Arnold Mathematical Journal (2024) 10:431–448
https://doi.org/10.1007/s40598-024-00245-w

RESEARCH CONTRIBUTION

Polynomial Superpotential for Grassmannian Gr(k, n) from a Limit of Vertex Function

Andrey Smirnov[1] · **Alexander Varchenko**[1]

Received: 19 September 2023 / Revised: 23 January 2024 / Accepted: 2 February 2024 /
Published online: 15 March 2024

Abstract

In this note, we discuss an integral representation for the vertex function of the cotangent bundle over the Grassmannian, $X = T^* \mathrm{Gr}(k, n)$. This integral representation can be used to compute the $\hbar \to \infty$ limit of the vertex function, where $\hbar$ denotes the equivariant parameter of a torus acting on X by dilating the cotangent fibers. We show that in this limit, the integral turns into the standard mirror integral representation of the A-series of the Grassmannian $\mathrm{Gr}(k, n)$ with the Laurent polynomial Landau–Ginzburg superpotential of Eguchi, Hori and Xiong.

Keywords Superpotentials · Vertex functions · J-functions · Landau–Ginzburg model

Mathematics Subject Classification 14G33 · 11D79 · 32G34 · 33C05 · 33E30

1 Introduction

1.1. The vertex functions has been introduced in Ref. [23] as generating functions counting rational quasimaps to Nakajima varieties. In this respect, the vertex function is the "quasimap" analog of the J-function in quantum cohomology. In this paper we consider the cohomological vertex function for cotangent bundle over Grassmannian $X = T^* \mathrm{Gr}(k, n)$. By definition, this function is a power series in the quantum parameter z with coefficients in the equivariant cohomology:

$$\mathsf{Vertex}(z) \in H^{\bullet}_{T}(X)[[z]],$$

✉ Andrey Smirnov
asmirnov@email.unc.edu

Alexander Varchenko
anv@email.unc.edu

1 Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA

where T is a torus acting on X, see Sect. 2. Let $\mathsf{V}(z)$ denote the coefficient of the fundamental class in the vertex function

$$\mathsf{V}(z) := \Big\langle \mathsf{Vertex}(z), [X] \Big\rangle,$$

where $\langle -, - \rangle$ stands for the standard pairing in the equivariant cohomology. The function $\mathsf{V}(z)$ is the analog of the so called A-series in quantum cohomology, which is defined similarly as the pairing of J-function with the fundamental class. The function $\mathsf{V}(z)$ depends non-trivially on the equivariant parameter $\hbar$, which corresponds to the torus acting on X by dilating the cotangent fibers. In this note, we describe the following result (Theorem 4.1):

Theorem 1.1 *In the non-equivariant specialization, one has the following limit:*

$$\lim_{\hbar\to\infty} \mathsf{V}(z/\hbar^n) = \frac{1}{(2\pi\sqrt{-1})^{k(n-k)}} \oint e^{\frac{1}{\epsilon}S(x,z)} \bigwedge_{i,j} \frac{dx_{i,j}}{x_{i,j}}, \tag{1.1}$$

where $S(x,z)$ denotes a Laurent polynomial in $k(n-k)$ variables $x = (x_{i,j})$ given by (4.1).

The integral in the right side of (1.1) denotes the constant term of the integrand, see Sect. 4.2 for the definition. The Laurent polynomial $S(x,z)$ appearing in the limit above is the well-known version of a *superpotential* for $\mathrm{Gr}(k,n)$. It first appeared in the work of Eguchi–Hori–Xiong [8] and was since then reconsidered and generalized by many researchers in vast literature on quantum cohomology of Grassmannians, see for instance Refs. [2, 3, 11, 15, 17, 18] for a very incomplete set of references. The right side of (1.1) is thus the well-known integral representation for the A-series of Grassmannian $\mathrm{Gr}(k,n)$. A closed combinatorial formula for the coefficients of this series is also known, it was first conjectured in Ref. [2] and later proved in Ref. [18], see Corollary 4.8 in Ref. [18] (our z is their q).

Informally speaking (1.1) means that in the limit $\hbar = \infty$ the cotangent directions of X do not contribute to the quasimap partition function and the vertex function of $T^*\,\mathrm{Gr}(k,n)$ degenerates to the J-function of $\mathrm{Gr}(k,n)$. The idea that $\hbar = \infty$ bridges the vertex functions with the J-functions of flag varieties is not new, see Ref. [34] or section 5 of Ref. [16]. However, the derivation of the Laurent superpotential of Ref. [8] from the limit of $\mathsf{V}(z)$ has not been documented well. The goal of this letter is to fill up this gap in the existing literature. The main technical tool which allows us to compute the limit is the integral representation for $\mathsf{V}(z)$ obtained previously by the authors in Theorem 3.2 of Ref. [30].

1.2. As an illustration, let us consider the statement of Theorem 1.1 in the simplest case.

Example. Let $X = T^*\mathbb{P}(\mathbb{C}^2)$. There is a two-dimensional torus $\mathsf{A} = (\mathbb{C}^\times)^2$ naturally acting on $\mathbb{C}^2$ by dilating the coordinate subspaces. We denote the by u_1 and u_2 the equivariant parameters. There is a one-dimensional torus $\mathbb{C}^\times_\hbar$ acting on X by dilation of the cotangent fibers, we denote the corresponding character by $\hbar$. Finally, there is a one-dimensional torus $\mathbb{C}^\times_\epsilon$ which acts on the source of the quasimaps $\mathbb{P}^1 \dashrightarrow X$. We denote the equivariant parameter corresponding to the character of $T_0\mathbb{P}^1$ by ϵ. By definition, the vertex function is a power series with coefficients in equivariant cohomology [23]:

$$\mathsf{Vertex}(z) \in H^\bullet_{\mathsf{A}\times\mathbb{C}^\times_\hbar\times\mathbb{C}^\times_\epsilon}(X)_{loc}[[z]],$$

where the subscript *loc* denotes localization with respect to $\mathbb{C}^\times_\epsilon$. In the basis of $H^\bullet_{\mathsf{A}\times\mathbb{C}^\times_\hbar\times\mathbb{C}^\times_\epsilon}(X)_{loc}$ given by the classes of torus fixed points $[1], [2] \in X^{\mathsf{A}}$ (those correspond to the coordinate lines in $\mathbb{C}^2$), we have closed formulas in terms of the Gauss hypergeometric functions:

$$\begin{aligned} \Big\langle \mathsf{Vertex}(z), [p_1] \Big\rangle &= {}_2F_1\Big(\frac{\hbar}{\epsilon}, \frac{u_2-u_1+\hbar}{\epsilon}; \frac{u_2-u_1+\epsilon}{\epsilon}; z\Big), \\ \Big\langle \mathsf{Vertex}(z), [p_2] \Big\rangle &= {}_2F_1\Big(\frac{\hbar}{\epsilon}, \frac{u_1-u_2+\hbar}{\epsilon}; \frac{u_1-u_2+\epsilon}{\epsilon}; z\Big). \end{aligned} \tag{1.2}$$

