



On Groups with Few Subgroups not in the Chermak–Delgado Lattice

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Abstract

We investigate the question of how many subgroups of a finite group are not in its Chermak–Delgado lattice. The Chermak–Delgado lattice for a finite group is a self-dual lattice of subgroups with many intriguing properties. Fasolă and Tărnăuceanu (Bull Aust Math Soc 107(3):451–455, 2023) asked how many subgroups are not in the Chermak–Delgado lattice and classified all groups with two or less subgroups not in the Chermak–Delgado lattice. We extend their work by classifying all groups with less than five subgroups not in the Chermak–Delgado lattice. In addition, we show that a group with less than five subgroups not in the Chermak–Delgado lattice is nilpotent. In this vein, we also show that the only non-nilpotent group with five or fewer subgroups in the Chermak–Delgado lattice is S_3 .

1 Introduction

In this paper, we examine the question of how many subgroups of a finite group G are not in the Chermak–Delgado lattice of G . Chermak and Delgado [6] first defined the Chermak–Delgado lattice in 1989. In 2022, Fasolă and Tărnăuceanu [9] investigated the question of how large the Chermak–Delgado lattice of a finite group can be. They classified all groups with at most 2 subgroups not in their Chermak–Delgado lattice.

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In Theorem C, we extend their work by classifying all groups with less than five subgroups not in their Chermak–Delgado lattice.

Throughout the paper, we will use the notation $H \leq G$ to mean that H is a subgroup of G . The Chermak–Delgado lattice is a sublattice of the subgroup lattice of G . To define the Chermak–Delgado lattice of a finite group G , which we write as $\mathcal{CD}(G)$, we first need to define a function m_G called the Chermak–Delgado measure. The function m_G takes as input a subgroup of G and returns the product of the order of the subgroup and the order of its centralizer in G , i.e.,

$$m_G(H) = |H| \cdot |\mathbf{C}_G(H)|.$$

It is surprising that the set of subgroups with maximum Chermak–Delgado measure have a very special property: they form a lattice, the Chermak–Delgado lattice.

For a finite group G , we write $m^*(G)$ for the maximum value of m_G on a finite group, i.e.,

$$m^*(G) = \max_{H \leq G} \{m_G(H)\}.$$

If the group is clear from context, we will shorten $m^*(G)$ to just m^* . If $H, K \leq G$ satisfy $m_G(H) = m_G(K) = m^*$, then $HK = \langle H, K \rangle$ and $m_G(HK) = m_G(H \cap K) = m^*$. Hence, the subgroups with maximum Chermak–Delgado measure form a sublattice of the subgroup lattice of G : this is the Chermak–Delgado lattice of G . The proof that $\mathcal{CD}(G)$ is a lattice can be found in Isaacs [11, 1.G].

Before stating our main results, we introduce some new notation. We write $\delta_{\mathcal{CD}}(G)$ for the number of subgroups of G not in $\mathcal{CD}(G)$. Hence, $\delta_{\mathcal{CD}}(G) = 0$, if all subgroups of G are in $\mathcal{CD}(G)$ and $\delta_{\mathcal{CD}}(G) = 1$ if there is single subgroup of G not in $\mathcal{CD}(G)$. The cyclic groups show that $\delta_{\mathcal{CD}}$ maps onto the natural numbers.

Fasolă and Tărnăuceanu classified all groups where $\delta_{\mathcal{CD}}(G) \leq 2$ [9, Theorem 1.1] and asked for such a classification when $\delta_{\mathcal{CD}}(G) > 2$. In this paper, we provide such a classification when $\delta_{\mathcal{CD}}(G) \leq 4$. To do this, we first investigate how $\delta_{\mathcal{CD}}(G)$ influences the structure of a finite group G .

Theorem A *Let G be a finite group. If $\delta_{\mathcal{CD}}(G) < 5$, then G is nilpotent.*

Of note, when $\delta_{\mathcal{CD}}(G) = 5$ and G is not nilpotent, we have the following theorem where S_3 is the symmetric group on 3 symbols.

Theorem B *Let G be a finite group that is not nilpotent. If $\delta_{\mathcal{CD}}(G) = 5$, then $G \cong S_3$.*

Using Theorem A together with a mixture of computational and theoretical results, we are able to complete the classification of groups with $\delta_{\mathcal{CD}}(G) < 5$.

Theorem C *Let G be a finite group. If $\delta_{\mathcal{CD}}(G) < 5$, then one of the following holds:*

1. $\delta_{\mathcal{CD}}(G) < 3$;
2. $\delta_{\mathcal{CD}}(G) = 3$ and G is cyclic of order either $p \cdot q$ or p^3 for primes p, q ;
3. $\delta_{\mathcal{CD}}(G) = 4$ and G is isomorphic to either a cyclic group of order p^4 for a prime p , $C_2 \times C_2$, or the extraspecial group of order 27 and exponent 9.

Hence, we extend the classification of groups with $\delta_{CD}(G) < 3$ to $\delta_{CD}(G) < 5$.

The rest of the paper proceeds as follows: In Sect. 2, we collect some basic properties of the Chermak–Delgado lattice. In Sect. 3, we prove Theorem A and in Sect. 4, we prove Theorem B. Sect. 5 contains our proof of Theorem C. Finally in our conclusion, we ask some questions motivated by the study of δ_{CD} .

2 Preliminary Material

In this section, we recall a few well-known properties of the Chermak–Delgado lattice for a finite group G . We also present some elementary properties of the Chermak–Delgado lattice for a few families of groups.

