RESEARCH CONTRIBUTION

Steady-State Flows of Ideal Incompressible Fluid with Velocity Pointwise Orthogonal to the Pressure Gradient

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Abstract

A new important relation between fluid mechanics and differential geometry is established. We study smooth steady solutions to the Euler equations with the additional property: the velocity vector is orthogonal to the gradient of the pressure at any point. Such solutions are called Gavrilov flows. We describe the local structure of Gavrilov flows in terms of the geometry of isobaric hypersurfaces. In the 3D case, we obtain a system of PDEs for axisymmetric Gavrilov flows and find consistency conditions for the system. Two numerical examples of axisymmetric Gavrilov flows are presented: with pressure function periodic in the axial direction, and with isobaric surfaces diffeomorphic to the torus.

Keywords Euler equations · Ideal fluid · Gavrilov flow · Geodesic vector field

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1 Introduction

In dimensions 2 and 3, the Euler equations

$$u \cdot \nabla u + \operatorname{grad} p = 0, \tag{1}$$

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$$\nabla \cdot u = 0 \tag{2}$$

describe steady-state flows of ideal incompressible fluid. The equations are also of some mathematical interest in an arbitrary dimension. Here, $u = \left(u_1(x), \ldots, u_n(x) \right)$ is a vector field on an open set $U \subset \mathbb{R}^n$ (the fluid velocity) and p is a scalar function on U (the pressure). We consider only smooth real solutions to the Euler equations, i.e., $u_i \in C^\infty(U)$ ($i = 1, \ldots, n$) and $p \in C^\infty(U)$ are assumed to be real functions (the term "smooth" is used as the synonym of "C^∞"). We say that a solution (u, p) to (1)–(2) is a *Gavrilov flow* if it satisfies

$$u \cdot \operatorname{grad} p = 0, \tag{3}$$

i.e., the velocity is orthogonal to the pressure gradient at all points of U. We use also the abbreviation GF for "Gavrilov flow". Equations (1)–(3) constitute the overdetermined system of first order differential equations: $n+2$ equations in $n+1$ unknown function. Therefore, every GF is an exception in a certain sense. Nevertheless, such flows exist and deserve study. Such flows satisfy the following important property: a pair of functions $(\tilde{u}, \tilde{p})$ given by

$$\tilde{u} = \varphi(p)u, \quad \operatorname{grad} \tilde{p} = \varphi^2(p)\operatorname{grad} p, \tag{4}$$

where $\varphi(p)$ is an arbitrary smooth function, is again a GF. This property underlies the following construction that will be called the *Gavrilov localization*. Given a GF in a domain U, let $p_0 \in U$ be a regular value of the function p such that $M_{p_0} = \{x \in U \mid p(x) = p_0\}$ is a compact hypersurface in U. Then, we can construct a compactly supported smooth solution to the Euler equations on the whole of $\mathbb{R}^n$ by choosing $\varphi(p)$ as a cutoff function with support in a small neighborhood of p_0. Indeed, the new velocity vector field $\tilde{u}$ and the gradient grad $\tilde{p}$ are supported in some compact neighborhood $\tilde{U} \subset U$ of the surface M_{p_0}, as is seen from (4), and we define $\tilde{u}$ as zero in $\mathbb{R}^n \setminus U$. Thus, the new pressure $\tilde{p}$ is constant on every connected component of $U \setminus \tilde{U}$. Since only the gradient grad p participates in (1)–(3), we can assume without lost of generality that $\tilde{p} = 0$ on the "exterior component" of $U \setminus \tilde{U}$. It is now clear that $\tilde{p}$ can be extended to a compactly supported function $\tilde{p} \in C^\infty(\mathbb{R}^n)$.

For some neighborhood $O(\mathcal{C})$ of the circle $\mathcal{C} = \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = R^2\}$, Gavrilov [6] proved the existence of a solution $u \in C^\infty\left(O(\mathcal{C})\setminus\mathcal{C}; \mathbb{R}^3 \right)$, $p \in C^\infty\left(O(\mathcal{C})\setminus\mathcal{C} \right)$ of the Euler equations satisfying (3), and such that for some regular value p_0 of the function p, the surface $M_{p_0} \subset O(\mathcal{C})\setminus\mathcal{C}$ is diffeomorphic to the torus. Gavrilov's formulas involve an arbitrary positive constant R, without lost of generality we set $R = 1$. Using the localization procedure described above, Gavrilov proved the existence of a solution $\tilde{u} \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$, $\tilde{p} \in C^\infty(\mathbb{R}^3)$ of the Euler equations supported in a small neighborhood of M_{p_0}. Thus, Gavrilov gave a positive answer to the long standing question: Is there a smooth compactly supported solution to the Euler equations on $\mathbb{R}^3$ that is not identically equal to zero? Unfortunately, [6] is written

in terse language and many technical details are omitted. Actually, the same idea is implemented in the subsequent article [3] by Constantin–La–Vicol. The latter paper starts with the so-called Grad–Shafranov ansatz that has appeared in plasma physics. Unlike [6], the article [3] involves a thorough analysis of nonlinear ODEs that arise while constructing a solution.

We emphasize that in both papers [3, 6] the existence of a Gavrilov *axisymmetric* smooth compactly supported flow on $\mathbb{R}^3$ is proved only. Indeed, [6] starts with the Euler equations in cylindrical coordinates for axisymmetric solutions. To authors' knowledge, The Grad–Shafranov ansatz is adapted to the study of axisymmetric solutions. Existence (or non existence) of compactly supported steady flows of another kind (i.e., not GFs) is discussed in [2, 8, 10, 11]. In 1986, Dombre, Frisch, Greene, Hénon, Mehr and Soward [4] discussed Chaotic streamlines in the A (Arnold) - B (Beltrami) - C (Childress) flows. After this discussion, one can also ask many open questions about Gavrilov flows and, in a sense, flows close to them. What is their Lagrangian structure? Do they have a positive Lyapunov exponent? Can they be used to amplify magnetic fields, etc.?

In the present article, GFs are considered up to the equivalence. We say that two GFs (u, p) and $(\tilde{u}, \tilde{p})$, defined on the same open set $U \subset \mathbb{R}^n$, are *equivalent* if (4) holds with a smooth non-vanishing function $\varphi(p)$. For example, (u, p) and $(-u, p)$ are equivalent GFs. We mostly study the structure of such a flow in a neighborhood of the hypersurface $M_{p_0} \subset U$ for a regular value p_0 of the pressure. The article is organized as follows.

In Sect. 2, we present a complete description of a GF in terms of first and second quadratic forms of isobaric hypersurfaces M_p. After the description is obtained, the Euler equations can be forgotten. Gavrilov was so kind as told his results to the second author and some other colleagues before the article [6] was published. While discussing Gavrilov's results, Bazaykin suggested an explicit example of a GF in any even dimension (private communication, 2018). The example was independently reproduced in the recent paper by Enciso, Peralta-Salas, and Torres de Lizaur [5, Proposition 2.1]. The example is discussed at the end of Sect. 2.

The following observation is widely used in [11]. If a solution (u, p) to the Euler equations is defined on the whole of $\mathbb{R}^3$ and sufficiently fast decays at infinity, then the quadratic form $(\nu \cdot u)(\xi \cdot u)$ integrates to zero over every 2D affine plane $P \subset \mathbb{R}^3$, where ν is the normal vector to P and ξ is an arbitrary vector parallel to P. For a GF, the two-dimensional integral over P can be reduced to a one-dimensional integral over the curve $P \cap M_p$. The reduction is presented in Sect. 3. This property of GFs is interesting by itself, but so far, we do not know any application of the property. Therefore, Sect. 3 can be skipped on first reading.

In Sect. 4, we obtain a system of PDEs for GFs in three dimensions. It is an overdetermined system: 4 equations in 3 unknown functions. The problem of deriving consistency conditions for the system is the main problem in the study of GFs. The problem remains open in the general case.

Sections 5–8 are devoted to axisymmetric GFs in the 3D case. Unlike [3, 6], our analysis of such flows is based on the well-known geometric fact: the equation for geodesics admits a first integral for surfaces of revolution, the Clairaut integral. In Sect. 5, we reduce the system of Sect. 4 to a simpler system of PDEs for axisymmetric GFs: two

equations in one unknown function of two variables, including also two functions of one variable p. Consistency conditions for the latter system are derived in Sect. 6.

In Sect. 7, we discuss a numerical method of constructing axisymmetric GFs. In the general case, our method gives only a local GF in a neighborhood of a given point. Nevertheless, global axisymmetric GFs can be found due to periodicity in the z-direction.

The existence of an axisymmetric GF in the open set $O(C)\backslash C$ is proved in [6] such that the pressure function $p(r, z)$ is smooth in a neighborhood of the point $(r, z) = (1, 0)$ that is a nondegenerate minimum point of p. We study such flows in Sect. 8. The corresponding system of PDEs can be solved in series.

We emphasize that [3, 6] only prove the existence of axisymmetric GFs but do not give numerical examples. In our opinion, numerical and geometric examples are of a great importance since they can lead to new hypotheses. In Sects. 7, 8, we present two geometric illustrations for better understanding axisymmetric GFs: one with isobaric surfaces diffeomorphic to a torus and second one periodic in the z-direction.

Some open questions on GFs are posed in Sect. 9.

2 Geometry of a Gavrilov flow

Let (u, p) be a GF on an open set $U \subset \mathbb{R}^n$. Integral curves of the vector field u are also called *particle trajectories*. The following statement immediately follows from (3).

Proposition 1 *The pressure p is constant on every integral curve of the vector field u.*

By Proposition 1, the Bernoulli law

$$|u|^2/2 + p = \text{const} \quad \text{along a particle trajectory}$$

splits, for a GF, into two conservation laws:

$$p = c = \text{const}, \quad |u| = C = \text{const} \quad \text{along a particle trajectory.} \tag{5}$$

We say that $x \in U$ is a *regular point* if grad $p(x) \neq 0$. The vector field u does not vanish at regular points as is seen from (1). The sets $M_{p_0} = \{x \in U \mid p(x) = p_0 = \text{const}\}$ will be called *isobaric hypersurfaces*. A particle trajectory starting at a point of an isobaric hypersurface M_p does not leave M_p "for ever". In the general case, an arbitrary closed subset of U can be an isobaric hypersurface M_p. But M_p is indeed a smooth hypersurface of $\mathbb{R}^n$ in a neighborhood of a regular point $x \in M_p$. We say that M_p is a *regular isobaric hypersurface* if it consists of regular points.

Recall that a vector field u on a manifold M with a Riemannian metric g is called a *geodesic vector field* if

$$\nabla_u u = 0, \tag{6}$$

where ∇ stands for the covariant derivative with respect to the Levi-Civita connection of (M, g). Integral curves of a geodesic vector field are geodesics. In the case of a GF, regular isobaric hypersurfaces $M_p \subset \mathbb{R}^n$ are considered with the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^n$.

Proposition 2 *Given a Gavrilov flow (u, p), the restriction of u to every regular isobaric hypersurface M_p is a non-vanishing geodesic vector field.*

Proof It consists of one line

$$\nabla_u u = P(u \cdot \nabla u) = -P(\operatorname{grad} p) = 0 \tag{7}$$

where P is the orthogonal projection onto the tangent hyperplane of M_p. On the left-hand side of (7), ∇ stands for the covariant derivative with respect to the Levi-Civita connection on M_p. But the second ∇ stands for the Euclidean gradient, the same operator as in the Euler equations (1)–(2). The first equality in (7) is the main relationship between intrinsic geometry of a hypersurface and geometry of the ambient space; it goes back to Gauss and is valid in a more general setting [9, Chapter VII, Proposition 3.1]. The second equality in (7) holds by (1), and the last equality holds by (3). □

Since the vector field u does not vanish at regular points, no integral curve of u living on a regular hypersurface M_p degenerates to a point. Thus, integral curves of u constitute a geodesic foliation of a regular part of any isobaric hypersurface.

Proposition 3 *Given a Gavrilov flow (u, p) on an open set $U \subset \mathbb{R}^n$, let us restrict the vector field u onto a regular isobaric hypersurface M_p, and let $\operatorname{div} u$ be the $(n-1)$-dimensional divergence of the restriction which is understood in the sense of intrinsic geometry of M_p. Then*

$$\operatorname{div} u = u(\log |\operatorname{grad} p|). \tag{8}$$

On the right-hand side of (8), the vector field u is considered as a differentiation of the algebra $C^\infty(M_p)$ of smooth functions on M_p.

