to the final slice with the kidneys of the same polarity. Thus, the number $J^{\alpha} = \#\{\partial_2^{+,\oplus} X(v)\} - \#\{\partial_2^{+,\ominus} X(v)\}$ did not change under all the surgeries above.

Now we are ready to investigate the 2-moderate bordisms $\mathbf{OB}_{\text{moderate} \leq 2}^{\text{emb}}(A)$ of embedded *oriented* blobs. Each connected blob can be oriented counterclockwise, i.e., coherently with the fixed orientation of the ambient A, or clockwise, i.e., opposite to the fixed orientation of A. We aim to show that the numbers $\mathsf{K}^{\alpha}, \mathsf{L}^{\alpha}$ from (2.11) define an isomorphism (2.13).

It is possible to attach *embedded/immersed* 1-handles in A only to *similarly oriented* blobs: the "twisted 1-handles" cannot be immersed in A.

We notice (see Fig. 6, the lower diagram) that attaching a narrow 1-handle, whose core is transversal to the field $\hat{v}$, to similarly oriented blobs contributes a new pair of points of the polarities $(+, \oplus, \downarrow)$ and $(+, \ominus, \uparrow)$ (the counterclockwise oriented blobs), or a new pair of points of the polarities $(+, \oplus, \uparrow)$ and $(+, \ominus, \downarrow)$ (the clockwise oriented blobs). Deleting a 1-handle contributes a new pair of points of the polarities $(-, \ominus, \uparrow)$ and $(-, \oplus, \downarrow)$ (the counterclockwise oriented blobs), or $(-, \ominus, \downarrow)$ and $(-, \oplus, \uparrow)$ (the clockwise oriented blobs); however, these changes affect only the convex loci $\partial_2^-(\sim)$ which play secondary roles. Thus, all these surgeries do not change the values $\mathsf{K}^{\alpha}, \mathsf{L}^{\alpha}$. Similarly, adding or deleting a 2-ball D such that $\partial_2^+ D(\hat{v}) = \emptyset$, does not affect the values of $\mathsf{K}^{\alpha}, \mathsf{L}^{\alpha}$.
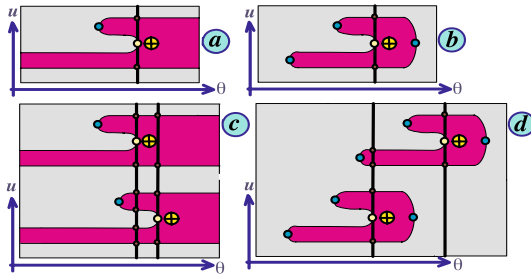
Fig. 5 Two portraits of a generator $\kappa \in \pi_1(\mathcal{F}^{\leq 2}, p)$, where the base point $p \in \mathcal{F}^{\leq 2}$ is modeled after the polynomial $p(u) = u^4 - 1$ in diagram (**a**) and by $p(u) = u^4 + 1$ in diagram (**b**). Diagrams (**c**) and (**d**) portray 2κ. In diagrams (**a**) and (**c**) that represent the case $\mathcal{T}(\hat{v}) = S^1$, the left and the right edges of the rectangle should be identified so that the shaded regions match. Note the polarity $\oplus$ of the tangent $\hat{v}$-trajectories with the combinatorial pattern $\omega = (\ldots 121 \ldots)$, where the number of 1's that precede 2 is odd

Thus, we will treat the counterclockwise and clockwise oriented blobs separately, recycling the our arguments in the non-oriented case. Each embedded connected blob $\alpha(X_0)$ is a disk or a disk with a number of holes. By deleting narrow 1-handles from $\alpha(X_0)$ as in Fig. 3, (a), or as in Fig. 6, we replace, via 2-moderate bordisms, each counterclockwise/clockwise oriented blob with holes by a similarly oriented topological 2-ball D_0. In the process, we keep the original values K^α, L^α. The tangency of ∂D_0 to $\hat{v}$ is still 2-moderate.

Now all the connected components of the new embedded surface $Y \subset A$ are 2-balls. As in the non-oriented case, by deleting 1-handles, each ball can be split into a number of balls, each of which has a single point from $\partial_2^+(\sim)$ at most. By a 2-surgery, we delete the balls which are convex with respect to $\hat{v}$ (i.e., they do not have singletons from $\partial_2^+(\sim)$). Now we a left with the balls that have a single point of the polarity from the list (2.10). Some of them are oriented counterclockwise, others clockwise.

Thus,

$$\mathsf{K} \times \mathsf{L} : \mathbf{OB}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \longrightarrow \mathbb{Z} \times \mathbb{Z}. \tag{2.14}$$

is an epimorphism.

Assuming that $\mathsf{K}^\alpha = 0 = \mathsf{L}^\alpha$ for some α, we see that this property allows to pair all the counterclockwise/clockwise oriented kidneys so that each pair by 1-surgery can be transformed into a ring and then eliminated as in Fig. 4. Therefore, $\mathsf{K} \times \mathsf{L}$ is a monomorphism. $\square$

As in the case of non-oriented blobs, using formulas (2.11), we get a surjective map

$$\mathcal{R}_{\mathsf{K},\mathsf{L}} : \mathbf{OB}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A) \rightarrow \mathbb{Z} \times \mathbb{Z} \approx \mathbf{OB}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A), \tag{2.15}$$

which serves as the right inverse to the obvious homomorphism

$$\mathbf{OB}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \rightarrow \mathbf{OB}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A).$$

Fig. 6 The region $\alpha(X)$ in the cylinder $A = \mathbb{R} \times S^1$ is shaded, the vector field $\hat{v}$ is vertical. Elementary cancellations, via surgery in A, of a pair tangencies of $\alpha(\partial X)$ to $\hat{v}$ of opposite polarities $\oplus$, $\ominus$ (the upper diagrams). Increasing the local connectivity of the region $\alpha(X)$ between two tangent trajectories of opposite polarities by a 1-surgery in A (the lower diagram). This operation introduces a new pair of points of opposite polarities $\oplus$, $\ominus$
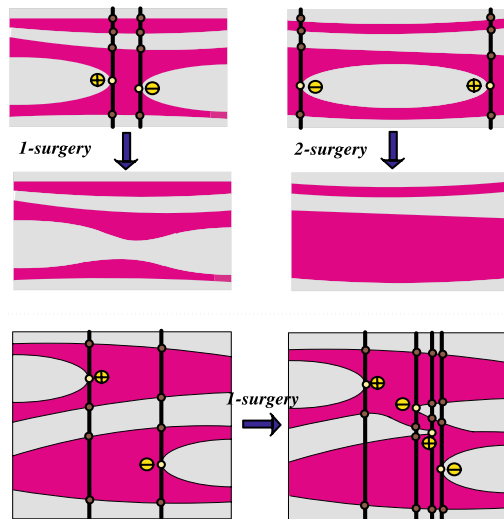
Therefore, $\mathbf{OB}^{\mathrm{imm}}_{\mathrm{moderate}\leq 2}(A)$ contains canonically $\mathbb{Z} \times \mathbb{Z}$.

Remark 2.1. The number $c^+(v) =_{\mathrm{def}} \#\{\partial_2^{+,\oplus}(v)\} + \#\{\partial_2^{+,\ominus}(v)\}$ may be interpreted as the *complexity* of the vector field v on X [12, 15]. We notice that

$$c^+(v) \geq \quad |\mathsf{J}^\alpha| =_{\mathrm{def}} |\#\{\partial_2^{+,\oplus}(v)\} - \#\{\partial_2^{+,\ominus}(v)\}|.$$

It is somewhat surprising that the invariant $\mathsf{J}^\alpha = \#\{\partial_2^{+,\oplus}(v)\} - \#\{\partial_2^{+,\ominus}(v)\}$ reflects more the local topology of the embedding α (or of the field $v = \alpha^*(\hat{v})$) than the global topology of the surface X: in fact, any integral value of J^α can be realized by a traversally generic field v on a 2-ball D which even admits a convex envelop! A portion of the boundary ∂D looks like a snake with respect to the field $\hat{v}$ of the envelop.

For any X, the effect of deforming a portion of the boundary $\partial X \subset A$ into a snake is equivalent to adding several spikes (an edge and a pair of a univalent and a trivalent vertices) to the graph $\mathcal{T}(v)$, the space of v-trajectories. Evidently, these operations do not affect $H_1(\mathcal{T}(v); \mathbb{Z}) \approx H_1(X; \mathbb{Z})$.

For example, for α as in Fig. 2, $\mathsf{J}^\alpha = 0$. If we subject α to an isotopy in A that introduces a snake-like pattern of Fig. 5(a), then for the new immersion α', the invariant $\mathsf{J}^{\alpha'} = 1$.

In contrast, the number $\#\{\partial_2^{+,\oplus}(v)\} + \#\{\partial_2^{+,\ominus}(v)\}$ has a topological significance for X. If X is the compliment to k disjoint balls in a closed orientable surface with g handles, then by [15], Lemma 1.1, $\#\{\partial_2^{+,\oplus}(v)\} + \#\{\partial_2^{+,\ominus}(v)\} \geq 4\,g - 4 + 2k$. $\diamond$

3 Invariants that Distinguish Between Bordisms of Immersions and Embeddings

Recall that $\mathcal{A}^\times : \mathbf{B}^{\times\mathrm{imm}}_{\mathrm{moderate}\leq 2}(A) \rightarrow \mathbf{B}^{\mathrm{imm}}_{\mathrm{moderate}\leq 2}(A)$ is a bijection. Therefore, without lost of generality, we deal now with immersions under additional assumption (2.8):
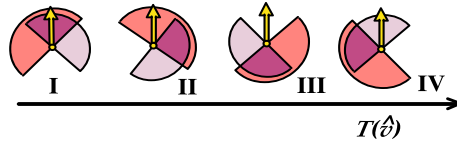
Fig. 7 Four configurations **I, II, III, IV** in the vicinity of a point $a \in A$ where two branches of $\alpha(\partial X)$ intersect transversally. The figure shows the α-images of neighborhoods in X of the two points $b_1, b_2 \in \alpha^{-1}(a) \cap \partial X$. Note the position of the vector $\hat{v}(a)$ relative to the darkly shaded sector. The trajectory space $\mathcal{T}(\hat{v})$ is oriented

we require that their self-intersections are transversal. We adopt the notations from Definition 2.9.

Consider an immersion $\alpha : X \to A$ with transversal self-intersections of $\alpha(\partial X)$ and no multiple self-intersections of more than two local branches. At each self-intersection $a \in \alpha(\partial X)$, there are exactly two points $a', a'' \in \partial X$ such that $\alpha(a') = \alpha(a'') = a$. Then we take two small neighborhoods $D_+(a')$ and $D_+(a'')$ of a' and a'', respectively, both diffeomorphic to a half-disk. The linearizations of their α-images produces two preferred half-spaces, $H_{a'}$ and $H_{a''}$, in the tangent space $T_a A$. Put $K_{a',a''} = H_{a'} \cap H_{a''}$. We denote by $K_{a',a''}^{\vee}$ the image of $K_{a',a''}$ under the central symmetry in $T_a A$ (see Fig. 7).

Using the sectors $K_{a',a''}$ and $K_{a',a''}^{\vee}$, we will divide the self-intersections $\{a\}$ into four types: type **I** occurs when the vector $\hat{v}(a)$ points in the interior of the sector $K_{a',a''}$, type **III** occurs when the vector $\hat{v}(a)$ points in the interior of the sector $K_{a',a''}^{\vee}$; type **II** and type **IV** arise when $\hat{v}(a)$ points in the interior of the complimentary to $K_{a',a''} \cup K_{a',a''}^{\vee}$ sectors. Type **II** corresponds to the case when the sector $K_{a',a''}$ is on the right of the line γ_a, while type **IV** corresponds to the case when the sector $K_{a',a''}$ is on the left of the line γ_a. Note that the types **II** and **IV** are "crudely symmetric" with respect to a reflection in A that has the $\hat{v}$-trajectory through a as a line of symmetry γ_a.

Consider a small disk D, centered on a crossing point a. We consider a diffeotopy $\{\Psi_\phi : A \to A\}_\phi$ whose final stage is the turn on a given angle $\phi_\star$ inside the concentric disk $D' \subset D$ and the identity outside D. Note that using Ψ_ϕ as a dial at a, we can change the type $\{$**I, II, III, IV**$\}$ of $\Psi_{\phi_\star}(\alpha(\partial X))$ at a at will. However, $\alpha_{\phi_\star}$, the composition of α with $\Psi_{\phi_\star}$, is not 2-moderately cobordant to α: in the process, the curves $\alpha_\phi(\partial X)$ may develop inflection points (cubic tangencies) to the foliation $\mathcal{F}(\hat{v})$. Moreover, the 2-moderate $\alpha_{\phi_\star}(\partial X)$ may develop inside D new quadratic tangencies to $\mathcal{F}(\hat{v})$. Thus, there is a subtle interplay between the count of crossings of a particular type from $\{$**I, II, III, IV**$\}$ and the invariants of the type J^α.

Let us find out which of these four types $\{$**I, II, III, IV**$\}$ or their combinations are invariant under non-oriented cobordisms of immersions with 2-moderate tangencies.

