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RESEARCH CONTRIBUTION

Solvability of Some Systems of Integro-differential Equations in Population Dynamics Depending on the Natality and Mortality Rates

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Abstract
We establish the existence of stationary solutions for certain systems of reaction–diffusion-type equations in the corresponding H^2 spaces. Our method relies on the fixed point theorem when the elliptic problem contains second-order differential operators with and without the Fredholm property, which may depend on the outcome of the competition between the natality and the mortality rates involved in the equations of the systems.

Keywords Solvability conditions · Non-Fredholm operators · Systems of integro-differential equations · Stationary solutions

Mathematics Subject Classification 35R09 · 35A01 · 35J91 · 35K91

1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F has the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the problem $Lu = f$ is solvable if and only if $\phi_k(f) = 0$ for a finite number of functionals ϕ_k from the dual space F^*. These properties of the Fredholm operators are widely used in various methods of the linear and nonlinear analysis.

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Elliptic equations considered in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property when the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see, e.g., [1, 19, 21]), which is the main result of the theory of linear elliptic problems. When dealing with unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, for the Laplace operator, $Lu = \Delta u$ considered in $\mathbb{R}^d$ the Fredholm property does not hold when the problem is studied either in Hölder spaces, such that $L : \mathbb{C}^{2+\alpha}(\mathbb{R}^d) \to \mathbb{C}^{\alpha}(\mathbb{R}^d)$ or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.
For linear elliptic equations studied in unbounded domains the Fredholm property is satisfied if and only if, in addition to the conditions stated above, the limiting operators are invertible (see [22]). In some simple cases, the limiting operators can be constructed explicitly. For example, when

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

with the coefficients of the operator having limits at infinity,

$$a_{\pm} = \lim_{x\to\pm\infty} a(x), \quad b_{\pm} = \lim_{x\to\pm\infty} b(x), \quad c_{\pm} = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are given by

$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Since the coefficients here are constants, the essential spectrum of the operator, which is the set of complex numbers λ for which the operator $L - \lambda$ does not possess the Fredholm property, can be found explicitly via the standard Fourier transform, such that

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

The limiting operators are invertible if and only if the origin does not belong to the essential spectrum.
For the general elliptic equations the analogous assertions are valid. The Fredholm property is satisfied if the essential spectrum does not contain the origin or when the limiting operators are invertible. Such conditions may not be written explicitly.
For the non-Fredholm operators, we may not apply the standard solvability conditions and in a general case the solvability relations are not known. However, the solvability conditions were obtained recently for some classes of operators. For example, consider the following problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, $d \in \mathbb{N}$ with a positive constant a. Here, the operator L and its limiting operators coincide. The corresponding homogeneous problem has a nontrivial bounded solution, such that the Fredholm property is not satisfied. Since the differential operator involved in (1.1) has constant coefficients, we are able to find the solution explicitly by applying the standard Fourier transform. In Lemmas 5 and 6 of [31], we derived the following solvability relations. Let $f(x) \in L^2(\mathbb{R}^d)$ and $xf(x) \in L^1(\mathbb{R}^d)$. Then, equation (1.1) has a unique solution in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

Here and below, S^d_r denotes the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Thus, although the Fredholm property is not satisfied for the operator, we are able to formulate the solvability relations similarly. Note that this similarity is only formal because the range of the operator is not closed. In the case when the operator involves a scalar potential, such that

$$Hu \equiv \Delta u + b(x)u = f,$$

we cannot apply the standard Fourier transform directly. However, the solvability relations in three dimensions can be obtained by virtue of the spectral and the scattering theory of Schrödinger type operators (see [27]). As in the constant coefficient case, the solvability conditions are expressed in terms of orthogonality to the solutions of the adjoint homogeneous equation. The solvability relations for several other examples of non-Fredholm linear elliptic operators were derived (see [11, 13, 22–24, 26, 27, 29–31]).
Solvability conditions are crucial in the analysis of the nonlinear elliptic problems. When then non-Fredholm operators are involved, in spite of some progress in the studies of the linear equations, the nonlinear non-Fredholm operators were analyzed only in few examples (see [6–8, 10, 12–14, 28, 31]). Evidently, this situation can be explained by the fact that the majority of the methods of linear and nonlinear analysis rely on the Fredholm property. Fredholm structures, topological invariants and applications were covered in [9]. In the present article, we study certain systems of nonlinear integro-differential reaction–diffusion-type equations, for which the Fredholm property may not be satisfied:

$$\frac{\partial u_k}{\partial t} = \Delta u_k + \int_\Omega G_k(x-y)F_k(u_1(y,t), u_2(y,t), \dots, u_{N_2}(y,t), y)dy + a_k u_k \tag{1.2}$$

for $1 \le k \le N_1$ and

$$\frac{\partial u_k}{\partial t} = \Delta u_k + \int_\Omega G_k(x-y)F_k(u_1(y,t), u_2(y,t), \dots, u_{N_2}(y,t), y)dy - a_k u_k \tag{1.3}$$

for $N_1+1 \le k \le N_2$. Here, $\{a_k\}_{k=1}^{N_2}$ are nonnegative if $1 \le k \le N_1$ and they are strictly positive when $N_1 + 1 \le k \le N_2$. Our domain $\Omega \subseteq \mathbb{R}^d$, $\quad d = 1, 2, 3$ which are the more physically relevant dimensions. Note that equations (1.2) describe the situation in the Population Dynamics in the Mathematical Biology when the natality rates are higher than the mortality ones for $a_k > 0$ and the cases when the mortality and natality rates balance each other for $a_k = 0$. On the other hand, equations (1.3) are important for understanding the situation when the mortality rates are higher than the natality ones. In the Population Dynamics the integro-differential problems are used to describe biological systems with the intra-specific competition and the nonlocal consumption of resources (see, e.g., [2, 4, 15, 25]). The stability issues for traveling fronts of reaction–diffusion-type equations with the essential spectrum of the linearized operator crossing the imaginary axis were also addressed in [3, 16]. Note that the single equation of (1.2) type has been studied in [28]. The reaction–diffusion-type problems in which in the diffusion term the Laplacian is replaced by the nonlocal operator with an integral kernel were discussed in [20]. Let us introduce

$$\mathcal{F}(u, x) := (F_1(u, x), F_2(u, x), \dots, F_{N_2}(u, x))^T.$$

The nonlinear terms of our system (1.2), (1.3) will satisfy the following regularity requirements.

Assumption 1.1 Let $1 \le k \le N_2$. Functions $F_k(u, x) : \mathbb{R}^{N_2} \times \Omega \to \mathbb{R}$ are satisfying the Caratheodory condition (see [18]), so that

$$|\mathcal{F}(u, x)|_{\mathbb{R}^{N_2}} \le K|u|_{\mathbb{R}^{N_2}} + h(x) \quad for \quad u \in \mathbb{R}^{N_2}, \quad x \in \Omega, \tag{1.4}$$

with a constant $K > 0$ and $h(x) : \Omega \to \mathbb{R}^+, \quad h(x) \in L^2(\Omega)$. Furthermore, they are Lipschitz continuous functions, so that for any $u^{(1),(2)} \in \mathbb{R}^{N_2}, \quad x \in \Omega :$

$$|\mathcal{F}(u^{(1)}, x) - \mathcal{F}(u^{(2)}, x)|_{\mathbb{R}^{N_2}} \le L|u^{(1)} - u^{(2)}|_{\mathbb{R}^{N_2}}, \tag{1.5}$$

with a constant $L > 0$.

Here and below, we use the notations for a vector $u := (u_1, u_2, \dots, u_{N_2})^T \in \mathbb{R}^{N_2}$ and its norm $|u|_{\mathbb{R}^{N_2}} := \sqrt{\sum_{k=1}^{N_2} u_k^2}$. The solvability of a local elliptic equation in a bounded domain in $\mathbb{R}^N$ was studied in [5]. The nonlinear function there was allowed to have a sublinear growth. Clearly, the stationary solutions of system (1.2), (1.3), if any exist, will satisfy the system of nonlocal elliptic equations