Proof We will show that Eq. (8) is equivalent to the incompressibility Eq. (2). To this end, we will rewrite the Eq. (2) in local curvilinear coordinates adapted to the foliation of U into isobaric hypersurfaces. Fix a regular point $x_0 \in U$ and set $p_0 = p(x_0)$. For $p \in \mathbb{R}$ sufficiently close to p_0, the isobar M_p is a regular hypersurface near x_0. Choose local curvilinear coordinates $(z^1, \ldots, z^{n-1})$ on the hypersurface M_{p_0}. Let

$$\mathbb{R}^{n-1} \supset \Omega \xrightarrow{r} \mathbb{R}^n, \quad r = r(z^1 \ldots, z^{n-1})$$

be the parametrization of M_{p_0} in these coordinates. Assume that $0 \in \Omega$ and $r(0) = x_0$. In some neighborhood of x_0, we introduce local curvilinear coordinates $(z^1, \ldots, z^n)$ in $\mathbb{R}^n$ as follows. Define the vector field

$$\xi = \frac{\operatorname{grad} p}{|\operatorname{grad} p|^2}. \tag{9}$$

For $(z^1, \ldots, z^{n-1}) \in \mathbb{R}^{n-1}$ sufficiently close to 0, let

$$R(z^1, \ldots, z^{n-1}; z^n), \quad (p_0 - \varepsilon < z^n < p_0 + \varepsilon)$$

be the integral curve of the vector field ξ starting at the point $r(z^1, \ldots, z^{n-1})$ at the initial moment $z^n = p_0$. Thus $R(z) \in \mathbb{R}^n$ is the solution to the Cauchy problem

$$\frac{\partial R}{\partial z^n}(z) = \xi\left(R(z)\right), \quad R(z^1, \ldots, z^{n-1}; p_0) = r(z^1, \ldots, z^{n-1}). \tag{10}$$

Obviously, R is a diffeomorphism between some neighborhood of the point $(0, \ldots, 0, p_0)$ and a neighborhood of x_0; therefore $(z^1, \ldots, z^n)$ constitute a local coordinate system in $\mathbb{R}^n$ near the point x_0. By our construction, R satisfies the identity $p\left(R(z)\right) = z^n$, which means that the coordinate z^n coincides with the pressure p. Nevertheless, we use the different notation for the coordinate since z^n is considered as an independent variable while p is a function on U. For every z^n sufficiently close to p_0, $(z^1, \ldots, z^{n-1})$ are local coordinates on the isobaric hypersurface $M_{z^n} = \{x \in U \mid p(x) = z^n\}$. Let

$$\mathrm{d}s_{z^n}^2 = g_{\alpha\beta}dz^\alpha dz^\beta, \quad g_{\alpha\beta} = \frac{\partial R}{\partial z^\alpha} \cdot \frac{\partial R}{\partial z^\alpha}$$

be the first quadratic form of M_{z^n}. We use the following convention: Greek indices vary from 1 to $n-1$ and the summation from 1 to $n-1$ is assumed over a repeating Greek index; while Roman indices vary from 1 to n with the corresponding summation rule. We also write the Euclidean metric of $\mathbb{R}^n$ in coordinates $(z^1, \ldots, z^n)$ as $\mathrm{d}s^2 = h_{ij}dz^i dz^j$, where $h_{ij} = \frac{\partial R}{\partial z^i} \cdot \frac{\partial R}{\partial z^j}$. Obviously, $h_{\alpha\beta} = g_{\alpha\beta}$, and

$$h_{\alpha n}(z) = \frac{\partial R}{\partial z^\alpha}(z) \cdot \frac{\partial R}{\partial z^n}(z) = \frac{\partial R}{\partial z^\alpha}(z) \cdot \frac{\mathrm{grad}\, p}{|\mathrm{grad}\, p|^2}\left(R(z)\right) = 0.$$

The last equality holds since the vector $\frac{\partial R}{\partial z^\alpha}(z)$ is tangent to the hypersurface M_{z^n} at the point $R(z)$ while the vector $\mathrm{grad}\, p(R(z))$ is orthogonal to M_{z^n} at the same point. Similarly, we get $h_{nn}(z) = \left|\frac{\partial R}{\partial z^n}(z)\right|^2 = \left|\mathrm{grad}\, p\left(R(z)\right)\right|^{-2}$. Thus,

$$(h_{ij}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & |\mathrm{grad}\, p|^{-2} \end{pmatrix}, \quad (h^{ij}) = (h_{ij})^{-1} = \begin{pmatrix} g^{\alpha\beta} & 0 \\ 0 & |\mathrm{grad}\, p|^2 \end{pmatrix}.$$

Let $\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\beta\delta}}{\partial z^\gamma} + \frac{\partial g_{\gamma\delta}}{\partial z^\beta} - \frac{\partial g_{\beta\gamma}}{\partial z^\delta}\right)$ be the Christoffel symbols of M_{z^n} in coordinates $(z^1, \ldots, z^{n-1})$ and G_{jk}^i be the Christoffel symbols of the Euclidean metric in coordinates $(z^1, \ldots, z^n)$. As follows from the above relations, the Christoffel symbols

satisfy

$$G^{\alpha}_{nn} = -\frac{1}{2} g^{\alpha\delta} \frac{\partial |\text{grad } p\,|^{-2}}{\partial z^{\delta}}, \quad G^{n}_{\beta n} = \frac{1}{2} |\text{grad } p\,|^2 \frac{\partial |\text{grad } p\,|^{-2}}{\partial z^{\beta}},$$

$$G^{n}_{nn} = \frac{1}{2} |\text{grad } p\,|^2 \frac{\partial |\text{grad } p\,|^{-2}}{\partial z^{n}},$$

$$G^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}, \quad G^{n}_{\beta\gamma} = -\frac{1}{2} |\text{grad } p\,|^2 \frac{\partial g_{\beta\gamma}}{\partial z^{n}}, \quad G^{\alpha}_{\beta n} = \frac{1}{2} g^{\alpha\delta} \frac{\partial g_{\beta\delta}}{\partial z^{n}}. \tag{11}$$

The velocity u can be represented in the chosen coordinates as $u(z) = u^{\alpha}(z)\frac{\partial R(z)}{\partial z^{\alpha}}$, since it is tangent to M_{z^n}. The velocity u can be thought as a smooth vector field either on the n-dimensional open set U or on each isobaric hypersurface M_{z^n} smoothly depending on the parameter z^n. We remain the notation $\nabla \cdot u$ for the n-dimensional divergence of u while the $(n-1)$-dimensional divergence of u on M_{z^n} will be denoted by div u. Thus,

$$\text{div } u = \nabla_{\alpha} u^{\alpha} = \frac{\partial u^{\alpha}}{\partial z^{\alpha}} + \Gamma^{\alpha}_{\alpha\beta} u^{\beta}. \tag{12}$$

By the same formula, $\nabla \cdot u = \frac{\partial u^i}{\partial z^i} + G^i_{ij} u^j$, since $u^n = 0$, this becomes

$$\nabla \cdot u = \frac{\partial u^{\alpha}}{\partial z^{\alpha}} + G^i_{i\beta} u^{\beta}.$$

In particular, the incompressibility Eq. (2) is written in the chosen coordinates as $\frac{\partial u^{\alpha}}{\partial z^{\alpha}} + G^i_{i\beta} u^{\beta} = 0$. Using this equation, (12) becomes div $u = -(G^i_{i\beta} - \Gamma^{\alpha}_{\alpha\beta})u^{\beta}$. By the formulas for Christoffel symbols, $G^i_{i\beta} - \Gamma^{\alpha}_{\alpha\beta} = -\frac{\partial (\log |\text{grad } p\,|)}{\partial z^{\beta}}$. Using this expression in the previous formula for div u, we get div $u = \frac{\partial (\log |\text{grad } p\,|)}{\partial z^{\beta}} u^{\beta}$. This is equivalent to (8). $\qquad \square$

Remark. Proposition 3 is not completely new, compare with [1, Theorem 3.4.12].

Let II be the second quadratic form of a regular isobaric hypersurface M_p. Recall that the second quadratic form depends on the choice of the unit normal vector to a hypersurface (the second quadratic form changes its sign if the unit normal vector N is replaced with $-N$). We choose the unit normal vector for a regular isobaric surface to be a positive multiple of grad p.

Proposition 4 *Given a Gavrilov flow (u, p), the restriction of the vector field u onto a regular isobaric hypersurface M_p satisfies*

$$\text{II}(u, u) = -|\text{grad } p\,|. \tag{13}$$

Proof We use the same local coordinates $(z^1, \ldots, z^n)$ as in the previous proof. To prove (13), we write down the Eq. (1) as $(u \cdot \nabla u)^i + (\text{grad } p)^i = 0$. Setting $i = \alpha$

here, we obtain nothing new; more precisely, we obtain the same result: integral curves of u are geodesics of M_p. Thus, we set $i = n$ in the latter formula

$$(u \cdot \nabla u)^n + (\text{grad } p)^n = 0.$$

By (9)–(10), $(\text{grad } p)^n = |\text{grad } p|^2$ and our equation becomes

$$(u \cdot \nabla u)^n = -|\text{grad } p|^2. \qquad (14)$$

By a well-known formula for covariant derivatives, $(u \cdot \nabla u)^n = u^i \left(\frac{\partial u^n}{\partial z^i} + G^n_{ij} u^j \right)$.
Since $u^n = 0$, this becomes $(u \cdot \nabla u)^n = G^n_{\alpha\beta} u^\alpha u^\beta$. Using $G^n_{\alpha\beta} = -\frac{1}{2} |\text{grad } p|^2 \frac{\partial g_{\alpha\beta}}{\partial z^n}$
from (11), we get

$$(u \cdot \nabla u)^n = -\frac{1}{2} |\text{grad } p|^2 \frac{\partial g_{\alpha\beta}}{\partial z^n} u^\alpha u^\beta. \qquad (15)$$

Differentiating the equality $g_{\alpha\beta} = \frac{\partial R}{\partial z^\alpha} \cdot \frac{\partial R}{\partial z^\beta}$ with respect to z^n, we obtain

$$\frac{\partial g_{\alpha\beta}}{\partial z^n} = \frac{\partial^2 R}{\partial z^\alpha \partial z^n} \cdot \frac{\partial R}{\partial z^\beta} + \frac{\partial^2 R}{\partial z^\beta \partial z^n} \cdot \frac{\partial R}{\partial z^\alpha}.$$

This can be written as

$$\frac{\partial g_{\alpha\beta}}{\partial z^n} = \frac{\partial}{\partial z^\alpha} \left(\frac{\partial R}{\partial z^n} \cdot \frac{\partial R}{\partial z^\beta} \right) + \frac{\partial}{\partial z^\beta} \left(\frac{\partial R}{\partial z^n} \cdot \frac{\partial R}{\partial z^\alpha} \right) - 2 \frac{\partial^2 R}{\partial z^\alpha \partial z^\beta} \cdot \frac{\partial R}{\partial z^n}.$$

Both expressions in parentheses are equal to zero, and we obtain

$$\frac{\partial g_{\alpha\beta}}{\partial z^n} = -2 \frac{\partial^2 R}{\partial z^\alpha \partial z^\beta} \cdot \frac{\partial R}{\partial z^n}. \qquad (16)$$

Let $N = \frac{\text{grad } p}{|\text{grad } p|} = |\text{grad } p| \frac{\partial R}{\partial z^n}$ be the unit normal vector of the hypersurface M_{z^n}.
By classic formulas of differential geometry (so called *derived formulas* [12]),

$$\frac{\partial^2 R}{\partial z^\alpha \partial z^\beta} = \Gamma^\gamma_{\alpha\beta} \frac{\partial R}{\partial z^\gamma} + b_{\alpha\beta} N,$$

where $b_{\alpha\beta}$ are coefficients of the second quadratic form for M_{z^n} in coordinates $(z^1, \ldots, z^{n-1})$. Taking the scalar product of this equality with N and using the orthogonality of N to $\frac{\partial R}{\partial z^\gamma}$, we obtain $\frac{\partial^2 R}{\partial z^\alpha \partial z^\beta} \cdot N = b_{\alpha\beta}$. Since $N = |\text{grad } p| \frac{\partial R}{\partial z^n}$,
we get $\frac{\partial^2 R}{\partial z^\alpha \partial z^\beta} \cdot \frac{\partial R}{\partial z^n} = |\text{grad } p|^{-1} b_{\alpha\beta}$. Using this equality, formula (16) becomes
$\frac{\partial g_{\alpha\beta}}{\partial z^n} = -2 |\text{grad } p|^{-1} b_{\alpha\beta}$. Substituting this value into (15), we get

$$(u \cdot \nabla u)^n = |\text{grad } p| b_{\alpha\beta} u^\alpha u^\beta = |\text{grad } p| \text{II}(u, u).$$

Inserting this expression into (14), we arrive to (13). $\square$

Although some special coordinates have been used in the proof, the resulting Eqs. (8) and (13) are of an invariant nature, i.e., independent of a coordinates choice.

All our arguments in this section are invertible, i.e., the following statement is valid.

Proposition 5 *The Euler–Gavrilov system* (1)–(3) *is equivalent to the system* (6), (8), (13). *More precisely, let a smooth vector field u and smooth real function p be defined on an open set $U \subset \mathbb{R}^n$. Choose a point $x_0 \in U$ such that* grad $p(x_0) \neq 0$. *Assume that integral curves of u are tangent to level hypersurfaces M_p and Eqs.* (6), (8), (13) *hold in some neighborhood of x_0. Then (u, p) is a Gavrilov flow in some neighborhood of x_0.*

Now, we discuss an easy example of GF which exists in any even dimension. Let $(x_1, \ldots, x_{2n})$ be Cartesian coordinates in $\mathbb{R}^{2n}$. Set

$$u_{2j-1}(x) = -x_{2j}, \; u_{2j}(x) = x_{2j-1} \; (j = 1, \ldots, n), \quad p(x) = \frac{1}{2} |x|^2. \quad (17)$$

It is easy to check that the Eqs. (1)–(3) hold in $\mathbb{R}^{2n}$. Observe that $|u|^2 = 2p$. Every $x \neq 0$ is a regular point. Isobaric hypersurfaces are spheres $M_p = \{x \in \mathbb{R}^{2n} : |x|^2 = 2p\}$. Integral curves of u (particle trajectories) are circles centered at the origin. Every sphere M_p is foliated by particle trajectories. This foliation coincides with the well-known Hopf fiber bundle $\mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$ of an odd-dimensional sphere over the complex projective space.

