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RESEARCH CONTRIBUTION

Normality of Two Families of Meromorphic Functions Concerning Partially Shared Values

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Abstract

In this paper, the normality of a family of meromorphic functions is deduced from the normality of a given family. Precisely, we have proved: Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a domain D, and $a,\ b,\ c$ be three finite complex numbers such that $a \neq 0$ and $b \neq c$. Suppose that $\mathcal{G}$ is normal in D such that no sequence in $\mathcal{G}$ converges locally uniformly to infinity in D. If $n \geq 3$ and for each function $f \in \mathcal{F}$ there exists $g \in \mathcal{G}$ such that $f^{'} - af^n$ and $g^{'} - ag^n$ partially share the values b and c, then $\mathcal{F}$ is normal in D. Further, examples are given to establish the sharpness of the result.

Keywords Normal family · Shared values · Meromorphic functions

Mathematics Subject Classification 30D45 · 30D30

1 Introduction and Main Results

Let D be a domain in the complex plane $\mathbb{C}$. A family $\mathcal{F}$ of meromorphic functions on D is said to be normal if from every sequence $\{f_n\}$ in $\mathcal{F}$, we can extract a subsequence $\{f_{n_k}\}$ which converges locally uniformly to f in D with respect to the spherical metric, where f is either a meromorphic function or identically equal to infinity in D. A family $\mathcal{F}$ is said to be normal at $z_0 \in D$ if it is normal in some neighborhood of z_0; thus, $\mathcal{F}$ is normal in D if and only if it is normal at each point $z \in D$. (see [14]).

Let f and g be two meromorphic functions in D and let $a \in \mathbb{C}$. We shall denote by $E(f, a)$ the set of zeros of $f - a$ (ignoring multiplicities). We say that f and g share the value a if $E(f, a) = E(g, a)$. Further, if $E(f, a) \subset E(g, a)$, we say that f and g share the value a partially (see [18]).

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According to Bloch's principle [14], any condition which reduces a meromorphic function in $\mathbb{C}$ to a constant is likely to force a family of meromorphic functions in a domain D to be normal. Although this principle as well as its converse does not hold in general (see, for example [2, 13]), still it serves as a guiding principle for obtaining normality criteria corresponding to Picard-type theorems and vice versa (see [1]).

In 1959, Hayman [5] proved that *if f is a meromorphic function in the complex plane, $a \in \mathbb{C}\backslash\{0\}$ and the differential polynomial $f' - af^n$, $n \geq 5$, does not assume a finite complex value in $\mathbb{C}$, then f is constant.* This result is not true for $n = 3, 4$ as shown by Mues [10]. In view of Bloch's principle, Hayman [6] in 1967 conjectured that there exists a normality criterion corresponding to this Picard-type theorem. Over the next few decades, the following normality criterion was established thereby proving the Hayman's conjecture.

Theorem 1.1 *Let $\mathcal{F}$ be a family of meromorphic (holomorphic) functions in a domain D, $n \in \mathbb{N}$ and a, b be two finite complex numbers such that $n \geq 3$ $(n \geq 2)$ and $a \neq 0$. If for each $f \in \mathcal{F}$, $f' - af^n \neq b$, then $\mathcal{F}$ is normal in D.*

The proof of Theorem 1.1 for meromorphic functions is due to S. Li [8], X. Li [9] and Langley [7] for $n \geq 5$, Pang [11] for $n = 4$, Chen and Fang [3] and Zalcman [17] for $n = 3$ independently and the proof of Theorem 1.1 for holomorphic functions is due to Drasin [4] for $n \geq 3$ and Ye [16] for $n = 2$.

In 2008, Zhang [19] considered the idea of shared values and proved the following.

Theorem 1.2 *Let $\mathcal{F}$ be a family of meromorphic (holomorphic) functions in D, $n \in \mathbb{N}$ and a, b be two finite complex numbers such that $n \geq 4$ $(n \geq 2)$ and $a \neq 0$. If for each pair of functions f and g in $\mathcal{F}$, $f' - af^n$ and $g' - ag^n$ share the value b, then $\mathcal{F}$ is normal in D.*

In this paper, we consider the related problems concerning two families of meromorphic functions and prove the following theorem:

Theorem 1.3 *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of holomorphic functions on a domain D, and a, b, c be three complex numbers such that $a \neq 0$ and $b \neq c$. Suppose that $\mathcal{G}$ is normal in D such that no sequence in $\mathcal{G}$ converges locally uniformly to infinity in D. If $n \geq 2$ and for each function $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f' - af^n$ and $g' - ag^n$ partially share the values b and c, then $\mathcal{F}$ is normal in D.*

In the following example, we show that the condition 'partial sharing of two values b and c' in Theorem 1.3 cannot be reduced to one.

