



Red Sizes of Quivers

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Abstract

In this article, we will expand on the notions of maximal green and reddening sequences for quivers associated with cluster algebras. The existence of these sequences has been studied for a variety of applications related to Fomin and Zelevinsky’s cluster algebras. Ahmad and Li considered a numerical measure of how close a quiver is to admitting a maximal green sequence called a *red number*. In this paper, we generalized this notion to what we call *unrestricted red numbers* which are related to reddening sequences. In addition to establishing this more general framework, we completely determine the red numbers and unrestricted red numbers for all finite mutation type of quivers. Furthermore, we give conjectures on the possible values of red numbers and unrestricted red numbers in general.

Keywords Reddening sequences · Maximal green sequences · Mutation sequence

Mathematics Subject Classification Primary 13F60

1 Introduction

Maximal green sequences were defined by Keller [15] and play a role in the theory of Fomin and Zelevinsky’s cluster algebras [11]. They also arise in representation theory, wall-crossing, and the computation of Donaldson–Thomas invariants. A survey of many results, properties, and the history of maximal green sequences can be found in [16]. While the primary focus of the study of maximal green sequences has been to

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determine when such a sequence exists, a natural question arises: if a quiver does not admit a maximal green sequence, then how close is it?

Recently, Ahmad and Li have considered a numerical measure of how close a quiver is to possessing a maximal green sequence [1]. This statistic is called the *red number* of the quiver and measures how many of the vertices in the quiver can be turned red via a sequence of green mutations. We further study this measure, as well as put forth an analogous measure related to the more general notion of reddening sequences (which are also referred to as green-to-red sequences). This more general concept of *unrestricted red numbers* measures how many of the vertices of a quiver can be turned red via any sequence of mutations.

In this paper, we will present the background definitions for quiver mutations in Sect. 2. In Sect. 3, we will establish the key concepts of this article: red numbers and unrestricted red numbers of quivers. We will present some foundational results regarding these new concepts as well as present a conjecture about the nature of (unrestricted) red numbers for arbitrary quivers. In Sect. 4, we will explicitly find the red numbers and unrestricted red numbers for quivers of finite mutation type, verifying the conjecture for finite mutation type quivers. These quivers play a vital role in the classification of cluster algebras, and the results will give definitive answers for this important family of quivers. We conclude in Sect. 5, providing some analysis of red number on quivers with few vertices and discussion of possible approaches to our conjectures using scattering diagrams and universal quivers.

2 Quiver Mutation

A *quiver* is a pair $Q = (V, E)$ where V is a set of *vertices* and E is multiset of *arrows* between two distinct vertices. That is, if $i \rightarrow j$ for $i, j \in V$ is an arrow, then we require that $i \neq j$ (i.e., no loops). Furthermore, we do not allow both $i \rightarrow j \in E$ and $j \rightarrow i \in E$ (i.e., no 2-cycles). Let us point out that E is a multiset, meaning multiple arrows between two particular vertices is permitted. Given a quiver $Q = (V, E)$, the corresponding *framed* quiver is $\hat{Q} = (V \sqcup F, E \sqcup E')$, where $F = \{i' : i \in V\}$ and $E' = \{i \rightarrow i' : i \in V\}$. In other words, the framed quiver is obtained by adding a new vertex i' for each $i \in V$ and then adding an arrow $i \rightarrow i'$. We retain the decomposition of the vertex set $V \sqcup F$. All elements of $V \sqcup F$ are vertices, but vertices in V are called *mutable*, while vertices in F are called *frozen*. The notion of a framed quiver will be essential for defining the red and green mutations.

We are now ready to define the process of quiver mutation which is a combinatorial ingredient in Fomin and Zelevinsky's cluster algebras [11]. The *mutation* of a quiver Q (framed quiver \hat{Q}) at a mutable vertex k is denoted $\mu_k(Q)$ ($\mu_k(\hat{Q})$) and is obtained by

- (1) adding an arrow $i \rightarrow j$ for any 2-path of vertices $i \rightarrow k \rightarrow j$,
- (2) reversing all arrows incident to k ,
- (3) deleting a maximal collection of 2-cycles as well as any arrow between frozen vertices.