Lemma 2.1 *Suppose G is a finite group. If $H \in \mathcal{CD}(G)$, then $\mathbf{Z}(G) \leq H$.*

Proof If $H \leq G$, then $\mathbf{C}_G(H) = \mathbf{C}_G(H\mathbf{Z}(G))$, and so if $H \in \mathcal{CD}(G)$, and so H attains the maximum Chermak–Delgado measure in G , then $\mathbf{Z}(G) \leq H$. \square

Corollary 2.2 *Suppose that a finite nontrivial group G is a p -group. Then, $1 \notin \mathcal{CD}(G)$.*

Proof A nontrivial p -group has a nontrivial center, and the result follows by Lemma 2.1. \square

The following result appears in McCulloch [12, Corollary 7].

Lemma 2.3 ([12]) *Suppose G is a finite group and $1 \in \mathcal{CD}(G)$. Then $\mathcal{CD}(G)$ contains no nontrivial p -group for any primes p .*

We will often use the contrapositive to Lemma 2.3, which we state below as a corollary.

Corollary 2.4 *Suppose G is a finite group. If a nontrivial p -group is in $\mathcal{CD}(G)$, then $1 \notin \mathcal{CD}(G)$.*

Another well-known result about Chermak–Delgado lattices of finite groups is the following by Brewster and Wilcox [5, Theorem 2.9].

Lemma 2.5 ([5]) *Let G and H be finite groups. Then, $\mathcal{CD}(G \times H) = \mathcal{CD}(G) \times \mathcal{CD}(H)$.*

Together with Corollary 2.2 above, this implies that in a finite nilpotent group, we can greatly restrict the types of groups that appear in the Chermak–Delgado lattice.

Corollary 2.6 *Let G be a finite nilpotent group. If p divides $|G|$, then p divides the order of H for every H in $\mathcal{CD}(G)$.*

Proof We can write G as $S \times K$ where S is a Sylow p -group of G and K is a p -complement of S . We know that $\mathcal{CD}(G) = \mathcal{CD}(S) \times \mathcal{CD}(K)$. Since $\mathcal{CD}(S)$ does not contain 1, we conclude that every element of $\mathcal{CD}(S)$ is a nontrivial p -group. Hence p divides any H in $\mathcal{CD}(G)$. \square

Another interesting result about the Chermak–Delgado lattice is that since $H \in \mathcal{CD}(G)$ if and only if $H^g \in \mathcal{CD}(G)$, we have that HH^g is a group for all $g \in G$. Recall that a subgroup H of a group G is called subnormal if

$$H = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_n = G.$$

Foguel showed that groups H that permute with their conjugates, i.e., HH^g is a subgroup, are subnormal [8]. Hence, subgroups in $\mathcal{CD}(G)$ are subnormal in G , which is also noted in [5, 7].

The following lemma is part of a theme we will see throughout the paper: certain conditions on G and $\mathcal{CD}(G)$ will exclude other groups from being in $\mathcal{CD}(G)$. We say that a set of subgroups $X = \{H_1, \dots, H_k : H_i \leq G\}$ of cardinality k witnesses that $\delta_{\mathcal{CD}}(G) \geq k$, if for each $H_i \in X$ we have that $H_i \notin \mathcal{CD}(G)$.

Lemma 2.7 *Let G be a finite group. If G has k subgroups of prime order, then $\delta_{\mathcal{CD}}(G) \geq k$.*

Proof If $k = 0$, then G is trivial and the result is trivially true. So suppose $k > 0$. If $1 \in \mathcal{CD}(G)$, then it follows from Lemma 2.3 that the k subgroups of prime order witness that $\delta_{\mathcal{CD}}(G) \geq k$. Suppose $1 \notin \mathcal{CD}(G)$ and suppose that $H, K \leq G$ are two distinct subgroups of prime order and $H, K \in \mathcal{CD}(G)$. Then, $1 = H \cap K \in \mathcal{CD}(G)$, a contradiction. Hence among the subgroups of prime order, at most one of them can be in $\mathcal{CD}(G)$. Thus, the other $k - 1$ subgroups of prime order, together with the identity witness that $\delta_{\mathcal{CD}}(G) \geq k$. □

The following lemma concerns generalized quaternion groups. Recall that a group is called a generalized quaternion group if $G \cong \langle a, b \mid a^{2^k}, b^4, a^k = b^2, a^b = a^{-1} \rangle$. We will write the generalized quaternion group of order m as Q_m . It is well-known that a finite p -group with a single subgroup of order p is either cyclic or generalized quaternion [13, 4.4]. We also need the following from Fasolă and Tărnăuceanu [9, The proof of Lemma 2.1].

Lemma 2.8 ([9]) *If G is a generalized quaternion 2-group, then*

$$|\mathcal{CD}(Q_{2^k})| = \begin{cases} 5 & \text{if } k = 3 \\ 1 & \text{if } k > 3. \end{cases}$$

For the next lemma, we will need to introduce some notation. For a group G and two subgroups H and K of G , we write $[[H : K]]_G$ for the set of subgroups between H and K , i.e.,

$$[[H : K]]_G = \{J \leq G : H \leq J \text{ and } J \leq K\}.$$

If the group G is clear from context, then we write $[[H : K]]$ for $[[H : K]]_G$. The following lemma from An [1, Theorem 3.4] and [2, Theorem 4.4] shows that when $H \in \mathcal{CD}(G)$, we can actually say something about $\mathcal{CD}(H)$.

Lemma 2.9 *Let G be a finite group and $H \leq G$. If $H \in \mathcal{CD}(G)$ then $\mathcal{CD}(H)$ is exactly the set of subgroups of H containing $\mathbf{Z}(H)$ and in $\mathcal{CD}(G)$, i.e.,*

$$\mathcal{CD}(H) = [[\mathbf{Z}(H) : H]] \cap \mathcal{CD}(G).$$

3 Proof of Theorem A

Our proof of Theorem A depends on a connection between the Sylow subgroups and the Chermak–Delgado lattice. One of the Sylow Theorems states that the total number of Sylow p -subgroups in G is 1 if and only if any of the Sylow p -subgroups is normal in G . The below lemma extends this observation.