A GF on $\mathbb{R}^{2n+1}$ can be obtained as a direct product of the flow (17) with a constant velocity flow. Namely,

$$u_{2j-1}(x) = -x_{2j}, \quad u_{2j}(x) = x_{2j-1} \; (j = 1, \ldots, n), \quad u_{2n+1}(x) = a = \text{const} \quad (18)$$

and $p(x) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$. Isobaric hypersurfaces are cylinders $\mathbb{S}^{2n-1} \times \mathbb{R}$, and particle trajectories are either circles (if $a = 0$) or helices (if $a \neq 0$).

In order to apply the Gavrilov localization to the flow (17), choose a compactly supported smooth function $\alpha : [0, \infty) \to \mathbb{R}$ such that $\alpha(r) = 0$ for $r \leq \varepsilon$ with some $\varepsilon > 0$ and define the function $\beta : [0, \infty) \to \mathbb{R}$ by $\beta(r) = -\int_r^\infty s \, \alpha^2(s) \, ds$. Then

$$\tilde{u}(x) = \alpha(|x|)u(x), \quad \tilde{p}(x) = \beta(|x|)$$

is a smooth compactly supported GF on the whole of $\mathbb{R}^{2n}$ satisfying $|\tilde{u}|^2 = \psi(\tilde{p})$ with a function ψ uniquely determined by α. In particular, if α is supported in $(r_0 - \delta, r_0 + \delta)$ for some $r_0 > \delta > 0$, then the velocity $\tilde{u}$ is supported in the spherical layer $\{x \in \mathbb{R}^{2n} : r_0 - \delta < |x| < r_0 + \delta\}$, and the pressure $\tilde{p}$ is supported in the ball $\{x \in \mathbb{R}^{2n} : |x| < r_0 + \delta\}$ with $\tilde{p} = \text{const}$ in the smaller ball $\{x \in \mathbb{R}^{2n} : |x| \leq r_0 - \delta\}$. Then we can take a linear combination of several such localized flows with disjoints supports. In particular, a periodic GF can be constructed in this way.

3 Plane sections of a Gavrilov flow

Let a sufficiently smooth solution (u, p) of the Euler equations (1)–(2) be defined on the whole of $\mathbb{R}^n$. Assume the solution to decay sufficiently fast at infinity together with first order derivatives (e.g., it can be a smooth compactly supported solution). Then, [11] the equality

$$\int_P \left(\xi \cdot u(x) \right) \left(\nu \cdot u(x) \right) dx = 0 \tag{19}$$

holds for every affine hyperplane $P \subset \mathbb{R}^n$ and every vector $\xi \in \mathbb{R}^n$ parallel to P, where ν is the normal vector to the hyperplane P and dx stands for the $(n-1)$-dimensional Lebesgue measure on P. Actually there are $n-1$ independent equations in (19) since the vector ξ can take $n-1$ linearly independent values from the space $\nu^\perp = \{\xi \in \mathbb{R}^n \mid \nu \cdot \xi = 0\}$.

For a GF, the Eq. (19), combined with the Gavrilov localization, yields an interesting statement. The following theorem can be easily generalized to an arbitrary dimension.

Theorem 6 *Let (u, p) be a smooth GF defined on the whole of $\mathbb{R}^3$ and sufficiently fast decaying at infinity together with first order derivatives. Let M_{p_0} be a regular isobaric surface and let an affine plane P_0 transversally intersect M_{p_0}. Then, for any p sufficiently close to p_0 and for any affine plane P sufficiently close to P_0,*

$$\int\limits_{M_p \cap P} \frac{1}{|\operatorname{grad} q(x)|} \left(\xi \cdot u(x) \right) \left(\nu \cdot u(x) \right) ds = 0, \tag{20}$$

where $q \in C^\infty(P)$ is the restriction of the function p to the plane P, ν is the unit normal vector to P, and ξ is an arbitrary vector parallel to P. The integration in (20) is performed with respect to the arc length ds of the curve $M_p \cap P$.

Proof Since M_{p_0} and P_0 intersect transversally, the same is true for M_p and P for any p close to p_0 and for any plane P close to P_0. We fix such p and P, set $q = p|_P$ and $\gamma = M_p \cap P$. Observe that γ is a regular curve on the plane P and the gradient $\operatorname{grad} q$ does not vanish in some neighborhood of γ. Therefore, the integral on the left-hand side of (20) is well defined.

We parameterize the curve γ by the arc length, $\gamma = \gamma(s)$. Then, we choose local coordinates (s, τ) in a neighborhood $U \subset P$ of γ in the same way as in the proof of Proposition 3. Namely, the coordinates are chosen so that $x(s, 0) = \gamma(s)$ and $q(x(s, \tau)) = p + \tau$. For every s_0, the coordinate line $\delta(\tau) = x(s_0, \tau)$ starts at $\gamma(s_0)$ orthogonally to γ with the initial speed $|\dot\delta(0)| = \frac{1}{|\operatorname{grad} q(\gamma(s_0))|}$. Therefore, the area form dx of the plane P is written in the chosen coordinates as $dx = \left(\frac{1}{|\operatorname{grad} q(\gamma(s))|} + o(\tau) \right) ds\, d\tau$.

Fix a smooth function $\mu : \mathbb{R} \to \mathbb{R}$ such that $\mu(r) = 0$ for $|r| \geq 1$, $\mu(r) > 0$ for $|r| < 1$, and $\int_{-1}^1 \mu(r)\, dr = 1$. For $\varepsilon > 0$, set $\alpha_\varepsilon(r) = \sqrt{\mu((r-c)/\varepsilon)}$. Using the latter

function, we define the localized GF $(\tilde{u}, \tilde{p})$ by $\tilde{u} = \alpha_\varepsilon(p)\, u$, $\operatorname{grad} \tilde{p} = \alpha_\varepsilon^2(p)\operatorname{grad} p$. Applying (19) to $(\tilde{u}, \tilde{p})$, we obtain $\int_P \alpha_\varepsilon(q(x))\Big(\xi \cdot u(x)\Big)\Big(v \cdot u(x)\Big)\, dx = 0$. In the chosen coordinates, this is written as

$$
\int_\gamma \int_{-\varepsilon}^{\varepsilon} \mu(\tau/\varepsilon)\left(\frac{1}{|\operatorname{grad} q(\gamma(s))|} + o(\tau)\right)\Big(\xi \cdot u(x(s, \tau))\Big)\Big(v \cdot u(x(s, \tau))\Big)\, d\tau\, ds = 0.
$$

$$(21)$$

The integrand can be represented as

$$
\mu(\tau/\varepsilon)\left(\frac{1}{|\operatorname{grad} q(\gamma(s))|} + o(\tau)\right)\Big(\xi \cdot u(x(s, \tau))\Big)\Big(v \cdot u(x(s, \tau))\Big)
$$
$$
= \mu(\tau/\varepsilon)\frac{1}{|\operatorname{grad} q(\gamma(s))|}\Big(\xi \cdot u(\gamma(s))\Big)\Big(v \cdot u(\gamma(s))\Big) + o(1).
$$

The variables s and τ are separated up to $o(\tau)$ on the right-hand side of the latter formula. Using this representation and $\int_{-\varepsilon}^{\varepsilon} \mu(\tau/\varepsilon)\, d\tau = \varepsilon$, we obtain from (21)

$$
\varepsilon \int_\gamma \frac{1}{|\operatorname{grad} q(\gamma(s))|}\Big(\xi \cdot u(\gamma(s))\Big)\Big(v \cdot u(\gamma(s))\Big)\, ds + o(\varepsilon) = 0.
$$

In the limit as $\varepsilon \to 0$, this gives (20). □

4 Differential equations for a Gavrilov flow

We consider the 3D case in this section. Let a GF (u, p) be defined on an open set of $\mathbb{R}^3$. Let (x, y, z) be Cartesian coordinates. We assume that, for $p \in (-p_0, p_0)$, the regular isobar surface M_p coincides with the graph of a smooth function

$$
z = f(p; x, y) \quad \Big((x, y) \in U\Big)
$$

$$(22)$$

for some open domain $U \subset \mathbb{R}^2$. The first and the second quadratic forms of M_p are

$$
I = (1 + f_x'^2)dx^2 + 2f_x' f_y'\, dx dy + (1 + f_y'^2)dy^2,
$$
$$
II = \pm\frac{1}{\sqrt{1 + f_x'^2 + f_y'^2}}\left(f_{xx}''\, dx^2 + 2f_{xy}''\, dx dy + f_{yy}''\, dy^2\right),
$$

$$(23)$$

where the sign depends on the choice of the unit vector normal to M_p. Christoffel symbols of this metric are

$$\Gamma^x_{xx} = \frac{f'_x f''_{xx}}{1 + f'^2_x + f'^2_y}, \quad \Gamma^x_{xy} = \frac{f'_x f''_{xy}}{1 + f'^2_x + f'^2_y}, \quad \Gamma^x_{yy} = \frac{f'_x f''_{yy}}{1 + f'^2_x + f'^2_y},$$
$$\Gamma^y_{xx} = \frac{f'_y f''_{xx}}{1 + f'^2_x + f'^2_y}, \quad \Gamma^y_{xy} = \frac{f'_y f''_{xy}}{1 + f'^2_x + f'^2_y}, \quad \Gamma^y_{yy} = \frac{f'_y f''_{yy}}{1 + f'^2_x + f'^2_y}.$$

(24)

Let (u^x, u^y) be geometric coordinates of the vector field u, i.e., $u = u^x(p; x, y)\frac{\partial}{\partial x} + u^y(p; x, y)\frac{\partial}{\partial y}$, where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are considered as coordinate vector fields tangent to the surface M_p. Using (24), we compute covariant derivatives

$$\nabla_x u^x = \frac{\partial u^x}{\partial x} + \frac{f'_x(f''_{xx}u^x + f''_{xy}u^y)}{1 + f'^2_x + f'^2_y}, \quad \nabla_y u^x = \frac{\partial u^x}{\partial y} + \frac{f'_x(f''_{xy}u^x + f''_{yy}u^y)}{1 + f'^2_x + f'^2_y},$$
$$\nabla_x u^y = \frac{\partial u^y}{\partial y} + \frac{f'_y(f''_{xx}u^x + f''_{xy}u^y)}{1 + f'^2_x + f'^2_y}, \quad \nabla_y u^y = \frac{\partial u^y}{\partial y} + \frac{f'_y(f''_{xy}u^x + f''_{yy}u^y)}{1 + f'^2_x + f'^2_y}.$$

(25)

In particular,

$$\operatorname{div} u = \nabla_x u^x + \nabla_y u^y = \frac{\partial u^x}{\partial x} + \frac{\partial u^y}{\partial y} + \frac{(f'_x f''_{xx} + f'_y f''_{xy})u^x + (f'_x f''_{xy} + f'_y f''_{yy})u^y}{1 + f'^2_x + f'^2_y}.$$

(26)

Substituting expressions (25) into the formula

$$\nabla_u u = (u^x \nabla_x u^x + u^y \nabla_y u^x)\frac{\partial}{\partial x} + (u^x \nabla_x u^y + u^y \nabla_y u^y)\frac{\partial}{\partial y},$$

we get

$$\nabla_u u = \left(u^x \frac{\partial u^x}{\partial x} + u^y \frac{\partial u^x}{\partial y} + \frac{f'_x}{1 + f'^2_x + f'^2_y}\left(f''_{xx}(u^x)^2 + 2f''_{xy}u^x u^y + f''_{yy}(u^y)^2\right)\right)\frac{\partial}{\partial x}$$
$$+ \left(u^x \frac{\partial u^y}{\partial x} + u^y \frac{\partial u^y}{\partial y} + \frac{f'_y}{1 + f'^2_x + f'^2_y}\left(f''_{xx}(u^x)^2 + 2f''_{xy}u^x u^y + f''_{yy}(u^y)^2\right)\right)\frac{\partial}{\partial y}.$$

Being a geodesic vector field, u solves the equation $\nabla_u u = 0$. We thus arrive to the system

$$u^x \frac{\partial u^x}{\partial x} + u^y \frac{\partial u^x}{\partial y} + \frac{f'_x}{1 + f'^2_x + f'^2_y}\left(f''_{xx}(u^x)^2 + 2f''_{xy}u^x u^y + f''_{yy}(u^y)^2\right) = 0,$$
$$u^x \frac{\partial u^y}{\partial x} + u^y \frac{\partial u^y}{\partial y} + \frac{f'_y}{1 + f'^2_x + f'^2_y}\left(f''_{xx}(u^x)^2 + 2f''_{xy}u^x u^y + f''_{yy}(u^y)^2\right) = 0.$$

(27)

Let us express $|\text{grad } p|$ in terms of the function f. Let $p(x, y, z)$ be the pressure in Cartesian coordinates. We have the identity $p(x, y, f(p; x, y)) = p$. Differentiate the identity to get

$$p_x'(x, y, f(p; x, y)) + p_z'(x, y, f(p; x, y))f_x'(p; x, y) = 0,$$
$$p_y'(x, y, f(p; x, y)) + p_z'(x, y, f(p; x, y))f_y'(p; x, y) = 0,$$
$$p_z'(x, y, f(p; x, y))f_p'(p; x, y) = 1.$$

