



Newton Polyhedra and Stratified Resolution of Singularities in the Class of Generalized Power Series

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Abstract

We generalize the construction of a toric variety associated with an integer convex polyhedron to construct generalized analytic varieties associated with polyhedra with not necessarily rational vertices. For germs of generalized analytic functions with a given Newton polyhedron Γ , the generalized analytic variety associated with Γ provides a stratified resolution of singularities of these functions; also ensuring a full resolution for almost all of them. Thus, this constructive and elementary approach replaces the non-effective previous proof of this result based on consecutive blow-ups.

Keywords Generalized power series · Newton polyhedra · Monomial maps · Resolution of singularities · Toric varieties

Mathematics Subject Classification 14E15 · 14M25 · 32S45 · 51M20 · 52B20

Introduction

This work deals with generalized analytic manifolds; that is, topological manifolds with boundary and corners endowed with a structure described by generalized analytic functions (cf., [5, §3]). Briefly, a generalized analytic function is a real-valued continuous function expressed in terms of a convergent power series whose exponents range over tuples of non-negative real numbers, as defined in [2]. Our main goals here are to exhibit generalized analytic manifolds by means of Newton polyhedra techniques, and

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then, to show how this construction solves not only the stratified resolution problem, in the sense of [7], but also provides a full resolution of singularities for almost all germs of generalized analytic functions with fixed Newton polyhedron. In doing so, we follow Khovanskii's exposition on Toric resolution of singularities as done in his seminal papers [3, 4].

Let us briefly recall the main steps behind the construction of the Toric resolution of singularities, as carried out in [3, 4] (see Varchenko's paper [12], where the local case is treated). Given any *admissible* finite collection of simplicial convex cones in the dual space of \mathbb{R}^n (nowadays commonly known as a *simple fan*), one can define an algebraic (analytic) manifold, completely described by the combinatorics of this object. Roughly, a simple fan is a finite collection of simplicial convex cones, which are primitive; namely, the volume of the parallelepiped formed by all primitive (non-multiple) integer vectors in its one-dimensional faces is one. All the computations and constructions involved can be read in simple linear algebra terms. Therefore, given any polynomial (or germ of an analytic function at the origin) with real or complex coefficients in several variables, one can always look at its Newton polyhedron, i.e., the convex hull of the exponents of the monomials appearing in it with non-zero coefficients. With this convex polyhedron, one associates a simple fan inscribed in the non-negative orthant. Consequently, one can construct the manifold associated with the latter. In the end, one gets that the algebraic (analytic) manifold thus defined, together with its projection into an open neighborhood of the origin, solves the desingularization problem for almost all polynomials (germs of analytic functions) with the same Newton polyhedron.

Adapting the construction just described, we show generalized analytic manifolds and morphisms between them explicitly. For this sake, we introduce a combinatorial object called a *pseudo-simple fan*. Grosso modo, a pseudo-simple fan is a nice partition of the non-negative orthant into simplicial convex cones (see Definition 6 below). We prove that each of these combinatorial objects can be realized as a generalized analytic manifold; namely, we have the following statement.

Theorem A *Any given pseudo-simple fan gives rise to a unique, up to isomorphism, generalized analytic manifold.*

We stress that, in our case, a required step done in the analogous construction in the standard (analytic) algebraic case is unnecessary; namely, the refinement process to get a simple fan from the initial one associated with a given Newton polyhedron. Primitiveness condition, crucial in the standard case, is absent in the framework of generalized analytic manifolds; the analytic meaning is that for generalized analytic manifolds, ramification maps are valid changes of coordinates. For instance, in one variable, given any positive real number α , the correspondence rule $x \mapsto x^\alpha := \exp(\alpha \ln x)$, well defined in the non-negative ray $\{x \in \mathbb{R} : x \geq 0\}$, defines an actual generalized analytic change-of-coordinates, with inverse given by $x \mapsto x^{1/\alpha}$.

Applying Theorem A to the case of the Newton polyhedron of a given generalized analytic function, we obtain a constructive and geometric proof of the so-called stratified resolution of singularities of generalized analytic functions.

Theorem B (Stratified resolution of singularities) *Let $f \in \mathbb{R}\{x^*\}$ be a germ of a generalized analytic function at the origin. Let $\Gamma = \Gamma(f)$ be the Newton polyhedron*

of f , and let us consider $M(\Gamma)$ the manifold associated with the Newton polyhedron of f , as well as its projection into the non-negative orthant \mathbb{R}_+^n , $\pi : M(\Gamma) \rightarrow \mathbb{R}_+^n$. Then, the map π is proper, and for any corner point p of the manifold $M(\Gamma)$, there is a local system of coordinates centered at p , such that the germ $(f \circ \pi)_p$ is a monomial multiplied by some unit; i.e., $(f \circ \pi)_p = x_p^\alpha U(x_p)$, where $\alpha \in \mathbb{R}_+^n$, $U \in \mathbb{R}\{x_p^*\}$, $U(0) \neq 0$.

We should say that this result was first proved in dimensions two and three in [9]. Recently, it has been proved in arbitrary dimension in [7] but using quite different methods. In addition, we notice that Theorem B does not ensure a full resolution of singularities for a given function, but tell us how its transform under the projection $\pi : M(\Gamma) \rightarrow \mathbb{R}_+^n$ looks like at the corners of the manifold $M(\Gamma)$. As an illustration, if $f(x, y) = (x - y)^2 + x^{10}$, then the map π in the statement of Theorem B turns out to be the standard blow-up at the origin (cf., Example 3), but it does not give the full resolution of singularities of f in the usual sense). Nevertheless, once a Newton polyhedron is fixed, we get, under plausible generic conditions, a full resolution for almost all germs of generalized analytic functions with such a Newton polyhedron.

Theorem C (On resolution of singularities) *Let $\Gamma \subset \mathbb{R}_+^n$ be a fixed Newton polyhedron. Then, for almost all germs of generalized analytic functions at the origin, whose Newton polyhedron coincides with Γ , the projection $\pi : M(\Gamma) \rightarrow \mathbb{R}_+^n$ ensures their full resolution of singularities. That is to say, for any point p on the boundary of the manifold $M(\Gamma)$, there is a local system of coordinates centered at p , such that the germ $(f \circ \pi)_p$ is a monomial multiplied by some unit; i.e., $(f \circ \pi)_p = x_p^\alpha U(x_p)$, where $\alpha \in \mathbb{R}_+^n$, $U \in \mathbb{R}\{x_p^*\}$, $U(0) \neq 0$.*

Note We add that our constructions, as they are straightforward satisfied when considering formal (non necessary convergent) generalized power series, could be applied to study the so-called *Generalized Quasi-analytic functions*, as recently introduced by Rolin and Servi in [10].