Lemma 3.1 *Let G be a finite group and let S be a Sylow subgroup of G . Then, S is subnormal in G if and only if S is normal in G .*

Proof A normal subgroup is also subnormal. Suppose now that S is subnormal in G , and that S is a p -group for a prime p . This means there is a chain of subgroups

$$S = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_n = G.$$

We note that S is a Sylow subgroup of all of the K_i . Since $S \triangleleft K_1$, we know that S is the unique Sylow p -subgroup of K_1 . Hence S is a characteristic subgroup of K_1 . Thus, S is normal in K_2 and by the same argument characteristic in K_2 . Continuing in this manner, we see that S is normal in G . □

Lemma 3.2 *Let G be a finite group. Then at most one Sylow subgroup of G is in $\mathcal{CD}(G)$. Furthermore, if a Sylow subgroup of G is in $\mathcal{CD}(G)$, then it is normal.*

Proof Since all subgroups in $\mathcal{CD}(G)$ are subnormal, we have by Lemma 3.1 that if a Sylow subgroup of G is in $\mathcal{CD}(G)$, then it is normal.

First suppose that $1 \in \mathcal{CD}(G)$. Now 1 is a Sylow subgroup of G for any prime not dividing $|G|$. It follows from Lemma 2.3 that no nontrivial Sylow subgroups of G are in $\mathcal{CD}(G)$, and so in this case, 1 is the unique Sylow subgroup in $\mathcal{CD}(G)$.

Suppose p is a divisor of $|G|$ and suppose a Sylow p -subgroup S of G is in $\mathcal{CD}(G)$. Write $|S| = p^k$. Since S is normal in G , we have that S is the unique Sylow p -subgroup of G . Let T be another Sylow subgroup of G , and so T is not a p -group. We show that T is not in $\mathcal{CD}(G)$.

By definition, $S \in \mathcal{CD}(G)$ means that $m^*(G) = |S| \cdot |\mathbf{C}_G(S)|$. Since $1 < \mathbf{Z}(S) \leq \mathbf{C}_G(S)$, we conclude that $p^{k+1} |m^*(G)$. Since S is a Sylow subgroup of G with $|S| = p^k$, we know that p^{k+1} does not divide $|\mathbf{C}_G(T)|$. And so p^{k+1} does not divide $|T| \cdot |\mathbf{C}_G(T)|$ and we conclude that T is not in $\mathcal{CD}(G)$. □

Corollary 3.3 *Let G be a finite group and write n for the number of nontrivial Sylow subgroups of G . Then $\delta_{\mathcal{CD}}(G) \geq n$.*

Proof If none of the nontrivial Sylow subgroups of G are in $\mathcal{CD}(G)$, then the nontrivial Sylow subgroups of G witness that $\delta_{\mathcal{CD}(G)} \geq n$.

Otherwise, suppose that a nontrivial Sylow subgroup S of G is in $\mathcal{CD}(G)$. By Lemma 3.2 and Corollary 2.4,

$$\{T : T \neq S \text{ is a nontrivial Sylow subgroup of } G\} \cup \{1\}$$

witnesses that $\delta_{\mathcal{CD}(G)} \geq n$. □

Another fact of Sylow theory is that the number of Sylow p -subgroup of a group G is congruent to 1 modulo p , and this number divides the index in G of a Sylow p -subgroup. Hence if a finite group G is not nilpotent, then it has at least four Sylow subgroups with equality if and only if G is a $\{2, 3\}$ -group and the Sylow 3-subgroup is normal.

Proof of Theorem A The result for $\delta_{\mathcal{CD}(G)} < 4$ follows from Corollary 3.3 and the fact that non-nilpotent groups always have at least four Sylow subgroups. Suppose $\delta_{\mathcal{CD}(G)} = 4$ and that G is not nilpotent. Let S be the Sylow 3-subgroup of G .

We first argue that $S \in \mathcal{CD}(G)$. Suppose by way of contradiction that $S \notin \mathcal{CD}(G)$. Then the four subgroups that are not in $\mathcal{CD}(G)$ are the three Sylow 2-subgroups of G and S . We conclude that $1 \in \mathcal{CD}(G)$. It follows from Lemma 2.3 that all of the nontrivial 2-groups and 3-groups of G are not in $\mathcal{CD}(G)$. Since $\delta_{\mathcal{CD}(G)} = 4$, G has exactly four total nontrivial 2-groups or 3-groups, namely the three Sylow 2-subgroups of G and S . So, all nontrivial Sylow subgroups must have prime order. Thus, $m^*(G) = m_G(1) = |G| = 6$. But S has a Chermak–Delgado measure of 9, a contradiction.

Therefore, $S \in \mathcal{CD}(G)$, and the four subgroups that are not in $\mathcal{CD}(G)$ are the three Sylow 2-subgroups of G and the identity subgroup. We conclude that $G \in \mathcal{CD}(G)$.

Note that any Sylow 2-subgroup of $C_G(S)$ is also a Sylow 2-subgroup of G . This is because $|S| \cdot |C_G(S)| = m^*(G)$ and since $G \in \mathcal{CD}(G)$, $|G|$ divides $m^*(G)$. Now, we have that $G = ST$ is a direct product where T is a Sylow 2-subgroup of $C_G(S)$, which implies that G is nilpotent, a contradiction. □

We can say more about the non-nilpotent case, as seen in the next section where we prove Theorem B.