We start with a key observation that guides us in this section.

Given a cobordism-immersion $B : W \to A \times [0, 1]$ between two immersions, $\alpha_0 : X_0 \to A \times \{0\}$ and $\alpha_1 : X_1 \to A \times \{1\}$, with transversal self-intersection of $B(\delta W)$ of multiplicity 2 at most, consider the curve ℓ where the surface $B(\delta W) \subset A \times [0, 1]$ self-intersects transversally. Either ℓ is:

(a) a simple segment in $A \times [0, 1]$ that connects a pair of crossings of $\alpha_0(\partial X_0)$, or

(b) a pair of crossings of $\alpha_1(\partial X_1)$, or

(c) a pair of crossings, one of which belongs to $\alpha_0(\partial X_0)$ and the other one to $\alpha_1(\partial X_1)$, or

(d) a simple loop in $A \times (0, 1)$.

The multiplicity of each intersection of ℓ with a $\hat{v}^\bullet$-trajectory $\hat{\gamma} \subset A \times [0, 1]$ is at least 2, and, according to Definition 2.9, cannot exceed 2. Therefore, each $\hat{v}^\bullet$-trajectory must be *transversal* to ℓ: otherwise, the multiplicity of the intersection $\ell \cap \hat{\gamma}$, where $\hat{\gamma}$ is tangent to ℓ, exceeds 2.

For each point $x \in \ell$, the two ℓ-localized branches, $B(\delta W_1)$ and $B(\delta W_2)$, of $B(\delta W)$ that intersect along ℓ divide the tubular neighborhood U_ℓ of ℓ into four chambers $S_x^{\mathbf{I}}, S_x^{\mathbf{II}}, S_x^{\mathbf{III}}, S_x^{\mathbf{IV}}$, where each point from the interior of $S_x^{\mathbf{I}}$ has two α-preimages in the vicinity of $\delta W_1 \cup \delta W_2$ in W, each point from the interior of $S_x^{\mathbf{III}}$ has no α-preimages in the vicinity of $\delta W_1 \cup \delta W_2$ in W, and each point from the interior of $S_x^{\mathbf{II}} \cup S_x^{\mathbf{IV}}$ has one α-preimage in the vicinity of $\delta W_1 \cup \delta W_2$ in W. Thanks to the transversality of the vector field $\hat{v}^\bullet$ to the curve ℓ, the vector field must point into the interior of one of the chambers; i.e., $\hat{v}^\bullet$ is not tangent to the branches $B(\delta W_1)$ and $B(\delta W_2)$. In $3D$, we cannot distinguish between the chambers $S_x^{\mathbf{II}}$ and $S_x^{\mathbf{IV}}$ without picking an orientation of ℓ. Thus, $S_x^{\mathbf{I}}, S_x^{\mathbf{III}}$, and $S_x^{\mathbf{II}} \cup S_x^{\mathbf{IV}}$ are geometrically distinct.

Now, $\hat{v}^\bullet$ helps to define the three types of ℓ (not to be confused with the four types (a)-(d)): if $\hat{v}^\bullet(x)$ points into $S_x^{\mathbf{I}}$ or $S_x^{\mathbf{III}}$, we say that the curve ℓ is of types **I** or **III**, respectively. Otherwise, ℓ is of the "mixed type" **II & IV**.

In all cases (a)-(d), $\hat{v}^\bullet$ preserves the three chamber types **I**, **III**, **II & IV** along ℓ. In fact, the case (d), in which ℓ is a loop, is irrelevant for what follows.

Let $x \cup y = \partial \ell$. Crucially, in cases (a)-(b) the type $S_x^{\mathbf{II}}$ turns into the type $S_y^{\mathbf{IV}}$, and $S_x^{\mathbf{IV}}$ turns into the type $S_y^{\mathbf{II}}$, while the types $S_x^{\mathbf{I}}$ and $S_y^{\mathbf{I}}$, $S_x^{\mathbf{III}}$ and $S_y^{\mathbf{III}}$ are the same.

Therefore, in cases (a)-(b), we get the following possible combinations: $S_x^{\mathbf{I}}$ and $S_y^{\mathbf{I}}$, or $S_x^{\mathbf{II}}$ and $S_y^{\mathbf{IV}}$, or $S_x^{\mathbf{III}}$ and $S_y^{\mathbf{III}}$, or $S_x^{\mathbf{IV}}$ and $S_y^{\mathbf{II}}$. In the case (c), all the types at x and y are the same.

Given an immersion $\alpha : X \to A$ as in Definition 2.9, let us denote by $\rho_{\mathbf{I}}(\alpha), \rho_{\mathbf{II}}(\alpha), \rho_{\mathbf{III}}(\alpha), \rho_{\mathbf{IV}}(\alpha)$ the number of crossings of $\alpha(\partial X)$ of the types **I**, **II**, **III**, **IV**, where the type of a crossing is determined by the configuration of the sector, formed by the two intersecting domains, and the vector $\hat{v}$ at the crossing.

For example, in Fig. 2, all the four crossings of $\alpha(\partial X)$ are of the different types **I**, **II**, **III**, **IV**; that is, $\rho_{\mathbf{I}}(\alpha) = 1$, $\rho_{\mathbf{II}}(\alpha) = 1$, $\rho_{\mathbf{III}}(\alpha) = 1$, $\rho_{\mathbf{IV}}(\alpha) = 1$.

These pairings between different crossings of $\alpha_0(\partial X_0)$ and $\alpha_1(\partial X_1)$, which are delivered by the curves $\{\ell\}$ of types (a), (b), (c), lead directly to the following conclusion.

Lemma 3.1 *If two 2-moderate immersions $\alpha_0 : X_0 \to A$ and $\alpha_1 : X_1 \to A$ are cobordant in $\mathbf{B}_{\mathrm{moderate} \leq 2}^{\times \mathrm{imm}}(A) \approx \mathbf{B}_{\mathrm{moderate} \leq 2}^{\mathrm{imm}}(A)$, then*

$$\rho_{\mathbf{I}}(\alpha_0) \equiv \rho_{\mathbf{I}}(\alpha_1) \mod 2,$$
$$\rho_{\mathbf{III}}(\alpha_0) \equiv \rho_{\mathbf{III}}(\alpha_1) \mod 2,$$
$$\rho_{\mathbf{II}}(\alpha_0) - \rho_{\mathbf{IV}}(\alpha_0) = \rho_{\mathbf{II}}(\alpha_1) - \rho_{\mathbf{IV}}(\alpha_1). \tag{3.1}$$

$\diamond$

Fig. 8 Basic immersions of two disks in the surface A and the intersection patterns **I, II, III, IV** from Fig. 7 they generate. As elements of $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq2}(A)$, the immersion on the left is minus the corresponding immersion on the right
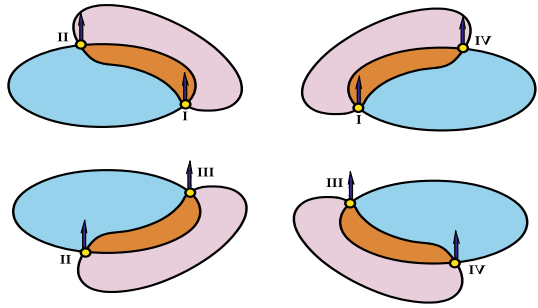
Thus, we produced three numerical invariants that have a potential to distinguish between the elements of the bordism $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq2}(A)$, modulo the ones that are in the image of the map $\mathbf{B}^{\text{emb}}_{\text{moderate}\leq2}(A) \to \mathbf{B}^{\text{imm}}_{\text{moderate}\leq2}(A)$.

Proposition 3.1 *For $A = \mathbb{R} \times [0, 1]$, the invariants $\rho_{\mathbf{I}}(\alpha)$, $\rho_{\mathbf{III}}(\alpha) \in \mathbb{Z}_2$, $\rho_{\mathbf{II}}(\alpha) - \rho_{IV}(\alpha) \in \mathbb{Z}$ define a homomorphism*

$$\mathcal{I}\rho : \mathbf{B}^{\text{imm}}_{\text{moderate}\leq2}(A)/\mathbf{B}^{\text{emb}}_{\text{moderate}\leq2}(A) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$$

whose image is an abelian subgroup $\mathcal{M} \approx \mathbb{Z}_2 \times \mathbb{Z}$ of index 2. The monomorphism $\phi : \mathcal{M} \hookrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ is delivered by the diagonal map $\mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{Z}_2$ and $\phi(\mathbb{Z}) = 2\mathbb{Z}$.

Proof The argument is based on the proof of Lemma 3.1 and on formulas (3.1).

Examining the types of crossings in the four diagrams from Fig. 8 that show the four immersions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, we get:

$$\begin{aligned}
\rho_{\mathbf{I}}(\alpha_1) &= 1, \ \rho_{\mathbf{III}}(\alpha_1) = 0, \ \rho_{\mathbf{II}}(\alpha_1) - \rho_{IV}(\alpha_1) = 1, \\
\rho_{\mathbf{I}}(\alpha_2) &= 1, \ \rho_{\mathbf{III}}(\alpha_2) = 0, \ \rho_{\mathbf{II}}(\alpha_2) - \rho_{\mathbf{IV}}(\alpha_2) = -1, \\
\rho_{\mathbf{I}}(\alpha_3) &= 0, \ \rho_{\mathbf{III}}(\alpha_3) = 1, \ \rho_{\mathbf{II}}(\alpha_3) - \rho_{\mathbf{IV}}(\alpha_3) = 1, \\
\rho_{\mathbf{I}}(\alpha_4) &= 0, \ \rho_{\mathbf{III}}(\alpha_4) = 1, \ \rho_{\mathbf{II}}(\alpha_4) - \rho_{\mathbf{IV}}(\alpha_4) = -1.
\end{aligned} \qquad (3.2)$$

Note that in $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq2}(A)$, $\alpha_2 = -\alpha_1$ and $\alpha_4 = -\alpha_3$.

Also, for $\alpha : X \to A$ from Fig. 2, where X is a torus with a hole, we have $\rho_{\mathbf{I}}(\alpha) = 1$, $\rho_{\mathbf{III}}(\alpha) = 1$, and $\rho_{\mathbf{II}}(\alpha_4) - \rho_{\mathbf{IV}}(\alpha_4) = 0$.

Consider now just these three immersions: α_1, α_3, and α. By linear algebra, applied to $\rho_{\mathbf{I}}(\sim)$, $\rho_{\mathbf{III}}(\sim)$, and $\rho_{\mathbf{II}}(\sim) - \rho_{\mathbf{IV}}(\sim)$ invariants of α_1, α_3, and α, the image of the map

$$\mathcal{I}\rho : \mathbf{B}^{\text{imm}}_{\text{moderate}\leq2}(A) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$$

is a subgroup of index 2 at most, which contains the $\mathbb{Z}$-module $\mathcal{M}$, spanned by the triples $(e, 0; 1)$, $(0, e; 1)$, $(e, e; 0)$, where $2e = 0$.

Lemma 3.2 *No loop $\mathcal{C}$ in $A = \mathbb{R} \times [0, 1]$ that bounds an immersed orientable surface Σ has an* odd *self-intersection number.*

Proof Indeed, if an immersion $\alpha : \Sigma \to A$ with $\alpha(\partial \Sigma) = \mathcal{C}$ exists, then the pull-back vector field $v = \alpha^{\dagger}(\hat{v}) \neq 0$. Thus, its index $\mathsf{ind}(v) = 0$. Recall that $deg(G)$, the degree of the Gauss normal map $G : \alpha(\partial \Sigma) \to S^1$ can be calculated by the Hopf–Gottlieb formula ([6, 8]): $deg(G) = \chi(\Sigma) - \mathsf{ind}(v) = \chi(\Sigma) = 1 - \beta_1(\Sigma)$, where the first Betti number $\beta_1(\Sigma) = 2g$ is even. On the other hand, by the Whiney formula [21] $deg(G) = \mu + N^+ - N^-$, where $\mu = \pm 1$ and N^+, N^- count the positive and negative crossings of $\mathcal{C}$ (this polarity of crossings is based on "global" considerations). So, each "kink" of $\mathcal{C}$ adds ± 1 to $deg(G)$. Odd number of self-crossings $N^+ + N^-$, by the Whiney formula, produces an even degree $deg(G)$, which contradicts to the fact that $1 - \beta_1(\Sigma)$ is an odd number. In fact, the minimal number of self-intersections an orientable immersed surface of genus g with a circular boundary may have is $2g + 2$ [7]. $\square$

Thus, by Lemma 3.2, none of the elements $(e, 0, 0)$, $(0, e, 0)$, $(0, 0, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ resides in the image of $\mathcal{I}\rho$. However, the elements $(e, 0, 1)$, $(0, e, 1)$, $(0, 0, 2) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ do belong to the $\mathcal{I}\rho$-image. Therefore, the $\mathcal{I}\rho$-image is $\mathcal{M} =_{\text{def}} \mathsf{span}_{\mathbb{Z}}\{(e, 0, 1), (0, e, 1), (0, 0, 2)\} \approx \mathbb{Z}_2 \times \mathbb{Z}$.