An example of a quiver and mutation is shown in Fig. 1.

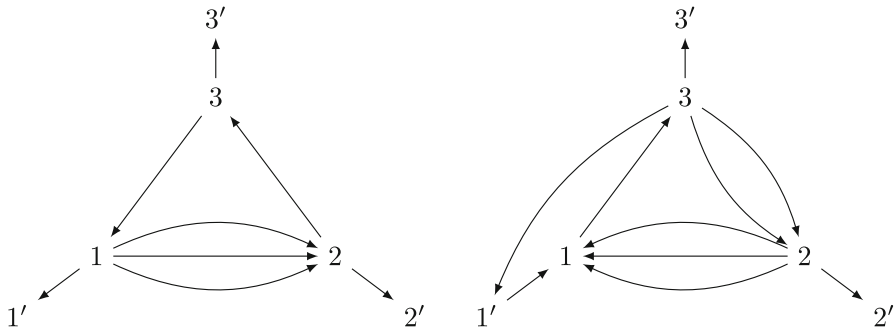


Fig. 1 A framed quiver \hat{Q} on the left and $\mu_1(\hat{Q})$ the result of a mutation on the right

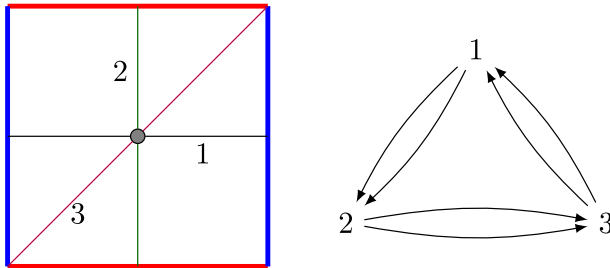


Fig. 2 An ideal triangulation of the once punctured torus on the left and its adjacency quiver on the right

Let i be a mutable vertex in $\mu_{j_\ell} \circ \dots \circ \mu_{j_2} \circ \mu_{j_1}(\hat{Q})$ for any sequence j_1, j_2, \dots, j_ℓ of mutable vertices. The vertex i is called *green* if there exists a frozen vertex j' and arrow $i \rightarrow j'$. The vertex i is called *red* if there exists a frozen vertex j' and arrow $j' \rightarrow i$. By a property known as *sign coherence*, a vertex is always green or red and never both. This is an important property in cluster algebra theory that was conjectured by Fomin and Zelevinsky [12]. It was first proven to hold by Derksen, Weyman, and Zelevinsky [7] (see [21] for another proof and [14] for a proof in a case of greater generality).

A sequence of vertices $(i_1, i_2, \dots, i_\ell)$ of a quiver Q is called a *reddening sequence* if all vertices of $\mu_{i_\ell} \circ \dots \circ \mu_{i_2} \circ \mu_{i_1}(\hat{Q})$ are red. If in addition for all $1 \leq j \leq \ell$, the vertex i_j is green in $\mu_{j-1} \circ \dots \circ \mu_{i_2} \circ \mu_{i_1}(\hat{Q})$, then the sequence $(i_1, i_2, \dots, i_\ell)$ is called a *maximal green sequence*. The definition of a maximal green sequence is due to Keller [15]. These sequences of mutations are purely combinatorial with the definition above, but they have numerous applications in cluster algebra theory and representation theory [16]. A sequence of vertices $(i_1, i_2, \dots, i_\ell)$ so that i_j is green in $\mu_{j-1} \circ \dots \circ \mu_{i_2} \circ \mu_{i_1}(\hat{Q})$ for all $1 \leq j \leq \ell$ is called a *green sequence*. With a green sequence, there is no condition on the final color of the vertices after applying the mutations.