In the *non-equivariant* limit $u_1 = u_2 = 0$, corresponding to "turning off" the action of torus A, the above functions coincide and give the coefficient of the vertex function at the fundamental class $[X]$, thus we obtain

$$\Big\langle \mathsf{Vertex}(z), [X] \Big\rangle = {}_2F_1\Big(\frac{\hbar}{\epsilon}, \frac{\hbar}{\epsilon}; 1; z\Big). \tag{1.3}$$

We denote this coefficient by $\mathsf{V}(z)$. Explicitly, we have

$$\mathsf{V}(z) = \sum_{d=0}^{\infty} \frac{(\hbar)_d^2}{(d!)^2\epsilon^{2d}}\, z^d, \quad \text{where} \quad (\hbar)_d = \hbar(\hbar+\epsilon)(\hbar+2\epsilon)\dots(\hbar+(d-1)\epsilon).$$

Noting that $\lim\limits_{\hbar\to\infty} (\hbar)_d/\hbar^d = 1$, we obtain

$$\lim_{\hbar\to\infty} \mathsf{V}(z/\hbar^2) = \sum_{d=0}^{\infty} \frac{z^d}{(d!)^2\epsilon^{2d}}.$$

Let $S(x,z) = x + z/x$, then

$$\oint \frac{\mathrm{d}x}{x}\, S(x,z)^d := [S(x,z)^d]_0 = \begin{cases} \dfrac{(d)!}{(d/2)!(d/2)!} z^d, & d \text{ is even} \\ 0, & d \text{ is odd} \end{cases}$$

where $[S(x,z)^d]_0$ denotes the constant term in x in the Laurent polynomial $S(x,z)^d$. Combining all this together, we can write

$$\lim_{\hbar\to\infty} \mathsf{V}(z/\hbar^2) = \sum_{d=0}^{\infty} \frac{1}{d!} \oint \frac{\mathrm{d}x}{x} S(x,z)^d = \oint \frac{\mathrm{d}x}{x}\, e^{\frac{S(x,z)}{\epsilon}},$$

which is in agreement with Theorem 1.1. A more straightforward way to compute this limit is to note that the hypergeometric function (1.3) has an integral representation:

$$\mathsf{V}(z) = \oint\limits_{|x|=\varepsilon} \frac{\mathrm{d}x}{x} \Big(1-x\Big)^{-\frac{\hbar}{\epsilon}} \Big(1-\frac{z}{x}\Big)^{-\frac{\hbar}{\epsilon}}, \tag{1.4}$$

where ε is any positive real number such that $|z| < \varepsilon < 1$. We note that change of variables $z \to z/\hbar^2$, $x \to x/\hbar$ together with change of the contour $\varepsilon \to \varepsilon/\hbar$ does not affect this condition for large $|\hbar|$. Thus, for large $|\hbar|$, we may have

$$\mathsf{V}(z/\hbar^2) = \oint\limits_{|x|=\varepsilon} \frac{\mathrm{d}x}{x} \Big(1-\frac{x}{\hbar}\Big)^{-\frac{\hbar}{\epsilon}} \Big(1-\frac{z}{x\hbar}\Big)^{-\frac{\hbar}{\epsilon}},$$

which allows us to compute the limit using elementary tools:

$$\lim_{\hbar\to\infty} \Big(1-\frac{x}{\hbar}\Big)^{-\frac{\hbar}{\epsilon}} \Big(1-\frac{z}{x\hbar}\Big)^{-\frac{\hbar}{\epsilon}} = e^{\frac{S(x,z)}{\epsilon}}.$$

1.3. In Sect. 2, we recall a combinatorial formula for the vertex functions generalizing (1.2) to the case of $X = T^*\,\mathrm{Gr}(k,n)$. In Sect. 3, we describe the analog of the integral representation (1.4) for this case. In Sect. 4, we use this integral representation to compute the $\hbar \to \infty$ limit of $\mathsf{V}(z)$ similarly to how it was done in the example above.

In our previous paper [30], we show that certain truncations of $\mathsf{V}(z)$ with parameters specialized to $\mathbb{Q}_p$ satisfy the Dwork type congruence relations. In Sect. 5, we show that a similar structure exists in the limit $\hbar \to \infty$.

2 The Vertex Function of $T^*\,\mathrm{Gr}(k,n)$

2.1. For $X = T^*\,\mathrm{Gr}(k,n)$, we consider the following explicit power series:

$$\langle \mathsf{Vertex}(z), [1,\dots,k]\rangle := \sum_{d=0}^{\infty} c_d(u_1,\dots,u_n,\hbar)\, z^d, \tag{2.1}$$

with the coefficients $c_d(u_1, \dots, u_n, \hbar) \in \mathbb{Q}(u_1, \dots, u_n, \hbar, \epsilon)$ given by

$$c_d(u_1, \dots, u_n, \hbar) = \sum_{\substack{d_1,\dots,d_k: \\ d_1+\dots+d_k=d}} \left(\prod_{i,j=1}^{k} \frac{(\epsilon - u_i + u_j)_{d_i-d_j}}{(\hbar - u_i + u_j)_{d_i-d_j}} \right) \left(\prod_{j=1}^{n} \prod_{i=1}^{k} \frac{(\hbar + u_j - u_i)_{d_i}}{(\epsilon + u_j - u_i)_{d_i}} \right), \tag{2.2}$$

where $(x)_d$ denotes the Pochhammer symbol with step ϵ:

$$(x)_d = \begin{cases} x(x+\epsilon)\dots(x+(d-1)\epsilon), & d > 0 \\ 1, & d = 0 \\ \dfrac{1}{(x-\epsilon)(x-2\epsilon)\dots(x+d\epsilon)}, & d < 0 \end{cases}$$

The degree d coefficient of this series counts (equivariantly) the number of degree d rational curves in X. More precisely, it is given by the equivariant integral

$$c_d(u_1, \dots, u_n, \hbar) = \int\limits_{[\mathsf{QM}_d(X,\infty)]^{\mathrm{vir}}} \omega^{vir}, \tag{2.3}$$

over the virtual fundamental class on moduli space $\mathsf{QM}_d(X, \infty)$ of quasimaps from $\mathbb{P}^1$ to X, which send $\infty \in \mathbb{P}^1$ to a prescribed torus fixed point $[1, \dots, k] \in X$, see Section 7.2 of Ref. [23] for definitions. Using the equivariant localization, the integral (2.3) reduces to the sum over the torus fixed points on $\mathsf{QM}_d(X, \infty)$ which gives the sum (2.2). We refer to Section 4.5 of Ref. [26] where this computation is done in some details.

The parameters $u_1, \dots, u_n, \hbar, \epsilon$ are the equivariant parameters of the torus $T = (\mathbb{C}^\times)^n \times \mathbb{C}^\times_\hbar \times \mathbb{C}^\times_\epsilon$ acting on the moduli space $\mathsf{QM}_d(X, \infty)$ in the following way:

- $(\mathbb{C}^\times)^n$ acts on $\mathbb{C}^n$ in a natural way, scaling the coordinates with weights $u_1, \dots, u_n$.
- The set of torus fixes points $X^{(\mathbb{C}^\times)^n}$ corresponds to k-subspaces in $\mathbb{C}^n$ spanned by any set of k coordinate lines. The fixed point $[1, \dots, k] \in X^{(\mathbb{C}^\times)^n}$ corresponds to the k-subspace spanned by the first k coordinate lines.
- $\mathbb{C}^\times_\hbar$ acts on X by scaling the cotangent fibers with weight $\hbar$.
- $\mathbb{C}^\times_\epsilon$ acts on the source of the quasimaps $C \cong \mathbb{P}^1$ fixing the points $0, \infty \in \mathbb{P}^1$. The parameter ϵ denotes the corresponding weight of the tangent space $T_0 C$.