4 Proof of Theorem B

If a finite group G is not nilpotent, then it has at least four Sylow subgroups, and if G has either four or five Sylow subgroups, then G is a $\{2, 3\}$ -group. In this section, we will sometimes use the non-standard notation H_p to denote a Sylow p -subgroup of G . This makes our proofs easier to read by reminding the reader of the prime associated with the Sylow p -subgroup, especially as the prime p will shift between certain lemmata. In addition, we wish to avoid confusion with S_p , the symmetric group on p symbols.

Lemma 4.1 *Suppose that a finite $\{2, 3\}$ -group G has exactly four Sylow 3-subgroups and a normal Sylow 2-subgroup H_2 . Then $\delta_{\mathcal{CD}(G)} > 5$.*

Proof Since G has four Sylow 3-subgroups, it is not nilpotent. By Theorem A, we conclude that $\delta_{\mathcal{CD}}(G) \geq 5$. The four Sylow 3-subgroups of G witness that $\delta_{\mathcal{CD}}(G) \geq 4$. By way of contradiction suppose that $\delta_{\mathcal{CD}}(G) = 5$.

Note that 4 divides $|H_2|$ which equals the index in G of a Sylow 3-subgroup.

Suppose $1 \in \mathcal{CD}(G)$. It follows from Lemma 2.3 that all of the nontrivial 2-groups and 3-groups of G are not in $\mathcal{CD}(G)$. Since $\delta_{\mathcal{CD}}(G) = 5$, G has exactly five total nontrivial 2-groups or 3-groups, namely the four Sylow 3-groups and the normal Sylow 2-group. We conclude that all of the nontrivial Sylow subgroups have prime order which contradicts $|H_2| \geq 4$.

Now suppose that $1 \notin \mathcal{CD}(G)$. By assumption, 1 and the four Sylow 3-subgroups are the only subgroups of G not contained in $\mathcal{CD}(G)$. Hence, G and H_2 are in $\mathcal{CD}(G)$. This means that $m^*(G) = |G| \cdot |\mathbf{Z}(G)|$ is divisible by $|G|$. However, $m^*(G) = |H_2| \cdot |\mathbf{C}_G(H_2)|$. Since H_2 is a 2-group, we conclude that $[H_2, H_3] = 1$ for a Sylow 3-subgroup H_3 of G . Then, $H_2H_3 = G$ is a direct product and is, thus, nilpotent, a contradiction. Hence, $\delta_{\mathcal{CD}}(G) > 5$. □

Lemma 4.2 *Suppose that a finite $\{2, 3\}$ -group G has exactly three Sylow 2-subgroups and a normal Sylow 3-subgroup H_3 . If $\delta_{\mathcal{CD}}(G) = 5$, then $1 \notin \mathcal{CD}(G)$.*

Proof Suppose by way of contradiction that $1 \in \mathcal{CD}(G)$. It follows from Lemma 2.3 that all of the nontrivial 2-groups and 3-groups of G are not in $\mathcal{CD}(G)$. Since $\delta_{\mathcal{CD}}(G) = 5$, G has at most five nontrivial 2-groups or 3-groups, including the three Sylow 2-subgroups and H_3 . And so G has at most two subgroups that are nontrivial 3-groups, and at most one subgroup that is a nontrivial 2-group, but not Sylow. This means that G is a $\{2, 3\}$ -group with $|G| \leq 2^23^2$. Considering each of the possible orders of G , $2 \cdot 3, 4 \cdot 3, 2 \cdot 9$, or $4 \cdot 9$, we see that in every case, the Sylow subgroups are abelian and one of them has Chermak–Delgado measure larger than $m^*(G) = |G|$. □

Lemma 4.3 *Suppose that a finite $\{2, 3\}$ -group G has exactly three Sylow 2-subgroups and a normal Sylow 3-subgroup H_3 . If $\delta_{\mathcal{CD}}(G) = 5$, then $H_3 \in \mathcal{CD}(G)$.*

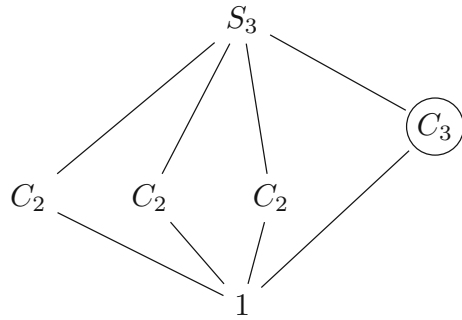
Proof By Lemma 4.2, we know that $1 \notin \mathcal{CD}(G)$. Hence 1, together with the three Sylow 2-subgroups witness that $\delta_{\mathcal{CD}}(G) \geq 4$. There is one other subgroup of G not in $\mathcal{CD}(G)$.

Suppose by way of contradiction that $H_3 \notin \mathcal{CD}(G)$. Let H_2 be a Sylow 2-subgroup. We claim that one of H_3 or H_2 must have prime order. Otherwise, if $1 < P < H_3$, and $1 < Q < H_2$, then $P, Q \in \mathcal{CD}(G)$, and so $1 = P \cap Q \in \mathcal{CD}(G)$, a contradiction. Let K be a Sylow subgroup of prime order p .

Then $K \cap \mathbf{Z}(G) = 1$, as otherwise, $K \leq \mathbf{Z}(G)$, and it would follow that the $\{2, 3\}$ -group G is nilpotent. And so $\mathbf{Z}(G)$ is a p' -group. Now $m^*(G) = |G| \cdot |\mathbf{Z}(G)| = p \cdot (p')^i$ for some i . But also $K < K\mathbf{Z}(G) < G$, and so $K\mathbf{Z}(G) \in \mathcal{CD}(G)$. But $K\mathbf{Z}(G)$ is abelian, and so $K\mathbf{Z}(G) \leq \mathbf{C}_G(K\mathbf{Z}(G))$, and so p^2 divides $|K\mathbf{Z}(G)| \cdot |\mathbf{C}_G(K\mathbf{Z}(G))| = m^*(G)$, a contradiction. □

We can now prove Theorem B, which states that for a non-nilpotent group if $\delta_{\mathcal{CD}}(G) = 5$, then $G \cong S_3$.