Structure of the Work

In Sect. 1, we introduce and recall the notations, definitions, and basic notions that we use throughout the text. Section 2 is devoted to the proof of Theorem A; that is to say, there we show how to realize pseudo-simple fans as generalized analytic manifolds. We also prove that for any such generalized analytic manifold, its projection into the non-negative orthant is well defined and proper (see Corollary 1). Although Theorem B is a direct consequence of the constructions carried out in Sect. 2, we include the details in Sect. 3 for completeness. In this same last section, we prove Theorem C; we elaborate on the aforementioned generic conditions, showing that, indeed, the construction we make ensures a full resolution of singularities for almost all germs of generalized analytic functions with the same Newton polyhedron.

Our exposition is mainly based on Chapter 8, §8.1–8.2, of Arnold et al. [1], where Varchenko’s construction of Toric resolution of singularities is covered, as presented in [12]. It is, in turn, an adaptation of Khovanskii’s original approach; cf., [3, 4]; which has also influenced our work.

1 Preliminaries

Throughout this manuscript, we use the multi-index notation. Let $n \in \mathbb{N}$ be a natural number. We denote by \mathbb{R}_+^n the set of n -tuples of non-negative real numbers and by $\mathbb{R}_{>0}^n$ the set of n -tuples of positive real numbers. Let $x = (x_1, x_2, \dots, x_n)$ be a tuple of n distinct variables. Given any two tuples $\alpha, \beta \in \mathbb{R}_+^n$, we write

$$\begin{aligned} x^\alpha &:= x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n, \\ \alpha + \beta &:= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n). \end{aligned}$$

And, we put $\alpha \leq \beta$ if and only if $\alpha_j \leq \beta_j$ for all $j \in \{1, 2, \dots, n\}$. This partial order relation is called the *division order* in \mathbb{R}_+^n . We stress that $\alpha \leq \beta$ if and only if the monomial x^α divides the other x^β . We also write $\langle \alpha, \beta \rangle := \alpha_1 \beta_1 + \alpha_2 \beta_2 + \dots + \alpha_n \beta_n$.

1.1 Newton Polyhedra

We say that a set $A \subset \mathbb{R}_+^n$ is *good* if there are well-ordered subsets of the non-negative real numbers $S_j \subset \mathbb{R}_+$, $j \in \{1, 2, \dots, n\}$, such that A is a subset of their Cartesian product; i.e., $A \subset S_1 \times S_2 \times \dots \times S_n$ (e.g., any subset A of the Cartesian product of n copies of the non-negative integers \mathbb{Z}_+ is a good set); cf., [2, §4]. The following basic properties of good sets are straightforward from the definition.

Proposition 1 [2, Lemma 4.2] *Let $A \subset \mathbb{R}_+^n$ be a good set. Then*

- (1) *The subset of A consisting of minimal elements with respect to the division order is a finite set.*
- (2) *The set $\{|\alpha| : \alpha \in A\}$ is a well-ordered subset of \mathbb{R}_+ . Moreover, for every non-negative real number $r \in \mathbb{R}_+$, the set $A_r := \{\alpha \in A : |\alpha| = r\}$ is finite.*

Definition 1 Let $A \subset \mathbb{R}_+^n$ be a good set. The *Newton polyhedron* of A is the convex hull in \mathbb{R}_+^n of the set $\cup_{\alpha \in A} (\alpha + \mathbb{R}_+^n)$. We denote by $\Gamma(A)$ the Newton polyhedron of A .

Moreover, the union of all compact faces of the Newton polyhedron of A is called the *Newton diagram* of A , and is denoted by $\Delta(A)$.

1.2 Formal Generalized Power Series

A *formal generalized power series* in the variables $x = (x_1, x_2, \dots, x_n)$ and with real coefficients is a power series $f(x) = \sum_{\alpha} f_{\alpha} x^{\alpha}$, such that its *support*; that is, the set of exponents occurring in f , $\text{Supp}(f) := \{\alpha \in \mathbb{R}_+^n : f_{\alpha} \neq 0\}$, is a good set. If the support of f is finite, we say that f is a *generalized polynomial*. We denote by $\mathbb{R}[[x^*]]$ and $\mathbb{R}[x^*]$ the local ring of all formal generalized power series in the variables x with real coefficients and the ring of all generalized polynomials, respectively.

Definition 2 Let $f \in \mathbb{R}[[x^*]]$ be a formal generalized power series. The *Newton polyhedron (diagram) of f* is the Newton polyhedron (diagram) of its support. We denote by $\Gamma(f)$ the Newton polyhedron of f (and by $\Delta(f)$ its Newton diagram).

Additionally, for any face γ of the Newton polyhedron of the power series f , the γ -part of f is the restriction of f to the intersection $\text{Supp}(f) \cap \gamma$; that is, $\sum_{\alpha \in \text{Supp}(f) \cap \gamma} f_{\alpha} x^{\alpha}$. If the face γ is compact, then the γ -part of f is a generalized polynomial. We also call the *principal part* of the power series f to its restriction to the intersection $\text{Supp}(f) \cap \Delta$; that is, the principal part of f is the finite sum $\sum_{\alpha \in \text{Supp}(f) \cap \Delta} f_{\alpha} x^{\alpha}$. We denote by f_{γ} the γ -part of the power series f , and by f_{Δ} we mean its principal part.

1.3 Monomial Maps

It is known (cf., [1, §8.1.2]) that any monomial map can be translated into a linear transformation and that this translation is a linear representation. Therefore, we can use simple linear algebra methods when dealing with monomial maps. Let us briefly recall the linear representation we refer to.