Example 1.4 Consider the two families $\mathcal{F} := \left\{f_j(z) = e^{jz} : j \in \mathbb{N}\right\}$ and $\mathcal{G} := \{1\}$ of holomorphic functions on $\mathbb{D}$. Note that $g_j' - g_j^2 \equiv -1$. Therefore, $f_j' - f_j^2 = -1 \Rightarrow g_j' - g_j^2 = -1$. But $\mathcal{F}$ fails to be normal at $z = 0$.

We demonstrate in the subsequent example that Theorem 1.3 fails to be true when $n = 1$. Therefore, the condition $n = 2$ is the best possible for Theorem 1.3.

Example 1.5 Consider the two families $\mathcal{F} := \{f_j(z) = jz : j \in \mathbb{N}\}$ and $\mathcal{G} := \{-1\}$ of holomorphic functions on $\mathbb{D}$. Then, clearly, $f_j^{'}(z) - f_j(z) = j(1-z) \neq 0$, and for each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'}(z) - f_j(z) = 1 \Rightarrow g_j^{'}(z) - g_j(z) = 1$. But $\mathcal{F}$ fails to be normal at $z = 0$.

The following example illustrates that Theorem 1.3 is not valid for the family of meromorphic functions when $n = 2$.

Example 1.6 Consider the two families

$$\mathcal{F} := \left\{f_j(z) = \frac{jz}{1 + jz^2} : j \in \mathbb{N}\right\}$$

and

$$\mathcal{G} := \{1\}$$

of meromorphic functions on $\mathbb{D}$. Take $a = -1$. Then, clearly, $f_j^{'}(z) - af_j^2(z) = \frac{j}{(1+jz^2)^2} \neq 0$ and for each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'}(z) - af_j^2(z) = 1 \Rightarrow g_j^{'}(z) - ag_j^2(z) = 1$. But $\mathcal{F}$ is not normal at $z = 0$ since $f_j(0) = 0$ and for $z \neq 0$, $f_j(z) \to 1/z$ as $n \to \infty$.

However, Theorem 1.3 can be extended to families of meromorphic functions provided that $n \geq 3$.

Theorem 1.7 *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a domain D, and $a,\ b,\ c$ be three finite complex numbers such that $a \neq 0$ and $b \neq c$. Suppose that $\mathcal{G}$ is normal in D such that no sequence in $\mathcal{G}$ converges locally uniformly to infinity in D. If $n \geq 3$ and for each function $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f^{'} - af^n$ and $g^{'} - ag^n$ partially share the values b and c, then $\mathcal{F}$ is normal in D.*

In the following example, we show that the condition 'partial sharing of two values b and c' in Theorem 1.7 cannot be reduced to one.

Example 1.8 Consider the two families

$$\mathcal{F} := \left\{f_j(z) = \frac{1}{jz} : j \in \mathbb{N}\right\}$$

and

$$\mathcal{G} := \left\{\frac{1}{z + \frac{1}{j^2} - 1} : j \in \mathbb{N}\right\}$$

of meromorphic functions on $\mathbb{D}$. Then for each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'} - f_j^3 = 0 \Rightarrow g_j^{'} - g_j^3 = 0$. Also, $g_j(z) \to g(z) = \frac{1}{z-1} \not\equiv \infty$. But $\mathcal{F}$ fails to be normal at $z = 0$.

For $n = 2$, we have the following weak version of the Theorem 1.7.

Theorem 1.9 *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a domain D such that each $f \in \mathcal{F}$ has neither simple zeros nor simple poles. Let a, b and c be three finite complex numbers such that $a \neq 0$ and $b \neq c$. Suppose that $\mathcal{G}$ is normal in D such that no sequence in $\mathcal{G}$ converges locally uniformly to infinity in D. If for each function $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f' - af^2$ and $g' - ag^2$ partially share the values b and c, then $\mathcal{F}$ is normal in D.*

Note that Example 1.6 also shows that the condition 'each $f \in \mathcal{F}$ has neither simple zeros nor simple poles' in Theorem 1.9 can not be omitted.

2 Lemmas and Proof of the Results

To prove our results, we need the following lemmas.