$$\Delta u_k + \int_\Omega G_k(x - y)F_k(u_1(y), u_2(y), \dots, u_{N_2}(y), y)dy + a_k u_k = 0, \quad a_k \ge 0,$$

for $1 \le k \le N_1$,

$$\Delta u_k + \int_\Omega G_k(x - y)F_k(u_1(y), u_2(y), \dots, u_{N_2}(y), y)dy - a_k u_k = 0, \quad a_k > 0$$

for $N_1 + 1 \le k \le N_2$. For the technical purposes, we introduce the auxiliary semi-linear problem

$$-\Delta u_k - a_k u_k = \int_\Omega G_k(x-y) F_k(v_1(y), v_2(y), \dots, v_{N_2}(y), y) dy, \quad a_k \ge 0 \tag{1.6}$$

if $1 \le k \le N_1$,

$$-\Delta u_k + a_k u_k = \int_\Omega G_k(x-y) F_k(v_1(y), v_2(y), \dots, v_{N_2}(y), y) dy, \quad a_k > 0 \tag{1.7}$$

if $N_1 + 1 \le k \le N_2$. We designate

$$(f_1(x), f_2(x))_{L^2(\Omega)} := \int_\Omega f_1(x) \bar{f}_2(x) dx, \tag{1.8}$$

with a slight abuse of notations in the case when these functions are not square integrable in Ω, like for example those used in the orthogonality relations of the assumption below. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x) \in L^\infty(\Omega)$, then the integral in the right side of (1.8) is well defined. Let us begin the article with the studies of the whole space case, such that $\Omega = \mathbb{R}^d$ and the corresponding Sobolev space is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})} := \sum_{k=1}^{N_2} \|u_k\|^2_{H^2(\mathbb{R}^d)} = \sum_{k=1}^{N_2} \{\|u_k\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u_k\|^2_{L^2(\mathbb{R}^d)}\},$$

where $u(x) : \mathbb{R}^d \to \mathbb{R}^{N_2}$. The primary obstacle in solving system (1.6), (1.7) is that operators $-\Delta - a_k : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),\ a_k \ge 0$ involved in its first part fail to satisfy the Fredholm property. The analogous situations in linear equations, which can be self-adjoint or non self-adjoint containing the non-Fredholm second-, fourth- and sixth-order differential operators or even systems of equations including the non-Fredholm operators have been studied actively in recent years (see [11, 24, 26, 27, 29, 30]). We manage to establish that system of equations (1.6), (1.7) defines a map $T_a : H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2}) \to H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})$, which is a strict contraction under the stated technical conditions. It gives a solution of the considered problem. The fact that this map is well defined is established in the proof of Theorem 1.3 below. We make the following assumption on the integral kernels involved in the nonlocal parts of system (1.6), (1.7).

Assumption 1.2 Let $G_k(x) : \mathbb{R}^d \to \mathbb{R},\ G_k(x) \in L^1(\mathbb{R}^d),\ 1 \le k \le N_2,\ 1 \le d \le 3$ and $1 \le m \le N_1 - 1,\ m \in \mathbb{N}$ with $N_1 \ge 2,\ N_2 > N_1$.

I) Let $a_k > 0,\ 1 \le k \le m$, assume that $xG_k(x) \in L^1(\mathbb{R}^d)$ and

$$\left(G_k(x), \frac{e^{\pm i\sqrt{a_k}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0 \quad if \quad d = 1, \tag{1.9}$$

$$\left(G_k(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0 \quad for \quad p \in S^d_{\sqrt{a_k}} \quad if \quad d = 2, 3. \tag{1.10}$$

II) Let $a_k = 0,\ m + 1 \le k \le N_1$, assume that $x^2 G_k(x) \in L^1(\mathbb{R}^d)$ and

$$(G_k(x), 1)_{L^2(\mathbb{R}^d)} = 0 \quad and \quad (G_k(x), x_s)_{L^2(\mathbb{R}^d)} = 0, \quad 1 \le s \le d. \tag{1.11}$$

III) Let $a_k > 0,\ N_1 + 1 \le k \le N_2$.
Let us use the hat symbol here and below to denote the standard Fourier transform, so that

$$\widehat{G_k}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G_k(x) e^{-ipx} dx, \quad p \in \mathbb{R}^d. \tag{1.12}$$