Let $h = (h_1, h_2, \dots, h_n): \mathbb{R}_{>0}^m \rightarrow \mathbb{R}_{>0}^n$ be a monomial map defined componentwise, at any point $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}_{>0}^m$, by the correspondence rule

$$h_j(x) = x^{\alpha^j} = x_1^{\alpha_1^j} x_2^{\alpha_2^j} \cdots x_m^{\alpha_m^j}, \quad \text{for some } \alpha^j = (\alpha_1^j, \alpha_2^j, \dots, \alpha_m^j) \in \mathbb{R}^m.$$

The *matrix of exponents* of the monomial map h is the matrix $\mathcal{E}(h) \in \text{Mat}_{n \times m}(\mathbb{R})$, whose arrows are the m -tuples $\alpha^1, \alpha^2, \dots, \alpha^n$

$$h \longmapsto \mathcal{E}(h) := \begin{bmatrix} \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_m^1 \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_m^2 \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_1^n & \alpha_2^n & \cdots & \alpha_m^n \end{bmatrix} \in \text{Mat}_{n \times m}(\mathbb{R}).$$

Let $g: \mathbb{R}_{>0}^l \rightarrow \mathbb{R}_{>0}^m$ and $h: \mathbb{R}_{>0}^m \rightarrow \mathbb{R}_{>0}^n$ be any two monomial maps. A straightforward computation shows that

$$\mathcal{E}(h \circ g) = \mathcal{E}(h) \cdot \mathcal{E}(g).$$

1.4 Cones, Projective Skeletons, and Pseudo-simple Fans

We now introduce the combinatorial-geometric objects that play the central role in constructing those generalized analytic manifolds announced in the introduction. We start by giving the definitions of cones, their projective skeletons (and their affine representatives), and the notion of a pseudo-simple fan. We restrict ourselves to considering cones inscribed in the non-negative orthant \mathbb{R}_+^n .

Definition 3 The *cone generated by the vectors* $\alpha^1, \alpha^2, \dots, \alpha^m \in \mathbb{R}_+^n$ is the cone consisting of linear combinations of these vectors with non-negative coefficients.

Let us notice that given any cone with a vertex at the origin, the union of all its one-dimensional faces is finite. Each one of these one-dimensional faces determines a point in the real projective space of dimension $n - 1$ in a natural way.

Definition 4 The *projective skeleton* of a cone with a vertex at the origin is the set of all points in the real projective space of dimension $n - 1$ determined by the union of all the faces of dimension one of the cone.

An *affine representative skeleton* of a cone is any choice of a complete ordered set of representatives of the projective skeleton contained in the non-negative orthant \mathbb{R}_+^n . We notice that any affine representative skeleton of a cone generates the cone itself.

Definition 5 A cone with a vertex at the origin is called *simplicial* if there is an affine representative skeleton that consists of linearly independent vectors in \mathbb{R}^n .

Definition 6 A *fan* is a finite set of cones, all of them with a vertex at the origin, satisfying the following properties:

- (1) Each face of a cone from the set also belongs to the set.
- (2) The intersection of any two cones from the set is a face of both.

We say that a fan is *pseudo-simple* if all its cones are simplices.

Definition 7 Let Σ_1 and Σ_2 be two pseudo-simple fans. We say that the fan Σ_1 is *inscribed* in the fan Σ_2 if each cone in Σ_1 is contained in some cone in Σ_2 .

1.5 Pseudo-simple Fans Associated with Newton Polyhedra

Let Γ be the Newton polyhedron of some good subset of the non-negative orthant \mathbb{R}_+^n . Let us denote by \mathbb{R}^{n*} the dual vector space of \mathbb{R}^n . In this space, we consider the *support function* of the Newton polyhedron Γ ; that is to say, the function $l_\Gamma: \mathbb{R}^{n*} \rightarrow \mathbb{R}$ that to each covector α in \mathbb{R}^{n*} associates

$$l_\Gamma(\alpha) := \min\{\langle \alpha, x \rangle : x \in \Gamma\}.$$

Let $\alpha \in \mathbb{R}_+^{n*}$ be a covector with non-negative components. The *trace* on the Newton polyhedron Γ of the covector α is the face of the polyhedron that coincides with the set

$$\{x \in \Gamma : \langle \alpha, x \rangle = l_\Gamma(\alpha)\}.$$

Given $\alpha, \beta \in \mathbb{R}_+^{n*}$ two covectors in the non-negative orthant, we say that they are *equivalent relative to a Newton polyhedron* Γ if they have the same trace on it. The closure of any equivalence class shapes a cone with vertex at the origin, dual to the face attached to it. Besides, the collection of all cones thus determined forms a fan, from which we obtain a pseudo-simple one (namely, the finite collection of all normal directions to the $(n - 1)$ -dimensional faces of Γ determines it; cf., Step 1 indicated in the proof of Theorem 1 in [4, §1.2, p. 2813]). Summing up, we have the following result.

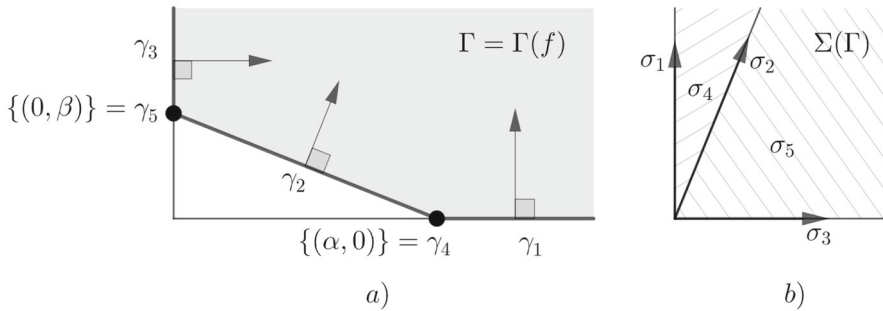


Fig. 1 **a** The Newton polygon of $f(x, y) = ax^\alpha + by^\beta$, and **b** the pseudo-simple fan associated with it

Proposition 2 Any given Newton polyhedron Γ determines a pseudo-simple fan in the dual space of \mathbb{R}^n .

The fan set in the previous proposition is called *the pseudo-simple fan associated with the Newton polyhedron Γ* , and we denote it by $\Sigma(\Gamma)$.

Example 1 Let us consider the generalized polynomial $f(x, y) = ax^\alpha + by^\beta$, where $a, b \in \mathbb{R}$ are non-zero real constants, and $\alpha, \beta \in \mathbb{R}_{>0}$ are positive exponents. The Newton polygon of f , say $\Gamma = \Gamma(f)$, has three edges (one-dimensional faces): $\gamma_1 = \{(x, y) \in \mathbb{R}^2 : x \geq \alpha, y = 0\}$, $\gamma_2 = \{(x, y) \in \mathbb{R}_+^2 : \beta x + \alpha y = \alpha\beta\}$ and $\gamma_3 = \{(x, y) \in \mathbb{R}^2 : x = 0, y \geq \beta\}$; and two vertices (zero-dimensional faces) $\gamma_4 = \{(\alpha, 0)\}$ and $\gamma_5 = \{(0, \beta)\}$ (see Fig. 1a).

