RESEARCH CONTRIBUTION



A Note on Contact Manifolds with Infinite Fillings

Zhengyi Zhou¹

Received: 20 November 2023 / Revised: 3 January 2024 / Accepted: 26 January 2024 / Published online: 1 March 2024 © Institute for Mathematical Sciences (IMS), Stony Brook University, NY 2024

Abstract

We use spinal open books to construct contact manifolds with infinitely many different Weinstein fillings in any odd dimension > 1, which were previously unknown for dimensions equal to 4n + 1. The argument does not involve understanding factorizations in the symplectic mapping class group.

Keywords Contact manifold · Weinstein filling · Spinal open book

1 Introduction

Contact manifolds arise naturally as convex boundaries of symplectic manifolds, it was known by Gromov [15] and Eliashberg [10] in the late 1980s that not all contact manifolds can be realized in such a way. Therefore understanding symplectic fillings of contact manifolds is a fundamental question in contact topology. Such questions were extensively studied by many researchers starting from the case of no fillings [7–9, 11, 12, 14, 24, 31], the case of unique (only for the topological type in many cases) fillings [5, 9, 13, 15, 25, 30, 32], and at the end of this spectrum, the case of infinitely many filings [2-4, 6, 27, 28]. As one can always blow up a symplectic filling to change the topology, we need to restrict to Liouville or Weinstein fillings for the question of infinite fillings. Unlike the no-filling and unique-filling situations, which typically depend on some rigidity arguments using pseudo-holomorphic curves, the construction of contact manifolds with infinitely many fillings usually uses the topological or flexible side of symplectic topology. The first contact manifold (beyond the trivial case of S^1) with infinitely many different Weinstein fillings was constructed by Ozbagci and Stipsicz [28] in dimension 3. Nowadays, there are many constructions with various constraints on the topology of fillings in dimension 3, see [2-4, 6]. Oba [27] generalized Ozbagci and Stipsicz's result to dimension 4n-1. Their constructions were based on the open book construction of contact manifolds and finding infinitely

Zhengyi Zhou zhyzhou@amss.ac.cn

¹ Morningside Center of Mathematics and Institute of Mathematics, AMSS, CAS, Beijing, China

many different factorizations by positive Dehn–Seidel twists of the monodromy in the symplectic mapping class group. Such an approach is most efficient in dimension 3, as the symplectic mapping class group agrees with the classical mapping class group is the case of surfaces. In higher dimensions, the symplectic mapping class group is different from the classical mapping class group in general, and much less is known. It is worth pointing out that Lazarev [19] constructed contact manifolds with many different Weinstein fillings in dimension ≥ 5 , where the number of fillings can be arbitrarily large, from a surgical perspective.

In this note, we give a new construction of contact manifolds with infinitely many different Weinstein fillings in any dimension. The construction is based on spinal open books—a generalization of contact open books, introduced by Lisi et al. [20]. The spinal open book was used by Baykur et al. [6] to construct contact 3-manifolds which admit infinitely many Weinstein fillings with arbitrarily big Euler characteristics and arbitrarily small signatures. Heuristically speaking, spinal open books arise as the contact boundary of a Lefschetz fibration over a general surface with boundary. Such contact manifolds, especially in dimension 4, were studied systematically in [20, 21, 23]. Moreover, there are notions of spinal open books, which fiber over Liouville domains of dimension higher than 2, see e.g. [22, 26]. In this note, we restrict to the case of the surface base.

Theorem 1.1 Let V be the plumbing of two T^*S^n along three points. Then the contact boundary $\partial(\Sigma_{1,1} \times V)$ has infinitely many different Weinstein fillings, where $\Sigma_{1,1}$ is a genus one Riemann surface with one boundary component, viewed as a Weinstein filling of S^1 .

We point out that our construction is local in nature, i.e. if V contains the domain in Theorem 1.1 as a symplectic subdomain and the homology computation in the proof works, then the conclusion can be drawn for $\partial(\Sigma_{1,1} \times V)$. For example, it holds for V in Theorem 1.1 taking boundary connected sum with any Weinstein domain. Moreover, similar phenomena hold for more general plumbings of spheres. When *n* is odd, i.e. when the contact manifold is of dimension 4k - 1, we can take V to be two T^*S^n plumbed at one point.