Thus,

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}}} \|G_k\|_{L^1(\mathbb{R}^d)}. \tag{1.13}$$

We introduce the following auxiliary quantities

$$M_k := \max\left\{\left\|\frac{\widehat{G_k}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}, \left\|\frac{p^2\widehat{G_k}(p)}{p^2 - a_k}\right\|_{L^\infty(\mathbb{R}^d)}\right\}, \quad 1 \le k \le m, \tag{1.14}$$

$$M_k := \max\left\{\left\|\frac{\widehat{G_k}(p)}{p^2}\right\|_{L^\infty(\mathbb{R}^d)}, \left\|\widehat{G_k}(p)\right\|_{L^\infty(\mathbb{R}^d)}\right\}, \quad m + 1 \le k \le N_1, \tag{1.15}$$

$$M_k := \max\left\{\left\|\frac{\widehat{G_k}(p)}{p^2 + a_k}\right\|_{L^\infty(\mathbb{R}^d)}, \left\|\frac{p^2\widehat{G_k}(p)}{p^2 + a_k}\right\|_{L^\infty(\mathbb{R}^d)}\right\}, \quad N_1 + 1 \le k \le N_2. \tag{1.16}$$

Evidently, expressions (1.14) and (1.15) are finite by virtue of Lemma A1 in one dimension and Lemma A2 for $d = 2, 3$ of [28] under Assumption 1.2 above. It can be easily verified that (1.16) are finite as well. Indeed, for $N_1 + 1 \le k \le N_2$ using (1.13), we have

$$\left|\frac{\widehat{G_k}(p)}{p^2 + a_k}\right| \le \frac{|\widehat{G_k}(p)|}{a_k} \le \frac{1}{(2\pi)^{\frac{d}{2}} a_k} \|G_k\|_{L^1(\mathbb{R}^d)} < \infty$$

as assumed. Similarly, via (1.13)

$$\left|\frac{p^2\widehat{G_k}(p)}{p^2+a_k}\right| \le |\widehat{G_k}(p)| \le \frac{1}{(2\pi)^{\frac{d}{2}}}\|G_k\|_{L^1(\mathbb{R}^d)} < \infty,$$

such that $M_k < \infty$ for $N_1+1 \le k \le N_2$ as well. This enables us to define

$$M := \max M_k, \quad 1 \le k \le N_2, \tag{1.17}$$

with M_k given by (1.14), (1.15) and (1.16). We have the following statement.

Theorem 1.3 *Let $\Omega = \mathbb{R}^d$, $d = 1, 2, 3$, Assumptions 1.1 and 1.2 hold and $\sqrt{2}(2\pi)^{\frac{d}{2}}ML < 1$.*
Then, the map $T_a v = u$ on $H^2(\mathbb{R}^d, \mathbb{R}^{N_2})$ defined by the system of equations (1.6), (1.7) has a unique fixed point $v_a(x) : \mathbb{R}^d \to \mathbb{R}^{N_2}$, which is the only stationary solution of problem (1.2), (1.3) in $H^2(\mathbb{R}^d, \mathbb{R}^{N_2})$.
This fixed point $v_a(x)$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)}(p) \cap supp\widehat{G_k}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for a certain $1 \le k \le N_2$.

Let us turn our attention to the studies of the analogous system of equations on the interval $\Omega = I := [0, 2\pi]$ with periodic boundary conditions for the solution vector function and its first derivative. We assume the following about the integral kernels involved in the nonlocal parts of problem (1.6), (1.7) in this case.