Evidently, $\mathbf{B}^{\text{emb}}_{\text{moderate} \leq 2}(A) \approx \mathbb{Z}$ is in the kernel of $\mathcal{I}\rho$. This completes the proof of Proposition 3.1. $\square$

Note that *oriented* immersed doodles also generate patterns similar to the ones in Fig. 7 at each intersection point $a \in A$. Indeed, let $\beta : \mathcal{C} \to A$ be an immersion as in (2.3). Consider the two local branches $\mathcal{C}'_a$ and $\mathcal{C}''_a$ of $\beta(\mathcal{C})$ at a that are transversal and oriented. The pair of tangent vectors u'_a and u''_a to $\mathcal{C}'_a$ and to $\mathcal{C}''_a$ that are consistent with the orientations, generate a particular *sector* in $T_a A$, the convex hull of u'_a and u''_a (see Fig. 9a).

By counting the four types of sectors in relation to $\hat{v}(x)$, which an oriented immersion β generates at its crossings $\{a\}$, we get the new quantities $\rho_{\mathsf{I}}(\beta)$, $\rho_{\mathsf{II}}(\beta)$, $\rho_{\mathsf{III}}(\beta)$, $\rho_{\mathsf{IV}}(\beta)$. The sector's number may be different by a permutation of four elements from the darkly shaded sectors in Fig. 7.

Therefore, the similar kind of invariants $\rho_{\mathsf{I}}(\beta) - \rho_{\mathsf{III}}(\beta)$, $\rho_{\mathsf{II}}(\beta) - \rho_{\mathsf{IV}}(\beta) \in \mathbb{Z}$ are available for the oriented doodles.

As in the case of blobs, these three quantities are invariants of the quotient $\mathbf{OC}^{\text{imm}}_{\text{moderate} \leq 2}(A)/\mathbf{OC}^{\text{emb}}_{\text{moderate} \leq 2}(A)$. The arguments that validate this claim are exactly the same as the ones that led to Lemma 3.1 for blobs. Moreover, for $A = \mathbb{R} \times [0, 1]$, these invariants are additive with respect to the connected sums $\uplus$ of oriented doodles.

Proposition 3.2 *For any oriented doodle* $\beta : \mathcal{C} \to A$, *the invariants* $\rho_{\mathsf{I}}(\beta) - \rho_{\mathsf{III}}(\beta)$, $\rho_{\mathsf{II}}(\beta) - \rho_{\mathsf{IV}}(\beta) \in \mathbb{Z}$ *define a surjective map*

$$\mathcal{I}\rho : \mathbf{OC}^{\text{imm}}_{\text{moderate} \leq 2}(A)/\mathbf{OC}^{\text{emb}}_{\text{moderate} \leq 2}(A) \to \mathbb{Z} \times \mathbb{Z}.$$

For $A = \mathbb{R} \times [0, 1]$, *the map* $\mathcal{I}\rho$ *is an epimorphism with respect to the group operation* $\uplus$.

Proof We recycle the list of ρ-invariants for the oriented boundaries of submersions α_1, α_3, and α from the proof of Proposition 3.1 and add to them the ρ-invariants of new doodles that do not bound immersions of surfaces. Differently oriented figures "∞", placed in A horizontally with respect to the vector field $\hat{v}$ by immersions $\beta_+, \beta_- : S^1 \to A$, realize the values $\{\rho_I(\beta_+) = 1, \ \rho_{III}(\beta_+) = 0, \ \rho_{II}(\beta_+) - \rho_{IV}(\beta_+) = 0\}$ and $\{\rho_I(\beta_-) = 0, \ \rho_{III}(\beta_-) = 1, \ \rho_{II}(\beta_-) - \rho_{IV}(\beta_-) = 0\}$. Consider another oriented doodle $\tilde{\beta} : S^1 \to A$ (not a boundary of an immersion) with a single self-intersection whose image is a "loop within a loop", symmetric with respect to the $\hat{v}$-trajectory through the self-intersection point. It realizes the values $\rho_I(\tilde{\beta}) = 0, \ \rho_{III}(\tilde{\beta}) = 0, \ \rho_{II}(\tilde{\beta}) = 1, \rho_{IV}(\tilde{\beta}) = 0$. Flipping the orientation of S^1, we realize the values $\rho_I(\bar{\tilde{\beta}}) = 0, \ \rho_{III}(\bar{\tilde{\beta}}) = 0, \ \rho_{II}(\bar{\tilde{\beta}}) = 0, \rho_{IV}(\bar{\tilde{\beta}}) = 1$. Therefore, by linear algebra, the map

$$\mathcal{I}\rho : \ \mathbf{OC}^{\text{imm}}_{\text{moderate}\leq 2}(A)/\mathbf{OC}^{\text{emb}}_{\text{moderate}\leq 2}(A) \to \mathbb{Z} \times \mathbb{Z}.$$

is surjective. For $A = \mathbb{R} \times [0, 1]$, the map $\mathcal{I}\rho$ is an epimorphism with respect to the group operation $\uplus$. $\qquad\square$

Let us introduce one useful operation/notation. Given an (oriented) immersion $\beta : \mathcal{C} \to A$ (doodles) and a non-negative number n, we denote by $\beta(n)$ the new immersion, obtained by surrounding $\beta(\mathcal{C})$ by n counterclockwise oriented nested simple loops. If n is negative, $\beta(n)$ denotes an immersion in which $\beta(\mathcal{C})$ is surrounded by n clockwise oriented nested simple loops. When talking about 2-moderate immersions, we assume that the surrounding simple loops are 2-moderate as well. Similarly, given an (oriented) immersion $\alpha : X \to A$ (blobs) and a non-negative n, $\alpha(n)$ denotes a submersion in which $\alpha(X)$ is surrounded by n nested (counterclockwise oriented) embedded disks whose boundaries are disjoint and 2-moderate. Again, for a negative n, we flip the orientations of the disks.

Let us list a few simple properties of these operations. We skip their straightforward validation. We consider all the immersions as elements of the appropriate (2-moderate) bordism groups, so that the following identities are understood as taking place in these groups:

$$\beta_1(n) \uplus \beta_2(n) = (\beta_1 \uplus \beta_2)(n),$$
$$(\beta(n))(m) = \beta(n + m); \tag{3.3}$$
$$\alpha_1(n) \uplus \alpha_2(n) = (\alpha_1 \uplus \alpha_2)(n),$$
$$(\alpha(n))(m) = \alpha(n + m). \tag{3.4}$$

We will use the operations $\odot_n : \beta \rightsquigarrow \beta(n)$ or $\odot_n : \alpha \rightsquigarrow \alpha(n)$, in combination with the addition $\uplus$, as main tools for generating new examples of immersions. In particular, we will apply $\odot_n$ to the immersed 2-moderate figures "∞" and "8", to get families of immersions "$\infty(n)$" and "8(n)" (see Fig. 9(a), which depicts the immersion $\infty(2) \uplus \overline{\infty}$). We distinguish between "∞" and "8" by their position in relation to the vector field $\hat{v}$, so that "∞" stands for the crossing of type I, III, while "8" for the crossing of type II, IV.
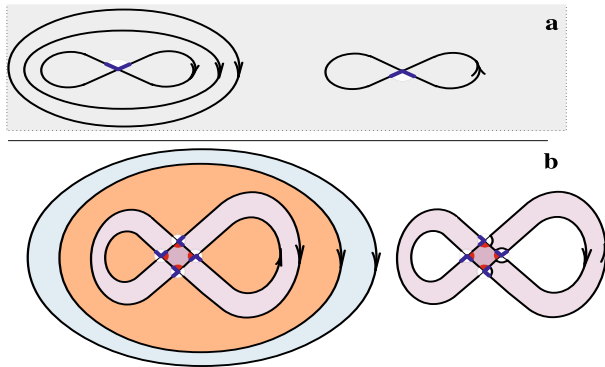
Fig. 9 a Oriented doodles $\infty(2) \uplus \overline{\infty}$; the two crossings are of the types I and III. **b** Oriented blobs $\infty-$ribbon(2) $\uplus \overline{\infty}-$ribbon; the four crossings on the left are of the types (**III**, II), (**IV**, III), (**I**, IV), (**II**, I), and on the right of the types (**III**, IV), (**IV**, I), (**I**, II), (**II**, III). The vector field $\hat{v}$ is vertical

We can improve the claim of Proposition 3.2 by analyzing algebraically and geometrically the kernel of the epimorphism $\mathcal{I}\rho$. With this goal in mind, we need to take a short detour into combinatorics that mimics the geometric situation we are facing.

• Let Θ be a finite set of "colors". In the near future, $\Theta = \{I, II, III, IV\}$ or $\Theta = \{I, III\}$.

Consider a finite collection of smooth 2-moderately embedded oriented loops $\mathcal{E}$ in A. We take a finite set of points Q in the complement to $\mathcal{E}$ (soon, Q will be the singular sets Σ_β of immersed doodles $\beta(\mathcal{C})$). In what follows, we consider the pairs $(Q, \mathcal{E})$ up to an ambient isotopy of A, or up to an ambient isotopy of A that preserves the foliation $\mathcal{F}(\hat{v})$.

We divide Q in complementary groups $Q_1, \ldots, Q_s$, each group being painted with a different color from the pallet Θ. This coloring is a part of the Q-structure.

The elements of Q are considered in pairs. We impose some restrictions $\mathcal{R}$ on the pairings of $P : Q \to Q$, where P is a free involution. These restrictions are expressed in terms of colors from the pallet Θ. Some of the restrictions will require that the elements of a particular color may be paired only with other elements the same color, in which case we assume that the cardinality $|Q_i|$ of the corresponding Q_i is even; other restrictions from $\mathcal{R}$ will pair elements of a particular color only with elements of another color, in which case $|Q_j| = |Q_k|$ for the corresponding sets Q_j, Q_k.

We denote the set of such pairings (i.e., free involutions on Q), subject to the restrictions from $\mathcal{R}$, by $\wp(Q, \mathcal{R})$.

• In one special case, we will pair elements of Q_I with elements of Q_{III}, elements of Q_{II} with elements of Q_{IV}. Let us denote this special set of rules by $\mathcal{R}^\odot$.

Employing $\mathcal{E}$, with each pairing $P \in \wp(Q, \mathcal{R})$ we associate a function $c(P)$ on the set of ordered pairs $\{(q, P(q))\}_{q \in Q}$ with values in $\mathbb{Z}$. The integer $c(q, P) =_{\text{def}} c(P)(q, P(q))$ counts the signed transversal intersections of an oriented path $\gamma(q, P(q)) \subset A$, connecting q to $P(q)$, with the union of oriented loops that form the 1-cycle $\mathcal{E}$. For homological reasons, $c(q, P)$ does not depend on the choice of the path $\gamma(q, P(q))$, provided $A = \mathbb{R} \times [0, 1]$.

We may view $c(q, P)$ as the linking number of the pair $(q, P(q))$ with the cycle $\mathcal{E}$. Note that $c(q, P) = -c(P(q), P)$.

This construction gives rise to a map

$$\mathbf{c}_{\mathcal{E}, P} : \mathsf{Graph}(P) \to \mathbb{Z}, \tag{3.5}$$

where $\mathsf{Graph}(P) \subset Q \times Q$ denotes the graph of the free involution $P : Q \to Q$, subject to the $\mathcal{R}$-constraints.

For any $\mathcal{R}$-admissible pairing $P : Q \to Q$, we always can choose the connecting paths $\{\gamma(q, P(q))\}_{q \in Q}$ so that they do not intersect each other and are transversal to $\mathcal{E}$. Without loss of generality, we may also assume that each path $\gamma(q, P(q))$ is 2-moderate.

Definition 3.1 ● For a fix set of rules $\mathcal{R}$, we call the triple $\{Q, P, \mathcal{E}\}$ irreducible if there exist a set of paths $\{\gamma(q, P(q))\}_{q \in Q}$ such that all the paths are disjoint and each path $\gamma(q, P(q))$ has all its intersections with $\mathcal{E}$ of the same sign. By definition, if for some q, the intersection $\gamma(q, P(q)) \cap \mathcal{E}$ is empty, the collection $\{Q, P, \mathcal{E}\}$ is reducible.

● For a fix set of rules $\mathcal{R}$, we call the collection $\{Q, P, \mathcal{E}\}$ primitive if it is irreducible and each colored set $Q_i \subset Q$ consists of exactly two elements, when $P(Q_i) = Q_i$, and of a single element when $P(Q_i) = Q_j$ for a $Q_j \neq Q_i$. $\diamond$

It is easy to see that, given two sets $(Q_1, P_1, \mathcal{E}_1)$ and $(Q_2, P_2, \mathcal{E}_2)$, their connected sum $\uplus$ in A produces a new triple $(Q_1 \uplus Q_2, \ P_1 \uplus P_2, \ \mathcal{E}_1 \uplus \mathcal{E}_2)$ by squeezing $(Q_1, \mathcal{E}_1)$ in $\mathbb{R} \times (0, 0.5)$ and $(Q_2, \mathcal{E}_2)$ in $\mathbb{R} \times (0.5, 1)$ and applying the pairing $P_1 \coprod P_2 \in \mathcal{R}$ to $Q_1 \uplus Q_2$.