The associated pseudo-simple fan with the Newton polygon Γ is the union of the cones that are defined in terms of the support function of Γ , as we explained above. Namely, to each edge, it is associated a one-dimensional cone (defined by the normal direction to such face); $\gamma_j \mapsto \sigma_j, j \in \{1, 2, 3\}$, where $\sigma_1 = \text{span}(\{(0, 1)\}) \cap \mathbb{R}_+^{2*}$, $\sigma_2 = \text{span}(\{(\beta, \alpha)\}) \cap \mathbb{R}_+^{2*}$ and $\sigma_3 = \text{span}(\{(1, 0)\}) \cap \mathbb{R}_+^{2*}$. While to the vertices γ_4 and γ_5 correspond the two-dimensional cones σ_4 and σ_5 as indicated in Fig. 1b). In addition, we notice that the projective skeletons associated with the edges are $[B_{\sigma_1}] = \{[0 : 1]\}$, $[B_{\sigma_2}] = \{[\beta : \alpha]\}$ and $[B_{\sigma_3}] = \{[1 : 0]\}$. Therefore, the corresponding projective skeletons to the two-dimensional cones σ_4 and σ_5 are $[B_{\sigma_4}] = \{[\beta : \alpha], [0 : 1]\}$ and $[B_{\sigma_5}] = \{[1 : 0], [\beta : \alpha]\}$, respectively.

2 Proof of Theorem A

In this section, we construct a generalized analytic manifold determined by the combinatorics and geometry of a given pseudo-simple fan. This manifold is covered by a finite (generalized analytic) atlas. Plus, there is a one-to-one correspondence between the charts of the atlas and the n -dimensional cones of the fan. Each chart is equal to the non-negative orthant \mathbb{R}_+^n (so the resulting manifold has boundary and corners). Of course, such a manifold's structure is characterized by how we glue together different charts. We claim that the kind of structures we get is generalized analytic. At this point, we use the curve selection result proved by van den Dries and Speissegger in [2,

p. 4416]. It permits us to conclude that the topological space we construct is indeed Hausdorff and that certain maps between these manifolds are proper.

2.1 Step-by-Step Construction of the Manifold Associated with a Pseudo-simple Fan

Let us fix a pseudo-simple fan Σ . Let σ be an n -dimensional cone in the fan Σ , and let $[B_\sigma]$ be the projective skeleton of σ . First of all, let us take an affine representative of the projective skeleton $[B_\sigma]$, say $B_\sigma = \{\alpha^1(\sigma), \alpha^2(\sigma), \dots, \alpha^n(\sigma)\} \subset \mathbb{R}_+^n$. At the end of this subsection, we prove that our construction does not depend on the choice we made. Besides, we associate with the cone σ a copy of the non-negative orthant \mathbb{R}_+^n , denoted by U_σ . We refer to (U_σ, B_σ) as the chart associated with the cone σ .

Let σ_1 and σ_2 be two n -dimensional cones of the fan Σ . Let $(U_{\sigma_1}, B_{\sigma_1})$ and $(U_{\sigma_2}, B_{\sigma_2})$ be the charts associated with σ_1 and σ_2 , respectively. We define a monomial map from the first chart into the second one. Concretely, the monomial map we look for is nothing but the one whose matrix of exponents coincides with the change-of-basis matrix that transforms the second basis B_{σ_2} into the first B_{σ_1} . We denote by $\varphi_{\sigma_1, \sigma_2}: U_{\sigma_1} \rightarrow U_{\sigma_2}$ this (change-of-coordinates) monomial map.

Definition 8 Let $p \in U_{\sigma_1}$ be a point in the first chart and let $q \in U_{\sigma_2}$ be a point in the second chart. We say that p is equivalent to q if the latter is the image of the former under the monomial map defined from the first chart into the second one; i.e., $\varphi_{\sigma_1, \sigma_2}(p) = q$.

The following result ensures that our construction makes sense as a topological manifold. It is an adaptation to our case of Lemma 8.2 proved in [1]. We can reproduce, almost verbatim, its proof; the only step that could fail is where it is used the classical curve selection lemma; cf., [6, §3]. Nevertheless, in the setting of generalized analytic manifolds, the analogous result also takes place; cf., [2, p. 4416], so we are done.

Lemma 1 (cf., Lemma 8.2 in [1]) Let $\varphi_{\sigma_1, \sigma_2}: U_{\sigma_1} \rightarrow U_{\sigma_2}$ be the monomial map between the two charts U_{σ_1} and U_{σ_2} . Let $\{p_j\}_{j=1}^\infty \subset U_{\sigma_1}$ be a sequence of points in the first chart satisfying:

- (1) there is a point p_0 in U_{σ_1} such that $\lim_{j \rightarrow \infty} p_j = p_0$,
- (2) the monomial map $\varphi_{\sigma_1, \sigma_2}$ is defined at each point of the sequence, and
- (3) there is a point q_0 in U_{σ_2} , such that $\lim_{j \rightarrow \infty} \varphi_{\sigma_1, \sigma_2}(p_j) = q_0$.

Then, the monomial map $\varphi_{\sigma_1, \sigma_2}$ is defined and nondegenerate at the limit point of the sequence.

Consequently, we conclude that the relation between points in any two ordered charts settled in Definition 8 is an equivalence relation. Then, arguing as in [1, p. 239], the manifold associated with the fan Σ is defined as the quotient space obtained by means of the previous equivalence relation. Let us recall the general arguments. From Lemma 1, we know that this topological space is Hausdorff. Moreover, the canonical inclusion of all the charts U_σ gives rise to an open cover of it and defines a homeomorphism from each one of these open sets into the non-negative orthant \mathbb{R}_+^n .

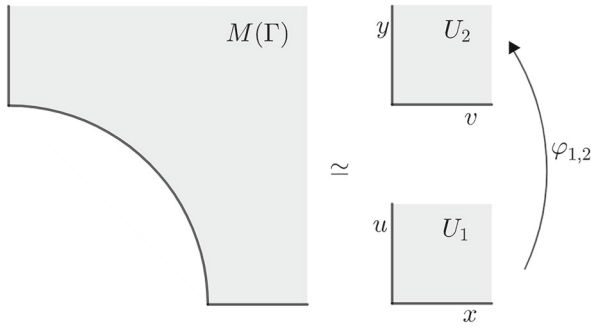


Fig. 2 Geometric picture of the generalized analytic manifold associated with the pseudo-simple fan $\Sigma(\Gamma)$

The transition maps connected with these homeomorphisms are invertible monomial maps. Hence, we have constructed a generalized analytic manifold determined by the pseudo-simple fan Σ .