Our strategy is similar to [27, 28]: the contact boundary $\partial(\Sigma_{1,1} \times V)$ is the trivial spinal open book over $\Sigma_{1,1}$, and any representation $\rho : \pi_1(\Sigma_{1,1}) \to \pi_0(\text{Symp}_c(V))$ such that $\rho(\partial \Sigma_{1,1}) = \text{id gives rise to a Weinstein filling by a V-fiber bundle over <math>\Sigma_{1,1}$. Sending a generator of $\pi_1(\Sigma_{1,1})$ to the identity always yields one such representation, hence every element ϕ of $\pi_0(\text{Symp}_c(V))$, i.e. the image of the other generator under ρ , induces a filling. Then by understanding the effect of ϕ on the homology of V, we can get infinite many fillings. In particular, we do not need to consider factorizations in the symplectic mapping class group.

Theorem 1.1 still uses classical topological invariants to differ symplectic fillings, a natural question is whether the phenomena of infinite fillings can be purely symplectic, that is:

Question 1.2 Are there contact manifolds with infinitely many different Weinstein/Liouville fillings with the same formal data, i.e. as the same differential/almost complex/almost Weinstein manifold (relative to the boundary)?

We expect the answer to the question to be yes, at least for dimensions high enough. However, to the best of the author's knowledge, we do not even know examples of contact manifolds with more than one smoothly same, but symplectically different fillings. Unlike Theorem 1.1, rigidity techniques, e.g. holomorphic curves or sheaves, must enter the picture to solve the above question. Using spinal open books, we have candidates for Question 1.2 at least for in dimension 4n + 1.

Question 1.3 Let $\phi \in \pi_0(\text{Symp}_c(V))$ be generated by the eighth powers of the Dehn– Seidel twist along any Lagrangian sphere in V, where dim V = 4n. Is the symplectic fiber bundle induced from $\pi_1(\Sigma_{1,1}) \to \pi_0 \text{Symp}_c(V)$ by sending one generator to ϕ and the other to id symplectomorphic to $\Sigma_{1,1} \times V$?

Clearly, the motivation behind such a question is the fact that eighth powers of the Dehn–Seidel twist are smoothly isotopic to identity in dimension 4n [17], yet not symplectically isotopic to identity [5, 29]. In dimensions 4 and 12, one can replace the eighth power with a square.

2 Proof

Let V be a Liouville domain and $\phi \in \pi_0(\text{Symp}_c(V))$. We can endow

$$\Sigma_{g,1} \times \partial V \cup_{S^1 \times \partial V} V_{\phi}$$

a contact structure by a generalization of the Thurston–Winkelnkemper construction, see [20, §2.3], where $\Sigma_{g,1}$ is a genus g surface with one boundary component and V_{ϕ} is the mapping torus $V \times [0, 1]/(x, 0) \sim (\phi(x), 1)$. This is a very special case of the spinal open book considered in [20], where the vertebrae ($\Sigma_{g,1}$ here) can have more boundary components and be disconnected. In this paper, we only consider the case of g = 1 and $\phi = id$. Then the contact manifold is the contact boundary $\partial(\Sigma_{1,1} \times V)$.

Lemma 2.1 [6, 20] Let Σ be a connected Riemann surface with boundary and V be a Weinstein domain. Any representation $\pi_1(\Sigma) \to \pi_0(\text{Symp}_c(V))$ mapping the boundary to id gives rise to a Weinstein filling of $\partial(\Sigma \times V)$, which is diffeomorphic to the V-bundle over Σ from $\pi_1(\Sigma) \to \pi_0(\text{Symp}_c(V))$.

More generally, if the monodromy of the spinal open book is ϕ and there exist $\phi_1, \psi_1, \ldots, \phi_g, \psi_g \in \text{Symp}_c(V)$ and τ_1, \ldots, τ_k are Dehn–Seidel twists along some exact Lagrangian spheres in V, such that

$$\phi = \prod \tau_i \prod [\phi_i, \psi_i].$$

Then the spinal open book given by $(\Sigma_{g,1}, V, \phi)$ is the contact boundary of a symplectic Lefschetz fibration over $\Sigma_{g,1}$ with *k* singular fibers. When *V* is Weinstein, the total space of the Lefschetz fibration is a Weinstein filling of the spinal open book.