From this, we get $p_x' = -\frac{f_x'}{f_p'}$, $p_y' = -\frac{f_y'}{f_p'}$, $p_z' = \frac{1}{f_p'}$. The derivative f_p' does not vanish on a regular isobaric surface (22). Thus,

$$|\text{grad } p|^2 = p_x'^2 + p_y'^2 + p_z'^2 = \frac{1 + f_x'^2 + f_y'^2}{f_p'^2}. \tag{28}$$

Substituting expressions (26) and (28) into (8), we arrive to the equation

$$\frac{\partial u^x}{\partial x} + \frac{\partial u^y}{\partial y} + \frac{(f_x'f_{xx}'' + f_y'f_{xy}'')u^x + (f_x'f_{xy}'' + f_y'f_{yy}'')u^y}{1 + f_x'^2 + f_y'^2}$$
$$= \frac{1}{2} \cdot \frac{f_p'^2}{1 + f_x'^2 + f_y'^2}\left[u^x \frac{\partial}{\partial x}\left(\frac{1 + f_x'^2 + f_y'^2}{f_p'^2}\right) + u^y \frac{\partial}{\partial y}\left(\frac{1 + f_x'^2 + f_y'^2}{f_p'^2}\right)\right].$$

After the obvious simplification, it becomes

$$\frac{\partial u^x}{\partial x} + \frac{\partial u^y}{\partial y} + \frac{1}{f_p'}(u^x f_{px}'' + u^y f_{py}'') = 0.$$

Substituting expressions (23) and (28) into (13), we arrive to the equation $f_{xx}''(u^x)^2 + 2f_{xy}''u^x u^y + f_{yy}''(u^y)^2 = \frac{1 + f_x'^2 + f_y'^2}{|f_p'|}$. Using this, Eq. (27) can be written as

$$u^x \frac{\partial u^x}{\partial x} + u^y \frac{\partial u^x}{\partial y} + \frac{f_x'}{|f_p'|} = 0, \quad u^x \frac{\partial u^y}{\partial x} + u^y \frac{\partial u^y}{\partial y} + \frac{f_y'}{|f_p'|} = 0.$$

We have proved the following

Proposition 7 *Let a Gavrilov flow (u, p) be defined on an open set of $\mathbb{R}^3$. Assume that for $p \in (-p_0, p_0)$ the isobaric surface M_p is regular and coincides with the graph of a smooth function $z = f(p; x, y)$ $\left((x, y) \in U \subset \mathbb{R}^2\right)$. Write the restriction of the vector field u to the surface M_p in the form $u = u^x(p; x, y)\frac{\partial}{\partial x} + u^y(p; x, y)\frac{\partial}{\partial y}$. Then the derivative f_p' does not vanish and the functions $f(p; x, y), u^x(p; x, y)$ and*

$u^y(p; x, y)$ *satisfy the equations*

$$\frac{\partial(f'_p u^x)}{\partial x} + \frac{\partial(f'_p u^y)}{\partial y} = 0, \tag{29}$$

$$u^x \frac{\partial u^x}{\partial x} + u^y \frac{\partial u^x}{\partial y} + \frac{f'_x}{|f'_p|} = 0, \tag{30}$$

$$u^x \frac{\partial u^y}{\partial x} + u^y \frac{\partial u^y}{\partial y} + \frac{f'_y}{|f'_p|} = 0, \tag{31}$$

$$f''_{xx}(u^x)^2 + 2f''_{xy}u^x u^y + f''_{yy}(u^y)^2 = \frac{1 + f'^2_x + f'^2_y}{|f'_p|}. \tag{32}$$

It is easy to check that the system (29)–(32) has the following solution:

$$f(x, y, p) = f(x, p) = -\sqrt{r^2(p) - x^2} \quad \left(-r(p) < x < r(p)\right), \tag{33}$$

$$u^x(x, y, p) = u^x(x, p) = \frac{\sqrt{r^2(p) - x^2}}{\sqrt{r(p)|r'(p)|}}, \quad u^y(x, y, p) = u^y(p) = b(p), \tag{34}$$

where $r(p)$ is a smooth positive function with non-vanishing derivative, and $b(p)$ is an arbitrary smooth function. For every p, the graph M_p of the function $(x, y) \mapsto f(x, p)$ is the half of the cylinder $\tilde{M}_p = \{(x, y, z) \mid x^2 + z^2 = r^2(p)\}$. Observe that $\tilde{M}_p$ is a surface of revolution around the y-axis. Thus, (33)–(34) is an axisymmetric GF. We will study axisymmetric Gavrilov flows in the next section. The solution (33)–(34) can be slightly modified by a change of Cartesian coordinates in $\mathbb{R}^3$. We do not know any solution to the system (30)–(33) different of the (modified) solution (33)–(34).

5 Axisymmetric Gavrilov flows

Let (r, z, θ) be cylindrical coordinates in $\mathbb{R}^3$ related to Cartesian coordinates (x_1, x_2, x_3) by $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, $x_3 = z$. We study a GF (u, p) invariant under rotations around the z-axis. The flow is defined in an open set $\tilde{U} \subset \{(r, z, \theta) : r > 0\}$ invariant under rotations around the z-axis. Such a rotationally invariant set $\tilde{U}$ is uniquely determined by the two-dimensional set $U = \tilde{U} \cap \{\theta = 0\} \subset \{(r, z) : r > 0\}$. For brevity we say that an axisymmetric GF is defined in U.

A regular isobaric surface M_p is a surface of revolution determined by its *generatrix* $\Gamma_p = M_p \cap U$. We parameterize the curve Γ_p by the arc length $r = R(p, t) > 0$, $z = Z(p, t)$,

$$R'^2_t + Z'^2_t = 1. \tag{35}$$

The variables (t, θ) serve as coordinates on the isobaric surface M_p. Since the vector field u is tangent to M_p, it is uniquely represented as $u = u^t(p, t)\frac{\partial}{\partial t} + u^\theta(p, t)\frac{\partial}{\partial \theta}$,

where $\left(u^t(p, t), u^\theta(p, t) \right)$ are *geometric coordinates* of u. *Physical coordinates* of u are defined by

$$u = u_r(r, z)e_r + u_z(r, z)e_z + u_\theta(r, z)e_\theta, \tag{36}$$

where e_r, e_z, e_θ are unit coordinate vectors. Physical and geometric coordinates are related by

$$u_r = R'_t u^t, \quad u_z = Z'_t u^t, \quad u_\theta = R u^\theta. \tag{37}$$

We are going to write down differential equations for a GF in terms of the functions (R, Z, u^t, u^θ). First of all, the first quadratic form of M_p in coordinates (t, θ) is

$$I = \mathrm{d}t^2 + R^2 \mathrm{d}\theta^2. \tag{38}$$

Christoffel symbols of the metric are $\Gamma^t_{\theta\theta} = -RR'_t$, $\Gamma^\theta_{t\theta} = \frac{R'_t}{R}$, $\Gamma^t_{tt} = \Gamma^t_{t\theta} = \Gamma^\theta_{tt} = \Gamma^\theta_{\theta\theta} = 0$. Using these formulas, we calculate

$$\nabla_t u^t = \frac{\partial u^t}{\partial t}, \quad \nabla_\theta u^t = -RR'_t u^\theta, \quad \nabla_t u^\theta = \frac{\partial u^\theta}{\partial t} + \frac{R'_t}{R} u^\theta, \quad \nabla_\theta u^\theta = \frac{R'_t}{R} u^t.$$

In particular,

$$\operatorname{div} u = \nabla_t u^t + \nabla_\theta u^\theta = \frac{\partial u^t}{\partial t} + \frac{R'_t}{R} u^t. \tag{39}$$

The restriction of u to M_p is a geodesic vector field, i.e., $\nabla_u u = 0$. This gives the system

$$u^t \frac{\partial u^t}{\partial t} - RR'_t (u^\theta)^2 = 0, \quad u^t \frac{\partial u^\theta}{\partial t} + \frac{2R'_t}{R} u^t u^\theta = 0. \tag{40}$$

Let us express $|\operatorname{grad} p|$ in terms of the functions $R(p, t)$ and $Z(p, t)$. Let $p = p(r, z)$ be the pressure in cylindric coordinates (it is independent of θ). We have the identity $p\left(R(p, t), Z(p, t) \right) = p$. Differentiating the identity with respect to p and t, we arrive to the linear algebraic system with unknowns $p'_r\left(R(p, t), Z(p, t) \right)$ and $p'_z\left(R(p, t), Z(p, t) \right)$

$$R'_p(p, t) p'_r\left(R(p, t), Z(p, t) \right) + Z'_p(p, t) p'_z\left(R(p, t), Z(p, t) \right) = 1,$$

$$R'_t(p, t) p'_r\left(R(p, t), Z(p, t) \right) + Z'_t(p, t) p'_z\left(R(p, t), Z(p, t) \right) = 0.$$

Solving the system, we have

$$p'_r\Big(R(p,t), Z(p,t)\Big) = J^{-1}(p,t)Z'_t(p,t),$$

$$p'_z\Big(R(p,t), Z(p,t)\Big) = -J^{-1}(p,t)R'_t(p,t),$$

where

$$J = \begin{vmatrix} R'_p & R'_t \\ Z'_p & Z'_t \end{vmatrix} \tag{41}$$

is the Jacobian of the transformation $r = R(p,t), \quad z = Z(p,t)$. The Jacobian does not vanish. This implies with the help of (35)

$$|\text{grad } p| = |J|^{-1}. \tag{42}$$

Substituting expressions (39) and (42) into (8), we arrive to the equation

$$\frac{\partial u^t}{\partial t} + \left(\frac{R'_t}{R} + \frac{J'_t}{J}\right)u^t = 0. \tag{43}$$

The second quadratic form of the surface M_p is expressed in coordinates (t, θ) by

$$II = -((R'_t Z''_{tt} - Z'_t R''_{tt})\, dt^2 + RZ'_t\, d\theta^2).$$

The sign on the right-hand side is chosen taking our agreement into account: the unit normal vector to the surface M_p must coincide with $\frac{\text{grad } p}{|\text{grad } p|}$. Recall that the curvature $\kappa = \kappa(p,t)$ of the plane curve $r = R(p,t), z = Z(p,t)$ is expressed, under the condition (35), by

$$\kappa = R'_t Z''_{tt} - Z'_t R''_{tt}. \tag{44}$$

Using the latter equality, the previous formula takes the form

$$II = -(\kappa\, dt^2 + RZ'_t\, d\theta^2). \tag{45}$$

By (42) and (45), the Eq. (13) takes the form

$$\kappa(u^t)^2 + RZ'_t(u^\theta)^2 = |J|^{-1}. \tag{46}$$

We have thus obtained the system of five Eqs. (35), (40a,b), (43), (46) in four unknown functions (R, Z, u^t, u^θ). The functions J and κ participating in the system are expressed through (R, Z) by (41) and (44), respectively. We proceed to the analysis of the system.

All isobaric surfaces M_p under consideration are assumed to be regular and connected. The Eq. (43) implies the following alternative for every p_0: either $u^t(p_0, t) \neq 0$ for all t or $u^t(p_0, t) \equiv 0$. The second case of the alternative is realized in the example (18) with $a = 0$. The converse statement is true at least partially: If $u^t(p_0, t) \equiv 0$, then M_{p_0} coincides with the cylinder $\{r = \text{const} > 0\}$ and particle trajectories living on M_{p_0} are horizontal circles. Indeed, if $u^t(p_0, t) \equiv 0$ then, as is seen from (35) and (40b), $R'_t(p_0, t) \equiv 0$ and $Z'_t(p_0, t) \equiv \pm 1$. Nevertheless, it is possible that $u^t(p_0, t) \equiv 0$ but $u^t(p, t) \neq 0$ for p close to p_0; the corresponding example can be constructed by a slight modification of (18).

To avoid degenerate cases of the previous paragraph, we additionally assume $u^t(p, t) \neq 0$ for all (p, t). Recall that we study flows up to the equivalence (4). In particular, GFs (u, p) and $(-u, p)$ are equivalent. Therefore, the latter assumption can be written without lost of generality in the form

$$u^t(p, t) > 0 \quad \text{for all} \quad (p, t). \tag{47}$$

Equation (40a) simplifies under the assumption (47) to the following one:

$$\frac{\partial u^\theta}{\partial t} + \frac{2R'_t}{R} u^\theta = 0. \tag{48}$$

If $u^\theta(p, t_0) = 0$ for some t_0, then (48) implies that $u^\theta(p, t) = 0$ for all t. On the other hand, assuming that $u^\theta(p, t) \neq 0$ for all t and for a fixed p, we can rewrite (48) in the form $\frac{\partial(\log u^\theta)}{\partial t} + \frac{\partial(\log R^2)}{\partial t} = 0$. From this we get

$$u^\theta(p, t) = \frac{b(p)}{R^2(p, t)} \tag{49}$$

with some function $b(p)$. This equality is also true for such p_0 that $u^\theta(p_0, t) = 0$ for all t, just by setting $b(p_0) = 0$. The equality (49) implies smoothness of the function b.