Fig. 1 Subgroup diagram of S_3 with a circle around the Chermak–Delgado sublattice, which consists of the single subgroup isomorphic to C_3 . This is the only non-nilpotent group with $\delta_{\mathcal{CD}}(G) = 5$



Proof of Theorem B From Lemma 4.1 we conclude that G is a $\{2, 3\}$ group with exactly three Sylow 2-subgroups, and a normal Sylow 3-subgroup H_3 . From Lemmata 4.2 and 4.3, we have that $H_3 \in \mathcal{CD}(G)$ and that the five subgroups of G not in $\mathcal{CD}(G)$ are the three Sylow 2-subgroups, the identity subgroup, and one other subgroup X . We will show that $X = G$. Suppose instead that $G \in \mathcal{CD}(G)$.

Note that any Sylow 2-subgroup of $C_G(H_3)$ is also a Sylow 2-subgroup of G . This is because $|H_3| \cdot |C_G(H_3)| = m^*(G)$ and since $G \in \mathcal{CD}(G)$, $|G|$ divides $m^*(G)$. And so now we have that $G = H_3H_2$ is a direct product where H_2 is a Sylow 2-subgroup of $C_G(H_3)$, which implies that G is nilpotent, a contradiction.

So $X = G$, and the three Sylow 2-subgroups, the identity subgroup, and G witness that $\delta_{\mathcal{CD}}(G) = 5$.

Let H_2 be a Sylow 2-subgroup of G . We argue that $|H_2| = 2$. Otherwise, there exists $1 < X \leq \mathbf{Z}(H_2)$ with $X < H_2$, and so $X \in \mathcal{CD}(G)$. Then $m^*(G) = |X| \cdot |C_G(X)| = |H_3| \cdot |C_G(H_3)|$, and so $|H_3|$ divides $|C_G(X)|$. This implies that $H_3 \leq C_G(X)$. Hence $[X, H_2] = 1$ and $[X, H_3] = 1$. Since $\mathbf{Z}(G) \leq H_3$, we have $X \leq \mathbf{Z}(G) \leq H_3$, a contradiction. Hence $|H_2| = 2$. Since $|H_2| = 2$, we note that $C_G(H_3)$ is a 3-group as otherwise a Sylow 2-subgroup would be contained in $C_G(H_3)$ which would imply that G is nilpotent. So $m^*(G) = |H_3| \cdot |C_G(H_3)|$ is a power of 3.

We now argue that H_2 is self-normalizing. Otherwise, let Y be a nontrivial Sylow 3-subgroup of $\mathbf{N}_G(H_2)$. So $1 < Y < H_3$ (note that $Y < H_3$ as otherwise H_3 and H_2 would normalize one another, and so $G = H_3H_2$ would be nilpotent). Then H_2Y is a subgroup, and $H_2 < H_2Y < G$, and so H_2Y is in $\mathcal{CD}(G)$, which contradicts the fact that $m^*(G)$ is a power of 3.

So H_2 is self-normalizing, and $3 = |G : \mathbf{N}_G(H_2)| = |G : H_2| = |H_3|$. So H_3 is cyclic of order 3, H_2 is cyclic of order 2, and $G \cong S_3$. \square

Figure 1 shows the subgroup diagram of S_3 .

5 Proof of Theorem C

Lemma 5.1 *Let G be a nonabelian nilpotent group. If $\delta_{\mathcal{CD}}(G) < 7$, then G is a p -group. If $\delta_{\mathcal{CD}}(G) < 6$, then G is a 2-group or a 3-group.*

Proof Suppose that there are at least two prime divisors of $|G|$. Corollary 2.6 tells us that no groups of prime-power order are in $\mathcal{CD}(G)$. Moreover, since G is a nonabelian nilpotent group, then at least one Sylow subgroup of G is nonabelian. Let p be a prime such that G has a nonabelian Sylow p -subgroup and let q be another prime divisor of G .

Let S be a nonabelian Sylow p -subgroup of G . Then S has at least $p + 2$ subgroups above the center, which means that S has at least $p + 4$ subgroups total (including $\mathbf{Z}(S)$ and 1). Thus G contains at least $p + 4$ distinct p -groups for the prime p . These p -groups, together with a nontrivial Sylow q -group T form a set of $p + 5$ subgroups that witness $\delta_{\mathcal{CD}}(G) \geq p + 5$.

Thus, we have

$$p + 5 \leq \delta_{\mathcal{CD}}(G) < 7.$$

This is a contradiction. We conclude that G must be divisible by a single prime.

Now suppose that $\delta_{\mathcal{CD}}(G) < 6$. The group G must be divisible by a single prime p . Recall that a nonabelian p -group of odd order must contain $p + 1$ subgroups of order p . By Lemma 2.7 this means that

$$p + 1 \leq \delta_{\mathcal{CD}}(G) < 6$$

and we conclude that $p < 4$. Hence if a nonabelian nilpotent group G has $\delta_{\mathcal{CD}}(G) < 6$, then G is a 2-group or a 3-group. □

We note that for an odd prime p , $\delta_{\mathcal{CD}}(Q_8 \times C_p) = 7$ and is a nonabelian nilpotent group. We also note that the extraspecial group of order 5^3 and exponent 25 satisfies $\delta_{\mathcal{CD}}(G) = 6$, so both bounds in Lemma 5.1 are sharp.