The full vertex function is a power series with coefficients in equivariant cohomology:

$$\mathsf{Vertex}(z) \in H^\bullet_T(X)_{loc}[[z]],$$

where loc denotes the equivariant localization with respect to torus $\mathbb{C}^\times_\epsilon$. Using the equivariant localization, we can expand $\mathsf{Vertex}(z)$ in the basis of $H^\bullet_{T\times\mathbb{C}^\times_\epsilon}(X)_{loc}$ given by the classes of torus fixed points. The power series (2.1) gives the coefficient

$\mathsf{Vertex}(z)$ at the "first" torus fixed point $[1,\ldots,k]$. Other coefficients have the same structure and can be obtained from (2.1) by permutations of parameters u_i.
2.2. In this paper, we consider the specialization of the equivariant parameters:

$$u_1 = 0, \ldots, u_n = 0, \tag{2.4}$$

which corresponds to non-equivariant limit when the action of the torus $(\mathbb{C}^\times)^n$ is "turned off". The coefficients of $\mathsf{Vertex}(z)$ at the torus fixed points all reduce to the same function (simply because without $(\mathbb{C}^\times)^n$-action these points are indistinguishable) which corresponds to the coefficient of the vertex function at the fundamental class:

$$\mathsf{V}(z) := \Big\langle \mathsf{Vertex}(z), [X] \Big\rangle \Big|_{u_1=0,\ldots,u_n=0}. \tag{2.5}$$

Thus, $\mathsf{V}(z)$ can be obtained by specializing the coefficients of the power series (2.1) at (2.4). We note that this specialization is non-trivial: already in the case of $T^*\,\mathrm{Gr}(2,4)$ the terms in the sum (2.2) have poles at $u_i = u_j$. The total sum (2.2) is, however, non-singular since the vertex function is an integral equivariant cohomology class (we recall that only $\mathbb{C}^\times_\epsilon$ - localization is required to define it).

3 Integral Representation of $\mathsf{V}(z)$

3.1. In this section, we describe an integral representation for the function (2.5)

$$\mathsf{V}(z) = \int_\gamma \Phi(x,z)\,\mathrm{d}x,$$

which has its origin in $3D$-mirror symmetry, we refer to Section 3 of Ref. [30] for more details.

3.2. Assume that $n \geqslant 2k$. Let v_i, $i = 1,\ldots,n-1$ be integers defined by

$$\mathsf{v}_i = \begin{cases} i, & i < k, \\ k, & k \leqslant i \leqslant n-k, \\ n-i, & n-k < i, \end{cases}$$

We denote by $\omega = \hbar/\epsilon$ and define the *superpotential* function:

$$\Phi(x,z) = \Big(\prod_{i=1}^{n-1}\prod_{j=1}^{\mathsf{v}_i} x_{i,j}\Big)^{-1+\omega} \Big(\prod_{m=1}^{\mathsf{v}_m} \prod_{1\leqslant i<j\leqslant \mathsf{v}_m} (x_{m,j}-x_{m,i})\Big)^{2\omega}$$
$$\times \Big(\prod_{i=1}^{n-2}\prod_{a=1}^{\mathsf{v}_i}\prod_{b=1}^{\mathsf{v}_{i+1}} (x_{i,a}-x_{i+1,b})\Big)^{-\omega} \Big(\prod_{i=1}^{k} (z_1 - x_{k,i})(z_2 - x_{n-k,i})\Big)^{-\omega}. \tag{3.1}$$

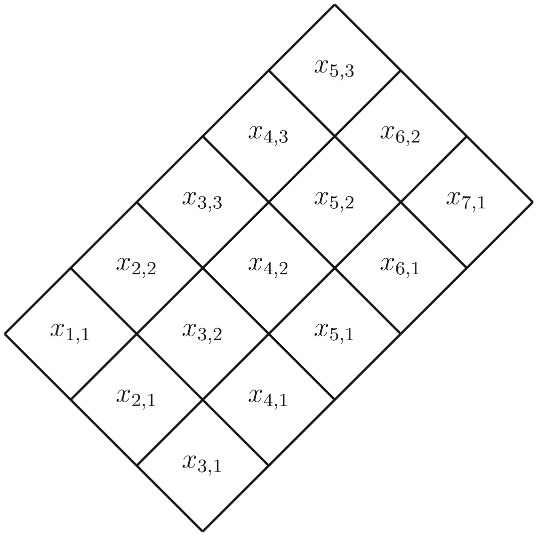


Fig. 1 Set of variables $x_{i,j}$ for $k = 4$ and $n = 8$

We note that this function is an example of the *master functions* in the theory of integral representations of the trigonometric Knizhnik–Zamolodchikov equations. In particular, (3.1) corresponds to the KZ equation associated with the weight subspace of weight [1,...,1] in the tensor product the k-th and $(n-k)$-th fundamental representations of $\mathfrak{gl}_n$, see Refs. [20, 31].

3.3. The dimension vector v_i and the variables $x_{i,j}$ have a convenient combinatorial visualization. Let us consider a $k \times (n-k)$ rectangle rotated counterclockwise by 45°, see Fig. 1. Note that in this picture, the number of boxes in i-th vertical column is exactly v_i. In this way, we may assign the variables $x_{i,j}$ to the boxes in this picture. We will order them as in Fig. 1. Note that the total number of variables $x_{i,j}$ equals to $\dim \mathrm{Gr}(k,n) = k(n-k)$. To a box (i, j) in Fig. 1, we assign a weight

$$m_{i,j} = (|i-k| + 2j - 1) \in \mathbb{N}. \tag{3.2}$$

This function ranges from $m_{k,1} = 1$ to $m_{n-k,k} = n-1$. The definition of $m_{i,j}$ is clear from Fig. 2. We have a partial ordering on the boxes (i, j) corresponding to

$$m_{k,1} < m_{k-1,1} = m_{k+1,1} < \cdots < m_{n-k,k}. \tag{3.3}$$

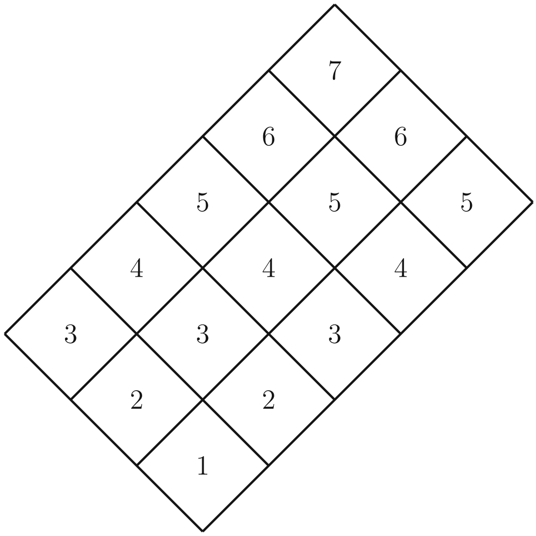
For a small real number $0 < \varepsilon \ll 1$, let us define the torus by the following equations:

$$\gamma_{k,n} \subset \mathbb{C}^{k(n-k)}, \quad |x_{i,j}| = m_{i,j}\varepsilon, \tag{3.4}$$

where i, j run through all possible values.