A quiver Q is said to be of *finite mutation type* if the set of all quivers that can be obtained from Q by iteratively applying a mutation is finite. A classification of quivers of finite mutation type was completed by Felikson et al. [8]. The main source of quivers of finite mutation type is adjacency quivers of ideal triangulations of bordered marked

surfaces [10]. A triangulation of the once punctured torus and its adjacency quiver that is known as the *Markov quiver* is shown in Fig. 2. The remaining quivers of finite mutation type either have two vertices or come from 11 exceptional types. The exceptional quiver of finite mutation type that will be of the most interest to us is X_7 , which is shown in Fig. 4 and originally found by Derksen and Owen [6].

3 Red Sizes of Quivers

3.1 Red Sizes

We now give two definitions originally proposed by Ahmad and Li [1]. A *general maximal green sequence* for a quiver Q is a green sequence $(i_1, i_2, \dots, i_\ell)$ of vertices of Q , so that $\mu_{i_\ell} \cdots \mu_{i_2} \mu_{i_1}(\widehat{Q})$ has the maximal possible number of red vertices among any possible applying green sequence of mutations. The *red size* of a quiver Q is the number of red vertices in $\mu_{i_\ell} \circ \cdots \circ \mu_{i_2} \circ \mu_{i_1}(\widehat{Q})$ for any general maximal green sequence $(i_1, i_2, \dots, i_\ell)$. We let $Red(Q)$ denote the red size of Q .

Let us make the analogous definitions for reddening sequences. A *general reddening sequence* for a quiver Q is a sequence $(i_1, i_2, \dots, i_\ell)$ of vertices of Q , so that $\mu_{i_\ell} \cdots \mu_{i_2} \mu_{i_1}(\widehat{Q})$ has that maximal possible number of red vertices obtainable by applying mutations to Q . The *unrestricted red size* of a quiver Q is the number of red vertices in $\mu_{i_\ell} \circ \cdots \circ \mu_{i_2} \circ \mu_{i_1}(\widehat{Q})$, where $(i_1, i_2, \dots, i_\ell)$ is a general reddening sequence. We let $uRed(Q)$ denote the unrestricted red size of Q .

We briefly discuss some basic facts which follow immediately from these definitions. First, a quiver Q admits a maximal green sequence if and only if $Red(Q) = |V(Q)|$. Similarly, a quiver Q admits a reddening sequence if and only if $uRed(Q) = |V(Q)|$. Also, for any Q , we have that $Red(Q) \leq uRed(Q)$. Equality $Red(Q) = uRed(Q)$ can hold (e.g., any quiver which admits a maximal green sequence, take for instance an acyclic quiver [2, Lemma 2.20]). Also, strict inequality is possible (e.g., any quiver which admits a reddening sequence but not a maximal green sequence, take for instance Muller’s example [20, Fig. 1]).

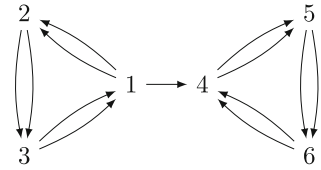
3.2 Triangular Extension

For any two quivers Q_1 and Q_2 , a *triangular extension* of Q_1 by Q_2 is any quiver Q with

$$\begin{aligned} V(Q) &= V(Q_1) \sqcup V(Q_2), \\ E(Q) &= E(Q_1) \sqcup E(Q_2) \sqcup E, \end{aligned}$$

where E is any set of arrows such that $i \rightarrow j \in E$ implies $i \in Q_1$ and $j \in Q_2$. The triangular extension plays an important role in the existence and construction of both maximal green and reddening sequences [3–5, 13]. These techniques can be adapted to construct general maximal green sequences or at least bound the red number. However, the case of general maximal green sequences behaves differently when compared with the case where a maximal green sequence exists.

Fig. 3 A triangular extension



Let Q be a triangular extension of Q_1 by Q_2 . It is known that in this case, $Red(Q) \geq Red(Q_1) + Red(Q_2)$ [1, Theorem 4.8]. Furthermore, it is true that Q admits a maximal green sequence if and only if both Q_1 and Q_2 admit maximal green sequences [5, Theorem 4.10]. It can be shown in a similar fashion that also $uRed(Q) \geq uRed(Q_1) + uRed(Q_2)$ (cf. [5, Remark 4.6]). Ahmad and Li have asked if $Red(Q) = Red(Q_1) + Red(Q_2)$, when Q is a triangular extension of Q_1 by Q_2 [1, Question 4.9]. We answer this question, as well as the analogous question for $uRed$, in the negative.