For a fixed set of rules $\mathcal{R}$, the operation $\uplus$ turns the set $\mathbf{D}(\mathcal{R})$ of all triples $\{Q, P, \mathcal{E}\}$, being considered up to an ambient isotopy of A that preserves $\mathcal{F}(\hat{v})$, into an abelian semi-group.

The associativity and commutativity of the operation $\uplus$ in $\mathbf{D}(\mathcal{R})$ can be validated as in Lemma 2.2.

For a given triple $(Q, \mathcal{R}, \mathcal{E})$, consider the ℓ_1-norm $\|\mathbf{c}_{\mathcal{E}, P}\|_{\ell_1}$ of the discrete function $\mathbf{c}_{\mathcal{E}, P} : \mathsf{Graph}(P) \to \mathbb{Z}$ in (3.5). It is independent of the choice of paths $\{\gamma(q, P(q))\}_{q \in Q}$. We introduce a non-negative integer by the formula

$$m(Q, \mathcal{R}, \mathcal{E}) =_{\mathrm{def}} \min_{P \in \wp(Q, \mathcal{R})} \left\{ \|\mathbf{c}_{\mathcal{E}, P}\|_{\ell_1} \right\}. \tag{3.6}$$

Evidently,

$$m(Q_1 \uplus Q_2, \ \mathcal{R}, \ \mathcal{E}_1 \uplus \mathcal{E}_2) \ \leq \ m(Q_1, \ \mathcal{R}, \ \mathcal{E}_1) + m(Q_2, \ \mathcal{R}, \ \mathcal{E}_2).$$

Thus, $m(Q, \mathcal{R}, \mathcal{E})$ resembles a semi-norm under the connected sum operation $\uplus$. However, in general, $m\left(\uplus_1^k \{Q, \mathcal{R}, \mathcal{E}\}\right) \neq k \cdot m(Q, \mathcal{R}, \mathcal{E})$. At the same time, for a primitive $\{Q, \mathcal{R}, \mathcal{E}\}$, we have $m\left(\uplus_1^k \{Q, \mathcal{R}, \mathcal{E}\}\right) = k \cdot m(Q, \mathcal{R}, \mathcal{E})$.

We denote by $\mathbf{D}_0(\mathcal{R})$ the sub-semigroup of $\mathbf{D}(\mathcal{R})$ consisting of the triples $\{Q, \mathcal{R}, \mathcal{E}\}$ with the property $\{m(Q, \mathcal{R}, \mathcal{E}) = 0\}$ and consider the quotients $\mathbf{D}(\mathcal{R})\big/\mathbf{D}_0(\mathcal{R})$.

Lemma 3.3 $\mathbf{D}(\mathcal{R})\big/\mathbf{D}_0(\mathcal{R})$ *is an abelian group with respect to the operation* $\uplus$.

If $\mathcal{R}$ allows for pairings within each given color, then any non-zero element of the group $\mathbf{D}(\mathcal{R})\big/\mathbf{D}_0(\mathcal{R})$ is of order 2.

Proof The only property of groups that needs a clarification is the existence of the negation (minus element). Let $\tau : A \to A$ be a mirror involution. For any triple $\{Q, P, \mathcal{E}\}$, we form the triple $\tau(\{Q, \bar{P}, \mathcal{E}\}$, where each pair $(q, P(q))$ is replaced in $\tau(\{Q, \bar{P}, \mathcal{E}\})$ by the pair $(\tau(P(q)), \tau(q))$ so that $\tau(q)$ is given the color of $P(q)$ and $\tau(P(q))$ is given the color of q. Consider the new triple $\{Q, P, \mathcal{E}\} \uplus \tau(\{Q, \bar{P}, \mathcal{E}\}$. Now the pairing $(q, \tau(q))$ is permissible. Since the orientations of $\tau(\mathcal{E})$ is opposite to the orientations of $\mathcal{E}$, the path γ_q that connects q and $\tau(q)$ has the property $\gamma_q \circ (\mathcal{E} \uplus \tau(\mathcal{E})) = 0$. Therefore, $\{Q, P, \mathcal{E}\} \uplus \tau(\{Q, \bar{P}, \mathcal{E}\} \in \mathbf{D}_0(\mathcal{R})$. Hence, $\tau(\{Q, P, \mathcal{E}\})$ is the negative of $\{Q, P, \mathcal{E}\}$ in $\mathbf{D}(\mathcal{R})\big/\mathbf{D}_0(\mathcal{R})$. This validates the first claim.

Consider the triple $\{Q, P, \mathcal{E}\} \uplus \{Q, P, \mathcal{E}\}$ and a point $q \in Q$. Let $\gamma(q, s) \subset \mathbb{R} \times [0, 0.4]$ be any smooth path that is transversal to $\mathcal{E}$ and connects q to a point s whose $\mathbb{R}$-coordinate is bigger than h, where the box $[-h, h] \times [0, 0.4]$ contains Q and $\mathcal{E}$. We think of the second copy of $\{Q, P, \mathcal{E}\}$ as being obtained by the 0.5-shift $Sh : \mathbb{R} \times [0, 0.5] \to \mathbb{R} \times [0.5, 1]$ to the right. By the hypotheses about $\mathcal{R}$, we may pair q and $Sh(q)$.

We can connect s to $Sh(s)$ by a path ω that resides outside the box $[-h, h] \times [0, 1]$. Then the path $\tau = \gamma(q, s) \cup \omega \cup \overline{Sh(\gamma(q, s))}$ (where $\overline{Sh(\gamma(q, s))}$ denotes the path $Sh(\gamma(q, s))$ with the reverse orientation) connects q to $Sh(q)$ and has a pair of adjacent intersections with $\mathcal{E} \cup Sh(\mathcal{E})$ of opposite signs. Therefore, $\{Q, P, \mathcal{E}\} \uplus \{Q, P, \mathcal{E}\} \in \mathbf{D}_0(\mathcal{R})$, and thus $\{Q, P, \mathcal{E}\}$ is an element of order 2 in $\mathbf{D}(\mathcal{R})\big/\mathbf{D}_0(\mathcal{R})$. $\qquad\square$

Lemma 3.4 *The correspondence $\{Q, \mathcal{R}, \mathcal{E}\} \rightsquigarrow m(\{Q, \mathcal{R}, \mathcal{E}\})$ defines a norm-like map $m : \mathbf{D}(\mathcal{R})\big/\mathbf{D}_0(\mathcal{R}) \to \mathbb{Z}_+$ such that $m(a \uplus b) \leq m(a) + m(b)$ and $m(a) = 0$ implies $a = 0$, where $a, b \in \mathbf{D}(\mathcal{R})\big/\mathbf{D}_0(\mathcal{R})$.*

Proof The verification of the claims is on the level of definitions. $\qquad\square$

- This ends the combinatorial detour. $\qquad\diamond$

We are in position to return to bordisms of 2-moderate immersions of oriented doodles and to state one of our main results.

Theorem 3.1 *Let $A = \mathbb{R} \times [0, 1]$ and $\mathcal{R} = \{\mathrm{I} \Leftrightarrow \mathrm{III}, \mathrm{II} \Leftrightarrow \mathrm{IV}\}$.*
- *There is an exact sequence of abelian groups*

$$0 \to \mathbf{K} \to \mathbf{OC}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A)\big/\mathbf{OC}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \xrightarrow{\mathcal{I}\rho} \mathbb{Z} \times \mathbb{Z} \to 0,$$

where the homomorphism $\mathcal{I}\rho$ is given by the two integral invariants, $\rho^{\mathrm{I}}(\sim) - \rho^{\mathrm{III}}(\sim)$ and $\rho^{\mathrm{II}}(\sim) - \rho^{\mathrm{IV}}(\sim)$.

- *There exists a monomorphism $\Theta : \mathbf{D}(\mathcal{R})\big/\mathbf{D}_0(\mathcal{R}) \hookrightarrow \mathbf{K}$ whose image $\mathbf{G}$ is spanned over $\mathbb{Z}$ by the doodles $\{\infty(n) \uplus \overline{\infty}\}$ and $\{8(n) \uplus 8\}$, where $n \in \mathbb{Z}, n \neq 0$. In fact, $\mathbf{G} \approx (\mathbb{Z})^\infty$.*

- $\mathbf{OC}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \approx \mathbb{Z} \times \mathbb{Z}$ *via an isomorphism as in ([2.14](#)).*

Fig. 10 Resolving crossings of types I or III into the figures "∞", "$\overline{\infty}$" and a pair of arcs via a 2-moderate cobordism (the lower figure). In contrast, resolving crossings of types II or IV into the figures "8", "$\overline{8}$" and a pair of arcs is impossible via a 2-moderate cobordism (the upper figure)
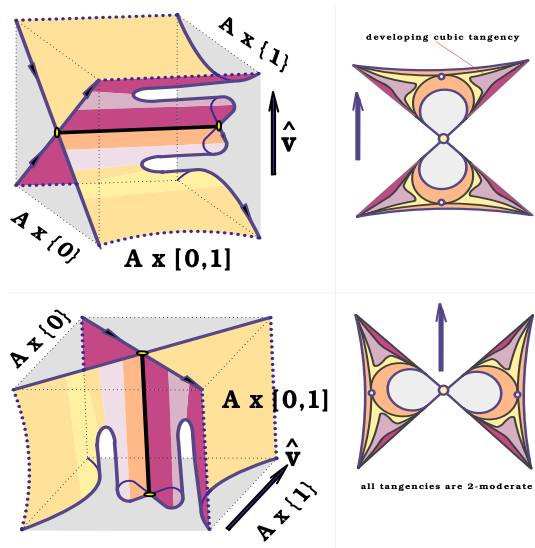
Proof We start with a finite set of oriented loops $\mathcal{C}$ and their 2-moderately generic (relatively to $\hat{v}$) immersion $\beta : \mathcal{C} \rightarrow A$. At each point of self-intersection $a \in \beta(\mathcal{C})$ of types I and III, in the vicinity of the crossing, we perform a 2-moderate surgery on $\beta(\mathcal{C})$ whose result $\beta^{\dagger}$ is shown in Fig. 10, lower diagram. The resulting oriented curves contain small figures "∞_a" and "$\overline{\infty}_a$", whose self-intersection resides at a. The bar denotes the flip of the orientation of ∞_a, determined by the orientations of the two branches of $\beta(\mathcal{C})$ at a. The type of the self-intersection of ∞_a in relation to $\hat{v}(a)$ is the same as the type of the original crossing of $\beta(\mathcal{C})$ at a with values in {I, III}; so ∞_a occurs for type I, and $\overline{\infty}_a$ for type III crossings. The rest of the oriented curves $\mathcal{C}'$ are immersed in A by a map β' which has the crossings of the types {II, IV} only. The immersions β and $\beta^{\dagger}$ are 2-moderately cobordant.

By a 2-surgery on $\beta(\mathcal{C}')$, we could replace the doodles $\mathcal{C}'$ by embedded doodles $\mathcal{C}''$ union several figures "8_a" for the crossings of types II and several figures "$\overline{8}_a$" for the crossings of types IV. Let us denote by $\beta^{\ddagger}$ the resulting immersion. Crucially, such a surgery, although being canonical, is not 2-moderate (see Fig. 10)! As a result, the immersions β and $\beta^{\ddagger}$ fail to be 2-moderately cobordant. This fact complicates our efforts.

In any case, via these canonical surgeries, we may replace any given immersion $\beta : \mathcal{C} \rightarrow A$ with the immersion $\beta^{\ddagger}$ which comprises the embedded doodles $\mathcal{C}'' \subset A$ disjoint union with a number of 2-moderately immersed figures "∞_a", "$\overline{\infty}_a$", and "8_a", "$\overline{8}_a$".

To summarize, for types I and III, we get the configurations "$\infty \uparrow_\vee$" and "$\overline{\infty} \uparrow_\wedge$"; for types II and IV, we either keep the original configurations "$\times \uparrow_<$" and "$\times ~_> \uparrow$" or, depending on the context, replace them with configurations "$8 \uparrow_<$" and "$8 ~_> \uparrow$". Here "$\uparrow$" mimics the direction of the vector field $\hat{v}$ in relation to the oriented figures "∞ and $\times$" and the symbols "$\vee, ~\wedge, ~<, ~>$" indicate the preferred sector (determined
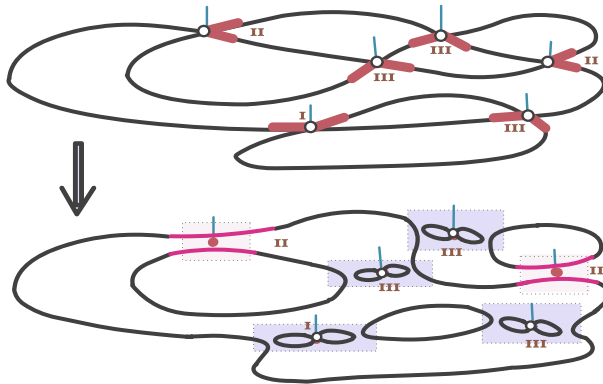
Fig. 11 Resolving an oriented immersed doodle into several figures "∞" (type I), "$\overline{\infty}$" (type III), and number of 2-moderate embedded loops. The figure illustrates the homomorphism $\mathrm{res}^{\mathrm{II},\mathrm{IV}}$ from (3.7). The vector field $\hat{v}$ is vertical

by the orientations of the two branches at the crossing a) in relation to $\hat{v}(a)$, i.e., in relation to "$\uparrow$".