Lemma 2.2 Let V_{ϕ} be the mapping torus, then we have short exact sequences

$$0 \rightarrow \ker(\phi_* - \mathrm{id}) \rightarrow H_*(V_{\phi}) \rightarrow \operatorname{coker}(\phi_* - \mathrm{id})[-1] \rightarrow 0.$$

Proof The homology of V_{ϕ} can be computed from the cone of $C_*(V) \xrightarrow{\phi_* - \mathrm{id}} C_*(V)$. We have an induced long exact sequence

$$\dots \longrightarrow H_*(V) \longrightarrow H_*(V_{\phi}) \longrightarrow H_{*-1}(V) \stackrel{\phi_* - \mathrm{id}}{\longrightarrow} H_{*-1}(V) \longrightarrow \dots$$

see [16, Example 2.48] for details. Then the claimed short exact sequence follows from the exactness of the above sequence. \Box

More generally, let $V_{\phi \vee id}$ be the *V*-fiber bundle over $S^1 \vee S^1$ (or homotopically equivalently over $\Sigma_{1,1}$), such that the monodromy over one S^1 is ϕ and is id over the other S^1 . Then we have a short exact sequence

$$0 \to \ker(\phi_* - \mathrm{id})|_{H_k(V;\mathbb{Z})} \to H_k(V_{\phi \lor \mathrm{id}};\mathbb{Z})$$

$$\to H_{k-1}(V;\mathbb{Z}) \oplus \operatorname{coker}(\phi_* - \mathrm{id})|_{H_{k-1}(V;\mathbb{Z})} \to 0$$

for $k \ge 1$. In particular, when V is a Weinstein domain of dimension 2n, then the cardinality of the torsion of $H_{n+1}(V_{\phi \lor id})$ will be at least that of $\operatorname{coker}(\phi_* - \operatorname{id})|_{H_n(V;\mathbb{Z})}$.

Lemma 2.3 (Picard–Lefschetz formula, [18, (6.3.3)]) Let L be an oriented Lagrangian *n*-sphere in an exact domain W and τ_L the Dehn–Seidel twist along L,¹ then $(\tau_L)_*$: $H_*(W; \mathbb{Z}) \rightarrow H_*(W; \mathbb{Z})$ is given by

$$(\tau_L)_*(c) = \begin{cases} c + (-1)^{\frac{(n+1)(n+2)}{2}} \langle c, [L] \rangle [L], & c \in H_n(W; \mathbb{Z}); \\ c, & c \in H_j(W; \mathbb{Z}), j \neq n \end{cases}$$

where $\langle \cdot, \cdot \rangle : H_n(W; \mathbb{Z}) \otimes H_n(W; \mathbb{Z}) \to \mathbb{Z}$ is the intersection product.

Recall the plumbing of two cotangent bundles T^*Q_1 and T^*Q_2 at one point is given by choosing balls B_i around the plumbing point in Q_i , and identifying their disc cotangent bundles $D^*B_i \subset D^*Q_i$ by a symplectomorphism which interchanges the zero-section and fiber directions. We refer readers to [1, §2.3] for details of the plumbing construction. Let V^{2n} be plumbing of two T^*S^n along three points. We use L_1, L_2 to denote the two Lagrangian spheres, oriented such that $\langle [L_1], [L_2] \rangle =$ $(-1)^{\frac{n(n+1)}{2}}$ 3. More precisely, we first choose an orientation of L_1 , then we orient L_2 by the induced orientation of the fiber $T_q^*L_1$, where q is an intersection point, since the fiber is identified with a neighborhood of q in L_2 in the plumbing construction. Then the intersection number of L_1, L_2 using the orientation on T^*L_1 induced from the orientation of L_1 is 3. The extra sign comes from the fact that the symplectic

¹ Strictly speaking, the Dehn–Seidel twist depends on a parametrization $S^n \simeq L$, which can yield different Dehn–Seidel twists in the symplectic mapping class group in general. However, such an ambiguity does not play a role in our discussions.

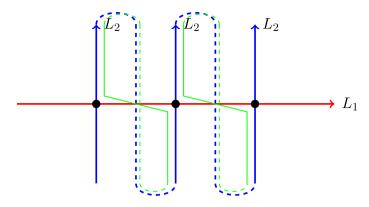


Fig. 1 The Lagrangian skeleton and homology basis

orientation (using $-d \sum p_i dq_i$, i.e. the standard symplectic orientation on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$) is different from the induced orientation on T^*L_1 from that of L_1 by $(-1)^{\frac{n(n+1)}{2}}$. Note that $\langle [L_1], [L_1] \rangle = \langle [L_2], [L_2] \rangle = 0$ when *n* is odd and $\langle [L_1], [L_1] \rangle = \langle [L_2], [L_2] \rangle = (-1)^{\frac{n(n+1)}{2}} 2$ when *n* is even. When n > 1, under the free basis $[L_1], [L_2]$ of $H_n(V^{2n}; \mathbb{Z})$, by the Picard–Lefschetz formula, the effect of the Dehn–Seidel twists τ_{L_1}, τ_{L_2} on $H_n(V^{2n}; \mathbb{Z})$ is given by

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

for *n* odd respectively, and

$$\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -3 & -1 \end{bmatrix}$$

for *n* even respectively.