Lemma 2.1 [12] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disk $\mathbb{D}$ such that all the zeros of $f \in \mathcal{F}$ are of multiplicity at least p and all the poles of $f \in \mathcal{F}$ are of multiplicity at least q. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Then, for every $\alpha \in (-p, q)$, there exist*

(a) points z_n in $\mathbb{D}$: $z_n \to z_0$;
(b) functions $f_n \in \mathcal{F}$;
(c) positive real numbers ρ_n : $\rho_n \to 0$

such that the re-scaled sequence $\left\{g_n(\zeta) = \rho_n^{\alpha} f_n(z_n + \rho_n\zeta)\right\}$ converges spherically locally uniformly on $\mathbb{C}$ to a non-constant meromorphic function g on $\mathbb{C}$ of finite order.

Lemma 2.2 [3] *Let f be a meromorphic function in $\mathbb{C}$, and let n be a positive integer. If $f^n f'$ does not assume a non-zero finite complex number in $\mathbb{C}$, then f is constant.*

Lemma 2.3 [15] *Let f be a meromorphic function in $\mathbb{C}$ and b be a non-zero complex number. If f has neither simple zero nor simple pole and $f'(z) \neq b$, then f is constant.*

Proof of the Theorem 1.3 We may consider D to be an open unit disk $\mathbb{D}$. Suppose that the family $\mathcal{F}$ is not normal at $z_0 \in \mathbb{D}$. Then by Lemma 2.1, there exist points $z_j \in \mathbb{D}$ with $z_j \to z_0$, a sequence of positive numbers $\rho_j \to 0$ and a sequence of functions $f_j \in \mathcal{F}$ such that

$$F_j(\zeta) = \rho_j^{\frac{1}{n-1}} f_j(z_j + \rho_j\zeta) \to F(\zeta) \tag{2.1}$$

is locally uniformly on $\mathbb{C}$, where F is a non-constant entire function of finite order. From (2.1), we have

$$\rho_j^{\frac{n}{n-1}}\{(f_j' - af_j^n)(z_j + \rho_j\zeta) - b\} = (F_j' - aF_j^n)(\zeta) - \rho_j^{\frac{n}{n-1}} b \to F'(\zeta) - aF^n(\zeta) \tag{2.2}$$

and

$$\rho_j^{\frac{n}{n-1}}\{(f_j^{'} - af_j^n)(z_j + \rho_j\zeta) - c\} = (F_j^{'} - aF_j^n)(\zeta) - \rho_j^{\frac{n}{n-1}}c \to F^{'}(\zeta) - aF^n(\zeta) \tag{2.3}$$

locally uniformly on $\mathbb{C}$.

For each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'} - af_j^n$ and $g_j^{'} - ag_j^n$ share the values b and c partially in $\mathbb{D}$. Since $\mathcal{G}$ is normal, there exists a subsequence in $\{g_j\}$, again denoted by $\{g_j\}$, that converges uniformly to a holomorphic function $g(z) \not\equiv \infty$ in some neighborhood of z_0.

Suppose $(F^{'} - aF^n) \not\equiv 0$ otherwise $\frac{-1}{n-1}\frac{1}{F^{n-1}} \equiv a\zeta + d$, for some $d \in \mathbb{C}$, which contradicts to the fact that F is an entire function and $n \geq 2$. Further, suppose that $(F^{'} - aF^n)(\zeta) \neq 0$, $\zeta \in \mathbb{C}$. Then $\frac{F^{'}}{F^n} \neq a$. By setting $F = 1/\phi$, we have $\phi^{n-2}\phi^{'} \neq -a$. When $n \geq 3$, ϕ is constant by Lemma 2.2 and when $n = 2$, ϕ is again constant by Hayman's alternative since $\phi \neq 0$ and $\phi^{'} \neq -a$. In both cases, ϕ is constant. This implies that F is constant, a contradiction. Thus, $(F^{'} - aF^n)$ has at least one zero.

Now we have two cases:

Case-I. $(g^{'} - ag^n)(z_0) \neq b$.

Suppose that $(F^{'} - aF^n)(\zeta_0) = 0$, for some $\zeta_0 \in \mathbb{C}$. From (2.2), by Hurwitz's theorem, there exists a sequence $\{\zeta_j\}$ with $\zeta_j \to \zeta_0$ such that for sufficiently large j

$$(F_j^{'} - aF_j^n)(\zeta_j) - \rho_j^{\frac{n}{n-1}}b = 0,$$

and thus

$$(f_j^{'} - af_j^n)(z_j + \rho_j\zeta_j) = b.$$

By hypothesis, we have $(g_j^{'} - ag_j^n)(z_j + \rho_j\zeta_j) = b$ and so $(g^{'} - ag^n)(z_0) = b$, a contradiction.