Lemma 5.2 *Let G be a nonabelian p -group and suppose that $3 \leq \delta_{\mathcal{CD}}(G) \leq 4$. Then G contains at least $p + 1$ subgroups of order p . Of these, p of the subgroups together with the identity witness that $\delta_{\mathcal{CD}}(G) \geq p + 1$.*

Proof If $p > 2$, then every nonabelian p -group contains at least $p + 1$ subgroups of order p .

If $p = 2$, and G contains a single subgroup of order 2, then G is generalized quaternion. Since the generalized quaternion 2-groups of order greater than 8 contain more than 7 subgroups and contain a single subgroup in their Chermak–Delgado lattices, we conclude that for such groups $\delta_{\mathcal{CD}}(G) > 4$. Moreover, $\delta_{\mathcal{CD}}(Q_8) = 1$. □

Hence for a nonabelian p -group G with $3 \leq \delta_{\mathcal{CD}}(G) \leq 4$ there are p subgroups of order p , which together with the identity witness that $\delta_{\mathcal{CD}}(G) \geq p + 1$. We will use this to derive a set of conditions for a subgroup of G to satisfy.

Lemma 5.3 *Let G be a nonabelian p -group and suppose $3 \leq \delta_{\mathcal{CD}}(G) \leq 4$. For every $K \leq G$, the following hold:*

- 1) *There are at most 4 subgroups of order p in K .*

2) There is a subgroup $Z \leq \mathbf{Z}(K)$ with $|Z| = p$ such that

$$|\{H \leq K : p^2 \leq |H| \text{ and } Z \not\leq H\}| \leq \begin{cases} 0 & p > 2 \\ 1 & p = 2. \end{cases}$$

3) If $|K| \geq p^3$, then $K \in \mathcal{CD}(K)$ and at most one subgroup H of K can satisfy $\mathbf{Z}(K) \leq H$ and $H \notin \mathcal{CD}(K)$.

Proof As noted in Lemma 5.2, the group G has at least $p + 1$ subgroups not in $\mathcal{CD}(G)$ consisting of the identity and the p subgroups of order p . 1) If K had more than 4 subgroups of order p , then it would have at least 5 such groups. Hence, G would have at least 5 subgroups of order p and by Lemma 2.7 we would have that $\delta_{\mathcal{CD}}(G) \geq 5$.

2) If K has a single subgroup of order p it is cyclic or generalized quaternion and all groups of order greater than p^2 contain the unique subgroup of order p . Now, suppose that K has $p + 1$ involutions. Hence, G has $p + 1$ witnesses from Lemma 5.2. There can only be one additional witness with order greater than p if $p = 2$.

Suppose that $p = 2$. Then, K contains the three subgroups $A = \langle a \rangle$, $B = \langle b \rangle$, and $C = \langle ab \rangle$ of order 2. If none of them are in $\mathcal{CD}(G)$, then the four subgroups not in $\mathcal{CD}(G)$ are $\{1, A, B, C\}$. Now $\langle a, b \rangle$ contains a central involution, and hence is a product of any two of the subgroups of order 2 and has order 4. Now $\langle a, b \rangle$ is in $\mathcal{CD}(G)$, and it follows that $\mathbf{Z}(G) = \langle a, b \rangle$. Moreover, since $\delta_{\mathcal{CD}}(G) = 4$ in this case, we conclude that every subgroup of G of order 4 or larger is in $\mathcal{CD}(G)$, and thus contains $\mathbf{Z}(G) = \langle a, b \rangle$.

Otherwise, let $Z \leq K$ of order 2 in $\mathcal{CD}(G)$. Since $Z \in \mathcal{CD}(G)$, we know that $Z = \mathbf{Z}(G)$ and hence $Z \leq \mathbf{Z}(K)$. Moreover, since the other two involutions and the identity would not be in $\mathcal{CD}(G)$, we have $\delta_{\mathcal{CD}}(G) \geq 3$ and, thus, at most one other subgroup of G (which must have order greater than 2) could not be in $\mathcal{CD}(G)$ and thus not contain Z .

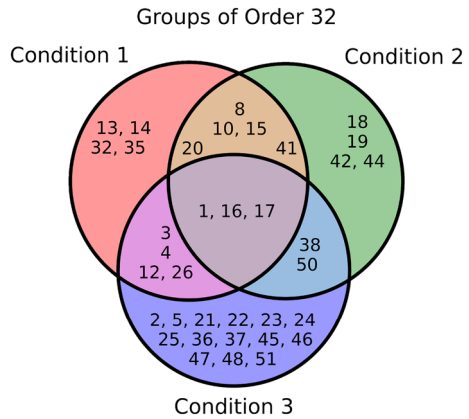
Suppose that $p > 2$. Then by Lemma 5.1, $p = 3$ and from 5.2 we note that exactly one subgroup of order 3 is in $\mathcal{CD}(G)$; call this subgroup H and note that $H = \mathbf{Z}(G)$. Then, every subgroup of G of order greater than 3 is in $\mathcal{CD}(G)$ and must contain H as a subgroup.

3) We note that if K is cyclic then $\mathbf{Z}(K) = K \in \mathcal{CD}(K)$. Suppose that $|K| \geq p^3$ is not cyclic and by way of contradiction that $K \notin \mathcal{CD}(K)$. By Lemma 2.9 we have that $K \notin \mathcal{CD}(G)$. Moreover, since $\mathcal{CD}(G)$ is closed under products we must have that at least p of the maximal subgroups of K (which have order $\geq p^2$) are not in $\mathcal{CD}(G)$. Hence $\delta_{\mathcal{CD}}(G) \geq 2(p + 1) > 5$. We conclude that $K \in \mathcal{CD}(K)$.