Proposition 3.1 *There exist quivers Q , Q_1 , and Q_2 such that Q is a triangular extension of Q_1 by Q_2 and $Red(Q) > Red(Q_1) + Red(Q_2)$. Similarly, there exist quivers Q , Q_1 , and Q_2 such that Q is a triangular extension of Q_1 by Q_2 and $uRed(Q) > uRed(Q_1) + uRed(Q_2)$.*

Proof We will give one example which establishes the proposition for both Red and $uRed$. Let Q be the quiver in Fig. 3 which is a triangular extension of $Q_1 = Q|_A$ and $Q_2 = Q|_B$ for $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. Here, we have that

$$Red(Q_1) = uRed(Q_1) = Red(Q_2) = uRed(Q_2) = 2,$$

since $Q|_A = Q|_B$ is the Markov quiver which is known not to admit a reddening sequence [17, Remark 3.4]. However, we find that $Red(Q) = uRed(Q) = 5$, since any one of the general maximal green sequences in Table 1 produces 5 red vertices. \square

We have found in Proposition 3.1 that red sizes do not respect triangular extensions in the same way that maximal green sequences do. In the case of red sizes, we can end up with more red vertices than one might expect. This observation leads directly into the next subsection where we conjecture it is always possible to find a sequence of mutations resulting in only 1 remaining green vertex for any connected quiver.

Table 1 General maximal green sequences for the quiver in Fig. 3

General maximal green sequence	Remaining green vertex
(3, 4, 1, 2, 4, 6, 5)	1
(1, 4, 2, 3, 5, 4, 6, 5, 6, 2, 5)	2
(3, 2, 1, 4, 5, 3, 6, 5, 6, 3, 5, 3, 5, 6)	3
(3, 1, 4, 2, 6, 5)	4
(3, 1, 4, 2, 6, 5, 4)	5
(6, 5, 4, 2, 3, 1, 3, 4, 1, 6, 3, 4)	6

3.3 Conjectures

Let us now make two conjectures. The first conjecture is in the spirit of Muller's result [20, Corollary 3.2.2.] and deals with the potential mutation invariance of the unrestricted red number.

Conjecture 3.2 *The unrestricted red size is mutation invariant. That is, $\text{uRed}(Q) = \text{uRed}(\mu_k(Q))$ for any k .*

Since the existence of a maximal green sequence is not mutation invariant [20], the above conjecture is false if $\text{uRed}(Q)$ is replaced by $\text{Red}(Q)$. We now make the second conjecture (with multiple versions of varying strength), and establishing special cases of this conjecture will be the focus of the rest of the paper. Even the weakest version of the following conjecture implies Conjecture 3.2.

Conjecture 3.3 *Q , which does not admit a reddening sequence:*

- (i) *we have $\text{Red}(Q) = |V(Q)| - 1$;*
- (ii) *we have $\text{Red}(Q) = |V(Q)| - 1$ and every vertex of Q can occur as the last remaining green vertex after the application of some general maximal green sequence;*
- (iii) *we have $\text{uRed}(Q) = |V(Q)| - 1$;*
- (iv) *we have $\text{uRed}(Q) = |V(Q)| - 1$ and every vertex of Q can occur as the last remaining green vertex after the application of some general reddening sequence.*