Proposition 3.1 *Assume that $|z_1| < \varepsilon$ and $(n-1)\varepsilon < |z_2|$, then the superpotential (3.1) has a single-valued branch on the torus $\gamma_{k,n}$, which is distinguished in the proof and which will be used in the paper.*

Fig. 2 The values of the weight function $m_{i,j}$



Proof Let us denote

$$L(x_{i,a}, x_{j,b}) = \begin{cases} (1 - x_{i,a}/x_{j,b})^{-\omega}, \; m_{i,a} < m_{j,b}, \; i \neq j \\ (x_{j,b}/x_{i,a} - 1)^{-\omega}, \; m_{i,a} > m_{j,b}, \; i \neq j. \end{cases} \tag{3.5}$$

Each of these ratios $x_{i,a}/x_{j,b}$, $x_{j,b}/x_{i,a}$ restricted to $\gamma_{k,n}$ has absolute value less than 1. We replace $(1 - x_{i,a}/x_{j,b})^{-\omega}$ on $\gamma_{k,n}$ with $\sum_{m=0}^{\infty} \binom{-\omega}{m} (-x_{i,a}/x_{j,b})^m$ and replace $(x_{j,b}/x_{i,a} - 1)^{-\omega}$ with $e^{-\pi\sqrt{-1}\omega} \sum_{m=0}^{\infty} \binom{-\omega}{m} (-x_{j,b}/x_{i,a})^m$.

Next, we denote $L(z_1, x_{k,a}) = (1 - z_1/x_{k,a})^{-\omega}$ and $L(z_2, x_{k,a}) = (1 - x_{n-k,a}/z_2)^{-\omega}$. On $\gamma_{k,n}$, we have $|x_{k,i}| \geqslant \epsilon$, and $|x_{n-k,i}| \leqslant |x_{n-k,k}| = n\epsilon$, therefore, $|z_1/x_{k,i}| < 1$ and $|x_{n-k,i}/z_2| < 1$. We replace on $\gamma_{k,n}$ the factor $(1 - z_1/x_{k,a})^{-\omega}$ with $\sum_{m=0}^{\infty} \binom{-\omega}{m} (-z_1/x_{k,a})^m$ and the factor $(1 - x_{n-k,a}/z_2)^{-\omega}$ with $\sum_{m=0}^{\infty} \binom{-\omega}{m} (-x_{n-k,a}/z_2)^m$.

Finally, we denote $\Delta(x_{m,i}, x_{m,j}) = (1 - x_{m,i}/x_{m,j})^{2\omega}$ for $1 \leqslant i < j \leqslant \mathsf{v}_m$. On $\gamma_{k,n}$, we have $|x_{m,i}/x_{m,j}| < 1$. We replace on $\gamma_{k,n}$ the factor $\Delta(x_{m,i}, x_{m,j})$ with $\sum_{m=0}^{\infty} \binom{2\omega}{m} (-x_{m,i}/x_{m,j})^m$. In these notations, we have

$$\begin{aligned} \Phi(x, z) = &\left(\prod_{i=1}^{n-1} \prod_{a<b} \Delta(x_{i,a}, x_{i,b}) \right) \times \left(\prod_{i=1}^{n-2} \prod_{a=1}^{\mathsf{v}_i} \prod_{b=1}^{\mathsf{v}_{i+1}} L(x_{i,a}, x_{i+1,b}) \right) \\ &\times \left(\prod_{i=1}^{k} L(z_1, x_{k,i}) L(z_2, x_{n-k,i}) \right) \times \prod_{i=1}^{n-1} \prod_{j=1}^{\mathsf{v}_i} x_{i,j}^{-1}, \end{aligned} \tag{3.6}$$

and for each factor a single-valued branch is chosen by replacing that factor with the corresponding power series. The product of those power series distinguishes a single-valued branch of $\Phi(x, z)$ on $\gamma_{k,n}$. □

Example For $X = T^* \operatorname{Gr}(2,4)$, we have

$$\begin{aligned}\Phi(x,z) = &(x_{11}x_{21}x_{22}x_{31})^{-1}(1-x_{21}/x_{22})^{2\omega}\\ &\times\big((x_{21}/x_{11}-1)(1-x_{21}/x_{31})(z_1/x_{21}-1)(1-x_{21}/z_2)\big)^{-\omega}\\ &\times\big((1-x_{11}/x_{22})(x_{31}/x_{22}-1)(z_1/x_{22}-1)(1-x_{22}/z_2)\big)^{-\omega}.\end{aligned}$$

From the previous proposition, the integral of $\Phi(x,z)$ over $\gamma_{k,n}$ is an analytic function of $z = z_1/z_2$ in the disc $|z| < \epsilon$.

Theorem 3.2 ([30]) *The function (2.5) has the following integral representation:*

$$\mathsf{V}(z) = \frac{\alpha}{(2\pi\sqrt{-1})^{k(n-k)}} \oint_{\gamma_{k,n}} \Phi(x,z) \bigwedge_{i,j} dx_{i,j}, \tag{3.7}$$

where $\Phi(x,z)$ is the branch of superpotential function (3.1) on the torus $\gamma_{k,n}$ chosen in Proposition 3.1, and $\alpha = e^{\pi\sqrt{-1}N\omega}$ is a normalization constant where N is the number of factors in (3.6) *having the form* $(x_{j,b}/x_{i,a}-1)^{-\omega}$.

Definition 3.3 Let $\gamma'_{k,m}$ be another contour defined by $|x_{i,j}| = R_{i,j}$ for $R_{i,j} \in \mathbb{R}$ such that the conditions $|z_1| < R_{1,1}$, $R_{n-k,k} < |z_2|$ and

$$m_{i,j} < m_{a,b} \Longrightarrow R_{i,j} < R_{a,b} \tag{3.8}$$

are satisfied for all pairs of indices (i,j) and (a,b). Then, we say that $\gamma'_{k,m}$ is equivalent to $\gamma_{k,m}$ and write $\gamma'_{k,m} \sim \gamma_{k,m}$.

Note that (3.7) remains invariant if we replace $\gamma_{k,m}$ by an equivalent $\gamma'_{k,m}$. This is simply because the evaluation of the integral over $\gamma'_{k,n}$ is again by computing the residues at $x_{i,j} = 0$, and the residues are computed in the same order as for $\gamma_{k,n}$, and therefore, the result remains the same.

3.4 Relation to 3*D*-Mirror Symmetry

Let us explain the origin of the superpotential function (3.1). The factors of (3.1) correspond to the edges of the quiver which describes the 3*D-mirror variety* $X^!$. For $X = T^*Gr(k,n)$, the quiver of $X^!$ is given in Sections 3.2–3.3 of Ref. [30], the correspondence between the factors of (3.1) and the edges of this quiver is also explained there.