If the invariants $\rho_{\mathrm{I}}(\beta) - \rho_{\mathrm{III}}(\beta) \in \mathbb{Z}$, $\rho_{\mathrm{II}}(\beta) - \rho_{\mathrm{IV}}(\beta) \in \mathbb{Z}$ vanish, then we can pair figures "8 $\uparrow_<$" of type II with figures "8 $_>\uparrow$" of type IV, figures "$\infty\ \uparrow_\vee$" of type I with figures "$\infty\ \uparrow_\wedge$" of type III.

Consider immersions in the kernel of $\mathcal{I}\rho$ that consist of several 2-moderate figures $\infty, \overline{\infty}, 8, \overline{8}$ disjoint union with 2-moderate embeddings. Thus, the number of figures ∞ is equal to the number of figures $\overline{\infty}$, and the number of figures 8 is equal to the number of figures $\overline{8}$. Let us denote by $\hat{\mathbf{G}}$ the set of such immersions, being considered up to the 2-moderate cobordisms. In particular, we focus on the subgroup $\mathbf{G} \subset \mathbf{K}$, spanned over $\mathbb{Z}$ by the elements $\{\infty(n) \uplus \overline{\infty}\}$ and $\{8(n) \uplus \overline{8}\}$, where $n \in \mathbb{Z}$, $n \neq 0$.

First, we need to verify that these elements are non-trivial and distinct in $\mathbf{K}$. Moreover, we will show that they are elements of infinite order in $\mathbf{K}$.

Consider a 2-moderate immersion $B : W \to A \times [0, 1]$ that delivers a cobordism between an immersion $\beta : \mathcal{C} \to A \times \{0\}$ and some embedding $\beta_1 : \mathcal{C}_1 \to A \times \{1\}$. Then, the self-intersection curves $\{\ell\}$ of $B(W)$ define a pairing $\{P : a \rightsquigarrow P(a)\}_{a\in\Sigma_\beta}$ consistent with the set of rules $\mathcal{R} = \{\mathrm{I} \Leftrightarrow \mathrm{III}, \mathrm{II} \Leftrightarrow \mathrm{IV}\}$.

On the other hand, for $\beta : \mathcal{C} \to A$ which represents an element of $\hat{\mathbf{G}}$, any pairing $\{P : a \rightsquigarrow P(a)\}_{a\in\Sigma_\beta}$ from $\mathcal{R}$ allows us to attach in $A \times [0, 1)$ 1-dimensional 2-moderate cross-handles $\infty_a \times I$ and $8_a \times I$ (the immersed images in $A \times [0, 1)$ of the handles $S_a^1 \times I$) to each pair $\infty_a \coprod \overline{\infty}_{P(a)} \subset A \times \{0\}$ or $8_a \coprod \overline{8}_{P(a)} \subset A \times \{0\}$. By a general position argument, these 2-moderate handles are disjoint. The union of these cross-handles with the embedded surface $\mathcal{C}_1 \times [0, 1]$ delivers an oriented cobordism

$$B : \coprod_a \{S_a^1 \times I\} \coprod \{\mathcal{C}_1 \times [0, 1]\} \to A \times [0, 1]$$

between β and β_1. It is important to attach the cross-handles with some care to satisfy the requirement for B to have only 2-moderate tangencies to the $\hat{v}^\bullet$-trajectories in $A \times [0, 1]$.

First, we need to avoid the intersections of the 1-dimensional singular cores of the immersed surfaces $B(\coprod_a \{S_a^1 \times I\})$ with the embedded surfaces $B(\mathcal{C}_1 \times [0, 1])$ since such intersections would contribute cubic tangencies at least to the $\hat{v}^\bullet$-trajectories.

It turns out that there are new combinatorial obstructions to B being 2-moderate. These new obstructions are described in the next lemma.

Lemma 3.5 *Let $A = \mathbb{R} \times [0, 1]$ and let $\mathcal{R} = \{I \Leftrightarrow III, II \Leftrightarrow IV\}$ or $\mathcal{R} = \{I \Leftrightarrow III\}$. Let $\mathcal{C}$ be a finite collection of oriented circles. Consider a 2-moderate immersion $\beta : \mathcal{C} \to A$ such that $\rho_I(\beta) = \rho_{III}(\beta)$ and $\rho_{II}(\beta) = \rho_{IV}(\beta)$. We denote by Σ_β its self-intersection locus of β.*

By replacing in $\beta(\mathcal{C})$ each crossing with the figures "∞_a", "$\overline{\infty}_a$" or "8_a" or "$\overline{8}_a$", we get a new 2-moderate immersion $\beta^{\ddagger}$ that is a disjoint union of these figures with a 2-moderate embedding $\beta'' : \mathcal{C}'' \to A$.

Then, $\beta^{\ddagger}$ is cobordant, via 2-moderate oriented immersions of surfaces, to an embedding β_1 if and only if $m(\Sigma_\beta, \mathcal{R}, \mathcal{C}'') = 0$ (see (3.6)), i.e., iff there exists a pairing $P \in \wp(\Sigma_\beta, \mathcal{R})$ whose function $\mathbf{c}_{\mathcal{E}, P} : \mathsf{Graph}(P) \to \mathbb{Z}$ is identically zero.

In the case $\mathcal{R} = \{I \Leftrightarrow III\}$, $\beta^{\ddagger}$ is 2-moderately cobordant to the original β.

Proof We will prove the claim for $\mathcal{R} = \{I \Leftrightarrow III, II \Leftrightarrow IV\}$; the arguments in the case $\mathcal{R} = \{I \Leftrightarrow III\}$ are very similar and simpler. As the Fig. 10 testifies, for this choice of $\mathcal{R}$, $\beta^{\dagger}$ is 2-moderately cobordant to the original β.

Let $B : W \to A \times [0, 1]$ be a 2-moderate immersion that delivers the oriented cobordism between the immersion β and some embedding β_1. We have seen that the self-intersection locus $\Xi_B = \coprod \ell$ of $B(W)$ delivers a particular pairing $P_B : \Sigma_\beta \to \Sigma_\beta$ that is consistent with $\mathcal{R}$.

We denote by I a closed interval. We replace the singular surface $B(W)$ with the disjoint union of narrow 2-moderate cross-handles

$$\left\{ B_a^{\ddagger} : S_a^1 \times I \to A \times [0, 1] \right\}_{a \in \Sigma_\beta / P_B}$$

whose images are $\infty_a \times I$, $\overline{\infty}_a \times I$, and $8_a \times I$, $\overline{8}_a \times I$, together with a number of narrow 2-moderate cross-tori

$$\left\{ B_\kappa^{\ddagger} : S_\kappa^1 \times S^1 \to A \times (0, 1) \right\}_\kappa$$

$(B_\kappa^{\ddagger}(S_\kappa^1 \times S^1)$ are diffeomorphic to $\infty_\kappa \times S^1$ or to $8_\kappa \times S^1)$, and a 2-moderate regular embedding $B'' : W'' \hookrightarrow A \times [0, 1]$ of an oriented surface W''. The trace of this cobordism in $A \times \{0\}$ produces the curves

$$\left(\coprod_{a \in \Sigma_\beta^I} \infty_a \right) \coprod \left(\coprod_{a \in \Sigma_\beta^{III}} \overline{\infty}_a \right) \coprod \left(\coprod_{a \in \Sigma_\beta^{II}} 8_a \right) \coprod \left(\coprod_{a \in \Sigma_\beta^{IV}} \overline{8}_a \right)$$

disjoint union with a *non-singular* curve $\mathcal{C}'' = B''(W'') \cap (A \times \{0\})$ as has been described above.

Consider now the cores $\{\delta =_{\mathsf{def}} \delta_{a, P(a)}\}_{a \in \Sigma_\beta}$ of the cross-handles and of the cross-torii. We notice that $\delta \cap B''(W'')$ must be empty in order to insure that B is 2-moderate. Indeed, if $x \in \delta \cap B''(W'')$, then x is a triple-intersection point for $B(W)$, which violates the 2-moderate property of B.

Note that the algebraic (i.e., signed) intersection number $\delta_{a, P(a)} \circ B''(W'')$ is a homological invariant for the relative homology class $[\delta_{a, P(a)}] \in H_1(A \times [0, 1], a \cup P_B(a); \; \mathbb{Z})$ of the path $\delta_{a, P(a)}$ and the relative homology class

$$[B''(W'')] \in H_2\big(A \times [0, 1], \; [A \setminus (a \cup P_B(a))] \times \{0\} \sqcup (A \times \{1\}); \; \mathbb{Z}\big)$$

of the relative 2-cycle $B''(W'')$. Since the relative homology class $[\delta_{a, P(a)}]$ and the relative homology class

$$[\gamma(a, P_B(a))] \in H_1(A \times [0, 1], \; a \cup P_B(a); \; \mathbb{Z})$$

are equal (the two paths are even homotopic relative to $a \cup P_B(a)$ in $A \times [0, 1]$), we conclude that the algebraic intersection number

$$\delta_{a, P(a)} \circ B''(W'') = \gamma(a, P_B(a)) \circ B''(W'') = \gamma(a, P_B(a)) \circ C''.$$

Therefore, if $\gamma(a, P_B(a)) \circ C_1 \neq 0$, we get a contradiction with the requirement $\delta_{a, P(a)} \cap B'(W') = \emptyset$. As a result, if $\gamma(a, P_B(a)) \circ C'' \neq 0$ for some $a \in \Sigma_\beta$, then the cobordism B is not 2-moderate.

On the other hand, assume that, for some $\mathcal{R}$-amenable involution $P : \Sigma_\beta \to \Sigma_\beta$, the intersection $\gamma(a, P(a)) \circ C'' = 0$ for all $a \in \Sigma_\beta$, where C'' is a 2-moderate non-singular component of the replacement $\beta^\ddagger$ of β. Then, let us show that β is 2-moderately cobordant to an embedding β_1.

Using the notations from the previous arguments, we start with the cobordism W which is the union $H \subset A \times [0, 1]$ of 2-moderate cross-handles, attached to $A \times \{0\}$ at points of Σ_β according to the coupling P, with the non-singular surface $C'' \times [0, 1]$. Since $\hat{v}^\bullet$ is transversal to the cores of the narrow cross-handles, we may ensure that H is 2-moderate.

Although the surfaces H and $S = C'' \times [0, 1]$ individually are 2-moderate, their union $W = H \cup S$ may be not.

However, if we assume that all the cores δ of cross-handles from H have the property $\delta \circ S = 0$, then we can correct the failure of $W = H \cup S$ to be 2-moderate. Indeed, if two adjacent intersections x, y of δ with the oriented surface S are of opposite signs, then we can attach a 1-handle $S^1 \times [x, y]$ to S whose core is the segment $[x, y] \subset \delta$ and whose interior $D^2 \times [x, y]$ engulfs the cross-handles $\infty_a \times [x, y]$, or $\overline{\infty}_a \times [x, y]$, or $8_a \times [x, y]$, or $\overline{8}_a \times [x, y]$. The result is a new embedded oriented surface S' which intersects δ_a at fewer points: the intersections x and y has been eliminated. The rest of the cores δ_b keep their old intersections with S. Continuing this elimination of the adjacent along the cores intersections of opposite signs with evolving non-singular surfaces, eventually we will get a non-singular oriented surface $\tilde{S}$ which is disjoint from all the cross-handles. The union of $\tilde{S}$ with all the cross-handles delivers the desired 2-moderate cobordism between the immersion β and some embedding β_1.

By the previous arguments, this recipe for constructing β_1 works when, for some free involution $P : \Sigma_\beta \to \Sigma_\beta$, we have the property $\gamma(a, P(a)) \circ \mathcal{C}'' = 0$ for all $a \in \Sigma_\beta$. This means exactly that there exists a pairing $P \in \wp(\Sigma_\beta, \mathcal{R})$ whose function $\mathbf{c}_{\mathcal{C}'', P} : \mathsf{Graph}(P) \to \mathbb{Z}$ is identically zero. In other words, the semi-norm $m(\Sigma_\beta, \mathcal{R}, \mathcal{C}'')$ must vanish. $\square$

We continue with the proof of Theorem 3.1. As before, in our notations, we suppress the choices of the paths $\{\gamma(q, P(q))\}_{q \in Q}$ (equivalently, we assume them to be oriented segments of lines).