When n = 1, $H_1(V; \mathbb{Z}) = \mathbb{Z}^4$ and $(\tau_{L_1})_*$, using the basis $[L_1]$, $[L_2]$ and two other cycles (with suitable orientation) glued from two arcs from L_1, L_2 , is given by

$$\begin{bmatrix} 1 & -3 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
 (2.1)

To see this, the Lagrangian skeleton of the plumbing is sketched below, where the dashed blue line is the part of L_2 that goes under L_1 . The extra middle crossing of the projection is necessary as the intersection points between L_1 , L_2 have the same orientation (Fig. 1).

The two green loops are the other two free generators of $H_1(V; \mathbb{Z})$, whose intersection numbers with L_1 are ± 1 from the middle intersection point of the green solid

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line with L_1 . It is clear that we can choose orientations of those two green loops to form a basis of $H_1(V; \mathbb{Z})$ such that $(\tau_{L_1})_*$ on $H_1(V; \mathbb{Z})$ is (2.1).

Proof of Theorem 1.1 Let γ_1, γ_2 be two loops in $\Sigma_{1,1}$, representing the bases of the fundamental group in the torus. We consider the representation $\rho : \pi_1(\Sigma_{1,1}) \rightarrow \pi_0(\text{Symp}_c(V)), \gamma_1 \mapsto \phi, \gamma_2 \mapsto \text{id}$. By Lemma 2.1, it gives rise to a filling of $\partial(\Sigma_{1,1} \times V)$, which is homotopically equivalent to $V_{\phi \vee \text{id}}$.

When n > 1 is odd, we take ϕ to be τ_{L_1} . Since ϕ_*^k on $H_n(V; \mathbb{Z})$ is given by

$$\begin{bmatrix} 1 & -3k \\ 0 & 1 \end{bmatrix}$$

Then by the discussion after Lemma 2.2, we know that $H_{n+1}(V_{\phi^k \vee id}; \mathbb{Z})$ has a torsion of $\mathbb{Z}/3k$. As a consequence, each *k* yields a different Weinstein filling.

When n = 1, we take ϕ to be τ_{L_1} . Then ϕ_*^k on $H_n(V; \mathbb{Z})$ is given by

[1]	-3k	-k	-k]
0	1	0	0	
0	0	1	0	·
0	0	0	1	

We know that $H_2(V_{\phi^k \vee id}; \mathbb{Z})$ has a torsion of \mathbb{Z}/k . As a consequence, each k yields a different Weinstein filling.

When *n* is even, we take ϕ to be $\tau_{L_1} \circ \tau_{L_2}$. Then ϕ_* on $H_n(V; \mathbb{Z})$ is given by

$$\begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix}$$

This matrix has positive eigenvalues $\lambda_1 = \frac{7+3\sqrt{5}}{2} > 1, \lambda_2 = \frac{7-3\sqrt{5}}{2} < 1$. As a consequence, we have

$$|\det((\phi_*)^k - \mathrm{id})| = |2 - \lambda_1^k - \lambda_2^k|,$$

which grows exponentially. The torsion of $H_{n+1}(V_{\phi^k \vee id})$ is of size $|2 - \lambda_1^k - \lambda_2^k|$, which yields infinitely many different fillings as before.

When *n* is odd, we can simply take *V* to be the plumbing of two T^*S^n at one point. Then $\tau_{L_1}^k$ acts on $H_n(V; \mathbb{Z})$ by

$$\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$$

which yields infinitely many fillings.

Acknowledgements The author is grateful to Fabio Gironella for productive discussions and interest in the project, and to Samuel Lisi, Takahiro Oba, Chris Wendl for helpful comments. The author would like to thank the referee for suggestions that improve the paper. The author is supported by National Natural Science Foundation of China under Grant No. 12288201 and 12231010.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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