Example 2 Let us consider the generalized polynomial $f(x, y) = ax^\alpha + by^\beta$ again. As we showed in Example 1, the pseudo-simple fan associated with the Newton polygon of f possesses two cones of dimension two, σ_4 and σ_5 . The respective projective skeletons are $[B_{\sigma_4}] = \{[\beta : \alpha], [0 : 1]\}$ and $[B_{\sigma_5}] = \{[1 : 0], [\beta : \alpha]\}$. Let us choose the (ordered bases) affine representatives $B_1 = \{(1, \alpha/\beta), (0, 1)\}$ and $B_2 = \{(1, 0), (\beta/\alpha, 1)\}$, and let us consider the charts (U_1, B_1) and (U_2, B_2) , where $U_1 = U_2 = \mathbb{R}_+^2$. Then, the monomial map from the first chart into the second, $\varphi_{1,2}: U_1 \rightarrow U_2$, is codified by the change-of-basis matrix

$$\mathcal{E}(\varphi_{1,2}) = \begin{bmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ \alpha/\beta & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\beta/\alpha \\ \alpha/\beta & 1 \end{bmatrix}.$$

Therefore, if (x, u) are coordinates in the first chart, and (v, y) are coordinates in the second one, the transition map between them is given by the correspondence rule

$$\varphi_{1,2}(x, u) = (u^{-\beta/\alpha}, x^{\alpha/\beta}u).$$

The manifold $M(\Gamma)$ thus defined is depicted in Fig. 2.

We claim that our constructions do not depend on the choice of affine representative skeletons. That is to say; if we had chosen any other pair of affine representative skeletons, then we would obtain a generalized analytic manifold isomorphic to the one just constructed. Indeed, let $\tilde{B}_1 = \{(\lambda\beta, \lambda\alpha), (0, \mu)\}$ and $\tilde{B}_2 = \{(v, 0), (\xi\beta, \xi\alpha)\}$ be any other affine representative skeletons of the cones σ_4 and σ_5 , respectively, where λ, μ, v , and ξ are some fixed real positive numbers. Repeating the process to construct the manifold $M(\Sigma)$, but using now the charts $(\tilde{U}_1, \tilde{B}_1)$ and $(\tilde{U}_2, \tilde{B}_2)$, we get another generalized analytic manifold $\tilde{M}(\Gamma)$. However, there is a quite natural isomorphism between the manifolds $\tilde{M}(\Gamma)$ and $M(\Gamma)$. Namely, it is given in local charts by the

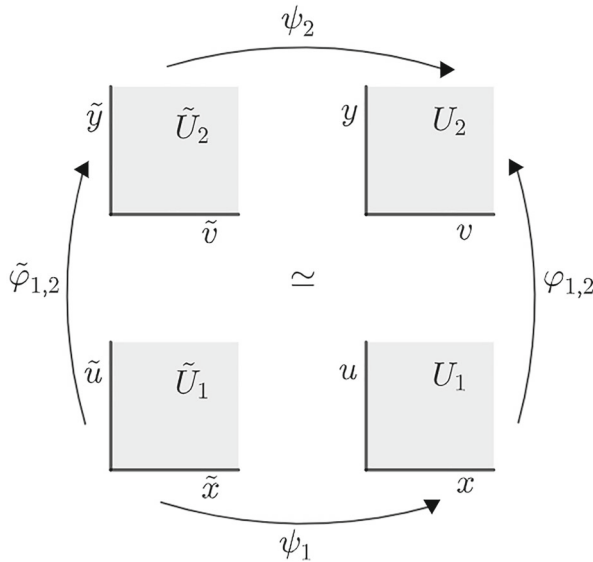


Fig. 3 Independence of the choice of affine representative skeletons

monomial maps $\psi_1: \tilde{U}_1 \rightarrow U_1$, and $\psi_2: \tilde{U}_2 \rightarrow U_2$ codified by the diagonal (change-of-basis) matrices

$$\mathcal{E}(\psi_1) = \begin{bmatrix} \lambda & \beta & 0 \\ 0 & \mu & \end{bmatrix}, \quad \mathcal{E}(\psi_2) = \begin{bmatrix} \nu & 0 \\ 0 & \xi\alpha \end{bmatrix},$$

respectively. It is straightforward to check that $\mathcal{E}(\varphi_{1,2}) \cdot \mathcal{E}(\psi_1) = \mathcal{E}(\psi_2) \cdot \mathcal{E}(\tilde{\varphi}_{1,2})$; i.e., $\varphi_{1,2} \circ \psi_1 = \psi_2 \circ \tilde{\varphi}_{1,2}$. Hence, the manifold $\tilde{M}(\Gamma)$ is isomorphic (as a generalized manifold) to the manifold $M(\Gamma)$, as claimed (see Fig. 3).

Let us prove, in general, that our constructions do not depend on the affine representative skeletons chosen. Do not forget that, as illustrated in the previous example, this is due to the very construction of such kind of manifolds: the monomial maps defining the structures on them correspond to change-of-basis matrices in an n -dimensional real vector space.

Lemma 2 (Independence of the choice of affine representative skeletons) *Let σ_1 and σ_2 be two n -dimensional cones of the same pseudo-simple fan Σ . Let $B_{\sigma_1}, \tilde{B}_{\sigma_1}$ be two affine representative skeletons of the first cone σ_1 , and let $B_{\sigma_2}, \tilde{B}_{\sigma_2}$ be two affine representative skeletons of the second cone σ_2 . Let us consider the pairs of charts $\{(U_{\sigma_1}, B_{\sigma_1}), (U_{\sigma_2}, B_{\sigma_2})\}$, and $\{(\tilde{U}_{\sigma_1}, \tilde{B}_{\sigma_1}), (\tilde{U}_{\sigma_2}, \tilde{B}_{\sigma_2})\}$, as well as the respective change-of-coordinates maps between them; $\varphi_{\sigma_1,\sigma_2}: U_{\sigma_1} \rightarrow U_{\sigma_2}$, and $\tilde{\varphi}_{\sigma_1,\sigma_2}: \tilde{U}_{\sigma_1} \rightarrow \tilde{U}_{\sigma_2}$. Then, there are invertible monomial maps $\psi_{\sigma_1}: \tilde{U}_{\sigma_1} \rightarrow U_{\sigma_1}$, and $\psi_{\sigma_2}: \tilde{U}_{\sigma_2} \rightarrow U_{\sigma_2}$, such that $\varphi_{\sigma_1,\sigma_2} \circ \psi_{\sigma_1} = \psi_{\sigma_2} \circ \tilde{\varphi}_{\sigma_1,\sigma_2}$.*