Given an element $\{Q, P, \mathcal{E}\}$ of the group $\mathbf{D}(\mathcal{R})$, we associate with it a 2-moderate immersion $\beta : \mathcal{C} \to A$ by replacing each point $q \in Q$ with a small figure $\infty, \overline{\infty}$, or $8, \overline{8}$ (depending on the type $\{\mathsf{I}, \mathsf{II}, \mathsf{III}, \mathsf{IV}\}$ of q), and keeping the 2-moderate embedding $\mathcal{E}$ for the role of $\mathcal{C}''$. By Lemma 3.5, β is cobordant via a 2-moderately cobordism $B : W \to A \times [0, 1]$ (whose self-intersection locus $\coprod \ell$ defines P) to an embedding if and only if the element $\{Q, P, \mathcal{E}\}$ is such that the function $\mathbf{c}_{\mathcal{E}, P} : \mathsf{Graph}(P) \to \mathbb{Z}$ is identically zero. Therefore, by the definition of $\mathbf{D}_0(\mathcal{R})$, this construction $\{Q, P, \mathcal{E}\} \rightsquigarrow \beta$ defines a monomorphism $\Theta : \mathbf{D}(\mathcal{R})/\mathbf{D}_0(\mathcal{R}) \hookrightarrow \mathbf{K}$. Its image is exactly the subgroup $\hat{\mathbf{G}}$ (by this group's definition).

Consider the subgroup $\mathbf{G} \subset \hat{\mathbf{G}}$ spanned over $\mathbb{Z}$ by the elements $\{\infty(n) \uplus \overline{\infty}\}$ and $\{8(n) \uplus \overline{8}\}$, where $n \in \mathbb{Z}$, $n \neq 0$. For any $\mathcal{R}$-admissible pairing P between the two crossing a and $P(a)$ of each figure, the intersection number $\gamma(a, P(a)) \circ \mathcal{C}''$ of any path $\gamma(a, P(a))$ with the concentric loops $\mathcal{C}''$ surrounding $\infty, \overline{\infty}$, or $8, \overline{8}$ is equal to $n \neq 0$. Thus, by Lemma 3.5, the elements $\{\infty(n) \uplus \overline{\infty}\}$ and $\{8(n) \uplus \overline{8}\}$ are non-trivial in $\mathbf{K}$ for all $n \neq 0$. Moreover, for the same reason, their k-multiple copies (with respect to $\uplus$) are not 2-moderately cobordant to an embedding. Therefore, the elements $\{\infty(n) \uplus \overline{\infty}\}$ and $\{8(n) \uplus \overline{8}\}$ are of the infinite order in $\mathbf{K}$. Along the same lines, for $n_1 \neq n_2 \neq 0$, no k-multiple of $\{\infty(n_1) \uplus \overline{\infty}\}$ is 2-moderately cobordant to $\{\infty(n_2) \uplus \overline{\infty}\}$. Since $\mathbf{K}$ is abelian, it follows that $\mathbf{G} \approx (\mathbb{Z})^\infty$.

In fact, $\hat{\mathbf{G}} = \mathbf{G}$. Indeed, consider a 2-moderate simple loop $\mathcal{L}$ that contains several figures $\infty, \overline{\infty}$, or $8, \overline{8}$ and does not contain any other simple loops. By a 2-moderate 1-surgery (the 1-handles must avoid the figures $\infty, \overline{\infty}, 8, \overline{8}$) on the domain $\mathcal{D}$ that bounds $\mathcal{L}$, we split $\mathcal{D}$ in several topological disks so that each of them contains a single figure. Let us call temporarily a nested group of simple loops which contain a single figure $\infty, \overline{\infty}$, or $8, \overline{8}$ a "block". If a block contains oppositely oriented simple loops, they can be eliminated by a 2-moderate surgery on the annulus that is bounded by them (see Fig. 4, the last two slices). Thus, we may assume that all the loops in a block are oriented coherently.

If a simple loop $\mathcal{L}'$ bounds a domain $\mathcal{D}'$ that contains several blocks, then again by a 2-moderate 1-surgery on $\mathcal{D}'$ we can split $\mathcal{D}'$ in a number of disks, each of which bounds a single new block. Proceeding this way, we transform any configuration of figures and simple loops into 2-moderately cobordant disjoint union of coherently oriented blocks. This transformation does not affect the original figures $\{\infty, \overline{\infty}, 8, \overline{8}\}$ and their pairings. Therefore, $\hat{\mathbf{G}} = \mathbf{G} \approx (\mathbb{Z})^\infty$.

The last claim of the theorem follows from the isomorphism (2.14), since any oriented embedded doodle in A is the boundary of an oriented embedded blob. This completes the proof of Theorem 3.1. $\square$

Let $\beta : \mathcal{C} \to A$ be a 2-moderate oriented immersion. By canonical resolutions of the types II and IV singularities "$\times \uparrow_<$" and "$\times \,_> \uparrow$", we eliminate them in β' (see Fig. 11). The result, $\beta^\dagger$, is a new 2-moderate embedding $\beta'' : \mathcal{C}'' \to A$ disjoint union with a number of immersions of the type "∞_a" and "$\overline{\infty}_a$". The resolution $\mathrm{res}_{\mathsf{II,IV}} : \beta \leadsto \beta^\dagger$ may change the 2-moderate bordism invariant $\rho_{\mathsf{II}}(\beta) - \rho_{\mathsf{IV}}(\beta)$ and thus may change the bordism class of β. These canonical resolutions produce a homomorphism

$$\mathrm{res}^{\mathsf{II,IV}} : \mathbf{OC}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A) \to \mathbf{OC}^{\mathrm{imm,\ I\&III}}_{\mathrm{moderate} \leq 2}(A), \tag{3.7}$$

where $\mathbf{OC}^{\mathrm{imm,\ I\&III}}_{\mathrm{moderate} \leq 2}(A)$ stands for the 2-moderate bordisms of oriented immersions whose crossings are of the types I and III only. The map $\mathrm{res}^{\mathsf{II,IV}}$ is evidently the right inverse of the obvious homomorphism $\mathbf{OC}^{\mathrm{imm,\ I\&III}}_{\mathrm{moderate} \leq 2}(A) \to \mathbf{OC}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A)$.

The treatment of $\mathbf{OC}^{\mathrm{imm,\ I\&III}}_{\mathrm{moderate} \leq 2}(A)$ is very similar to our treatment of $\mathbf{OC}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A)$, although the case of $\mathbf{OC}^{\mathrm{imm,\ I\&III}}_{\mathrm{moderate} \leq 2}(A)$ is easier to analyze.

Theorem 3.2 *Let $A = \mathbb{R} \times [0, 1]$ and $\mathcal{R}^\circ = \{\mathsf{I} \Leftrightarrow \mathsf{III}\}$.*

- *There exists an exact sequence of abelian groups:*

$$0 \to \mathbf{K}^{\mathsf{I\ \&\ III}} \to \mathbf{OC}^{\mathrm{imm,\ I\ \&\ III}}_{\mathrm{moderate} \leq 2}(A)/\mathbf{OC}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \xrightarrow{\mathcal{I}\rho^{\mathsf{I\ \&\ III}}} \mathbb{Z} \to 0,$$

where the epimorphism $\mathcal{I}\rho^{\mathsf{I\ \&\ III}}$ is given by the integral invariant $\rho^{\mathsf{I}}(\sim) - \rho^{\mathsf{III}}(\sim)$.

- *The kernel $\mathbf{K}^{\mathsf{I\ \&\ III}}$ is generated by the doodles $\{\infty(n) \coprod \overline{\infty}\}_{n \in \mathbb{Z}; \, n \neq 0}$ and is isomorphic to the group $\mathbf{D}(\mathcal{R}^\circ)/\mathbf{D}_0(\mathcal{R}^\circ) \approx (\mathbb{Z})^\infty$.*
- *Thus, any 2-moderate oriented doodle β with the crossings of the types I and III only and such that $\rho_{\mathsf{I}}(\beta) = \rho_{\mathsf{III}}(\beta)$, is 2-moderately cobordant to an embedding if and only if its image $\Theta(\beta) \in \mathbf{D}_0(\mathcal{R}^\circ)$ (i.e., the "semi-norm" $m(\Sigma_\beta, \mathcal{R}^\circ, \mathcal{C}') = 0$).*

Proof Since the oriented figure "∞" has the crossing of type I, and "$\overline{\infty}$" has the crossing of type III, the map $\mathcal{I}\rho^{\mathsf{I\ \&\ III}} = \rho_{\mathsf{I}}(\sim) - \rho_{\mathsf{III}}(\sim)$ is an epimorphism.

By Lemma 3.5, any 2-moderate immersion $\beta : \mathcal{C} \to A$ with the crossings of the types I and III only and such that $\rho_{\mathsf{I}}(\beta) = \rho_{\mathsf{III}}(\beta)$ is cobordant in $\mathbf{OC}^{\mathrm{imm,\ I\ \&\ III}}_{\mathrm{moderate} \leq 2}(A)$ to an embedding β_1 if and only if the singular set Σ_β admits a free involution $P \in \wp(\Sigma_\beta, \mathcal{R}^\circ)$ with the following property: for any $a \in \Sigma_\beta$, the algebraic intersection number $\gamma(a, P(a)) \circ \mathcal{C}' = 0$. Here the embedded doodle $\mathcal{C}'$ is the doodle $\mathcal{C}''$ from the proof of Theorem 3.1. Equivalently, $m(\Sigma_\beta, \mathcal{R}^\circ, \mathcal{C}') = 0$.

For any triple $\{Q, P, \mathcal{E}\}$, representing an element of $\mathbf{D}(\mathcal{R}^\circ)$, we construct a 2-moderate immersion $\beta(\{Q, P, \mathcal{E}\})$ by placing a small figure ∞ or $\overline{\infty}$ at each point $q \in Q$ (depending on whether q is of color I or III) and using the embedded doodle $\mathcal{E}$ as a part of the immersion. The role of the involution $P : Q \to Q$ is tenuous: it makes sure that the colors of q and $P(q)$ are different. By Lemma 3.5, $\beta(\{Q, P, \mathcal{E}\})$ is cobordant to an embedding if and only if there is a triple $\{Q, \tilde{P}, \mathcal{E}\}$ for which $m(\{Q, \tilde{P}, \mathcal{E}\}) = 0$; in other words, when $\{Q, \tilde{P}, \mathcal{E}\} \in \mathbf{D}_0(\mathcal{R}^\circ)$.

Therefore, the correspondence $\{Q, P, \mathcal{E}\} \rightsquigarrow \beta(\{Q, P, \mathcal{E}\})$ gives rise to a monomorphism

$$\Theta : \mathbf{D}(\mathcal{R}^\circ)/\mathbf{D}_0(\mathcal{R}^\circ) \to \mathbf{K}^{\mathsf{I} \, \& \, \mathsf{III}}.$$

On the other hand, we have seen that, for any 2-moderate immersion $\beta \in \mathbf{K}^{\mathsf{I} \, \& \, \mathsf{III}}$, the immersion $\beta^\dagger$ (see Fig. 11) from Lemma 3.5 is 2-moderately cobordant to β. Evidently, $\beta^\dagger$ is in the image of Θ. As a result, Θ is an epimorphism, and thus, an isomorphism.

Exactly as in the proof of Theorem 3.1 and with the help of Lemma 3.5, we verify that all the oriented doodles $\{\infty(n) \coprod \overline{\infty}\}_{n \in \mathbb{Z}}$ are distinct elements of infinite order in $\mathbf{K}^{\mathsf{I} \, \& \, \mathsf{III}}$ and generate a subgroup $(\mathbb{Z})^\infty$. $\qquad\square$

Let "∞-ribbon" denotes the immersion of a band $S^1 \times I$ in A, shaped as a fat figure ∞ in relation to $\hat{v}$ (see Fig. 9(b)). The symbol "∞-ribbon(n)" stands for the ∞-ribbon surrounded by n concentric nested disks. Similarly, the "8-ribbon" is the immersion of a band $S^1 \times I$ in A, shaped as a fat figure 8 in relation to $\hat{v}$ and 8-ribbon(n) denotes the 8-ribbon surrounded by n concentric nested disks.

In the case of non-oriented blobs, we get the following result:

Theorem 3.3 *Let $A = \mathbb{R} \times [0, 1]$.*
- *There is an exact sequence of abelian groups:*

$$0 \to \mathbf{M} \to \mathbf{B}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A) \big/ \mathbf{B}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \xrightarrow{\mathcal{I}\rho} \mathbb{Z}_2 \times \mathbb{Z} \to 0,$$

where $\mathbf{B}^{\mathrm{emb}}_{\mathrm{moderate} \leq 2}(A) \overset{J}{\approx} \mathbb{Z}$ is a direct summand of $\mathbf{B}^{\mathrm{imm}}_{\mathrm{moderate} \leq 2}(A)$.
- *The homomorphism $\mathcal{I}\rho$ is defined by the invariants*

$$\rho_{\mathbf{I}}(\sim), \rho_{\mathbf{III}}(\sim) \in \mathbb{Z}_2, \ \rho_{\mathbf{II}}(\sim) - \rho_{\mathbf{IV}}(\sim) \in \mathbb{Z}$$

(which realize the subgroup $\mathbb{Z}_2 \times \mathbb{Z} \hookrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$).
- *The kernel $\mathbf{M}$ contains the group $(\mathbb{Z}_2)^\infty$ generated by blobs $\{\infty$-ribbon$(n) \uplus \overline{\infty}$-ribbon$\}_{n \in \mathbb{Z}}$ $(n \neq 0)$ as in Fig. 9(b).*

Proof Thanks to Proposition 3.1, to prove the theorem we need only to show that the kernel $\mathbf{M}$ contains the group $(\mathbb{Z}_2)^\infty$. It suffices to exhibit infinitely many distinct elements of $\mathbf{M}$ of order 2.