Substituting the expression (49) into (40a), we get $\frac{\partial(u^t)^2}{\partial t} = 2 \frac{R'_t}{R^3} b^2(p)$, that can be written in the form $\frac{\partial(u^t)^2}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{b^2(p)}{R^2}\right)$. From this, we obtain

$$u^t(p, t) = \frac{\sqrt{d(p)R^2(p, t) - b^2(p)}}{R(p, t)}, \tag{50}$$

where $d(p)$ is a smooth function satisfying

$$d(p)R^2(p, t) - b^2(p) > 0. \tag{51}$$

By (38), $|u|^2 = (u^t)^2 + R^2(u^\theta)^2$. Substituting values (49)–(50), we get

$$|u(p, t)|^2 = d(p). \tag{52}$$

We have thus discovered the important phenomenon: for an axisymmetric GF, all particles living on an isobaric surface M_p move with the same speed. In other words, constants c and C in the Bernoulli law (5) can be expressed through each other. The phenomenon is actually expected since it holds for the Grad — Shafranov ansatz [3]. Most likely, the phenomenon is absent for a general (not axisymmetric) GF, at least we cannot derive a relation like (52) from (29)–(32).

Studying GFs up to equivalence, we can multiply $u(p, t)$ by a non-vanishing smooth function $\varphi(p)$. This opportunity was already used to fix the sign of u^t in (47). We still have the freedom of multiplying $u(p, t)$ by a positive smooth function $\varphi(p)$ together with the corresponding change of the pressure. Choosing $\varphi(p) = d(p)^{-1/2}$ and denoting the new GF by (u, p) again, we simplify (52) to the following one:

$$|u(p, t)|^2 = 1. \tag{53}$$

The inequality (51) becomes now $|b(p)| < R(p, t)$, and formula (50) takes the form

$$u^t(p, t) = \frac{\sqrt{R^2(p, t) - b^2(p)}}{R(p, t)}. \tag{54}$$

We continue our analysis under assumptions (47) and (53). In virtue of (47), the Eq. (43) can be written in the form $\frac{\partial(\log u^t)}{\partial t} + \frac{\partial(\log(|J|R))}{\partial t} = 0$. This implies

$$u^t(p, t) = \frac{\alpha^2(p)}{|J(p, t)|R(p, t)} \tag{55}$$

with some positive smooth function $\alpha(p)$. Comparing (54) and (55), we arrive to the equation $\sqrt{R^2(p, t) - b^2(p)} \, |J(p, t)| = \alpha^2(p)$.

Finally, we simplify the Eq. (46). Substituting expressions (49) and (54) for u^θ and u^t into (46), we obtain $\kappa R(R^2 - b^2) + b^2 Z'_t = R^3|J|^{-1}$. Expressing $|J|$ from (55) and substituting the expression into the latter formula, we arrive to the equation $\kappa R(R^2 - b^2) + b^2 Z'_t = \frac{R^3 \sqrt{R^2 - b^2}}{\alpha^2}$. We have thus proved the following

Theorem 8 *Let an axisymmetric Gavrilov flow (u, p) be defined on an open set $U \subset \{(r, z) \mid r > 0\}$. Assume that every isobaric surface M_p is regular and connected. For the surface of revolution M_p, let $r = R(p, t) > 0$, $z = Z(p, t)$ be the arc length parametrization of the generatrix Γ_p of M_p. Assume also that in the representation $u = u^t(p, t)\frac{\partial}{\partial t} + u^\theta(p, t)\frac{\partial}{\partial \theta}$ the function $u^t(p, t)$ does not vanish. Then*

(i) there exists a smooth positive function $d(p)$ such that $|u|^2 = d(p)$. Replacing (u, p) with an equivalent Gavrilov flow and denoting the new flow by (u, p) again, we can assume without lost of generality that $u^t(p, t) > 0$ and

$$|u|^2 = 1. \tag{56}$$

(ii) *under the assumption* (56)*, the functions* $R(p, t)$ *and* $Z(p, t)$ *satisfy the equations*

$$R_t'^2 + Z_t'^2 = 1, \tag{57}$$

$$\sqrt{R^2 - b^2(p)} \left| R_p' Z_t' - R_t' Z_p' \right| = \alpha^2(p), \tag{58}$$

$$\kappa R \left(R^2 - b^2(p) \right) + b^2(p) Z_t' = \frac{R^3 \sqrt{R^2 - b^2(p)}}{\alpha^2(p)} \tag{59}$$

with some smooth functions $\alpha(p) > 0$ *and* $b(p)$*, where* $\kappa = \kappa(p, t)$ *is the curvature of the plane curve* $r = R(p, t)$, $z = Z(p, t)$*. The functions* $u^t(p, t)$ *and* $u^\theta(p, t)$ *are expressed through* $\left(R(p, t), b(p) \right)$ *by*

$$u^t = \frac{\sqrt{R^2 - b^2(p)}}{R}, \quad u^\theta = \frac{b(p)}{R^2}. \tag{60}$$

Let us make some remarks on Theorem 8.

1. First, we attract reader's attention to the hypothesis: M_p are connected surfaces. Otherwise functions $\alpha(p)$ and $b(p)$ can be different on different connected components.

2. Equations (57)–(60) are invariant under some transformations. First, the parameter t is defined up to a shift, i.e., nothing changes after the replacement $R(p, t) = \tilde{R}\left(p, t + t_0(p) \right)$, $Z(p, t) = \tilde{Z}\left(p, t + t_0(p) \right)$. Second, the equations are invariant under the changes $Z(p, t) = \tilde{Z}(p, t) + z_0$ and $Z(p, t) = -\tilde{Z}(p, t)$, $\kappa = -\tilde{\kappa}$, which mean a vertical shift of the origin in $\mathbb{R}^3$ and the change of the direction of the z-axis, respectively.

3. Compared with Proposition 7, Theorem 8 has an important advantage. The unknown functions (f, u^x, u^y) are not separated in the system (29)–(32) and, probably, cannot be separated. On the other hand, Eqs. (57)–(59) involve only the functions (R, Z) that determine isobaric surfaces M_p (the equations involve also $\alpha(p)$ and $b(p)$ that appear as integration constants). If the system (57)–(59) was solved, the velocity vector field u would be determined by explicit formulas (60). Of course, the simplification is possible due to the Clairaut integral for the equation of geodesics on a surface of revolution. Although the Clairaut integral is not mentioned in our proof of Theorem 8, formulas (60) are actually equivalent to the Clairaut integral.

Recall that physical coordinates (u_r, u_z, u_θ) of the velocity vector field u are defined by (36). The function u_θ is called *the swirl*. As follows from (37) and (60),

$$(u_r^2 + u_z^2)(r, z) = 1 - \frac{\beta(p)}{r^2}, \quad u_\theta(r, z) = \frac{\sqrt{\beta(p)}}{r}. \tag{61}$$

Theorem 8 has two other useful forms.

Case A. Assume, under hypotheses of Theorem 8, that the curve Γ_p is the graph of a function $r = f(p, z)$. Then, $f'_p \neq 0$ and Eqs. (57)–(59) are equivalent to the system

$$f'^2_p = \frac{\alpha(p)(1 + f'^2_z)}{f^2 - b^2(p)}, \tag{62}$$

$$\alpha(p) f f''_{zz} = b^2(p) f'^2_p - f^3 f'_p (1 + f'^2_z) \tag{63}$$

with the same functions $\alpha(p) > 0$ and $b(p)$. The function f must satisfy $f(z, p) > |b(p)|$. The velocity vector u is now represented as $u = u^z(z, p)\frac{\partial}{\partial z} + u^\theta(z, p)\frac{\partial}{\partial \theta}$, where the functions u^z and u^θ are expressed through (f, α, b) by $u^z = \mathrm{sgn}(u^z)\frac{\sqrt{\alpha(p)}}{f|f'_p|}$, $u^\theta = \frac{b(p)}{f^2}$ and $\mathrm{sgn}(u^z) = \pm 1$ is the sign of u^z that is assumed do not vanish.

Case B. Assume, under hypotheses of Theorem 8, that the curve Γ_p is the graph of a function $z = g(p, r)$. Then, Eqs. (57)–(59) are equivalent to the system

$$\left(r^2 - b^2(p)\right) g'^2_p - \alpha(p)(1 + g'^2_r) = 0, \tag{64}$$

$$r\alpha(p) g''_{rr} + b^2(p) g'_r g'^2_p + r^3 g'_p (1 + g'^2_r) = 0 \tag{65}$$

with the same functions $\alpha(p) > 0$ and $b(p)$. The function $g(p, r)$ is considered for $r \in \left(r_1(p), r_2(p)\right)$ with $|b(p)| < r_1(p)$. The velocity vector u is now represented as $u = u^r(p, r)\frac{\partial}{\partial r} + u^\theta(p, r)\frac{\partial}{\partial \theta}$, where the functions u^r and u^θ are expressed through (g, α, b) by $u^r = \mathrm{sgn}(u^r)\frac{\sqrt{\alpha(p)}}{r|g'_p|}$, $u^\theta = \frac{b(p)}{r^2}$ and $\mathrm{sgn}(u^r) = \pm 1$ is the sign of u^r that does not vanish.

6 Consistency conditions

We will first recall some basic facts from theory of first order PDEs following [14, Part I, Section 14]. Let us consider the system of two first order PDEs

$$F(x, y, z, p, q) = 0, \quad G(x, y, z, p, q) = 0, \tag{66}$$

where $z = z(x, y)$ is an unknown function and $p = z'_x$, $q = z'_y$. We assume F and G to be sufficiently smooth functions defined for $(x, y, z) \in U$ and for all $(p, q) \in \mathbb{R}^2$, where $U \subset \mathbb{R}^3$ is an open set. The system (66) is supplied with the initial condition

$$z(x_0, y_0) = z_0 \tag{67}$$

for a point $(x_0, y_0, z_0) \in U$. The *Jacobi brackets* (sometimes also called *Mayer brackets*) of functions $F(x, y, z, p, q)$ and $G(x, y, z, p, q)$ are defined by

$$[F, G] = (F'_x + p F'_z)G'_p - (G'_x + p G'_z)F'_p + (F'_y + q F'_z)G'_q - (G'_y + q G'_z)F'_q.$$

$$(68)$$

The system (66) is said to be an *involutory system* if $[F, G] \equiv 0$ for $(x, y; z, p, q) \in U \times \mathbb{R}^2$. The system (66) is said to be a *complete system on the open set* $U \times \mathbb{R}^2$ if the equation

$$[F, G] = 0 \tag{69}$$

is an algebraic corollary of the system (66), i.e., if (69) holds for $(x, y; z, p, q) \in U \times \mathbb{R}^2$ satisfying (66). The Eq. (69) is called the *consistency condition* (or *integrability condition*) for the system (66). In the case of a complete system, for an arbitrary point $(x_0, y_0, z_0) \in U$, the initial value problem (66)–(67) has a unique solution at least in some neighborhood of the point (x_0, y_0). Several methods are known for the numerical solution of the IVP (66)–(67), the Mayer method is the most popular one [14].

We return to axisymmetric GFs. Under certain additional conditions, systems (62)–(63) and (64)–(65) are equivalent. We study the system (62)–(63) as the simplest one. Recall that the system is considered in a neighborhood of a regular point, where the transform $(p, z) \mapsto (f(p, z), z)$ is one-to-one. Therefore the derivative f_p' does not vanish. Note that only b^2 is involved in (62)–(63), not the function b itself. To simplify our formulas a little bit, we introduce the function $\beta(p) = b^2(p) \geq 0$ and rewrite the system (62)–(63) as

$$(f^2 - \beta)f_p'^2 - \alpha(1 + f_z'^2) = 0, \tag{70}$$

$$\alpha f f_{zz}'' - \beta f_p'^2 + f^3 f_p'(1 + f_z'^2) = 0. \tag{71}$$

The function f is assumed to satisfy the inequality

$$f(p, z) > \sqrt{\beta(p)}. \tag{72}$$

Given functions $\alpha(p) > 0$ and $\beta(p) \geq 0$, (70)–(71) is an overdetermined system of two PDEs in one unknown function $f(p, z)$. The overdeterminess is caused by the circumstance mentioned in Introduction: a GF is defined by the overdetermined system (1)–(3). We pose the question: *What conditions should be imposed on* $(\alpha(p), \beta(p))$ *for solvability of the system* (70)–(71) *at least locally, i.e., in a neighborhood of a given point* (p_0, z_0)? For a fixed p, (71) can be considered as a second order ODE with an unknown function $f_p(z) = f(p, z)$. Observe that the variable z does not explicitly participate in (71). As well known [13, Ssection 15.3], such an equation can be reduced to a first order ODE. The observation is realized by the following statement.

Lemma 9 *Let C^1-functions $\alpha(p) > 0$ and $\beta(p) \geq 0$ be defined on an interval (p_1, p_2), where $-\infty \leq p_1 < p_2 \leq \infty$. Then the following statements are valid.*

1. Let $f(p, z)$ be a solution to the system (70)–(71) *on a rectangle*

$$(p, z) \in (p_1, p_2) \times (z_1, z_2), \tag{73}$$

and let the inequality (72) be valid on the rectangle. Then there exists a C^1-function $\gamma(p)$ on the interval (p_1, p_2) satisfying the equation

$$f^2(\varepsilon f^2 + \gamma)^2(1 + f_z'^2) - 4\alpha(f^2 - \beta) = 0 \tag{74}$$

and the inequalities

$$0 < \varepsilon f^2 + \gamma \le \frac{2\alpha^{1/2}\sqrt{f^2 - \beta}}{f}, \tag{75}$$

where $\varepsilon = \pm 1$ is the sign of f_p' that does not vanish.