Proof By definition, the sets

$$B_{\sigma_1} = \{\alpha^1(\sigma_1), \alpha^2(\sigma_1), \dots, \alpha^n(\sigma_1)\}, \quad \tilde{B}_{\sigma_1} = \{\tilde{\alpha}^1(\sigma_1), \tilde{\alpha}^2(\sigma_1), \dots, \tilde{\alpha}^n(\sigma_1)\},$$

$$B_{\sigma_2} = \{\alpha^1(\sigma_2), \alpha^2(\sigma_2), \dots, \alpha^n(\sigma_2)\}, \quad \tilde{B}_{\sigma_2} = \{\tilde{\alpha}^1(\sigma_2), \tilde{\alpha}^2(\sigma_2), \dots, \tilde{\alpha}^n(\sigma_2)\},$$

are ordered bases of \mathbb{R}^n contained in the non-negative orthant \mathbb{R}_+^n . Moreover, for each $j \in \{1, 2\}$, and each $k \in \{1, 2, \dots, n\}$, we know that¹ the vectors $\alpha^k(\sigma_j)$ and $\tilde{\alpha}^k(\sigma_j)$ are colinear in \mathbb{R}^n . Therefore, for each $j \in \{1, 2\}$ and each $k \in \{1, 2, \dots, n\}$, there is a positive real number $\lambda_j^k > 0$, such that

$$\lambda_j^k \alpha^k(\sigma_j) = \tilde{\alpha}^k(\sigma_j).$$

Let D_{σ_j} be the diagonal matrix whose entries correspond to the positive numbers $\{\lambda_j^k\}_{k=1}^n$; that is

$$D_{\sigma_j} := \begin{bmatrix} \lambda_j^1 & 0 & \cdots & 0 \\ 0 & \lambda_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_j^n \end{bmatrix}, \quad j \in \{1, 2\}.$$

Then, we define the monomial maps $\psi_{\sigma_1}: \tilde{U}_{\sigma_1} \rightarrow U_{\sigma_1}$, and $\psi_{\sigma_2}: \tilde{U}_{\sigma_2} \rightarrow U_{\sigma_2}$ as the monomial maps whose matrices of exponents are given by the matrices D_{σ_1} , and D_{σ_2} , respectively. Thus, recalling the definition of the (change of coordinates) monomial maps $\varphi_{\sigma_1, \sigma_2}$, and $\tilde{\varphi}_{\sigma_1, \sigma_2}$, it is straightforward that $\varphi_{\sigma_1, \sigma_2} \circ \psi_{\sigma_1} = \psi_{\sigma_2} \circ \tilde{\varphi}_{\sigma_1, \sigma_2}$, as desired. \square

In conclusion, Lemma 2 ensures that our construction does not depend on the affine representative skeletons chosen for each projective cone in the fan Σ . In the end, we get isomorphic generalized analytic manifolds.

The generalized analytic manifold constructed above is called the (*generalized analytic*) manifold associated with the pseudo-simple fan Σ . We denote it by $M(\Sigma)$.

2.2 Maps Between Manifolds Associated with Pseudo-simple Fans

Let us show how we can define maps between this class of generalized analytic manifolds.

Proposition 3 *Let Σ_1 and Σ_2 be two pseudo-simple fans. Let us suppose that Σ_1 is inscribed in Σ_2 . Let $M_1 = M(\Sigma_1)$ and $M_2 = M(\Sigma_2)$ be the manifolds associated with the fans Σ_1 and Σ_2 , respectively. Then, there is a well-defined generalized analytic map from the manifold M_1 into the manifold M_2 .*

¹ Recall that, by definition of affine representative skeletons, both vectors $\alpha^k(\sigma_j)$ and $\tilde{\alpha}^k(\sigma_j)$ are representatives of the same point in the real projective space $\mathbb{R}P^{n-1}$; i.e., $[\alpha^k(\sigma_j)] = [\tilde{\alpha}^k(\sigma_j)] \in \mathbb{R}P^{n-1}$.

Proof We define the map from M_1 to M_2 locally; that is to say, we define it on each chart of the first manifold. Let U be a chart of the manifold M_1 . Then, to U corresponds an n -dimensional cone in the first fan Σ_1 , say $\sigma_1 \in \Sigma_1$. By hypothesis, Σ_1 is inscribed in Σ_2 ; so, there is an n -dimensional cone σ_2 in Σ_2 that contains the cone σ_1 . Moreover, the cone σ_2 determines a chart V of the manifold M_2 . Thus, as discussed before, we consider the monomial map defined from U to V . We point out that the exponents occurring in such a map belong all to the non-negative orthant, since the first cone σ_1 is contained in the second σ_2 . Therefore, it defines a generalized analytic map from the chart U into the other V . To finish the proof, we must check that these local monomial maps define a global well-defined generalized analytic map from the first manifold M_1 to the second M_2 . However, this is clear using the matrices of exponents of monomial maps recalled in Sect. 1.3. □

The proof of Theorem A is done. Finally, among the maps we have shown, there is a particular subclass when one additional condition is considered. Namely, if we suppose that the union of the cones of the first fan contains the union of the cones of the second fan, then the map between them is proper. This statement can be proved by reproducing verbatim the proof given in [1] of its Theorem 8.1; cf., [1, pp. 241–242].

Lemma 3 *Let Σ_1 and Σ_2 be two pseudo-simple fans. Let us suppose that Σ_1 is inscribed in Σ_2 . Let $M_1 = M(\Sigma_1)$ and $M_2 = M(\Sigma_2)$ be the manifolds associated with the fans Σ_1 and Σ_2 , respectively, and let $h : M_1 \rightarrow M_2$ be the map constructed in Proposition 3. Then, if the union of the cones of the first fan contains the union of the cones of the second fan, the map h is proper and invertible on an everywhere dense subset of M_1 . The converse is also true.*

Corollary 1 *Let Σ be a pseudo-simple fan, and let $M(\Sigma)$ be the manifold associated with it. Then, there is a proper map from $M(\Sigma)$ into the non-negative orthant \mathbb{R}_+^n . Moreover, this map is invertible on an everywhere dense subset of $M(\Sigma)$.*

Before going forward, let us come back once again to Example 1, and let us see how its projection into the non-negative quadrant looks like.