Consider the immersion $\beta_\bullet : X_\bullet \to A$ of a band $X_\bullet = S^1 \times I$, whose image is shaped as ∞-ribbon (see Fig. 9, (b)). With the help of $\beta_\bullet$, the orientation of $X_\bullet$ (and thus of $\partial X_\bullet$) is induced by the preferred orientation of A. The image $\beta_\bullet(\partial X_\bullet)$ has four self-intersections of distinct colors $\{\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}\}$ as blobs (see Fig. 7) and of distinct colors $\{\mathsf{I}, \mathsf{II}, \mathsf{III}, \mathsf{IV}\}$ (as doodles). Alternatively, we could use the immersion $\beta_\bullet : X_\bullet = D^2 \coprod D^2 \to A$ whose image has also four self-intersections of distinct colors $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}$ and of distinct colors $\mathsf{I}, \mathsf{II}, \mathsf{III}, \mathsf{IV}$ (see Fig. 12).

We add to $\beta_\bullet$ the regular embeddings of n coherently oriented concentric disks so that the image of the smallest disk in A contains properly the ∞-ribbon $\beta_\bullet(X_\bullet)$.

We claim that ∞-ribbon(n) is a non-trivial element of $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A)$. Assume to the contrary that the ∞-ribbon(n) is the boundary of a solid 2-moderate cobordism $B : W \to A \times [0, 1)$. Then the self-intersection curves $\ell \subset B(\delta W)$ cannot connect the unique self-intersection point of $\beta_\bullet(\partial X_\bullet)$ of type **I** to itself. Thus, no pairing consistent with

$$\mathcal{R}^\bullet =_{\text{def}} \{\mathbf{I} \Leftrightarrow \mathbf{I}, \ \mathbf{III} \Leftrightarrow \mathbf{III}, \ \mathbf{II} \Leftrightarrow \mathbf{IV}\}$$

is available, and no such 2-moderate cobordism B exists.

Moreover, for $n \neq m$, ∞-ribbon$(n) \neq \infty$-ribbon(m) in $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A)/$ $\mathbf{B}^{\text{emb}}_{\text{moderate}\leq 2}(A)$. Indeed, if

$$\infty-\text{ribbon}(n) = \infty-\text{ribbon}(m) \uplus \alpha' \text{ in } \mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A), \qquad (3.8)$$

where $\alpha' : X' \hookrightarrow A$ is a 2-moderate embedding whose image is separated from the rest of the blobs by a $\hat{v}$-trajectory, then the 2-moderate solid cobordism $B : W \to A \times [0, 1)$ between the two blobs in the RHS and LHS of equality (3.8) is available.

Consider the self-intersection curve $\ell \subset B(\delta W)$ that connects the unique self-intersection point a of type **I** in $\infty-\text{ribbon}(n)$ to the unique self-intersection point b of type **I** in $\infty-\text{ribbon}(m)$. Since B is an immersion, the preimage $B^{-1}(\ell)$ must be union of several arcs, so that $B : B^{-1}(\ell) \to \ell$ is an immersion. By the construction of ℓ, two of these arcs, ℓ_1 and ℓ_2, belong to the boundary δW, while the rest of the arcs start at n points in the interior of W and terminate at m points in the interior of W. For $m \neq n$, at least one of these arcs must hit δW at some point c. Therefore, the point $B(c) \in \ell$ belongs to triple-intersection locus of $B(\delta W)$, a contradiction with the assumption that $B : \delta W \to A \times [0, 1]$ is 2-moderate.

Note that ∞-ribbon(n) does not belong to the kernel of $\mathcal{I}\rho$ since its $\rho_{\mathsf{I}} = 1$. At the same time, all the $\mathcal{I}\rho$-invariants of ∞-ribbon$(n) \uplus \overline{\infty}$-ribbon do vanish; thus, it belongs to the kernel $\mathcal{I}\rho$.

We recycle the previous arguments and examine the $\mathcal{R}^\bullet$-permissible pairings between the four self-intersections of type **I** of the ribbons $\infty-\text{ribbon}(n) \uplus \infty-\text{ribbon}$ and $\infty-\text{ribbon}(m) \uplus \infty-\text{ribbon}$. These pairings are delivered by a pair of curves $\ell \subset B(\delta W)$ and $\ell' \subset B(\delta W)$ that belong to the self-intersection locus of $B(\delta W)$. Thus, for $m \neq n$, $B(\delta W)$ must develop triple intersections in $A \times [0, 1]$. We conclude that

$$\infty-\text{ribbon}(n) \uplus \infty-\text{ribbon} \neq \infty-\text{ribbon}(m) \uplus \infty-\text{ribbon} \quad \text{mod } \mathbf{B}^{\text{emb}}_{\text{moderate}\leq 2}(A).$$

On the other hand, if the mirror image (with respect to a $\hat{v}$-trajectory) of a non-oriented blob $\alpha : X \to A$ is isotopic to the original blob α, then $\alpha \uplus \alpha = 0$ in $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A)$. Here the isotopy is assumed to preserve the oriented foliation $\mathcal{F}(\hat{v})$. Indeed, in such symmetric case, $\alpha \uplus \alpha$ is the boundary of the 2-moderate cobordism $B : W \to A \times [0, 1)$, traced by a rotation in $A \times [0, 1)$ of $\alpha(X) \subset \mathbb{R} \times [0, 0.5] \subset A$

on the angles $0 \leq \phi \leq \pi$ around the axis $\gamma = \mathbb{R} \times \{0.5\} \subset A \times \{0\} \subset A \times [0, 1)$. Thus,

$$\left(\infty-\text{ribbon}(n) \uplus \infty-\text{ribbon}\right) \uplus \left(\infty-\text{ribbon}(n) \uplus \infty-\text{ribbon}\right) = 0$$

in $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A)$ and hence in $\mathbf{M}$. Therefore, $\{\infty-\text{ribbon}(n) \uplus \infty-\text{ribbon}\}_n$ deliver distinct elements of order 2 in $\mathbf{M}$.

Thanks to the homomorphism $\mathcal{R}_J : \mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A) \to \mathbf{B}^{\text{emb}}_{\text{moderate}\leq 2}(A)$ from (2.9), $\mathbf{B}^{\text{emb}}_{\text{moderate}\leq 2}(A) \overset{J}{\approx} \mathbb{Z}$ is a direct summand of $\mathbf{B}^{\text{imm}}_{\text{moderate}\leq 2}(A)$. □

Remark 3.1 We do not know whether $\mathbf{M} \approx (\mathbb{Z}_2)^\infty$ and we do not have geometric models for all the generators of $\mathbf{M}$, beyond the ones that have been constructed above and the similar ones from (3.10), based on Fig. 12. The main difficulty is that we do not have *localized* resolutions of singularities, present in the boundaries of immersed blobs, into canonical singular (like a small ∞-ribbon) and non-singular parts. This prevents us from establishing an analogue of Lemma 3.5 for blobs. ◇

In the case of *oriented* blobs, many new invariants counting self-intersections of their boundaries arise. In principle, there are 16 ways to combine the orientation independent types $\mathbf{I} - \mathbf{IV}$ with the orientation dependent types $\mathsf{I} - \mathsf{IV}$. As before, the curves ℓ that belong to the self-intersections of the δ-boundaries of solid cobordisms W, define 8 parings π from the set $\mathcal{R}^{\bullet\bullet}$ between these 16 types:

$$(\mathbf{I}, \mathsf{I}) \Leftrightarrow (\mathbf{I}, \mathsf{III}), \ (\mathbf{I}, \mathsf{II}) \Leftrightarrow (\mathbf{I}, \mathsf{IV}), \ (\mathbf{II}, \mathsf{I}) \Leftrightarrow (\mathbf{IV}, \mathsf{III}), \ (\mathbf{II}, \mathsf{II}) \Leftrightarrow (\mathbf{IV}, \mathsf{IV}),$$
$$(\mathbf{III}, \mathsf{I}) \Leftrightarrow (\mathbf{III}, \mathsf{III}), \ (\mathbf{III}, \mathsf{II}) \Leftrightarrow (\mathbf{III}, \mathsf{IV}), \ (\mathbf{IV}, \mathsf{I}) \Leftrightarrow (\mathbf{II}, \mathsf{III}), \ (\mathbf{IV}, \mathsf{II}) \Leftrightarrow (\mathbf{II}, \mathsf{IV}).$$
$$(3.9)$$

These pairings from $\mathcal{R}^{\bullet\bullet}$ follow two rules: 1) start with a bold Roman number, indexing a sector that belongs to a pair of intersecting blobs at a point a where their boundaries intersect (see Fig. 7); subject the sector to a reflection with respect to the vector $\hat{v}(a)$, then the new sector acquires new bold Roman number; 2) the second Roman number that indexes the orientations of two intersecting boundaries, is changed to the Roman number that indexes the two flipped orientations of the boundary curves.

Consider two subsets, $\mathbf{A}$ and $\mathbf{B}$, of the 16 patterns, each subset containing 8 patterns: $\mathbf{A}$ consists of patterns whose second index is I or II, while $\mathbf{B}$ consists of patterns whose second index is III or IV. We notice that the pairings from (3.9) couple only elements from $\mathbf{A}$ with elements from $\mathbf{B}$.

For a given oriented blob $\alpha : X \to A$, counting the crossings of each of the 16 types κ from the list (3.9), produces a number ρ_κ. Each of the 8 pairings $\pi \in \mathcal{R}^{\bullet\bullet}$ from the list (3.9), with the help of the self-intersection curves $\ell \subset \delta W$, generates an oriented cobordism invariant $\rho_\kappa - \rho_{\pi(\kappa)} \in \mathbb{Z}$.

Together, $\{\rho_\kappa - \rho_{\pi(\kappa)}\}_\kappa$ produce a homomorphism

$$\mathcal{I}\rho^{\bullet\bullet} : \mathbf{OB}^{\text{imm}}_{\text{moderate}\leq 2}(A) \longrightarrow (\mathbb{Z})^8.$$

Examining the pairings π from the list (3.9) between different types of intersection patterns, we notice that π couples crossings of two *similarly oriented* (clockwise or

counterclockwise) blobs with crossings of two *similarly oriented* blobs, and couples crossings of two *oppositely oriented* blobs with crossings of two *oppositely oriented* blobs. Therefore, $\mathcal{I}\rho^{\bullet\bullet}$ is the direct product of two homomorphisms $\mathcal{I}\rho^{\bullet}$ and $\overline{\mathcal{I}\rho^{\bullet}}$.

Theorem 3.4 *For $A = \mathbb{R} \times [0, 1]$, there is an exact sequence of abelian groups:*

$$0 \to \mathbf{OM} \to \mathbf{OB}^{\text{imm}}_{\text{moderate}\leq 2}(A)\big/\mathbf{OB}^{\text{emb}}_{\text{moderate}\leq 2}(A) \overset{\mathcal{I}\rho^{\bullet} \times \overline{\mathcal{I}\rho^{\bullet}}}{\longrightarrow}$$

$$(\mathbb{Z})^4 \times (\mathbb{Z})^4 \overset{\Sigma \times \Sigma}{\longrightarrow} \mathbb{Z}_2 \times \mathbb{Z}_2 \to 0,$$

where $\mathbf{OB}^{\text{emb}}_{\text{moderate}\leq 2}(A) \overset{K \times L}{\approx} \mathbb{Z} \times \mathbb{Z}$ is a direct summand of $\mathbf{OB}^{\text{imm}}_{\text{moderate}\leq 2}(A)$. The homomorphism Σ takes each 4-vector to the sum of its components modulo 2.

The kernel $\mathbf{OM}$ contains the subgroup $(\mathbb{Z})^{\infty}$, generated by the blobs $\{Y_{(\bar{\mu},\nu)}(n, 0)\}$, $\{Y_{(\bar{\mu},\nu)}(0, m)\}$ from (3.10) (see Fig. 12) and indexed by the elements of the set

$$\big\{m, n \in \mathbb{Z}; \ m, n \neq 0; \ (\bar{\mu}, \nu) \in \tilde{\mathbf{A}} =_{\text{def}} \{\mathbf{I}, \mathbf{II}\} \times \{\mathsf{I}, \mathsf{II}\}\big\}.$$

Proof Consider two smooth simple loops (say, ellipcii), $\mathcal{C}_1$ and $\mathcal{C}_2$, in $A = \mathbb{R} \times [0, 1]$ such that they intersect transversally only at a pair of points a and b, where the vectors $\hat{v}(a)$ and $\hat{v}(b)$ are transversal to both curves (see Fig. 12). We pick any pair of indexes $(\bar{\mu}(a), \nu(a))$, where $\bar{\mu}(a)$ from the list $\{\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}\}$ and $\nu(a)$ from the list $\{\mathsf{I}, \mathsf{II}, \mathsf{III}, \mathsf{IV}\}$. This choice of $(\bar{\mu}(a), \nu(a))$ determines orientations (counterclockwise or clockwise) of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ as well as one of the four possible choices of the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ in A that bound $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Only one of the four choices will produce a pair of compact domains. To address this complication, we trim the non-compact domains by bounding them with very big concentric circles, disjoint from $\mathcal{C}_1 \cup \mathcal{C}_2$.