2. Conversely, let $f(p, z) > 0$ and $\gamma(p)$ satisfy (70) and (74)–(75). Assume additionally that, for every $p \in (p_1, p_2)$, the derivative $f_z'(p, z)$ is not identically equal to zero on any interval $(z_1', z_2') \subset (z_1, z_2)$. Then, f solves (70)–(71) on the rectangle (73).

Proof First of all we find from (70)

$$f_p' = \varepsilon \frac{\alpha^{1/2}\sqrt{1 + f_z'^2}}{\sqrt{f^2 - \beta}}, \tag{76}$$

where $\varepsilon = \pm 1$ is the sign of f_p', and substitute the expression into (71)

$$f_{zz}'' = \frac{\beta(1 + f_z'^2)}{f(f^2 - \beta)} - \varepsilon \frac{f^2(1 + f_z'^2)^{3/2}}{\alpha^{1/2}\sqrt{f^2 - \beta}}. \tag{77}$$

Let us show that Eqs. (76) and (77) imply

$$\frac{\partial}{\partial z}\left(\frac{\sqrt{f^2 - \beta}}{f\sqrt{1 + f_z'^2}} - \varepsilon \frac{f^2}{2\alpha^{1/2}}\right) = 0. \tag{78}$$

To this end we implement the differentiation in (78). The result can be written as

$$\left(f_{zz}'' - \frac{\beta(1 + f_z'^2)}{f(f^2 - \beta)} + \varepsilon \frac{f^2(1 + f_z'^2)^{3/2}}{\alpha^{1/2}\sqrt{f^2 - \beta}}\right)f_z' = 0. \tag{79}$$

By (77), the left-hand side of (79) is identically zero. This proves (78). The Eq. (78) means the existence of a function $\gamma(p)$ such that $\frac{\sqrt{f^2 - \beta}}{f\sqrt{1 + f_z'^2}} - \varepsilon \frac{f^2}{2\alpha^{1/2}} = \frac{\gamma}{2\alpha^{1/2}}$. This can be written in the form

$$2\alpha^{1/2}\sqrt{f^2 - \beta} = f\sqrt{1 + f_z'^2}(\varepsilon f^2 + \gamma). \tag{80}$$

By (72), $f > 0$ and $f^2 - \beta > 0$. Therefore (80) implies the inequalities (75). Squaring the Eq. (80), we get (74). We have proved the first statement of the lemma.

The second statement of the lemma is proved by reversing presented arguments with the following additional remark. To pass from (79) to (77), we need to remove the factor f_z' on the left-hand side of (79). To do this, it suffices to assume that for every $p \in (p_1, p_2)$, the derivative $f_z'(p, z)$ is not identically zero on any interval $(z_1', z_2') \subset (z_1, z_2)$. □

By Lemma 9, the system (70)–(71) is equivalent to the following one:

$$
\begin{aligned}
(f^2 - \beta) f_p'^2 - \alpha(1 + f_z'^2) = 0, \\
f^2(\varepsilon f^2 + \gamma)^2(1 + f_z'^2) - 4\alpha(f^2 - \beta) = 0.
\end{aligned}
\tag{81}
$$

Introducing the notations $\pi = f_p'$, $\zeta = f_z'$, we write the system as

$$
F(p, z, f, \pi, \zeta) = 0, \quad G(p, z, f, \pi, \zeta) = 0,
\tag{82}
$$

where

$$
\begin{aligned}
F(p, z, f, \pi, \zeta) &= (f^2 - \beta)\pi^2 - \alpha(\zeta^2 + 1), \\
G(p, z, f, \pi, \zeta) &= f^2(\varepsilon f^2 + \gamma)^2(\zeta^2 + 1) - 4\alpha(f^2 - \beta).
\end{aligned}
\tag{83}
$$

Assume that $\alpha(p) > 0$, $\beta(p) \geq 0$ and $\gamma(p)$ are defined and smooth on an interval (p_1, p_2), where $-\infty \leq p_1 < p_2 \leq \infty$. Then, F and G are defined and smooth in $U \times \mathbb{R}^2$, where

$$
U = \left\{ (p, z, f) \in \mathbb{R}^3 \mid p_1 < p < p_2, \ f > \sqrt{\beta}, \ 0 < \varepsilon f^2 + \gamma < 2\sqrt{\alpha}\sqrt{f^2 - \beta}/f \right\}.
\tag{84}
$$

The functions F and G are actually independent of z and depend on p through the functions $\alpha(p)$, $\beta(p)$, $\gamma(p)$ only. Up to notations, the system (82) is of the form (66).

Theorem 10 *Given C^1-functions $\alpha(p) > 0$, $\beta(p) \geq 0$ and $\gamma(p)$ on an interval (p_1, p_2), define F and G by (83) and consider the system of PDEs (82), where $f = f(p, z)$ is an unknown function and $\pi = f_p'$, $\zeta = f_z'$. Define an open set $U \subset \mathbb{R}^3$ by (84). Then,*

1. *The system (82) is complete on $U \times \mathbb{R}^2$ if the functions α, β, γ satisfy*

$$
\alpha(p) = \alpha_0 e^{3p} \quad \text{with some constant } \alpha_0 > 0,
\tag{85}
$$
$$
\beta' + 2\varepsilon\gamma' + 3\beta + \varepsilon\gamma = 0,
\tag{86}
$$
$$
\gamma\beta' - 2\beta\gamma' + 3\beta\gamma - 4\varepsilon\alpha + \varepsilon\gamma^2 = 0,
\tag{87}
$$

where either $\varepsilon = 1$ or $\varepsilon = -1$ and $\beta + \varepsilon\gamma \neq 0$. But (82) is not an involutory system.

2. *Conversely, assume* (82) *to be a complete system on* $U \times \mathbb{R}^2$, *where the open set* $U \subset \mathbb{R}^3$ *is defined by* (84). *Assume additionally that for every* $p_0 \in (p_1, p_2)$ *there exists* z_0 *such that* (82) *has a solution* $f(p, z)$ *in a neighborhood of* (p_0, z_0) *satisfying* $f'_z(p_0, z_0) \neq 0$. *Then, the functions* α, β, γ *satisfy* (85)–(87) *with some* $\varepsilon = \pm 1$.

Proof To agree (66) and (82), we need to change the variables in (66) as follows: $x := p$, $y := z$, $z := f$, $p := \pi$ and $q := \zeta$. Then, the formula (68) takes the form

$$[F, G] = (F'_p + \pi F'_f)G'_\pi - (G'_p + \pi G'_f)F'_\pi + (F'_z + \zeta F'_f)G'_\zeta - (G'_z + \zeta G'_f)F'_\zeta. \tag{88}$$

We find the derivatives by differentiating (83):

$$
\begin{array}{ll}
F'_p = -\beta'\pi^2 - \alpha'(\zeta^2 + 1), & G'_p = 2\gamma' f^2(\varepsilon f^2 + \gamma)(\zeta^2 + 1) - 4\alpha'(f^2 - \beta) + 4\alpha\beta', \\
F'_z = 0, & G'_z = 0, \\
F'_f = 2f\pi^2, & G'_f = 2f(\varepsilon f^2 + \gamma)^2(\zeta^2 + 1) + 4\varepsilon f^3(\varepsilon f^2 + \gamma)(\zeta^2 + 1) - 8\alpha f, \\
F'_\pi = 2(f^2 - \beta)\pi, & G'_\pi = 0, \\
F'_\zeta = -2\alpha\zeta, & G'_\zeta = 2f^2(\varepsilon f^2 + \gamma)^2\zeta.
\end{array}
\tag{89}
$$

Since $G'_\pi = F'_z = G'_z = 0$, the formula (88) simplifies to the following one:

$$[F, G] = -G'_p F'_\pi - \pi G'_f F'_\pi + \zeta F'_f G'_\zeta - \zeta G'_f F'_\zeta. \tag{90}$$

Substituting values (89) into (90), we obtain

$$
\begin{aligned}
\frac{1}{4}[F, G] = {}& -\pi(f^2 - \beta)\left(\gamma' f^2(\varepsilon f^2 + \gamma)(\zeta^2 + 1) - 2\alpha'(f^2 - \beta) + 2\alpha\beta'\right) \\
& -\pi^2 f(f^2 - \beta)\left((\varepsilon f^2 + \gamma)^2(\zeta^2 + 1) + 2\varepsilon f^2(\varepsilon f^2 + \gamma)(\zeta^2 + 1) - 4\alpha\right) \\
& +\pi^2\zeta^2 f^3(\varepsilon f^2 + \gamma)^2 + \zeta^2\alpha f\left((\varepsilon f^2 + \gamma)^2(\zeta^2 + 1) + 2\varepsilon f^2(\varepsilon f^2 + \gamma)(\zeta^2 + 1) - 4\alpha\right).
\end{aligned}
\tag{91}
$$

The right-hand side of (91) is a 7th degree polynomial in f and the coefficient at f^7 is $-\pi^2(2\zeta^2 + 3) \neq 0$. Thus, (82) is not an involutory system. Now, we prove that (82) is a complete system. To this end we derive from (82)–(83)

$$\zeta^2 = \frac{4\alpha(f^2 - \beta)}{f^2(\varepsilon f^2 + \gamma)^2} - 1 \tag{92}$$

and $\pi = \varepsilon \dfrac{\alpha^{1/2}\sqrt{\zeta^2 + 1}}{\sqrt{f^2 - \beta}}$, where $\varepsilon = \pm 1$ is the sign of π. We find from two last equalities

$$\pi = \frac{2\varepsilon\alpha}{f(\varepsilon f^2 + \gamma)}. \tag{93}$$

Substituting expressions (92)–(93) into (91), we obtain

$$
\frac{1}{16\alpha^2}[F,G] = -\varepsilon \frac{2(f^2-\beta)}{f(\varepsilon f^2+\gamma)}\left(\frac{\gamma'(f^2-\beta)}{\varepsilon f^2+\gamma} - \frac{\alpha'}{2\alpha}(f^2-\beta) + \frac{\beta'}{2}\right)
$$
$$
- \frac{4\alpha(f^2-\beta)}{f(\varepsilon f^2+\gamma)^2}\left(\frac{f^2-\beta}{f^2} + 2\varepsilon\frac{f^2-\beta}{\varepsilon f^2+\gamma} - 1\right) + f\left(\frac{4\alpha(f^2-\beta)}{f^2(\varepsilon f^2+\gamma)^2} - 1\right)
$$
$$
+ f\left(\frac{4\alpha(f^2-\beta)}{f^2(\varepsilon f^2+\gamma)^2} - 1\right)\left(\frac{f^2-\beta}{f^2} + 2\varepsilon\frac{f^2-\beta}{\varepsilon f^2+\gamma} - 1\right). \tag{94}
$$

We are interested in the case when $[F,G] = 0$. Equating the right-hand side of (94) to zero and multiplying the resulting equality by $f^3(\varepsilon f^2+\gamma)^3$, we arrive to the equation

$$
-2\varepsilon f^2(f^2-\beta)(\varepsilon f^2+\gamma)\left(\gamma'(f^2-\beta) - \frac{\alpha'}{2\alpha}(f^2-\beta)(\varepsilon f^2+\gamma) + \frac{\beta'}{2}(\varepsilon f^2+\gamma)\right)
$$
$$
- 4\alpha(f^2-\beta)\left((f^2-\beta)(\varepsilon f^2+\gamma) + 2\varepsilon f^2(f^2-\beta) - f^2(\varepsilon f^2+\gamma)\right)
$$
$$
+ 4\alpha f^2(f^2-\beta)(\varepsilon f^2+\gamma) - f^4(\varepsilon f^2+\gamma)^3
$$
$$
+ \left(4\alpha(f^2-\beta) - f^2(\varepsilon f^2+\gamma)^2\right)
$$
$$
\times \left((f^2-\beta)(\varepsilon f^2+\gamma) + 2\varepsilon f^2(f^2-\beta) - f^2(\varepsilon f^2+\gamma)\right) = 0. \tag{95}
$$

The left-hand side of the Eq. (95) is a polynomial of 10th degree in f. It is almost unbelievable, but the degree of the polynomial can be decreased to 4. Namely, the Eq. (95) is equivalent to the following one:

$$
-f^2(f^2-\beta)(\varepsilon f^2+\gamma)\left[\left(3 - \frac{\alpha'}{\alpha}\right)f^4 + \left(2\varepsilon\gamma' - \varepsilon\frac{\alpha'}{\alpha}\gamma + \frac{\alpha'}{\alpha}\beta + \beta' + 4\varepsilon\gamma\right)f^2\right.
$$
$$
\left. + \left(-2\varepsilon\beta\gamma' + \varepsilon\frac{\alpha'}{\alpha}\beta\gamma + \varepsilon\gamma\beta' - 4\alpha + \gamma^2\right)\right] = 0. \tag{96}
$$

Indeed, a simple (though rather cumbersome) calculation shows that the polynomials on the left-hand sides of (95) and (96) are identically equal. By (84), the factor $f^2(f^2-\beta)(\varepsilon f^2+\gamma)$ does not vanish on U. Therefore, the Eq. (96) is equivalent to the following one:

$$
\left(3 - \frac{\alpha'}{\alpha}\right)f^4 + \left(2\varepsilon\gamma' - \varepsilon\frac{\alpha'}{\alpha}\gamma + \frac{\alpha'}{\alpha}\beta + \beta' + 4\varepsilon\gamma\right)f^2
$$
$$
+ \left(-2\varepsilon\beta\gamma' + \varepsilon\frac{\alpha'}{\alpha}\beta\gamma + \varepsilon\gamma\beta' - 4\alpha + \gamma^2\right) = 0. \tag{97}
$$

Equating coefficients of the polynomial on the left-hand side of (97) to zero, we arrive to the system of ODEs

$$
\begin{aligned}
&\alpha' - 3\alpha = 0, \\
&\beta' + 2\varepsilon\gamma' + 3\beta + \varepsilon\gamma = 0, \\
&\varepsilon\gamma\beta' - 2\varepsilon\beta\gamma' + 3\varepsilon\beta\gamma - 4\alpha + \gamma^2 = 0.
\end{aligned}
\tag{98}
$$

This is equivalent to (85)–(87).