For the Nakajima quiver varieties, the superpotential function (3.1) is constructed by the same procedure if the quiver for the 3D-mirror $X^!$ is known. For the Nakajima quiver varieties of type A, which include cotangent bundles over partial flag varieties as special cases, a conjectural description of the 3D-mirrors was given by physicists. It is explained for instance in Ref. [9]. We expect that the results of this note and of Ref. [30] have straightforward generalizations to these cases.

The $3D$-mirror symmetry conjecture is formulated on the level of K-theory rather than cohomology. Recall, that the quantum difference equations [25] are the K-theoretic generalizations of quantum differential equations in quantum cohomology. The $3D$-mirror symmetry conjecture claims that the quantum difference equations for X and $X^!$ are equivalent. The K-theoretic vertex functions of X and $X^!$ provide two different bases of solutions to the this common system of q-difference equations. For cotangent bundles over Grassmannians this conjecture was proved by Dinkins in Ref. [5] and for full flag varieties in Ref. [6]. For the hypertoric varieties, this result is obtained in Ref. [33].

An alternative definition of $3D$-mirror symmetry postulates the equality of the elliptic stable envelopes [1] of X and $X^!$. This idea was first proposed in Ref. [24] and later examined for various cases of X in Ref. [27–29]. It was shown in Ref. [13, 14] that the elliptic stable envelope of X determines the corresponding quantum difference equation of X and vice versa. This established an equivalence between the two definitions of $3D$-mirror symmetry.

Theorem 1.1 says that the mirror description of J-function for $\mathrm{Gr}(k,n)$ arises as a double limit of $3D$-mirror symmetry. In the first limit, one considers the cohomological limit of K-theoretic vertex functions for $T^*\mathrm{Gr}(k,n)$. In this limit, the $3D$-mirror symmetry description of these functions [5] degenerates to the integral representation (3.7). In the second limit $\hbar \to \infty$, we obtain Theorem 1.1.

4 The Limit $\hbar \to \infty$

4.1 Polynomial Superpotential

Let Γ be an oriented graph, with vertices given by boxes inside the $k \times (n-k)$ Young diagram, plus two extra vertices corresponding to z_1 and z_2, see Fig. 3. The edges of the graph are defined as follows: every two adjacent boxes are connected by an edge. Each edge is oriented in the direction of decrease of weight function $m_{i,j}$, which is defined by (3.2). Two additional edges are from $x_{k,1}$ to z_1 and from z_2 to $x_{n-k,k}$, Fig. 3. Given an edge e of Γ, we denote by $h(e)$ and $t(e)$ the corresponding head and tail. We define the following Laurent polynomial:

$$S(x,z) = \sum_{e \in \mathrm{edges}(\Gamma)} \frac{x_{h(e)}}{x_{t(e)}}. \tag{4.1}$$

Example For $k=1$, we obtain

$$S(x,z) = \frac{z_1}{x_{1,1}} + \frac{x_{1,1}}{x_{2,1}} + \frac{x_{2,1}}{x_{3,1}} + \cdots + \frac{x_{n-2,1}}{x_{n-1,1}} + \frac{x_{n-1,1}}{z_2}. \tag{4.2}$$

Substituting $z_1 = q$, $z_2 = 1$, and introducing new variables by

$$x_{1,1} = a_1 a_2 \ldots a_{n-1}, \quad x_{2,1} = a_1 a_2 \ldots a_{n-2}, \quad \ldots, \quad x_{n-1,1} = a_1,$$

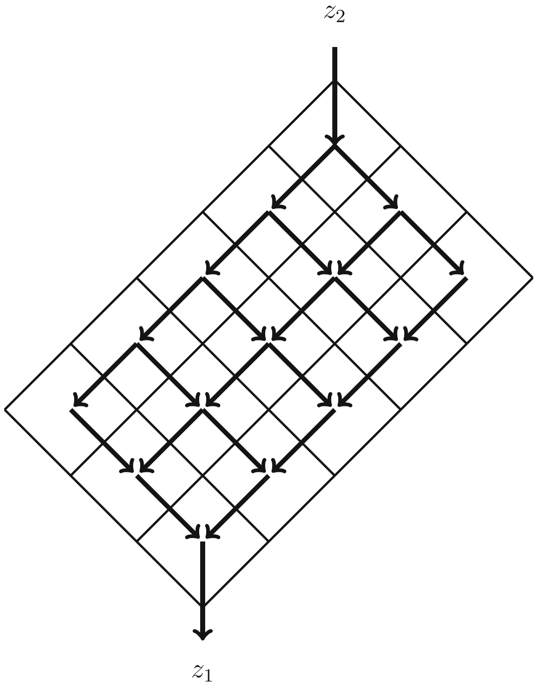


Fig. 3 The graph Γ associated with $X = T^* \mathrm{Gr}(k, n)$

we arrive at the standard Givental's superpotential of projective space

$$S(x) = a_1 + a_2 + \cdots + a_{n-1} + \frac{q}{a_1 \cdots a_{n-1}}.$$

Example For $X = T^* \mathrm{Gr}(2, 4)$, we obtain

$$S(x, z) = \frac{z_1}{x_{2,1}} + \frac{x_{2,1}}{x_{1,1}} + \frac{x_{2,1}}{x_{3,1}} + \frac{x_{3,1}}{x_{2,2}} + \frac{x_{1,1}}{x_{2,2}} + \frac{x_{2,2}}{z_2}. \tag{4.3}$$

4.2 Exponential Integral

For $S(x, z)$ defined by (4.1), we consider the power series:

$$\frac{1}{(2\pi\sqrt{-1})^{k(n-k)}} \oint e^{\frac{1}{\epsilon} S(x,z)} \bigwedge_{i,j} \frac{\mathrm{d}x_{i,j}}{x_{i,j}} = \sum_{d=0}^{\infty} \frac{\left[S(x, z)^d\right]_0}{d! \epsilon^d}, \tag{4.4}$$

where $\left[S(x, z)^d\right]_0$ denotes the constant term of the Laurent polynomial $S(x, z)^d$ in variables $x = (x_{i,j})$. From the structure of the superpotential (4.1), it is easy to see that $\left[S(x, z)^d\right]_0$ is a monomial in $z = z_1/z_2$ and thus (4.4) is a power series in z.

Example For $X = T^* \mathbb{P}^{n-1}$, the superpotential is given by (4.2). In this case, elementary computations shows that $\left[S(x, z)^d\right]_0$ is non-vanishing only if the degree d is of the form $d = nm$ for some $m \in \mathbb{N}$. In this case, we have

$$\left[S(x,z)^{mn}\right]_0 = \frac{(z_1/z_2)^m (nm)!}{(m!)^n}.$$

We thus conclude that

$$\oint e^{\frac{1}{\epsilon}S(x,z)} \bigwedge_{i,j} \frac{\mathrm{d}x_{i,j}}{x_{i,j}} = \sum_{m=0}^{\infty} \frac{z^m}{(m!)^n \epsilon^{mn}}.$$

Example For $X = T^*\mathrm{Gr}(2,4)$, the superpotential is given by (4.3). In this case, $\left[S(x,z)^d\right]_0$ is non-vanishing only if $d = 4m$, in which case

$$\left[S(x,z)^{4m}\right]_0 = \frac{(2m)!(4m)!}{(m!)^6} \frac{z^m}{\epsilon^{4m}}.$$