Suppose that H and J are subgroups of K with $\mathbf{Z}(K) \leq H$ and $\mathbf{Z}(K) \leq J$ such that neither H nor J are in $\mathcal{CD}(K)$. Then H and J are not in $\mathcal{CD}(G)$ by Lemma 2.9. Since $K \in \mathcal{CD}(K)$, we conclude that $\mathbf{Z}(K) < H$ and $\mathbf{Z}(K) < J$ and thus the orders of both H and J are greater than or equal to p^2 . Hence, H, J , the identity, and p of the subgroups of order p in G would witness that $\delta_{\mathcal{CD}}(G) \geq p + 3 \geq 5$. \square

Lemma 5.3 gives us conditions on the types of subgroups a 2-group or 3-group G can have if $3 \leq \delta_{\mathcal{CD}}(G) \leq 4$. This allows us to specify what types of subgroups G can have of a given order. For example, a 2-group G with $3 \leq \delta_{\mathcal{CD}}(G) \leq 4$ cannot contain

Fig. 2 Groups of order 32 that satisfy at least one of conditions 1), 2), and 3) of Lemma 5.3 identified by their SmallGroup index among groups of order 32, e.g., 1 corresponds to SmallGroup(32,1) which is the cyclic group of order 32



the dihedral group of order 8 as a subgroup because the dihedral group of order 8 has 5 involutions. Similarly, it cannot contain the elementary abelian group of order 8 as a subgroup. This means all of its subgroups of order 8 are either cyclic, quaternion, or isomorphic to $C_4 \times C_2$. We can computationally then search over all of the 2-groups of a given order to look for potential subgroups of a 2-group G with $3 \leq \delta_{\mathcal{C}\mathcal{D}}(G) \leq 4$.

We will pause to introduce the reader to another family of p -groups. These groups are commonly written as $M_{p^k} = \langle a, b | a^{p^{k-1}}, b^p, a^b = a^{p^{k-2}+1} \rangle$. For an odd prime p , the groups M_{p^k} are the only nonabelian p -groups with cyclic maximal subgroups.

Lemma 5.4 *Let G be a 2-group and suppose $3 \leq \delta_{\mathcal{C}\mathcal{D}}(G) \leq 4$. Then all subgroups of order 32 in G are isomorphic to one of the following groups: C_{32} , $C_{16} \times C_2$, or M_{32} .*

Proof Using GAP [10], we can sort through all groups of a given order that meet the conditions of Lemma 5.3. For groups of order 32, we obtain the Venn diagram in Fig 2. Code to do this in the GITHUB repo: https://github.com/7cocke/chermak_delgado_It5. Different authors wrote the code independently of each other in the two languages. □

We could continue our calculations and would see that there are 3 possible subgroups of order 64 which are C_{64} , $C_{32} \times C_2$ and M_{64} . Similarly, we would see that there are 3 possible subgroups of order 128, i.e., the groups C_{128} , $C_{64} \times C_2$, M_{128} . There are also 3 possible subgroups of order 256. This observation motivates the following theorem whose proof relies heavily on the exhaustive work of Berkovich and Janko [3].

Theorem 5.5 *Let G be a p -group with order $|G| = p^{k+1}$ for $k > 4$. Suppose that G contains at most $p+1$ subgroups of order p and that every maximal subgroup of G is isomorphic to one of C_{p^k} , $C_{p^{k-1}} \times C_p$, and M_{p^k} . Then G is isomorphic to one of the groups $C_{p^{k+1}}$, $C_{p^k} \times C_p$, or $M_{p^{k+1}}$.*

Proof If G is abelian, then it must be isomorphic to either C_{p^k} or $C_{p^{k-1}} \times C_p$.

The rest of the proof will cite a number of results from the encyclopedic work of Berkovich and Janko [3] which classifies p -groups all of whose subgroups of index p^2 are abelian.

Suppose that all maximal subgroups of G are abelian. Then G is minimal nonabelian. Berkovich and Janko [3, 3.1] attribute the classification of minimal nonabelian p -groups to Rédei. Our group G contains at most $p + 1$ subgroups of order p and has order greater than p^4 . If G were minimal nonabelian, it would have to be isomorphic to $M_{p^{k+1}}$.

Hence G must contain a nonabelian maximal subgroup. Thus G is an A_2 -group, i.e., a group all of whose subgroups of index p^2 are abelian. Moreover, all proper subgroups of G would be metacyclic. If G itself were not metacyclic, then it would be minimal nonmetacyclic. Berkovich and Janko [3, 1.1(I)] cite Blackburn [4, 3.2] for the classification of minimal nonmetacyclic groups, none of which could be our group G . Hence the group G is a metacyclic A_2 -group. Such groups are classified by Berkovich and Janko [3, 5.2]. Again, since we have restricted the maximal subgroups of G , it cannot be any of these groups.

We conclude that G is either abelian or isomorphic to $M_{p^{k+1}}$. □

For 3-groups, Lemma 5.3 can be used to establish the following.

Lemma 5.6 *Let G be a nonabelian 3-group with $\delta_{\mathcal{CD}}(G) = 4$. Then all subgroups of order 243 in G are isomorphic to one of C_{243} , $C_{81} \times C_3$, or M_{243} .*

These are the same 3 types of groups that occur in Theorem 5.5. This means that as in the 2-group case, for a 3-group with $\delta_{\mathcal{CD}}(G) = 4$, there are at most three isomorphism classes of subgroups for every order greater than 243.

Theorem 5.7 *If G is a nonabelian 2-group and $\delta_{\mathcal{CD}}(G) \leq 4$, then G is the quaternion group of order 8.*

Proof If G has a single involution, then G is generalized quaternion and we note that $m^*(G) = \left(\frac{|G|}{2}\right)^2$. If $|G| > 8$, then $\mathcal{CD}(G)$ is a single subgroup and $\delta_{\mathcal{CD}}(G) > 4$.