It remains to discuss the passage from (97) to (98). Of course, (98) implies (97). This proves the first statement of Theorem 10. To prove that (97) implies (98), we need for each $p_0 \in (p_1, p_2)$ to have at least three distinct z_1, z_2, z_3 such that the values $f^2(p_0, z_1)$, $f^2(p_0, z_2)$, $f^2(p_0, z_3)$ are pairwise different and (97) holds at (p_0, z_1), (p_0, z_2) and (p_0, z_3). The existence of such z_1, z_2, z_3 is guaranteed by the hypothesis of the second assertion of Theorem 10: For each $p_0 \in (p_1, p_2)$, there exists z_0 such that the system (82) has a solution $f(p, z)$ in a neighborhood of (p_0, z_0) satisfying $f'_z(p_0, z_0) \neq 0$.   □

Remark Roughly speaking, Theorem 10 means that the relations (85)–(87) constitute the consistency condition for the system (82). Nevertheless, we emphasize that two statements of Theorem 10 are not exactly converse to each other. For example, the axisymmetric GF (18) (isobaric surfaces are cylinders and particle trajectories are either circles or spiral lines) corresponds to the solution $f(p, z) = 2p^{1/2}$ to the system (81) with $(p_1, p_2) = (0, \infty)$ and

$$
\alpha(p) = 4a_0, \quad \beta(p) = 4(1 - a_0)p, \quad \gamma(p) = 4(a_0 - p) \quad (a_0 = \text{const}, \, 0 < a_0 \leq 1).
\tag{99}
$$

The functions (99) do not satisfy the consistency conditions (85)–(87). For this solution, $f'_z \equiv 0$ and the second statement of Theorem 10 does not apply.

An analog of Lemma 9 is valid for the system (64)–(65) with minor changes (the variables (p, z) are replaced by (p, r), the function f is replaced by g, etc.). The corresponding system of two first order PDEs looks as follows:

$$
\begin{aligned}
&(r^2 - \beta)g'^2_p - \alpha(1 + g'^2_r) = 0, \\
&r^2(\gamma - \tau r^2)^2(1 + g'^2_r) - 4\alpha(r^2 - \beta)g'^2_r = 0,
\end{aligned}
\tag{100}
$$

where $\alpha(p) > 0$, $\beta(p) \geq 0$ and $\gamma(p)$ are the same functions as in (81) and $\tau = \pm 1$ is the sign of g'_p that does not vanish. An analog of Theorem 10 is valid for the system (100) with the same consistency conditions (85)–(87).

7 Local and *z*-periodic axisymmetric Gavrilov flows

Here, we discuss the numerical method for constructing axisymmetric GFs on the base of the system (81). For the initial condition $f(p_0, z_0) = f_0$, we can assume without

lost of generality that $p_0 = z_0 = 0$ since the system (81) is invariant under the change $p \to p + \text{const}$ and $z \to z + \text{const}$. Thus, the initial condition for f is

$$f(0, 0) = f_0. \tag{101}$$

First, we will find the functions $\alpha(p), \beta(p), \gamma(p)$. The function α is given explicitly by (85) with an arbitrary constant $\alpha_0 = \alpha(0) > 0$. The functions $\beta(p) \geq 0$ and $\gamma(p)$ solve the system (86)–(87) supplied with the initial conditions

$$\beta(0) = \beta_0, \quad \gamma(0) = \gamma_0. \tag{102}$$

In particular, $\beta_0 \geq 0$. For the possibility to write the system (86)–(87) in the form $\beta' = B(\beta, \gamma)$, $\gamma' = \Gamma(\beta, \gamma)$, we have to impose the restrictions

$$\beta \geq 0, \quad \beta + \varepsilon\gamma \neq 0. \tag{103}$$

In particular, β_0 and γ_0 must satisfy these inequalities. Let (p_1, p_2) be the maximal interval such that the solution to the Cauchy problem (86)–(87), (102) exists on (p_1, p_2) and satisfies (103). Here, $p_1 = p_1(\alpha_0, \beta_0, \gamma_0) < 0 < p_2 = p_2(\alpha_0, \beta_0, \gamma_0)$. By Theorem 10, the system (81) is complete on $U \times \mathbb{R}^2$, where $U \subset \mathbb{R}^3$ is defined by (84). General theory [13, Chapter 1, Section 14] guarantees the existence and uniqueness of a solution to the IVP (81), (101) for any $(0, 0, f_0) \in U$ at least in some neighborhood of $(p_0, z_0) = (0, 0)$. Thus, a solution $f(p, z)$ to the IVP (81), (101) exists in some neighborhood of $(0, 0)$ and is uniquely determined by 5 constants $(\alpha_0, \beta_0, \gamma_0, f_0, \varepsilon)$ chosen so that

$$\alpha_0 > 0, \quad \beta_0 \geq 0, \quad \beta_0 + \varepsilon\gamma_0 \neq 0, \quad f_0 > 0, \quad \varepsilon = \pm 1,$$
$$f_0 > \beta_0^{1/2}, \quad 0 < \varepsilon f_0^2 + \gamma_0 < 2\alpha_0^{1/2} f_0^{-1} \sqrt{f_0^2 - \beta_0}. \tag{104}$$

The inequalities on the second line of (104) come from (84). Let us denote this unique solution by $f(p, z; \alpha_0, \beta_0, \gamma_0, f_0, \varepsilon)$.

In the general case, $f(p, z; \alpha_0, \beta_0, \gamma_0, f_0, \varepsilon)$ is a *local solution*, i.e., is defined in some neighborhood $U(\alpha_0, \beta_0, \gamma_0, f_0, \varepsilon) \subset \mathbb{R}^2$ of the point $(0, 0)$. Nevertheless, for some values of the parameters, it can happen that $(p_1', p_2') \times \mathbb{R} \subset U(\alpha_0, \beta_0, \gamma_0, f_0, \varepsilon)$ for some $-\infty \leq p_1' < 0 < p_2' \leq \infty$, in such a case we speak on a *global solution* defined on $(p_1', p_2') \times \mathbb{R}$. Global solutions are of particular interest. Unfortunately, so far we have neither necessary nor sufficient conditions on the parameters $(\alpha_0, \beta_0, \gamma_0, f_0, \varepsilon)$ for the existence of a global solution.

Everything said above in this section is valid for the numerical method based on the system (100).

Global solutions most often appear due to periodicity with the help of the following

Lemma 11 *Let a solution $f(p, z)$ of the system* (81) *be defined on a rectangle $(p_1, p_2) \times (z_1, z_2)$ and satisfy* (72). *Assume the existence of $(p_0, z_0) \in (p_1, p_2) \times$*

(z_1, z_2) *such that* $f'_z(p_0, z_0) = 0$ *and* $f''_{zz}(p_0, z_0) \neq 0$. *Then, at least in some neighborhood of* p_0, *the solution is symmetric with respect to* z_0, *i.e.,*

$$f(p, z) = f(p, -z + 2z_0) \quad \text{for } p \in (p_0 - \delta, p_0 + \delta) \text{ with some } \delta > 0. \quad (105)$$

Hence, f *can be extended to a solution defined in* $(p_0 - \delta, p_0 + \delta) \times (z_0 - \Delta, z_0 + \Delta)$ *such that* $(z_1, z_2) \subset (z_0 - \Delta, z_0 + \Delta)$.

Proof By the implicit function theorem, there exists a smooth function $z = \zeta(p)$ defined for $p \in (p_0 - \delta, p_0 + \delta)$ with some $\delta > 0$ such that $\zeta(p_0) = z_0$ and

$$f'_z(p, \zeta(p)) = 0. \quad (106)$$

We are going to prove that $\zeta(p)$ is actually a constant function. To this end we differentiate (106): $f''_{pz}(p, \zeta(p)) + f''_{zz}(p, \zeta(p)) \zeta'(p) = 0$. By choosing a smaller δ, we can assume that $f''_{zz}(p, \zeta(p)) \neq 0$. Thus, to prove the equality $\zeta' = 0$, we have to demonstrate that $f''_{pz}(p, \zeta(p)) = 0$. To this end we differentiate the first equation of the system (81) with respect to z (recall that α and β are independent of z) $(f^2 - \beta)f'_p f''_{pz} + f(f'_p)^2 f'_z - \alpha f'_z f''_{zz} = 0$. Setting $z = \zeta(p)$ here and using (106), we obtain $(f^2 - \beta)f'_p f''_{pz}|_{z=\zeta(p)} = 0$. Since $f'_p \neq 0$ and $f^2 - \beta > 0$, this implies $f''_{pz}(p, \zeta(p)) = 0$. We have thus proved

$$f'_z(p, z_0) = 0 \quad \text{for } p \in (p_0 - \delta, p_0 + \delta). \quad (107)$$

The system (81) is invariant under the change $z \to 2z_0 - z$. Therefore (107) implies (105). □

Now, under hypotheses of Lemma 11, assume the existence of a second point $(p'_0, z'_0) \in (p_1, p_2) \times (z_1, z_2)$ $(z_0 \neq z'_0)$ such that $f'_z(p'_0, z'_0) = 0$ and $f''_{zz}(p'_0, z'_0) \neq 0$. An analog of (105) holds for z'_0 with some $\delta' > 0$. Assume additionally that

$$(\tilde{p}_1, \tilde{p}_2) = (p_0 - \delta, p_0 + \delta) \cap (p'_0 - \delta', p'_0 + \delta') \neq \emptyset.$$

Using symmetries with respect to z_0 and z'_0, we extend $f(p, z)$ to a global z-periodic solution defined on $(\tilde{p}_1, \tilde{p}_2) \times \mathbb{R}$. The period is equal to $|z_0 - z'_0|$ if $p_0 = p'_0$, otherwise the period is equal to $2|z_0 - z'_0|$.

An example of z-periodic GF is presented on Fig. 1. The solution $f(p, z) = f(p, z; \alpha_0, \beta_0, \gamma_0, f_0, \varepsilon)$ to the IVP (81), (101) was computed for $\alpha_0 = 1$, $\beta_0 = 0.01$, $\gamma_0 = 0.5$, $f_0 = 0.97$, $\varepsilon = 1$. The solution to the Cauchy problem (86)–(87), (102) exists for $-0.07 < p < 7.48$. For $0 < p < p_c \approx 0.25$, the flow turns out to be z-periodic with the period approximately equal to 2.8. Graphs Γ_i of functions $r = f(p_i, z)$ for $p_i = 0.02\, i$ and $i = 0, \ldots, 8$ are drawn on the left-hand side of Fig. 1. Each curve Γ_i is the generatrix of the isobaric surface M_{p_i}. The velocity vector field u is drawn on the right picture for Γ_0 and Γ_8. We used (61) for computing physical coordinates of u. The two-dimensional vector field $u_r e_r + u_z e_z$ is tangent to Γ_0 and Γ_8, and the vector field $u_\theta e_\theta$ (the swirl) is orthogonal to the plane of the picture. The latter vector field

Fig. 1 Periodic axisymmetric GF
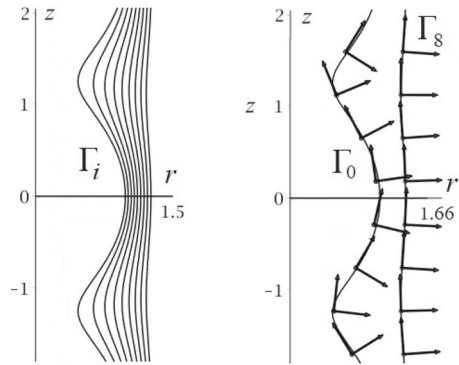
is drawn by vectors orthogonal to Γ_0 and Γ_8 in order to avoid 3D pictures. In order to make a nice picture, both fields $u_r e_r + u_z e_z$ and $u_\theta e_\theta$ are drawn in the scale $1:4$, i.e., $|u| = 0.25$ on the picture. But we remember that actually $|u| = 1$.

As we have mentioned in the remark written after the proof of Theorem 10, the axisymmetric GF (18) is an exception in a certain sense: it cannot be obtained in the scope of Theorem 10. We can state now the important phenomenon: the simplest axisymmetric GF (18) constitutes the main obstacle for our numerical method. As is seen from the Fig. 1, for p close to the critical value p_c, the flow is close to the z-independent solution (18): isobaric surfaces are close to circular cylinders and particle trajectories are close to spiral lines intersecting parallels approximately at the angle of $\pi/4$. The hypothesis $f''_{zz}(p_0, z_0) \neq 0$ of Lemma 11 is violated at $p_0 = p_c$. All our numerics become very unstable when p approaches p_c. The flow shown on Fig. 1 exists for $p > p_c$ as well, but it does not need to be periodic with the same period ≈ 2.8 for $p > p_c$.