Example 3 Let us consider the same generalized polynomial as in Example 1; i.e., $f(x, y) = ax^\alpha + by^\beta$, and let $M(\Gamma)$ be the manifold associated with the pseudo-simple fan $\Sigma(\Gamma)$, as described in Example 2. That is to say, we consider the generalized analytic structure arising from the atlas $\{(U_1, B_1), (U_2, B_2)\}$, where $B_1 = \{(1, \alpha/\beta), (0, 1)\}$ and $B_2 = \{(1, 0), (\beta/\alpha, 1)\}$ are affine representative skeletons of the two-dimensional cones σ_4 , and σ_5 , respectively. Then, the local pictures of the projection map $\pi : M(\Gamma) \rightarrow \mathbb{R}_+^2$ are given by the monomial maps $\pi_1 : U_1 \rightarrow \mathbb{R}_+^2$ and $\pi_2 : U_2 \rightarrow \mathbb{R}_+^2$ codified by the matrices

$$\mathcal{E}(\pi_1) = \begin{bmatrix} 1 & 0 \\ \alpha/\beta & 1 \end{bmatrix}, \quad \mathcal{E}(\pi_2) = \begin{bmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{bmatrix},$$

respectively. Equivalently

$$\pi_1(x, u) = (x, x^{\alpha/\beta}u), \quad \pi_2(v, y) = (vy^{\beta/\alpha}, y).$$

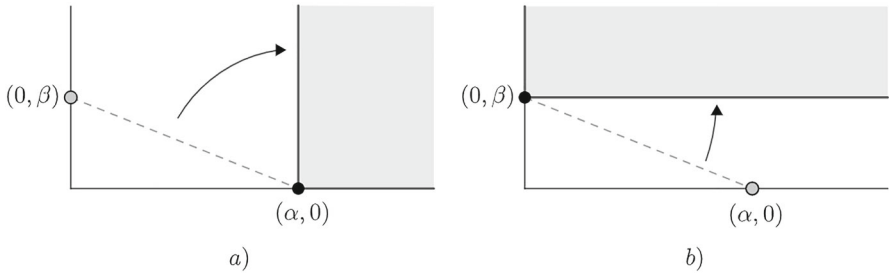


Fig. 4 Newton polygons of the composition $f \circ \pi$ at both corner points of the manifold $M(\Gamma)$

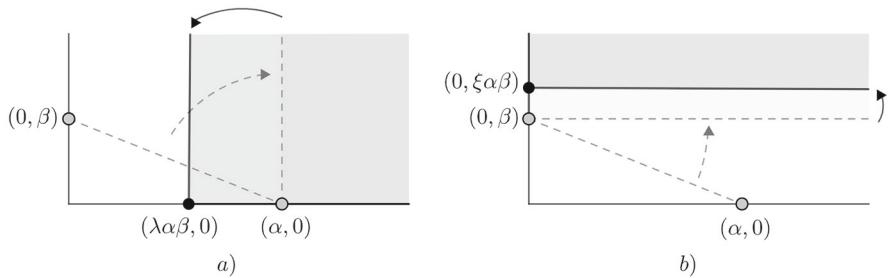


Fig. 5 Newton polygons of the composition $f \circ \tilde{\pi}$ at both corner points of the manifold $\tilde{M}(\Gamma)$

To end this example, let us compute the composition of the initial generalized polynomial $f(x, y) = ax^\alpha + by^\beta$ followed by the projection of the manifold $M(\Gamma)$ into the non-negative quadrant:

$$f \circ \pi_1(x, u) = x^\alpha(a + bu^\beta), \quad f \circ \pi_2(v, y) = y^\beta(av^\alpha + b).$$

Finally, we observe that both Newton polygons $\Gamma(f \circ \pi_1)$ and $\Gamma(f \circ \pi_2)$ are quadrants (see Fig. 4).

Remark 1 If we had chosen another system of coordinates; i.e., another generalized analytic structure, as done at the end of Example 2, the result would be essentially the same. Indeed, let $\tilde{M}(\Gamma)$ be the generalized analytic manifold determined by the atlas $\{(\tilde{U}_1, \tilde{B}_1), (\tilde{U}_2, \tilde{B}_2)\}$, where $\tilde{B}_1 = \{(\lambda\beta, \lambda\alpha), (0, \mu)\}$ and $\tilde{B}_2 = \{(v, 0), (\xi\beta, \xi\alpha)\}$. Then, its projection into the non-negative quadrant, $\tilde{\pi}: \tilde{M}(\Gamma) \rightarrow \mathbb{R}_+^2$, in local coordinates is given by the correspondence rules $\tilde{\pi}_1(\tilde{x}, \tilde{u}) = (\tilde{x}^{\lambda\beta}, \tilde{x}^{\lambda\alpha}\tilde{u}^\mu)$, and $\tilde{\pi}_2(\tilde{v}, \tilde{y}) = (\tilde{v}^\nu\tilde{y}^{\xi\beta}, \tilde{y}^{\xi\alpha})$. Hence

$$f \circ \tilde{\pi}_1(\tilde{x}, \tilde{u}) = x^{\lambda\alpha\beta}(a + bu^{\mu\beta}), \quad f \circ \tilde{\pi}_2(\tilde{v}, \tilde{y}) = y^{\xi\alpha\beta}(av^{\nu\alpha} + b).$$

Therefore, at both corner points of the manifold $\tilde{M}(\Gamma)$, the composition $f \circ \tilde{\pi}$ is also a monomial multiplied by some unit. Both Newton polygons corresponding to the local pictures $f \circ \tilde{\pi}_1$ and $f \circ \tilde{\pi}_2$ are drafted in Fig. 5a, b, respectively.

3 Proofs of Theorems B and C

Let us finally prove the statements on resolution of singularities announced in the introduction. In the proof of Theorem B, we are about to present, the most basic conditions that would lead to the proof of Theorem C are suggested. In consequence, after proving Theorem B, we shall finish our work verifying that such natural conditions are in fact generic.