With these choices in place, the intersection point b acquired a unique type $(\bar{\mu}(b), \nu(b)) \in \{\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}\} \times \{\mathsf{I}, \mathsf{II}, \mathsf{III}, \mathsf{IV}\}$. Thus, with the help of oriented $\mathcal{D}_1$ and $\mathcal{D}_2$, we get a permutation map Π of the set $\{\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}\} \times \{\mathsf{I}, \mathsf{II}, \mathsf{III}, \mathsf{IV}\}$ to itself.

By a direct inspection, we see that Π preserves the two subsets $\mathbf{A} = \{\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}\} \times \{\mathsf{I}, \mathsf{II}\}$ and $\mathbf{B} = \{\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}\} \times \{\mathsf{III}, \mathsf{IV}\}$. Moreover, Π is a *free* involution on $\mathbf{A}$ and on $\mathbf{B}$. In particular, the involution Π generates 4 pairs-orbits in $\mathbf{A}$ and 4 pairs-orbits in $\mathbf{B}$.

In contrast, the involutions

$$\pi : \{\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}\} \times \{\mathsf{I}, \mathsf{II}, \mathsf{III}, \mathsf{IV}\} \to \{\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}\} \times \{\mathsf{I}, \mathsf{II}, \mathsf{III}, \mathsf{IV}\}$$

from the list $\mathcal{R}^{\bullet\bullet}$ pair elements of $\mathbf{A}$ to elements of $\mathbf{B}$. As a result, the invariants $\{\rho_\kappa - \rho_{\pi(\kappa)}\}_{\kappa \in \mathbf{A}}$ do not vanish for the blobs $\mathcal{D}_1$ and $\mathcal{D}_2$ whose crossings are indexed by $(\bar{\mu}(a), \nu(a))$ and $(\bar{\mu}(b), \nu(b))$, both from $\mathbf{A}$ or $\mathbf{B}$. Moreover, $\{\rho_{(\bar{\mu}(a), \nu(a))} = 1 = \rho_{(\bar{\mu}(b), \nu(b))}\}$ for any of the 8 choices of $(\bar{\mu}(a), \nu(a)) \in \mathbf{A}$ and for any of the 8 choices of $(\bar{\mu}(a), \nu(a)) \in \mathbf{B}$. Therefore, each choice of $(\bar{\mu}(a), \nu(a)) \in \mathbf{A}$ or $(\bar{\mu}(a), \nu(a)) \in \mathbf{B}$ produces a vector $\vec{z}_{(\bar{\mu}(a), \nu(a))} \in (\mathbb{Z})^{16}$ with exactly two components equal to 1 and the rest of components equal to 0. The pairs of 1's in each vector are indexed by either elements of $\mathbf{A}$ or by elements of $\mathbf{B}$. Again, by a direct inspection, the 8×16
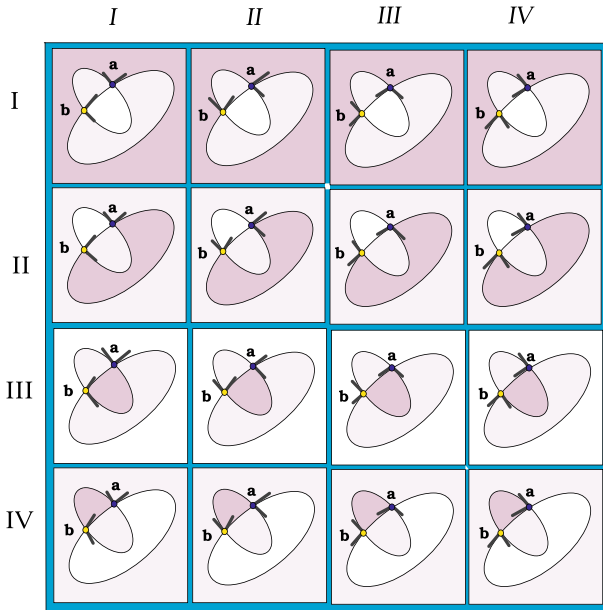
Fig. 12 The table lists 16 possible types at the crossing a. Vertically are displayed blobs' types **I, II, III, IV**, horizontally doodles' types I, II, III, IV. The vector field $\hat{v}$ is vertical. The corresponding types at the second crossing b define the pairing Π. The two bars at each crossing show the orientations of the intersecting curves. The shadings indicate the two choices of interior or exterior for each ellipse. Any non-compact domain is assumed to be bounded by a big circle, not shown

matrix T, formed by these vectors is of the rank 8. It consists of four 4×8 blocks $T_{11}, T_{12}, T_{21}, T_{22}$, where $T_{11} = T_{22}$ and $T_{12} = \mathbf{0} = T_{21}$. In turn, T_{11} consists of two 4×4 blocks, the first of which has 1's along the diagonal, and the second one has 1's along the anti-diagonal; the rest of entries are zeros.

Therefore, the image of $\mathbf{OB}^{\mathrm{imm}}_{\mathrm{moderate}\leq 2}(A) \big/ \mathbf{OB}^{\mathrm{emb}}_{\mathrm{moderate}\leq 2}(A) \xrightarrow{\mathcal{I}\rho^{\bullet} \times \overline{\mathcal{I}\rho^{\bullet}}} (\mathbb{Z})^4 \times (\mathbb{Z})^4$ contains the lattice spanned by the 8 vectors $\{\vec{z}_{(\bar{\mu}(a),\, v(a))}\}_{(\bar{\mu}(a),\, v(a)) \in \mathbf{A} \cup \mathbf{B}}$.

On the other hand, by Lemma 3.2, no vector in $(\mathbb{Z})^4$ whose sum of components is odd (in particular, with a single component 1) can be realized by an immersion of a compact surface $\alpha : X \to A$, since the total number of crossings $\sum_{\kappa}(\rho_{\kappa} + \rho_{\pi(\kappa)})$ for $\alpha(\partial X)$ must be even, and $\sum_{\kappa}(\rho_{\kappa} - \rho_{\pi(\kappa)}) \equiv \sum_{\kappa}(\rho_{\kappa} + \rho_{\pi(\kappa)}) \mod 2$.

Thus, the image of $\mathcal{I}\rho^{\bullet} \times \overline{\mathcal{I}\rho^{\bullet}}$ is the kernel of the homomorphism $(\mathbb{Z})^4 \times (\mathbb{Z})^4 \xrightarrow{\Sigma \times \Sigma} \mathbb{Z}_2 \times \mathbb{Z}_2$.

For any element $(\bar{\mu}, v) \in \mathbf{A}$, consider the two blobs $X_{(\bar{\mu}, v)}$ in Fig. 12 for which the crossing at a has the type $(\bar{\mu}, v)$. Then, the crossing at b has the type $\Pi(\bar{\mu}, v) \in \mathbf{A}$.

There are unique $\pi, \pi' \in \mathcal{R}^{\bullet\bullet}$ (see (3.9)) such that $\pi(\bar{\mu}, v) \in \mathbf{B}$ and $\pi'(\Pi(\bar{\mu}, v)) \in \mathbf{B}$. Moreover, $\pi'(\Pi(\bar{\mu}, v)) = \Pi(\pi(\bar{\mu}, v))$.

We denote by $X_{(\bar{\mu}, v)}(n)$ the blob $X_{(\bar{\mu}, v)}$, surrounded by n big concentric coherently oriented discs. Then the blob

$$Y_{(\bar{\mu},\nu)}(n,m) =_{\mathsf{def}} X_{(\bar{\mu},\nu)}(n) \uplus X_{\pi(\bar{\mu},\nu)}(m) \tag{3.10}$$

is in the kernel of $\mathcal{I}\rho^{\bullet} \times \overline{\mathcal{I}\rho^{\bullet}}$. Moreover, for $m \neq n$, $Y_{(\bar{\mu},\nu)}(n,m)$ is an element of infinite order in **OM**. The validation of this claim is similar to the one used in the proof of Theorem 3.3. First, we show that $Y_{(\bar{\mu},\nu)}(n,m)$ is non-trivial element in **OM**. Assume to the contrary that there is a solid cobordism $B : W \to A \times [0,1)$ that bounds $Y_{(\bar{\mu},\nu)}(n,m)$. Consider the self-intersection curve $\ell \subset B(\delta W)$ that connects the unique self-intersection point a of type $(\bar{\mu},\nu) \in X_{(\bar{\mu},\nu)}(n)$ to the unique self-intersection point a' of type $\pi(\bar{\mu},\nu)$ in $X_{\pi(\bar{\mu},\nu)}(m)$. Since B is an immersion, the preimage $B^{-1}(\ell)$ must be union of several arcs. By the construction of ℓ, two of these arcs, ℓ_1 and ℓ_2, belong to the boundary δW, while the rest of the arcs start at n points in the interior of W and terminate at m points in the interior of W. For $m \neq n$, at least one of these arcs must hit δW at some point c. Therefore, the point $B(c) \in \ell$ belongs to triple-intersection locus of $B(\delta W)$, a contradiction with the assumption that $B : \delta W \to A \times [0,1]$ is 2-moderate. The demonstration that $Y_{(\bar{\mu},\nu)}(n,m)$ is an element of infinite order is similar. □

Conjecture 3.1 *For $A = \mathbb{R} \times [0,1]$, consider the exact sequence of abelian groups:*

$$0 \to \mathbf{OM} \to \mathbf{OB}^{\mathsf{imm}}_{\mathsf{moderate}\leq 2}(A)/(\mathbb{Z} \times \mathbb{Z}) \xrightarrow{\mathcal{I}\rho^{\bullet} \times \overline{\mathcal{I}\rho^{\bullet}}} (\mathbb{Z})^4 \times (\mathbb{Z})^4 \xrightarrow{\Sigma \times \Sigma} \mathbb{Z}_2 \times \mathbb{Z}_2 \to 0$$

from Theorem 3.4. The kernel **OM** *is $\mathbb{Z}$-generated by the blobs $Y_{(\bar{\mu},\nu)}(n,m)$ from (3.10).*

For the set of rules $\mathcal{R}^{\bullet\bullet}$, there is an isomorphism

$$\Theta^{\bullet\bullet} : \mathbf{D}(\mathcal{R}^{\bullet\bullet})/\mathbf{D}_0(\mathcal{R}^{\bullet\bullet}) \hookrightarrow \mathbf{OM}.$$

With the help of $\Theta^{\bullet\bullet}$, the elements of **OM** *acquire "norms" $m(Q, \mathcal{R}^{\bullet\bullet}, \mathcal{E}) \in \mathbb{Z}_+$ as in (3.6).* ◇

To formulate the next conjecture, we introduce very informally a new topological space $\mathcal{G}_{\mathsf{moderate}\leq 2}$. Its points are the 2-moderate functions $f \in \mathcal{F}_{\mathsf{moderate}\leq 2}$ together with a "coupling" τ of their zeros. This τ may "link" either two simple zeros of f, or a double zero to itself, or a double zero to a pair of distinct simple zeros. One needs to describe also the deformation rules for τ, as a function f deforms within the space $\mathcal{F}_{\mathsf{moderate}\leq 2}$.

The space $\mathcal{G}_{\mathsf{moderate}\leq 2}$ admits a slightly different interpretation. Let $\mathcal{Q}_{\mathsf{moderate}\leq 2} \subset \mathcal{F}_{\mathsf{moderate}\leq 2}$ be the subspace of functions with either two simple zeros, or a single double zero, or with no zeros at all. Then any $f \in \mathcal{F}_{\mathsf{moderate}\leq 2}$ is a finite product $\prod_i f_i$, where $f_i \in \mathcal{Q}_{\mathsf{moderate}\leq 2}$ and the order of the multipliers is unimportant. Of course, the presentation $f = \prod_i f_i$ is far from being unique. For example, we always can replace $\prod_i f_i$ by a product $\prod_i h_i f_i$, where $\{h_i\}_i$ have no zeros and $\prod_i h_i = 1$.

We would like to think of points of the hypothetical space $\mathcal{G}_{\mathsf{moderate}\leq 2}$ as the presentations of functions $f \in \mathcal{F}_{\mathsf{moderate}\leq 2}$ as such unordered products, being considered up to the equivalence $\prod_i f_i \sim \prod_i h_i f_i$, where the positive functions $\{h_i\}$ are such that $\prod_i h_i = 1$.

Conjecture 3.2 *Let $A = \mathbb{R} \times [0, 1]$. There exists a topological space $\mathcal{G}_{\mathsf{moderate} \leq 2}$ (whose construction is sketched above) and a surjective map $q : \mathcal{G}_{\mathsf{moderate} \leq 2} \to \mathcal{F}_{\mathsf{moderate} \leq 2}$, given by $q(f, \tau) = f$, with finite fibers and such that the fundamental group*

$$\pi_1(\mathcal{G}_{\mathsf{moderate} \leq 2}, pt) \approx \mathbf{B}^{\mathsf{imm}}_{\mathsf{moderate} \leq 2}(A).$$

The isomorphism is delivered by an analogue Q^{imm} of the map J^{imm} from Theorem 2.1 so that $q \circ Q^{\mathsf{imm}} = J^{\mathsf{imm}}$. ◇

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