Case-II. $(g^{'} - ag^n)(z_0) = b$.

By using (2.3) instead of (2.2) in Case-I, we obtain $(g^{'} - ag^n)(z_0) = c\ (\neq b)$ which is not true. This completes the proof. □

Proof of the Theorem 1.7 We may consider D to be an open unit disk $\mathbb{D}$. Suppose that the family $\mathcal{F}$ is not normal at $z_0 \in \mathbb{D}$. Then there exists a sequence $\{f_n\} \subset \mathcal{F}$ which has no locally convergent subsequence at z_0. Thus, by Lemma 2.1, there exist points $z_j \in \mathbb{D}$ with $z_j \to z_0$, a sequence of positive numbers $\rho_j \to 0$, and a sequence of functions in $\{f_j\}$ again denoted by $\{f_j\}$ such that

$$F_j(\zeta) = \rho_j^{\frac{1}{n-1}} f_j(z_j + \rho_j\zeta) \to F(\zeta) \tag{2.4}$$

locally uniformly on $\mathbb{C}$ with respect to spherical metric, where F is a non-constant meromorphic function on $\mathbb{C}$ of finite order.

From (2.4), we have

$$\left(F_j^{'} - aF_j^n\right)(\zeta) - \rho_j^{\frac{n}{n-1}} b = \rho_j^{\frac{n}{n-1}}\{(f_j^{'} - af_j^n)(z_j + \rho_j\zeta) - b\} \to F^{'}(\zeta) - aF^n(\zeta) \tag{2.5}$$

and

$$\left(F_j^{'} - aF_j^n\right)(\zeta) - \rho_j^{\frac{n}{n-1}} c = \rho_j^{\frac{n}{n-1}}\{(f_j^{'} - af_j^n)(z_j + \rho_j\zeta) - c\} \to F^{'}(\zeta) - aF^n(\zeta) \tag{2.6}$$

spherically locally uniformly on $\mathbb{C}$ except possibly at the poles of F.

For each $f_j \in \mathcal{F}$, there exists $g_j \in \mathcal{G}$ such that $f_j^{'} - af_j^n$ and $g_j^{'} - ag_j^n$ partially share the values b and c in $\mathbb{D}$. Since $\mathcal{G}$ is normal, there exists a subsequence in $\{g_j\}$, again denoted by $\{g_j\}$, that converges uniformly to a meromorphic function $g(z) \not\equiv \infty$ in some neighborhood of z_0.

Claim. $(F^{'} - aF^n)(\zeta_0) = 0$, for some $\zeta_0 \in \mathbb{C}$.

Suppose that $(F^{'} - aF^n)(\zeta) \neq 0$. Then $\frac{F^{'}}{F^n} \neq a$. By setting $F = 1/\phi$, $\phi^{n-2}\phi^{'} \neq -a$. By Lemma 2.2, ϕ and so F is constant, a contradiction. This proves the claim.

Now we have three cases:

Case-I. $(g^{'} - ag^n)(z_0) \neq b, \infty$.

By Claim, $(F^{'} - aF^n)(\zeta_0) = 0$, for some $\zeta_0 \in \mathbb{C}$. Since $(F^{'} - aF^n) \not\equiv 0$, otherwise $\frac{-1}{n-1}\frac{1}{F^{n-1}} \equiv a\zeta + d$, for some $d \in \mathbb{C}$, which contradicts to the fact that F is a non-constant meromorphic function and $n \geq 3$, by (2.5), there exists a sequence $\{\zeta_j\}$ with $\zeta_j \to \zeta_0$ such that for sufficiently large j, $(f_j^{'} - af_j^n)(z_j + \rho_j\zeta_j) = b$. By assumption, we have $(g_j^{'} - ag_j^n)(z_j + \rho_j\zeta_j) = b$ and so $(g^{'} - ag^n)(z_0) = b$, a contradiction.

Case-II. $(g^{'} - ag^n)(z_0) = b$.

Using (2.6) instead of (2.5) in Case-I, we obtain $(g^{'} - ag^n)(z_0) = c\ (\neq b)$, which is not true.

Case-III. $(g^{'} - ag^n)(z_0) = \infty$.