Thus, we obtain

$$\oint e^{\frac{1}{\epsilon}S(x,z)} \bigwedge_{i,j} \frac{\mathrm{d}x_{i,j}}{x_{i,j}} = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^6 \epsilon^{4m}} z^m.$$

4.3 The Vertex Function in $\hbar = \infty$ Limit

Theorem 4.1 *Let* $\mathsf{V}(z)$ *be the function (2.5), then*

$$\lim_{\hbar \to \infty} \mathsf{V}(z/\hbar^n) = \frac{1}{(2\pi\sqrt{-1})^{k(n-k)}} \oint e^{\frac{1}{\epsilon}S(x,z)} \bigwedge_{i,j} \frac{dx_{i,j}}{x_{i,j}},$$

where the integral is defined by (4.4) and $S(x,z)$ *is the polynomial superpotential (4.1).*

Proof By Theorem 3.2, we have

$$\mathsf{V}(z) = \frac{\alpha}{(2\pi\sqrt{-1})^{k(n-k)}} \oint_{\gamma_{k,n}} \Phi(x,z) \bigwedge_{i,j} \mathrm{d}x_{i,j},$$

where the contour $\gamma_{k,n}$ is defined by (3.4) and $\Phi(x,z)$ is the branch of the superpotential (3.6) distinguished by Proposition 3.1. It will be convenient to define

$$\tilde{L}(x_{i,a}, x_{j,b}) = \begin{cases} (1 - x_{i,a}/x_{j,b})^{-\omega}, & m_{i,a} < m_{j,b}, \\ (1 - x_{j,b}/x_{i,a})^{-\omega}, & m_{i,a} > m_{j,b}, \end{cases} \tag{4.5}$$

which differ from (3.5) by a factor

$$\tilde{L}(x_{i,a}, x_{j,b}) = \begin{cases} L(x_{i,a}, x_{j,b}), & m_{i,a} < m_{j,b}, \\ e^{-\pi\sqrt{-1}\hbar/\epsilon} L(x_{i,a}, x_{j,b}), & m_{i,a} > m_{j,b}, \end{cases}$$

Recall that $\alpha = e^{\pi\sqrt{-1}N\hbar/\epsilon}$ where N is the total number of factors in $\Phi(x, z)$ for which $\tilde{L}(x_{i,a}, x_{j,b})/L(x_{i,a}, x_{j,b}) = e^{-\pi\sqrt{-1}\hbar/\epsilon}$. Thus, in these notations,

$$\mathsf{V}(z) = \frac{1}{(2\pi\sqrt{-1})^{k(n-k)}} \oint\limits_{\gamma_{k,n}} \tilde{\Phi}(x, z) \bigwedge_{i,j} \frac{\mathrm{d}x_{i,j}}{x_{i,j}}, \tag{4.6}$$

where

$$\tilde{\Phi}(x, z) = \Big(\prod_{i=1}^{n-1} \prod_{a<b} \Delta(x_{i,a}, x_{i,b})\Big)\Big(\prod_{i=1}^{n-2} \prod_{a=1}^{\mathsf{v}_i} \prod_{b=1}^{\mathsf{v}_{i+1}} \tilde{L}(x_{i,a}, x_{i+1,b})\Big)$$
$$\Big(\prod_{i=1}^{k} \tilde{L}(z_1, x_{k,i}) L(z_2, x_{n-k,i})\Big).$$

In this integral, we rescale the variables by: $z_1 \to z_1, \quad z_2 \to z_2\hbar^n$. Since in our notations $z = z_1/z_2$, this is equivalent to substitution $z \to z/\hbar^n$ in the left side of (4.6). Let $\gamma'_{k,n}(\hbar)$ be a contour defined by

$$|x_{i,j}| = m_{i,j}\varepsilon|\hbar|^{m_{i,j}}$$

assuming that $|\hbar| > 1$, we have

$$m_{i,j} < m_{a,b} \implies m_{i,j}\varepsilon|\hbar|^{m_{i,j}} < m_{a,b}\varepsilon|\hbar|^{m_{a,b}}$$

for all pairs (i, j) and (a, b). By assumption of Proposition 3.1 $|z_1| < |x_{1,1}|$ and $|x_{n-k,k}| < |z_2\hbar^n|$ on $\gamma'_{k,n}(\hbar)$. Therefore, $\gamma'_{k,n}(\hbar) \sim \gamma_{k,n}$ in the sense of Definition 3.3. Thus,

$$\mathsf{V}(z/\hbar^n) = \frac{1}{(2\pi\sqrt{-1})^{k(n-k)}} \oint\limits_{\gamma'_{k,n}(\hbar)} \tilde{\Phi}(x, z) \bigwedge_{i,j} \frac{\mathrm{d}x_{i,j}}{x_{i,j}}. \tag{4.7}$$

Now, in this integral, we change the variables of integration by $x_{i,j} = y_{i,j}\hbar^{m_{i,j}}$. The contour $\gamma'_{k,n}(\hbar)$ in the variables $y_{i,j}$ is given by $|y_{i,j}| = m_{i,j}\varepsilon$, i.e., in the coordinates $y_{i,j}$ we integrate over the original contour $\gamma_{k,n}$. Overall, we obtain

$$\mathsf{V}(z/\hbar^n) = \frac{1}{(2\pi\sqrt{-1})^{k(n-k)}} \oint\limits_{\gamma_{k,n}} \tilde{\Phi}(y, z) \bigwedge_{i,j} \frac{\mathrm{d}y_{i,j}}{y_{i,j}}, \tag{4.8}$$

where

$$\tilde{\Phi}(y,z) = \Big(\prod_{i=1}^{n-1}\prod_{a<b} \Delta(y_{i,a}\hbar^{m_{i,a}}, y_{i,b}\hbar^{m_{i,b}})\Big)\Big(\prod_{i=1}^{n-2}\prod_{a=1}^{\mathsf{v}_i}\prod_{b=1}^{\mathsf{v}_{i+1}} \tilde{L}(y_{i,a}\hbar^{m_{i,a}}, y_{i+1,b}\hbar^{m_{i+1,b}})\Big)$$
$$\times\Big(\prod_{i=1}^{k} \tilde{L}(z_1, y_{k,i}\hbar^{m_{k,i}})\tilde{L}(z_2\hbar^n, y_{n-k,i}\hbar^{m_{n-k,i}})\Big). \tag{4.9}$$

We have

$$\tilde{L}(y_{i,a}\hbar^{m_{i,a}}, y_{i+1,b}\hbar^{m_{i+1,b}}) = \begin{cases} \Big(1-(y_{i,a}/y_{i+1,b})/(\hbar^{m_{i+1,b}-m_{i,a}})\Big)^{-\hbar/\epsilon}, & m_{i,a} < m_{i+1,b}, \\ \Big(1-(y_{i+1,b}/y_{i,a})/(\hbar^{m_{i,a}-m_{i+1,b}})\Big)^{-\hbar/\epsilon}, & m_{i,a} > m_{i+1,b}. \end{cases}$$