8 Structure of an axisymmetric Gavrilov flow in a neighborhood of a minimum point of the pressure

In [6], the existence of a pair $(\tilde{u}, \tilde{p})$ is proved such that (a) $(1, 0)$ is a non-degenerate minimum point of the function $\tilde{p} = \psi(r, z) \in C^\infty(U)$ and $\psi(1, 0) = 0$, where $U \subset \{(r, z) \mid r > 0\}$ is a neighborhood of $(1, 0)$; (b) $(\tilde{u}, \tilde{p})$ is an axisymmetric GF in $U \setminus \{(1, 0)\}$; (c) the split Bernoulli law for the flow is of the simplest form

$$|\tilde{u}|^2 = 3\tilde{p}. \tag{108}$$

The flow $(\tilde{u}, \tilde{p})$ does not satisfy the normalization condition (56). To apply our equations, we must replace $(\tilde{u}, \tilde{p})$ with an equivalent GF (u, p) such that $|u|^2 = 1$. By the definition (4) of equivalent GFs, $u = \varphi(\tilde{p})\tilde{u}$, $\operatorname{grad} p = \varphi^2(\tilde{p})\operatorname{grad} \tilde{p}$ with some non-vanishing function $\varphi(\tilde{p})$. As is seen from (108), $\varphi^2(\tilde{p}) = 1/|\tilde{u}|^2 = 1/3\,\tilde{p}$. Thus,

$$p(r, z) = \frac{1}{3} \ln \psi(r, z) + C \quad (C = \text{const}), \tag{109}$$

Hence, the pressure function gets a logarithmic singularity at the point $(1, 0)$ after normalization. Comparing (85) and (109), we conclude that

$$\alpha = c\,\psi \quad (c = \mathrm{const} > 0). \tag{110}$$

Since $p \to -\infty$ as $(r, z) \to (1, 0)$, it is natural to choose initial conditions for the system (86)–(87) at $p = -\infty$. The conditions are

$$\beta(-\infty) = \lim_{p \to -\infty} \beta(p) = 1/3, \quad \gamma(-\infty) = \lim_{p \to -\infty} \gamma(p) = -1. \tag{111}$$

Indeed, for p "close" to $-\infty$, Γ_p is a "small" closed curve around $(1, 0)$. The tangent line to Γ_p is vertical at some point $(r_1, z_1) = \Big(r_1(p), z_1(p) \Big) \in \Gamma_p$. In a neighborhood of (r_1, z_1), the curve Γ_p is the graph of a function $r = f(p, z)$ solving the system (81) and satisfying $f'_z(p, z_1) = 0$. Setting $z = z_1$ in the second equation of (81), we get

$$f^2(p, z_1)\Big(\varepsilon f^2(p, z_1) + \gamma(p)\Big)^2 - 4\alpha(p)\Big(f^2(p, z_1) - \beta(p)\Big) = 0. \tag{112}$$

In view of our assumption $f^2(p, z) - \beta(p) > 0$, see (72), the factor $\Big(f^2(p, z_1) - \beta(p)\Big)$ remains bounded when $p \to -\infty$. Also $\alpha(p) = \alpha_0 e^{3p} \to 0$ as $p \to -\infty$. Thus, the second term on the left-hand side of (112) runs to 0 as $p \to -\infty$. The same is true for the first term. Taking into account that $f^2(p, z_1) \to 1$ as $p \to -\infty$, we obtain $\gamma(-\infty) = -\varepsilon$.

Since $\varepsilon = \pm 1$ is the sign of f'_p in (112), by a similar analysis of the system (100), we demonstrate that $\varepsilon = 1$ in our setting. This proves the second equality in (111). The first equality in (111) is proved similarly.

The "Cauchy problem" (86)–(87), (111) (with $\varepsilon = 1$) is easily solved in series $\beta = \frac{1}{3} + \sum_{k=1}^{\infty} \beta_k \alpha^k$ and $\gamma = \frac{1}{3} + \sum_{k=1}^{\infty} \gamma_k \alpha^k$. Equations (86)–(87) imply some recurrent relations that allow us to compute all coefficients. In particular,

$$\beta = \frac{1}{3} - \frac{7}{6}\alpha + \frac{13}{72}\alpha^2 - \frac{133}{1728}\alpha^3 + \frac{575}{13824}\alpha^4 - \frac{2077}{82944}\alpha^5 + \frac{37}{2304}\alpha^6 + \dots, \tag{113}$$

$$\gamma = -1 + \alpha - \frac{1}{8}\alpha^2 + \frac{7}{144}\alpha^3 - \frac{115}{4608}\alpha^4 + \frac{67}{4608}\alpha^5 - \frac{7}{768}\alpha^6 + \dots. \tag{114}$$

The series converge for all real α. We omit the proof of the convergence which is not easy.

The system (81) can be equivalently written in terms of the function $\psi(r, z)$. Indeed, as is seen from (109), a solution $f(p, z)$ to the system (81) is related to ψ by $\frac{1}{3} \ln \psi(f(p, z), z) + C = p$. Starting with this equation, we derive from (81)

the system

$$2c\psi_r' - 3r(r^2 + \gamma) = 0,$$

$$c^2\psi_z'^2 - 9c(r^2 - \beta)\psi + \frac{9}{4}r^2(r^2 + \gamma)^2 = 0, \tag{115}$$

where c is the constant from (110). By the change $\psi = \frac{1}{c}\tilde{\psi}$ of the unknown function, (115) is transformed to the same system with $c = 1$. Therefore we can assume $c = 1$ without lost of generality, i.e.,

$$2\psi_r' - 3r(r^2 + \gamma) = 0,$$

$$\psi_z'^2 - 9(r^2 - \beta)\psi + \frac{9}{4}r^2(r^2 + \gamma)^2 = 0. \tag{116}$$

The equality (110) becomes $\alpha = \psi$. The function $\psi(r, z)$ is defined and smooth in a neighborhod of the point $(1, 0)$ and satisfies $\psi(1, 0) = 0$. Let us show that $\psi(r, z)$ is an even function of z. Indeed, the second equation of the system (116) gives $\psi_z'^2(1, 0) + \frac{9}{4}r^2(1 + \gamma(-\infty))^2 = 0$. Since $\gamma(-\infty) = -1$ by (111), we obtain

$$\psi_z'(1, 0) = 0. \tag{117}$$

Then, we differentiate the first equation of the system (116) with respect to z

$$2\psi_{rz}'' - 3r\gamma_z' = 0. \tag{118}$$

Since γ depends on p only, $\gamma_z' = \gamma' p_z'$, where $\gamma' = d\gamma/dp$. Together with (109), this gives $\gamma_z' = \frac{\gamma'\psi_z'}{3\psi}$. Substituting this expression into (118) and setting $z = 0$, we arrive to the linear first-order ODE for the function $\psi_z'(r, 0)$:

$$2\frac{d\psi_z'(r, 0)}{dr} - 3\frac{r\gamma'(p(r, 0))}{3\psi(r, 0)}\psi_z'(r, 0) = 0. \tag{119}$$

The coefficient $\frac{r\gamma'(p(r,0))}{3\psi(r,0)}$ of the equation is a bounded smooth function in a neighborhood of $r = 1$ as is seen from (114). Together with the initial condition (117), the equation (119) implies $\psi_z'(r, 0) = 0$. The system (116) is invariant under the change $z \to -z$. Therefore, the equality $\psi_z'(r, 0) = 0$ implies that $\psi(r, z)$ is an even function of z.

The system (116) allows us to compute term-by-term all Taylor coefficients of the function $\psi(r, z)$ at the point $(1, 0)$. In particular,

$$\psi(r, z) = \frac{3}{2}(r - 1)^2 + \frac{3}{2}z^2 + \frac{9}{4}(r - 1)^3 + \frac{9}{4}(r - 1)z^2 + \frac{57}{32}(r - 1)^4$$

$$+ \frac{45}{16}(r - 1)^2z^2 + \frac{33}{32}z^4 + \frac{9}{8}(r - 1)^5 + \frac{9}{4}(r - 1)^3z^2 + \frac{9}{4}(r - 1)z^4 + \cdots \tag{120}$$
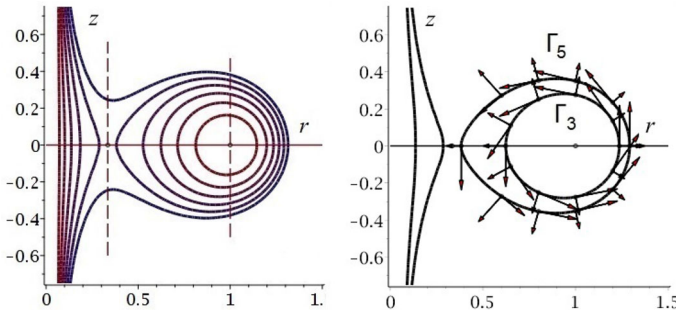
Fig. 2 Structure of GF in a neighborhood of a minimum point of the pressure

Second- and third-order terms on the right-hand side of (120) are easily derived from (116), and we used Maple for computing 4th and 5th order terms.

This GF is shown on Fig. 2. In our calculations we used the 5th order segment of the Taylor series of the function ψ, i.e., we ignored the remainder denoted by dots on the right-hand side of (120). Six isolines $\Gamma_i = \{(r, z) : \psi(r, z) = \psi_i = 0.04\,i\}$ ($i = 1, \ldots, 6$) are drawn on the left-hand side of Fig. 2. Each of the curves $\Gamma_1, \ldots, \Gamma_5$ consists of two connected components while Γ_6 has one component. The same curves Γ_i are isolines of the pressure function, see (109). We set $C = -\frac{1}{3} \ln \psi_1$, thus the formula (109) becomes $p(r, z) = \frac{1}{3} \ln \frac{\psi(r,z)}{\psi_1}$. Each curve Γ_i is the generatrix of the isobaric surface M_{p_i}, where $p_i = \frac{1}{3} \ln \frac{\psi_i}{\psi_1}$. In our case, $p_i = \frac{1}{3} \ln i$. The velocity vector field u is drawn on the right picture for Γ_3 and Γ_5. Observe the interesting phenomenon on Fig. 2: besides the minimum point $(1, 0)$, the function $\psi(r, z)$ has the saddle point at $(r, z) = (1/3, 0)$. The saddle point disappears when the 5th degree polynomial (120) is replaced with the corresponding 6th degree polynomial, and again appears at the same point for the 7th degree polynomial. We have no idea whether the phenomenon is essential for this type GFs, or it is just an artefact of ignoring higher degree terms in (120).

9 Some open questions

In our opinion, the main open question is: are there GFs on $\mathbb{R}^3$ which are not axisymmetric? From an analytical point of view, this is a question on the consistency conditions for the system (29)–(32). The most expected answer to the question is "yes". However, the question is not easy because of the following. For an axisymmetric GF, the constants c and C in the Bernoulli law (5) can be expressed through each other. In other words, all particles living on an isobaric surface M_p move with the same speed in the case of an axisymmetric GF. Most likely, this statement is not true for a general (not axisymmetric) GF, at least we cannot prove it by local reasoning. But maybe simple global arguments will help answer the question. For instance, if there exists a particle trajectory dense in M_p, then $|u| = \text{const}$ on M_p.

We mostly studied GFs locally in a neighborhood of a regular point. The most interesting questions relate to GFs with regular compact isobaric hypersurfaces $M_p \subset \mathbb{R}^n$. As mentioned in Introduction, application of the Gavrilov localization to such a flow gives a compactly supported GF on the whole of $\mathbb{R}^n$. Since a compact regular isobaric hypersurface M_p is endowed with a non-vanishing tangent vector field u, the Euler characteristic of M_p is equal to zero. In the most important 3D-case this means that M_p is diffeomorphic to the two-dimensional torus $\mathbb{T}^2$. The restriction of u onto M_p is a non-vanishing geodesic vector field. There are Riemannian metrics on $\mathbb{T}^2$ admitting a non-vanishing geodesic vector field, the corresponding example can be found in the class of so called double-twisted products [7]. But we are interested in metrics on $\mathbb{T}^2$ induced from the Euclidean metric of $\mathbb{R}^3$ by an embedding $i : \mathbb{T}^2 \subset \mathbb{R}^3$. Apart from surfaces of revolution, we do not know any example related to the following problem.

Problem 1 *Classify triples (i, u, λ), where $i : \mathbb{T}^2 \subset \mathbb{R}^3$ is an embedding of the torus, u is a non-vanishing geodesic vector field on $\mathbb{T}^2$ endowed with the Riemannian metric induced from the Euclidean metric of $\mathbb{R}^3$ by the embedding i, and $\lambda > 0$ is a smooth function on $\mathbb{T}^2$ satisfying the equations*

$$div\, u = u(\log \lambda), \quad I I(u, u) = -\lambda, \tag{121}$$

where $I I$ is the second quadratic form of $\mathbb{T}^2$.

Equations (121) are obtained from (8) and (13) by setting $\lambda = |\text{grad } p|$.

In this paper, we did not discuss the behavior of a GF near a critical point of the pressure. Such a discussion could be of great interest. For instance, it makes sense to study a GF (u, p) with the Morse function p, i.e., all critical points of p are non-degenerate. For such a flow, M_p is still a regular hypersurface for a regular value of the pressure; but M_p undergoes a Morse surgery when p changes near a critical value of the pressure. Which Morse surgeries are compatible with the Euler equations?

Data availability: Not applicable.

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