Proof of Theorem B Let $f = \sum_{\alpha} f_{\alpha} x^{\alpha} \in \mathbb{R}\{x^*\}$ be a germ of a convergent generalized power series. Let $\Gamma = \Gamma(f)$ be its Newton polyhedron. Let $\Sigma(\Gamma)$ be the pseudo-simple fan associated with the Newton polyhedron Γ , and let us denote by $M(\Gamma) := M(\Sigma(\Gamma))$ the manifold associated with the pseudo-simple fan $\Sigma(\Gamma)$.

First of all, Corollary 1 makes sure that the projection of the manifold $M(\Gamma)$ into the non-negative orthant \mathbb{R}_+^n , $\pi: M(\Gamma) \rightarrow \mathbb{R}_+^n$, is proper. Therefore, to accomplish our goal, we have to verify that, at each corner point p of the manifold $M(\Gamma)$, the germ of the composition $f \circ \pi$ is a monomial multiplied by some unit. Let (U, B) be a chart in $M(\Gamma)$. By definition, the local picture of the map π in the chart U , say $\pi|_U: U \rightarrow \mathbb{R}_+^n$, is codified by the change-of-basis matrix that transforms the standard basis of \mathbb{R}^n , $B_0 := \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$ into the basis $B = \{\alpha^1, \alpha^2, \dots, \alpha^n\}$. That is to say, the monomial map $\pi|_U$ is defined in terms of the matrix $\mathcal{E}(\pi|_U) = (a_{ij})$ whose columns are the ordered column-vectors of the basis B ; i.e., $a_{ij} = \alpha_i^j$ for all $i, j \in \{1, 2, \dots, n\}$. We stress that the composition $f \circ \pi|_U(x)$ is a monomial multiplied by some unit at the origin in U whenever the principal part of f is transformed into a germ of the same style. That is to say, it is enough to show that the composition $f_{\Delta} \circ \pi|_U(x)$ is a monomial multiplied by some unit, where f_{Δ} is the principal part of f . However, a direct computation shows that under the projection $\pi|_U$, each monomial x^{α} , with $\alpha \in \mathbb{R}_+^n$, is transformed into the monomial $x_1^{(\alpha^1, \alpha)} x_2^{(\alpha^2, \alpha)} \dots x_n^{(\alpha^n, \alpha)}$. Therefore, recalling the definition of the pseudo-simple fan $\Sigma(\Gamma)$ (see Sect. 1.5), we conclude that $f_{\Delta} \circ \pi|_U(x) = x^{\alpha_0} \tilde{f}(x)$, for some convergent power series $\tilde{f} \in \mathbb{R}\{x^*\}$, such that $\tilde{f}(0) \neq 0$. The proof of Theorem B is complete. \square

3.1 Proof of Theorem C

From now on, $\Gamma \subset \mathbb{R}_+^n$ will denote a fixed Newton polyhedron. Looking carefully into the previous arguments, we notice that, for any germ of a generalized analytic function at the origin, its stratified resolution of singularities, as done above, is ruled by its Newton diagram; i.e., by the union of all compact faces of its Newton polyhedron. Besides, such a monomialization process would be a full resolution of singularities of a given germ f , whenever the local monomial description of the composition $f \circ \pi$ remained true at any point on the boundary of the manifold $M(\Gamma)$, and not just at the corner ones. However, those points where could fail this belong to the zero-level set of the γ -parts of the germ f , intersected with the positive orthant $\mathbb{R}_{>0}^+$, where γ ranges over all compact faces of the Newton polyhedron Γ . Said that, next, we estab-

simplest conditions under which the previous construction provides a full resolution for germs of generalized analytic functions whose Newton polyhedron is exactly Γ .

Definition 9 Let $f \in \mathbb{R}[[x^*]]$ be a formal generalized power series, such that its Newton polyhedron coincides with Γ . We say that the principal part of the power series f is Γ -nondegenerate if none of the generalized polynomials f_γ has a zero in the positive orthant $\mathbb{R}_{>0}^n$, where the index γ ranges over all the compact faces of the Newton polyhedron Γ .

In consequence, all we need to do is to prove that being Γ -nondegenerate is a generic condition in the set of all principal parts of formal generalized power series, whose Newton polyhedron coincides with Γ .

Indeed, let γ be a compact face of the Newton polyhedron Γ , and let $f \in \mathbb{R}[[x^*]]$ be a generalized polynomial that is a γ -part; i.e., the generalized polynomial f is a finite sum of monomials $f = \sum_{\alpha \in \gamma} f_\alpha x^\alpha$, where all the exponents belong to the compact face γ .

Before continuing, we remark that we can assume that for any exponent $\alpha \in \gamma$, it happens that $n + 1 \leq \min\{\alpha_j : j \in \{1, 2, \dots, n\}\}$ (if were not the case, it is enough to multiply f by some monomial x^{α^0} , with $\alpha^0 \in \mathbb{R}_+^n$). In consequence, we can assume that the generalized polynomial f defines a C^k -differentiable function, with $n \leq k$.

The generalized polynomial f has also one special property: there are a tuple $\beta \in \mathbb{R}_+^n$ and a positive real number $\nu > 0$, such that it can be rewritten as follows:

$$f = \sum_{\langle \beta, \alpha \rangle = \nu} f_\alpha x^\alpha.$$

Then, a straightforward computation shows us that it satisfies the following *generalized Euler identity*:

$$\nu \cdot f = \beta_1 x_1 \frac{\partial f}{\partial x_1} + \beta_2 x_2 \frac{\partial f}{\partial x_2} + \dots + \beta_n x_n \frac{\partial f}{\partial x_n}.$$

Thus, we observe that the zero-level set of f contains the intersection of the zero-level sets of all its partial derivatives in the positive orthant $\mathbb{R}_{>0}^n$. Plus, it is clear that if the generalized polynomial is Γ -nondegenerate, then, the intersection of the zero-level sets of all its partial derivatives is empty in the positive orthant $\mathbb{R}_{>0}^n$. It follows that the set of all γ -parts that do not have any zeros at all in the positive orthant, is a subset of all the γ -parts which are regular in the positive orthant $\mathbb{R}_{>0}^n$. Therefore, arguing as in Lemma 6.1 in [1, §6.2, pp. 188–189], but using Morse–Sard Theorem; cf., [8, 11] (we apply the C^k -version of the statement, with $n \leq k$), instead of Bertini–Sard Theorem as done in [1], we finally get that: in the space of all generalized polynomials which are γ -parts, the subset of all Γ -nondegenerate ones is everywhere dense. Therefore, we are done.

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