Note that the powers of $\hbar$ appearing in these factors are positive integers. Thus, we compute

$$\lim_{\hbar\to\infty} L(y_{i,a}\hbar^{m_{i,a}}, y_{i+1,b}\hbar^{m_{i+1,b}}) = \begin{cases} e^{\frac{1}{\epsilon}\frac{y_{i,a}}{y_{i+1,b}}}, & m_{i,a} = m_{i+1,b}-1, \\ e^{\frac{1}{\epsilon}\frac{y_{i+1,b}}{y_{i,a}}}, & m_{i,a} = m_{i+1,b}+1, \\ 1, & \text{else.} \end{cases}$$

We also have

$$\tilde{L}(z_1, y_{k,i}\hbar^{m_{k,i}}) = \Big(1-(z_1/y_{k,i})/\hbar^{m_{k,i}}\Big)^{-\hbar/\epsilon},$$
$$\tilde{L}(z_2\hbar^n, y_{n-k,i}\hbar^{m_{n-k,i}}) = \Big(1-y_{n-k,i}\hbar^{m_{n-k,i}-n}/z_2\Big)^{-\hbar/\epsilon}.$$

with $m_{k,i} = 2i-1$ and $m_{n-k,i} = n-2k+2i-1$, therefore,

$$\lim_{\hbar\to\infty} \tilde{L}(z_1, y_{k,i}\hbar^{m_{k,i}}) = \begin{cases} e^{\frac{1}{\epsilon}\frac{z_1}{y_{k,1}}}, & i=1, \\ 1 & i\neq 1 \end{cases}$$

and

$$\lim_{\hbar\to\infty} \tilde{L}(z_2\hbar^n, y_{n-k,i}\hbar^{m_{n-k,i}}) = \begin{cases} e^{\frac{1}{\epsilon}\frac{y_{n-k,k}}{z_2}}, & i=k, \\ 1 & i\neq k \end{cases}$$

Finally,

$$\Delta(y_{i,a}\hbar^{m_{i,a}}, y_{i,b}\hbar^{m_{i,b}}) = (1-y_{i,a}/y_{i,b}/(\hbar^{m_{i,b}-m_{i,a}}))^{2\hbar/\epsilon},$$

and since $m_{i,b} - m_{i,a} \geqslant 2$ for $b > a$, we have

$$\lim_{\hbar \to \infty} \Delta(y_{i,a}\hbar^{m_{i,a}}, y_{i,b}\hbar^{m_{i,b}}) = 1.$$

In summary, the limit $\hbar \to \infty$ of a factor in (4.9) is non-trivial only if it corresponds to an edge of the graph Γ and

$$\lim_{\hbar \to \infty} \tilde{\Phi}(y, z) = \prod_{e \in \text{edges}(\Gamma)} e^{\frac{1}{\epsilon} \frac{y_{h(e)}}{y_{t(e)}}} = e^{\frac{S(y,z)}{\epsilon}}. \tag{4.10}$$

Finally, the point-wise limit (4.10) on the compact set $\gamma_{k,n}$ is uniform, the limit commutes with the integration and from (4.8), we obtain

$$\begin{aligned} \lim_{\hbar \to \infty} \mathsf{V}(z/\hbar^n) &= \lim_{\hbar \to \infty} \frac{1}{(2\pi\sqrt{-1})^{k(n-k)}} \oint_{\gamma_{k,n}} \tilde{\Phi}(y, z) \bigwedge_{i,j} \frac{\mathrm{d}y_{i,j}}{y_{i,j}} \\ &= \frac{1}{(2\pi\sqrt{-1})^{k(n-k)}} \oint_{\gamma_{k,n}} e^{\frac{S(y,z)}{\epsilon}} \bigwedge_{i,j} \frac{\mathrm{d}y_{i,j}}{y_{i,j}}. \end{aligned}$$

□

5 Dwork Congruences

Let $\Delta(k, n) = \mathsf{N}(S(x, z)) \subset \mathbb{R}^{k(n-k)}$ be the Newton polygon of the Laurent polynomial (4.1) in variables $x = (x_{i,j})$. Let $f_{i,j}$ with $i = 1, \ldots, k, \;\; j = 1, \ldots, n-k$ denote the standard basis in $\mathbb{R}^{k(n-k)}$. The elements $f_{i,j}$ correspond to boxes in $k \times (n-k)$ diagram in Fig. 3. From (4.1), we see that $\Delta(k, n)$ is the convex hull of the vectors:

$$\begin{aligned} &f_{1,1}, \quad f_{i,j+1} - f_{i-1,j+1}, \quad i = 2, \ldots, k, \quad j = 0, \ldots, n-k-1 \\ &-f_{k,n-k}, \quad f_{i,j+1} - f_{i,j}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n-k-1. \end{aligned}$$

This polytop was has been considered in many publications, in particular it is known to be *reflexive* see Theorem 3.1.3 in Ref. [2]. We recall that the origin (0, 0) is the only integral point in a reflexive polytop, see for instance Ref. [22] for an overview.

Theorem 5.1 *Let p be a prime number. Let us consider the power series:*

$$\mathsf{F}(z) = \sum_{d=0}^{\infty} \left[S(x, z)^d \right]_0 \in \mathbb{Z}[[z]], \tag{5.1}$$

and a system of its polynomial truncations:

$$\mathsf{F}_s(z) = \sum_{d=0}^{p^s - 1} \left[S(x, z)^d \right]_0 \in \mathbb{Z}[z], \quad s = 0, 1, 2, \ldots$$

Then, for every $s \geqslant 1$, one has a congruence

$$\frac{\mathsf{F}(z)}{\mathsf{F}(z^p)} = \frac{\mathsf{F}_s(z)}{\mathsf{F}_{s-1}(z^p)} \mod p^s$$

In particular, the polynomials $\mathsf{F}_s(z)$ satisfy the Dwork type congruences:

$$\frac{\mathsf{F}_{s+1}(z)}{\mathsf{F}_s(z^p)} = \frac{\mathsf{F}_s(z)}{\mathsf{F}_{s-1}(z^p)} \mod p^s, \quad s = 1, 2, \ldots$$

Proof The proof follows from Theorem 1.1 in Ref. [21], after simple modifications. Let $S(x, 1)$ denote the superpotential (4.1) with $z_1 = z_2 = 1$. Clearly, this Laurent polynomial has the same Newton polygon $\mathsf{N}(S(z, 1)) = \mathsf{N}(S(x, z)) = \Delta(k, n)$. Let us consider

$$\mathsf{M}(\xi) = \sum_{d=0}^{\infty} [S(x, 1)^d]_0 \, \xi^d, \quad \mathsf{M}_s(\xi) = \sum_{d=0}^{p^s-1} [S(x, 1)^d]_0 \, \xi^d.$$

By Theorem 1.1 in Ref. [21], these functions satisfy the desired congruences:

$$\frac{\mathsf{M}(\xi)}{\mathsf{M}(\xi^p)} = \frac{\mathsf{M}_s(\xi)}{\mathsf{M}_{s-1}(\xi^p)} \mod p^s, \quad \frac{\mathsf{M}_{s+1}(\xi)}{\mathsf{M}_s(\xi^p)} = \frac{\mathsf{M}_s(\xi)}{\mathsf{M}_{s-1}(\xi^p)} \mod p^s. \tag{5.2}$$

From the structure of the superpotential $S(x, z)$, it is clear that

$$[S(x, z)^d]_0 = [S(x, 1)^d]_0 z^{\frac{d}{n}}.$$

In particular, this coefficient is equal to zero unless n divides d. From this, we find

$$\mathsf{F}(z) = \sum_{d=0}^{\infty} \left[S(x, 1)^d \right]_0 z^{\frac{d}{n}} = \mathsf{M}(z^{\frac{1}{n}}),$$

and similarly $\mathsf{F}_s(z) = \mathsf{M}_s(z^{\frac{1}{n}})$. The theorem follows from (5.2) after substitution $\xi \to z^{\frac{1}{n}}$. □

For further discussion of Dwork congruences for vertex functions and solutions of qKZ equations, we refer to Refs. [30, 32, 35, 36].