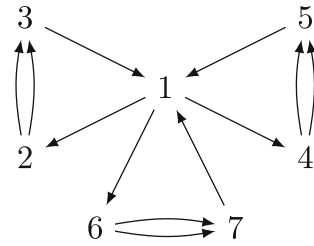
If Q is the disjoint union of two quivers which do not admit a reddening sequence, then $\text{Red}(Q) \leq \text{uRed}(Q) \leq |V(Q)| - 2$. So, the connectedness hypothesis is needed in Conjecture 3.3; otherwise, one must treat connected components of the quiver individually. Also, as previously stated it is the case that Conjecture 3.3 implies Conjecture 3.2. To see this, start by assuming part (iii) of Conjecture 3.3, which is the weakest version of the conjecture, implying that $\text{uRed}(Q) = |V(Q)|$ or $\text{uRed}(Q) = |V(Q)| - 1$ for any connected quiver Q . Note for Conjecture 3.2 we can restrict to connected quivers with out loss of generality. So, the quiver Q either admits a reddening sequence and has $\text{uRed}(Q) = |V(Q)|$ or else does not admit a reddening sequence and has $\text{uRed}(Q) = |V(Q)| - 1$. Since the existence of a reddening sequence is mutation invariant, this would imply that the unrestricted red size is also mutation invariant.

From Table 1, we can verify that part (ii) of Conjecture 3.3, which is the strongest version of the conjecture, holds for the quiver in Fig. 3. These general maximal green sequences were found with ad hoc methods, but we believe this example is instructive. One thing to observe about the quiver Q in Fig. 3 is that in $Q \setminus \{v\}$ contains the Markov quiver as an induced subquiver for any vertex v . It then follows that $Q \setminus \{v\}$ does not have a maximal green sequence [20, Theorem 1.4.1]. So, to find a general maximal green sequence leaving only the vertex v green, we must mutate at v at least once.

4 Finite Mutation Type and Red Size

It is known exactly which quivers of finite mutation type admit a maximal green sequence, which turns out to be equivalent to admitting a reddening sequence for this

Fig. 4 The quiver X_7



class of quivers, by the classification completed by Mills [18]. If a quiver Q is of finite mutation type, then $Red(Q) = |V(Q)|$ if and only if Q is not mutation equivalent to X_7 [23] or a quiver from a once punctured closed marked surface [17].

Lemma 4.1 *If Q is mutation equivalent to X_7 , then*

$$Red(Q) = uRed(Q) = 6 = |V(X_7)| - 1$$

and every vertex of X_7 is the last remaining green vertex after the application of some general maximal green sequence.

Proof It is known that neither of the two quivers in the mutation class of X_7 admit a reddening sequence. This result can be found in [23]. Thus, it suffices to provide a green sequence for each vertex v of both quivers in the mutation class of X_7 that results in 6 red vertices with v being that last remaining red vertex. Let us consider X_7 with vertices as labeled in Fig. 4. General maximal green sequences each of which has only one remaining green vertex are given in Table 2. After considering the automorphism group of the quiver X_7 we can conclude the lemma holds for X_7 . Mutating the quiver X_7 at vertex 1 gives the only other quiver in the mutation class of X_7 . For this other quiver, general maximal green sequences are shown in Table 3. \square

Table 2 General maximal green sequences for X_7

General maximal green sequence	Remaining green vertex
(2, 4, 6, 3, 5, 7)	1
(2, 4, 5, 3, 7, 1, 3, 5, 2, 1, 4, 3, 5, 1)	6
(1, 3, 2, 3, 5, 2, 3, 6, 4, 2, 6, 4, 3, 1, 2, 4, 5)	7

Table 3 General maximal green sequences for $\mu_1(X_7)$

General maximal green sequence	Remaining green vertex
(3, 5, 7, 2, 4, 6, 3, 5, 7)	1
(1, 2, 4, 5, 3, 7, 1, 3, 5, 2, 1, 4, 3, 5)	6
(3, 2, 3, 5, 2, 3, 6, 4, 2, 6, 4, 3, 1, 2, 4, 5, 2)	7

Theorem 4.2 *If Q is a quiver of finite mutation type which does not admit a reddening sequence, then*

$$\text{Red}(Q) = \text{uRed}(Q) = |V(Q)| - 1$$

and every vertex of Q is the last remaining green vertex after the application of some general maximal green sequence.