Assumption 1.4 Let $G_k(x) : I \to \mathbb{R}$, $G_k(x) \in C(I)$ with $G_k(0) = G_k(2\pi)$, $1 \le k \le N_2$, where $N_1 \ge 3$, $1 \le m < q \le N_1 - 1$, $m, q \in \mathbb{N}$ and $N_2 > N_1$.
I) Let $a_k > 0$ and $a_k \ne n^2$, $\quad n \in \mathbb{Z} \quad$ if $\quad 1 \le k \le m$.
II) Let $a_k = n_k^2$, $n_k \in \mathbb{N}$ and

$$\left(G_k(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}}\right)_{L^2(I)} = 0 \quad if \quad m+1 \le k \le q. \tag{1.18}$$

III) Let $a_k = 0$ and

$$(G_k(x), 1)_{L^2(I)} = 0 \quad for \quad q+1 \le k \le N_1. \tag{1.19}$$

IV) Let $a_k > 0$, $N_1 + 1 \le k \le N_2$.
Let $F_k(u, 0) = F_k(u, 2\pi)$ for $u \in \mathbb{R}^{N_2}$ and $1 \le k \le N_2$.
For the function on our $[0, 2\pi]$ interval, $G_k(x) : I \to \mathbb{R}$, $G_k(0) = G_k(2\pi)$, we introduce the Fourier transform as

$$G_{k,\,n} := \int_0^{2\pi} G_k(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n \in \mathbb{Z}, \tag{1.20}$$

such that

$$G_k(x) = \sum_{n=-\infty}^{\infty} G_{k,\,n} \frac{e^{inx}}{\sqrt{2\pi}}.$$

Clearly, the estimate

$$\|G_{k,\,n}\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}} \|G_k\|_{L^1(I)} \tag{1.21}$$

is valid. Let us define the following technical expressions

$$P_k := \max\left\{ \left\| \frac{G_{k,\,n}}{n^2 - a_k} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,\,n}}{n^2 - a_k} \right\|_{l^\infty} \right\}, \quad 1 \le k \le m, \tag{1.22}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,\,n}}{n^2 - n_k^2} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,\,n}}{n^2 - n_k^2} \right\|_{l^\infty} \right\}, \quad m+1 \le k \le q, \tag{1.23}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,\,n}}{n^2} \right\|_{l^\infty}, \left\| G_{k,\,n} \right\|_{l^\infty} \right\}, \quad q+1 \le k \le N_1, \tag{1.24}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,\,n}}{n^2 + a_k} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,\,n}}{n^2 + a_k} \right\|_{l^\infty} \right\}, \quad N_1+1 \le k \le N_2. \tag{1.25}$$

By virtue of Lemma A3 of [28] under our Assumption 1.4, the quantities given by (1.22), (1.23) and (1.24) are finite. It can be trivially checked that (1.25) are finite as well. Evidently, for $N_1 + 1 \le k \le N_2$ by means of (1.21), we obtain

$$\left| \frac{G_{k,\,n}}{n^2 + a_k} \right| \le \frac{|G_{k,\,n}|}{a_k} \le \frac{1}{a_k\sqrt{2\pi}} \|G_k\|_{L^1(I)} < \infty$$

via one of our assumptions. Using (1.21), we derive

$$\left| \frac{n^2 G_{k,\,n}}{n^2 + a_k} \right| \le |G_{k,\,n}| \le \frac{1}{\sqrt{2\pi}} \|G_k\|_{L^1(I)} < \infty.$$

Thus, $P_k < \infty$ if $N_1 + 1 \le k \le N_2$ as well. This allows us to define

$$P := \max P_k, \quad 1 \le k \le N_2 \tag{1.26}$$

with P_k given by formulas (1.22), (1.23), (1.24) and (1.25). For the purpose of the studies of the existence of stationary solutions of our system, we use the corresponding function space

$$H^2(I) = \{v(x) : I \to \mathbb{R} \mid v(x), v''(x) \in L^2(I), \quad v(0) = v(2\pi), \quad v'(0) = v'(2\pi)\},$$

aiming at $u_k(x) \in H^2(I),\ 1 \le k \le m$ and $N_1 + 1 \le k \le N_2$ as well. Let us use the following auxiliary constrained subspaces

$$H_k^2(I) := \left\{ v \in H^2(I) \mid \left(v(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}} \right)_{L^2(I)} = 0 \right\}, \quad n_k \in \mathbb{N}, \quad m+1 \le k \le q,$$