Then, clearly, $g(z_0) = \infty$. Suppose that z_0 is a pole of g with multiplicity $k \geq 1$. Then, for sufficiently large j, g_j has exactly $l \leq k$ distinct poles $z_j^1, \ldots, z_j^l$ in $D(z_0, r)$ with multiplicities $\alpha_1, \ldots, \alpha_l$ respectively such that $z_j^i \to z_0$ $(i = 1, \ldots, l)$ and $\sum_{i=1}^{l} \alpha_i = k$. Renumbering if possible, we may assume that the number l and multiplicities α_i, $i = 1, \ldots, l$ are independent of j. Now set

$$H_j(z) := g_j(z)\prod_{i=1}^{l}(z - z_j^i)^{\alpha_i}.$$

Then the functions H_n are holomorphic in $D(z_0, r)$ and $H_n \to H$ on $D(z_0, r/2)\setminus\{z_0\}$, where $H(z) = g(z)(z - z_0)^k$ is holomorphic on $D(z_0, r)$. Note that $H(z_0) \neq 0, \infty$. Hence by maximum principle, $H_n \to H$ on $D(z_0, r/2)$.

We have

$$
\begin{aligned}
g_j'(z) &= \left(H_j(z)\prod_{i=1}^{l}(z-z_j^i)^{-\alpha_i}\right)' \\
&= H_j'(z)\prod_{i=1}^{l}(z-z_j^i)^{-\alpha_i} - H_j(z)\sum_{i=1}^{l}\alpha_i(z-z_j^i)^{-\alpha_i-1}\prod_{s\neq i}(z-z_j^s)^{-\alpha_s} \\
&= \prod_{i=1}^{l}(z-z_j^i)^{-\alpha_i-1}\left(H_j'\prod_{i=1}^{l}(z-z_j^i) - H_j(z)\sum_{i=1}^{l}\alpha_i\prod_{s\neq i}(z-z_j^s)\right). \quad (2.7)
\end{aligned}
$$

Then

$$
g_j'(z) - ag_j^n(z) - b = K_j(z)\prod_{i=1}^{l}(z-z_j^i)^{-\alpha_i-1}, \quad (2.8)
$$

where

$$
\begin{aligned}
K_j(z) =& H_j'\prod_{i=1}^{l}(z-z_j^i) - H_j(z)\sum_{i=1}^{l}\alpha_i\prod_{s\neq i}(z-z_j^s) \\
&- aH_j^n(z)\prod_{i=1}^{l}(z-z_j^i)^{-\alpha_i(n-1)+1} - b\prod_{i=1}^{l}(z-z_j^i)^{\alpha_i+1}. \quad (2.9)
\end{aligned}
$$

Since $H(z_0)\neq 0,\infty$, we have

$$
\begin{aligned}
K_j(z) &\to H'(z)(z-z_0)^l - H(z)k(z-z_0)^{l-1} - \frac{aH^n(z)}{(z-z_0)^{k(n-1)-l}} - b(z-z_0)^{k+l} \\
&= \frac{1}{(z-z_0)^{k(n-1)-l}}\Big\{H'(z)(z-z_0)^{k(n-1)} - kH(z)(z-z_0)^{k(n-1)-1} \\
&\quad -aH^n(z) - b(z-z_0)^{nk}\Big\} \quad (2.10)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(H'(z)(z-z_0)^{k(n-1)} - kH(z)(z-z_0)^{k(n-1)-1} - aH^n(z) - b(z-z_0)^{nk}\right)_{z=z_0} \\
&= -aH^n(z_0)\neq 0. \quad (2.11)
\end{aligned}
$$

Therefore, $K_j(z)$ and so $g_j'(z)-ag_j^n(z)-b$ has no zeros in some neighborhood of z_0. By assumption, we find that $f_j'(z)-af_j^n(z)-b$ has no zero in some neighborhood of z_0. By Theorem 1.1, the sequence $\{f_j\}$ is normal at z_0, a contradiction. □

Proof of the Theorem 1.9 Following the proof of Theorem 1.7, we only need to prove that $F^{'} - aF^2 \not\equiv 0$ and $F^{'} - aF^2$ has at least one zero. Suppose that $F^{'} - aF^2 \equiv 0$. Then $(\frac{1}{F})^{'} \equiv a$ which implies that $\frac{1}{F} \equiv a\zeta + d$, for some $d \in \mathbb{C}$, which contradicts the fact that F has no simple pole. Next, suppose that $F^{'} - aF^2 \neq 0$. Then $\frac{F^{'}}{F^2} \neq a$. We set $F = 1/\phi$, $\phi^{'} \neq -a$. By Lemma 2.3, ϕ and so F is a constant, a contradiction. □

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Declarations

Data Availability Not applicable.

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