Proof The only quivers of finite mutation type which do not admit reddening sequences are X_7 and quivers which arise from ideal triangulations of once punctured closed surfaces. The case of X_7 is handled by Lemma 4.1. So, it remains to take a quiver Q such that Q is a adjacency quiver of an ideal triangulation of once punctured closed surface.

Let (S, M) be a marked surface and T any triangulation of this surface. Now, let Q_T denote the quiver associated with this triangulation. Let α be an arc in T , with v_α the associated vertex in Q_T . Since the surface S does not admit a reddening sequence, we know that it must be a closed surface with exactly one marked point. Then consider the surface obtained by cutting along α . The result is a new surface, with two additional boundary components each with a single marked point. It is possible that this cutting procedure disconnects the surface. In this case, note that each connected component of the surface has one boundary component. We will denote this new marked surface (S^α, M^α) .

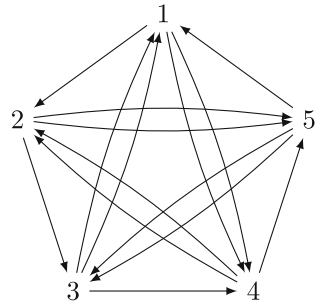
This also produces a triangulation of the surface (S^α, M^α) , which we will denote T^α . Now, we consider the associated quiver Q_{T^α} . This is exactly the induced subquiver of Q_T obtained by deleting the vertex v_α . Obtaining a quiver from a triangulation of different surface by deleting in this way is a known process and can be constructed in more generality when considering surfaces that have boundary components and additional punctures (see e.g., [19, Lemma 9.10]).

Since Q_{T^α} is a quiver associated with a triangulation of a surface with at least one boundary component (the one created in the above procedure), it admits a maximal green sequence by [18]. We will denote the composition of mutations corresponding to this sequence by σ .

We will now apply σ to Q_T . This will be a sequence of green mutations, and the resulting quiver, $\sigma(Q_T)$, will have exactly one green vertex namely v_α . Hence, $\text{uRed}(Q_T) \geq \text{Red}(Q_T) \geq n - 1$ and we have $\text{uRed}(Q_T) = \text{Red}(Q_T) = n - 1$. Moreover, the vertex v_α that remains green can be chosen arbitrarily, as this cutting procedure can occur for any arc, α , in the triangulation T . Hence, the theorem is proven. \square

Remark 4.3 The general maximal green sequences constructed for X_7 and adjacency quivers of ideal triangulations of once punctured closed surfaces never mutate at the one vertex that remains green. This means that every proper induced subquiver admits a maximal green sequence. Therefore with respect to taking induced subquivers, finite mutation type quivers not admitting maximal green sequence are minimal examples of quivers not admitting maximal green sequences.

Fig. 5 The McKay quiver which has red size equal to 4



We have the following immediate corollary since we have now verified Conjecture 3.3 for quivers of finite mutation type and have already discussed how the conjecture implies Conjecture 3.2.

Corollary 4.4 *If Q is a quiver of finite mutation type, then*

$$Red(Q) = uRed(Q) = uRed(\mu_k(Q)) = Red(\mu_k(Q))$$

for mutation in any direction k .

5 Conclusion

We have proven Conjecture 3.2 and the strongest version of Conjecture 3.3 for quivers of finite mutation type. To conclude, we discuss a few more cases our conjectures hold along with a few approaches to our conjectures in general. Also, we indicate where some potential obstacles are encountered.

5.1 Quivers with Few Vertices

Now, we verify that $Red(Q) \geq |V(Q)| - 1$ for quivers with $1 \leq |V(Q)| \leq 4$. In each case, we are able to find a sufficiently large acyclic induced subquiver.

Proposition 5.1 *If $|V(Q)| \leq 4$, then $Red(Q) \geq |V(Q)| - 1$.*

Proof If $|Q| = 1$ or $|Q| = 2$, then Q is acyclic and has a maximal green sequence. If $|Q| = 3$, then choosing a maximal green sequence in any induced subquiver on 2 vertices will demonstrate that $Red(Q) \geq 2$. If $|Q| = 4$, then Q must contain an induced acyclic subquiver on 3 vertices. A maximal green sequence for this induced acyclic subquiver on 3 vertices will show $Red(Q) \geq 3$. \square

Let us give one more small example involving a quiver which does not have a maximal green sequence, but for which we can verify we can mutate all but one vertex to red.