Remark The above theorem implies an infinite factorization:

$$\mathsf{F}(z) = \prod_{i=0}^{\infty} \frac{\mathsf{F}_s(z^{p^i})}{\mathsf{F}_{s-1}(z^{p^{i+1}})} \mod p^s.$$

Acknowledgements We thank Thomas Lam for very useful comments. Work of A. Smirnov is partially supported by NSF grant DMS-2054527 and by the RSF under grant 19-11-00062. Work of A. Varchenko is partially supported by NSF grant DMS-1954266.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of Interest On behalf of all the authors, the corresponding author states that there is no conflict of interest.

References

1. Aganagic, M., Okounkov, A.: Elliptic stable envelopes. J. Am. Math. Soc. **34**(1), 79–133 (2021)
2. Batyrev, V., Ciocan-Fontanine, I., Kim, B., van Straten, D.: Conifold transitions and mirror symmetry for Calabi-Yau complete intersections in Grassmannians. Nucl. Phys. B **514**, 640–666 (1998)
3. Castronovo, M.: Fukaya category of Grassmannians: rectangles, Adv. Math., **372**, (2020)
4. Danilenko, I.: Quantum differential equation for slices of the affine Grassmannian, arXiv:2210.17061
5. Dinkins, H.: Symplectic Duality of $T^*\mathrm{Gr}(k, n)$. Math. Res. Lett. **29**, 3 (2022)
6. Dinkins, H.: 3D mirror symmetry of the cotangent bundle of the full flag variety. Lett. Math. Phys. **112**, 100 (2022)
7. Dwork, B.: p-adic cycles. Publ. Math. de lHÉS **37**, 27–115 (1969)
8. Eguchi, T., Hori, K., Xiong, C.-S.: Gravitational quantum cohomology. Int. J. Mod. Phys. A **12**, 1743–1782 (1997)
9. Gaiotto, D., Koroteev, P.: On three dimensional quiver gauge theories and integrability. JHEP, **126**, (2013)
10. Givental, A.: Equivariant Gromov-Witten invariants. Int. Math. Res. Notices **13**, 613–663 (1996)
11. Galkin, S., Golyshev, V., Iritani, H.: Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures. Duke Math. J. **165**(11), 2005–2077 (2016)
12. Igusa, J.: Class number of a definite quaternion with prime discriminant. Proc. Natl. Acad. Sci. USA **44**(4), 312–314 (1958)
13. Kononov, Y., Smirnov, A.: Pursuing quantum difference equations II: 3D mirror symmetry. IMRN **2023**, 13290–13331 (2023)
14. Kononov, Y.: Elliptic Stable Envelopes and 3D Mirror Symmetry, PhD Thesis, Columbia University, 1–82, (2021)
15. Korff, C., Stroppel, C.: The $\widehat{sl}(n)_k$-WZNW fusion ring: a combinatorial construction and a realisation as quotient of quantum cohomology. Adv. Math. **225**(1), 200–268 (2010)
16. Koroteev, P., Pushkar, P., Smirnov, A., Zeitlin, A.: Quantum K-theory of Quiver varieties and many-body systems. Sel. Math. New Ser. **27**, 87 (2021)
17. Lam, T., Templier, N.: The mirror conjecture for minuscule flag varieties, arXiv:1705.00758
18. Marsh, R.J., Rietsch, K.: The B-model connection and mirror symmetry for Grassmannians. Adv. Math. **366**, 107027, 131 (2020)
19. Maulik, D., Okounkov, A.: Quantum groups and quantum cohomology, Astérisque, t. 408, Société Mathématique de France, 1–277, (2019)
20. Markov, Y., Varchenko, A.: Hypergeometric solutions of trigonometric KZ equations satisfy dynamical difference equations. Adv. Math. **166**(1), 100–147 (2002)
21. Mellit, A., Vlasenko, M.: Dwork's congruences for the constant terms of powers of a Laurent polynomial. Int. J. Number Theory **12**(2), 313–321 (2016)
22. Nill, B.: Reflexive Polytopes - Combinatorics and Convex Geometry, https://personales.unican.es/santosf/anogia05/slides/Nill-anogia05.pdf
23. Okounkov, A.: Lectures on K-theoretic computations in enumerative geometry, volume 24 of IAS/Park City Math. Ser., pages 251–380. Am. Math. Soc., Providence, RI, (2017)
24. Okounkov, A.: Enumerative symplectic duality, MSRI workshop "Structures in Enumerative Geometry" (the talk is accessible from MSRI web-page)

25. Okounkov, A., Smirnov, A.: Quantum difference equation for Nakajima varieties. Invent. Math. **229**, 1203–1299 (2022)
26. Pushkar, P., Smirnov, A., Zeitlin, A.: Baxter Q-operator from quantum K-theory. Adv. Math. **360**, 12 (2016)
27. Rimanyi, R., Smirnov, A., Varchenko, A., Zhou, Z.: Three-dimensional mirror self-symmetry of the cotangent bundle of the full flag variety SIGMA, 15 : **093**, **22**, (2019)
28. Rimanyi, R., Smirnov, A., Varchenko, A., Zhou, Z.: 3D-mirror symmetry and elliptic stable envelopes. IMRN **13**, 10016–10094 (2021)
29. Rimányi, R., Weber, A.: Elliptic classes of Schubert varieties via Bott-Samelson resolution. J. Topol. **13**(3), 1139–1182 (2020)
30. Smirnov, A., Varchenko, A.: The p-adic approximations of vertex functions via 3D-mirror symmetry, arXiv:2302.03092
31. Schechtman, V., Varchenko, A.: Arrangements of hyperplanes and Lie algebra homology. Invent. Math. **106**, 139–194 (1991)
32. Schechtman, V., Varchenko, A.: Solutions of KZ differential equations modulo p. Ramanujan J. **48**(3), 655–683 (2019)
33. Smirnov, A., Zhou, Z.: 3D-mirror symmetry and quantum K-theory of hypertoric varieties, Adv. Math. **395**, (2022)
34. Tarasov, V., Varchenko, A.: Landau-Ginzburg mirror, quantum differential equations and qKZ difference equations for a partial flag variety. J. Geom. Phys. **184**(23), 1–58 (2022)
35. Varchenko, A.: Dwork-type congruences and p-adic KZ connection, Essays in Geometry, Dedicated to Norbert A'Campo, (2023) EMS Press, 781–812, ESBN 978-3-98547-02-2
36. Varchenko, A., Zudilin, W.: Ghosts and congruences for p^s-appoximations of hypergeometric periods, J. Aust. Math. Soc., First View , pp. 1–32 https://doi.org/10.1017/S1446788723000083