Suppose by way of contradiction that G is not the quaternion group of order 8. We must have that $3 \leq \delta_{\mathcal{CD}}(G) \leq 4$. As noted, in Lemma 5.4 the only subgroups of order 32 in G would be isomorphic to C_{32} , $C_{16} \times C_2$, or M_{32} . By Theorem 5.5, the only subgroups of order 64 in G would be isomorphic to C_{64} , $C_{32} \times C_2$ or M_{64} . Continuing in this manner, we see that G itself would be isomorphic to either C_{2^k} , $C_{2^{k-1}} \times C_2$ or M_{2^k} . However, none of these groups satisfy $\delta_{\mathcal{CD}}(G) \leq 4$.

Thus $|G| \leq 32$. A computational search using either GAP or MAGMA shows that there is no such nonabelian 2-group. □

We note that the dihedral group of order 8 has exactly 5 groups not contained in $\mathcal{CD}(G)$, i.e., $\delta_{\mathcal{CD}}(G) = 5$.

Theorem 5.8 *If G is a nonabelian 3-group, then $\delta_{\mathcal{CD}}(G) \geq 4$ with equality if and only if G is the extraspecial group of order 27 and exponent 9.*

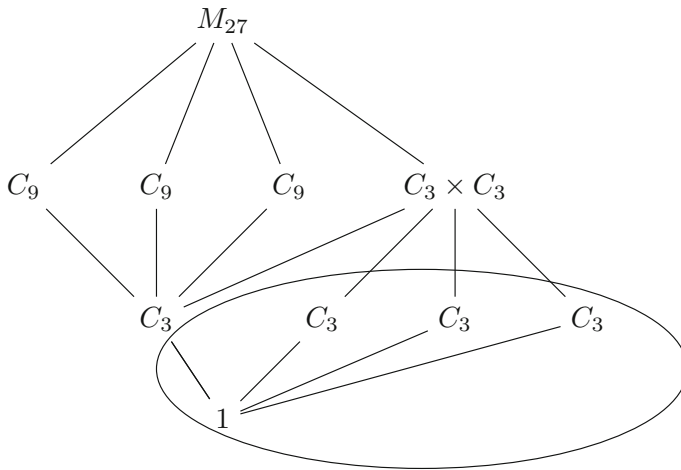


Fig. 3 The subgroup diagram of $G = M_{27}$, i.e., the extraspecial group of order 27 and exponent 9. The four subgroups not in $\mathcal{CD}(G)$ are contained within the ellipse in the figure. This is the only nonabelian group with $\delta_{\mathcal{CD}}(G) = 4$

Proof Using the same code as in Lemma 5.4 we see that all of the subgroups of G of order 243 are isomorphic to C_{243} , $C_{81} \times C_3$, or M_{243} . By Theorem 5.5, either $|G| \leq 81$ or G itself is isomorphic to one of C_{3^k} , $C_{3^{k-1}} \times C_3$, or M_{3^k} . Regardless, G does not satisfy $\delta_{\mathcal{CD}}(G) = 4$ except for when G is M_{27} which is the extraspecial group of order 27 and exponent 9. \square

In Fig 3, we display the subgroup diagram of M_{27} the only nonabelian nilpotent group with $\delta_{\mathcal{CD}}(G) = 4$.

Combining Theorems 5.7 and 5.8 provides the proof of Theorem C.

Proof of Theorem C If G is nonabelian, then G is either a 2-group or a 3-group by Lemma 5.1. Combining Theorem 5.7 and 5.8, we see that G must be the extraspecial group of order 27 and exponent 9.

If G is abelian, then we know that $\delta_{\mathcal{CD}}(G)$ is equal to the number of subgroups of G minus 1. Counting the subgroups of abelian groups finishes the proof. \square

6 Conclusion

We have extended the results by Fasolă and Tărnăuceanu [9] by classifying all finite groups with $3 \leq \delta_{\mathcal{CD}}(G) \leq 4$. Obvious questions exist about classifying all groups, where $\delta_{\mathcal{CD}}(G) = k$ for $k \geq 5$. In particular, for $\delta_{\mathcal{CD}}(G) = 5$ combining Theorem B and Lemma 5.1 means one only needs to classify nonabelian 2-groups and 3-groups with $\delta_{\mathcal{CD}}(G) = 5$.

Corollary 3.3 establishes that $\delta_{\mathcal{CD}}(G)$ is greater than the number of distinct prime divisors of a group G . Is it the case that for a fixed k we can also bound the multiplicity of a prime divisor of G ? We ask this in three distinct, but highly related questions:

Question 1 Let n be a positive integer not equal to 8. If G has order n , then is $\delta_{\mathcal{CD}}(G) \geq \delta_{\mathcal{CD}}(C_n)$, i.e., the number of proper divisors of n ?

Question 2 Let G be a finite group. If G has exactly k prime divisors up to multiplicity, is it the case that $\delta_{\mathcal{CD}}(G)$ is bounded by a function depending on k ?

Question 3 Let G be a finite group, p a prime divisor of $|G|$ and k an integer. Is

$$\delta_{\mathcal{CD}}(G) \geq \begin{cases} k & p \text{ is odd,} \\ k-2 & p = 2 \end{cases}$$

when G has a subgroup of p^k ?

Theorem A shows that when $\delta_{\mathcal{CD}}(G) < 5$, we have that G is nilpotent. One can ask the same ques for solvable groups.

Question 4 What is the maximal value of m , such that if a finite group satisfies $\delta_{\mathcal{CD}}(G) < m$, then G must be solvable?

We note that Theorem C shows that there is no nonabelian group G with $\delta_{\mathcal{CD}}(G) = 3$.

Question 5 Is there a positive integer m such that for every $k \geq m$ there is a nonabelian group G with $\delta_{\mathcal{CD}}(G) = m$?

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