with the goal of having $u_k(x) \in H_k^2(I),\ m+1 \le k \le q$. Also,

$$H_0^2(I) := \{ v \in H^2(I) \mid (v(x), 1)_{L^2(I)} = 0 \}.$$

Our goal is to have $u_k(x) \in H_0^2(I),\ q+1 \le k \le N_1$. The constrained subspaces introduced above are Hilbert spaces as well (see, e.g., Chapter 2.1 of [17]). The resulting space used for establishing the existence of solutions $u(x) : I \to \mathbb{R}^{N_2}$ of system (1.6), (1.7) will be the direct sum of the spaces given above, so that

$$H_c^2(I,\ \mathbb{R}^{N_2}) := \oplus_{k=1}^{m} H^2(I) \oplus_{k=m+1}^{q} H_k^2(I) \oplus_{k=q+1}^{N_1} H_0^2(I) \oplus_{k=N_1+1}^{N_2} H^2(I).$$

The corresponding Sobolev norm is given by

$$\|u\|^2_{H_c^2(I,\ \mathbb{R}^{N_2})} := \sum_{k=1}^{N_2} \{ \|u_k\|^2_{L^2(I)} + \|u_k''\|^2_{L^2(I)} \}$$

with $u(x) : I \to \mathbb{R}^{N_2}$. We establish that the system of equations (1.6), (1.7) in such case defines a map on the space given above, which will be a strict contraction under the stated conditions.

Theorem 1.5 *Let $\Omega = I$, Assumptions 1.1 and 1.4 hold and $2\sqrt{\pi} PL < 1$.*
Then, the map $\tau_a v = u$ on $H_c^2(I,\ \mathbb{R}^{N_2})$ defined by the system of equations (1.6), (1.7) has a unique fixed point $v_a(x) : I \to \mathbb{R}^{N_2}$, the only stationary solution of problem (1.2), (1.3) in $H_c^2(I,\ \mathbb{R}^{N_2})$.
This fixed point $v_a(x)$ is nonzero provided the Fourier coefficients $G_{k,\ n} F_k(0, x)_n \ne 0$ for a certain $1 \le k \le N_2$ and some $n \in \mathbb{Z}$.

Note that the constrained subspaces $H_k^2(I)$ and $H_0^2(I)$ involved in the direct sum of spaces $H_c^2(I, \mathbb{R}^{N_2})$ are such that the Fredholm operators

$$-\frac{d^2}{dx^2} - n_k^2 : H_k^2(I) \to L^2(I) \quad \textit{and} \quad -\frac{d^2}{dx^2} : H_0^2(I) \to L^2(I)$$

possess the trivial kernels.

We conclude the article with the studies of our system of equations in the layer domain, which is the product of the two sets, such that one is the I interval with periodic boundary conditions as in the previous part of the work and another is the whole space of dimension either one or two, namely $\Omega = I \times \mathbb{R}^d = [0, 2\pi] \times \mathbb{R}^d,\ d = 1, 2$ and $x = (x_1, x_\perp)$, with $x_1 \in I$ and $x_\perp \in \mathbb{R}^d$. The cumulative Laplacian in this context will be $\Delta := \frac{\partial^2}{\partial x_1^2} + \sum_{s=1}^{d} \frac{\partial^2}{\partial x_{\perp,\ s}^2}$. The corresponding Sobolev space for our problem will be $H^2(\Omega,\ \mathbb{R}^{N_2})$ of vector functions $u(x) : \Omega \to \mathbb{R}^{N_2}$, so that for $1 \leq k \leq N_2$

$$u_k(x),\ \Delta u_k(x) \in L^2(\Omega), \quad u_k(0, x_\perp) = u_k(2\pi, x_\perp), \quad \frac{\partial u_k}{\partial x_1}(0, x_\perp) = \frac{\partial u_k}{\partial x_1}(2\pi, x_\perp),$$

with $x_\perp \in \mathbb{R}^d$. It is equipped with the norm

$$\|u\|^2_{H^2(\Omega,\ \mathbb{R}^{N_2})} = \sum_{k=1}^{N_2} \{\|u_k\|^2_{L^2(\Omega)} + \|\Delta u_k\|^2_{L^2(\Omega)}\}.$$

Analogously to the whole space case discussed in Theorem 1.3, the operators $-\Delta - a_k : H^2(\Omega) \to L^2(\Omega)$ for $a_k \geq 0,\ 1 \leq k \leq N_1$ do not possess the Fredholm property. Let us establish that problem (1.6), (1.7) in such case defines a map $t_a : H^2(\Omega,\ \mathbb{R}^{N_2}) \to H^2(\Omega,\ \mathbb{R}^{N_2})$, which is a strict contraction under the appropriate technical conditions stated below.