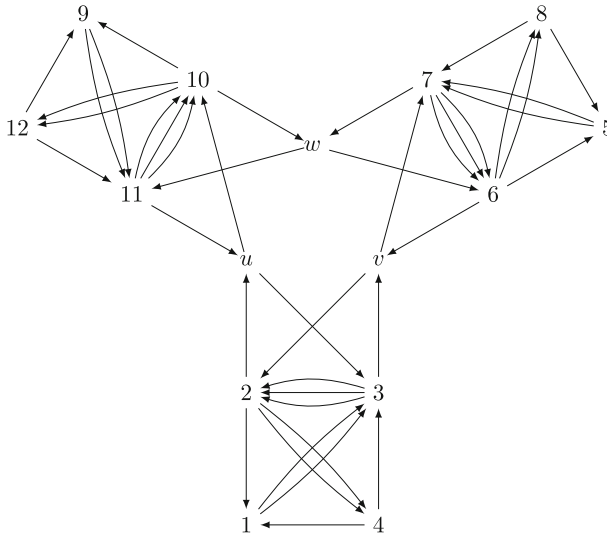


Fig. 6 A 3-universal quiver of Fomin et al. [9]

Example 5.2 In Fig. 5, we have the McKay quiver, which is a quiver that Brüstle, Dupont, and Pérotin have shown does not admit a maximal green sequence [2, Example 8.2]. A general maximal green sequence for this quiver is

$$(1, 4, 2, 5, 3),$$

which results in only the vertex 2 remaining green. By the symmetry of the quiver, we can conclude any vertex is the last remaining green vertex of some general maximal green sequence.

5.2 Scattering Diagrams and Universal Quivers

The (cluster) scattering diagrams of Gross et al. are a powerful tool that have been used to prove many important results in cluster algebra theory [14]. Using scattering diagrams, Muller [20] was able to show if a quiver admits a maximal green (reddening) sequence, then any induced subquiver also admits a maximal green (reddening) sequence. Additionally, it was established by Muller using scattering diagrams that the existence of a reddening sequences is a mutation invariant. The fact that the chamber of the scattering diagram corresponding to the case when all the vertices are red is distinguished as the all negative orthant was used in showing these results.

It would be desirable to show the mutation invariance in Conjecture 3.2 as well as a statement of the form: if $\text{Red}(Q) \geq |V(Q)| - \alpha$ for some constant α , then $\text{Red}(Q') \geq |V(Q')| - \alpha$ whenever Q' is an induced subquiver of Q (or the analogous statement with uRed in place of Red). We have not been able to obtain such a result due in part to the fact that without a reddening sequence, we do not end at the all negative

orthant. Rather, the corresponding path ends in some chamber spanned by a collection of g -vectors we have less control over. Moreover, what prevents a maximal green or reddening sequence from existing are complicated regions of the scattering diagram which contain infinitely many walls. Such regions are still not fully understood even for quivers with few vertices (see e.g., [22, Sect. 6.7]).

Provided we had such a statement with $\alpha = 1$, we could approach proving our conjecture by attempting to verify it on a *universal collection* of quivers as defined by Fomin, Igusa, and Lee [9]. A universal collection of quivers has the property that every quiver is an induced subquiver of a quiver mutation equivalent to some quiver in the collection. In any case, universal collections give a source of quivers which necessarily do not admit reddening sequences. An example of a universal quiver in given in Fig. 6.

We can produce general maximal green sequences for this quiver, leaving only one vertex green, but we do not have a systematic approach. The resulting quiver after a general maximal green sequence can be complicated unlike the case for a maximal green sequence which is related to the initial quiver in a simple manner [2, Proposition 2.10]. In fact, it empirically appears that by iteratively performing mutation at random green vertices will result in a single remaining green vertex.

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