Assumption 1.6 Let $G_k(x) : \Omega \to \mathbb{R},\ G_k(x) \in C(\Omega) \cap L^1(\Omega),\ G_k(0, x_\perp) = G_k(2\pi, x_\perp)$ and $F_k(u, 0, x_\perp) = F_k(u, 2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$, $u \in \mathbb{R}^{N_2},\ d = 1, 2$ and $1 \leq k \leq N_2$. Let $N_1 \geq 3,\ 1 \leq m < q \leq N_1 - 1$ with $m, q \in \mathbb{N}$ and $N_2 > N_1$.
I) Assume for $1 \leq k \leq m$, we have $n_k^2 < a_k < (n_k + 1)^2,\ n_k \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\},\ x_\perp G_k(x) \in L^1(\Omega)$ and

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{a_k - n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \leq n_k \quad for \quad d = 1, \tag{1.27}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{a_k - n^2}}, \quad |n| \leq n_k \quad for \quad d = 2. \tag{1.28}$$

II) Assume for $m + 1 \leq k \leq q$ we have $a_k = n_k^2,\ n_k \in \mathbb{N},\ x_\perp^2 G_k(x) \in L^1(\Omega)$ and

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{n_k^2 - n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \leq n_k - 1 \ for \ d = 1, \tag{1.29}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{n_k^2 - n^2}}, \quad |n| \leq n_k - 1 \quad for \quad d = 2, \tag{1.30}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad \left(G_k(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}} x_{\perp,\ s}\right)_{L^2(\Omega)} = 0, \tag{1.31}$$

with $1 \leq s \leq d$.

III) Assume for $q+1 \le k \le N_1$ we have $a_k = 0, \quad x_\perp^2 G_k(x) \in L^1(\Omega)$ and

$$(G_k(x), 1)_{L^2(\Omega)} = 0, \quad (G_k(x), x_{\perp,\, s})_{L^2(\Omega)} = 0, \quad 1 \le s \le d. \tag{1.32}$$

IV) Let $a_k > 0,\ N_1 + 1 \le k \le N_2$.

Let us use the Fourier transform for the functions on such a product of sets, namely

$$\widehat{G_{k,\, n}}(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G_k(x_1, x_\perp) e^{-inx_1} dx_1 \tag{1.33}$$

with $p \in \mathbb{R}^d,\ n \in \mathbb{Z},\ 1 \le k \le N_2$. Clearly,

$$\|\widehat{G_{k,\, n}}(p)\|_{L^\infty_{n,p}} := \sup_{\{p \in \mathbb{R}^d,\ \ n \in \mathbb{Z}\}} |\widehat{G_{k,\, n}}(p)| \le \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G_k\|_{L^1(\Omega)}. \tag{1.34}$$

We introduce the following auxiliary expressions

$$R_k := \max\left\{ \left\| \frac{\widehat{G_{k,\, n}}(p)}{p^2 + n^2 - a_k} \right\|_{L^\infty_{n,p}}, \ \left\| \frac{(p^2+n^2)\widehat{G_{k,\, n}}(p)}{p^2 + n^2 - a_k} \right\|_{L^\infty_{n,p}} \right\}, \quad 1 \le k \le m, \tag{1.35}$$

$$R_k := \max\left\{ \left\| \frac{\widehat{G_{k,\, n}}(p)}{p^2 + n^2 - n_k^2} \right\|_{L^\infty_{n,p}}, \ \left\| \frac{(p^2+n^2)\widehat{G_{k,\, n}}(p)}{p^2 + n^2 - n_k^2} \right\|_{L^\infty_{n,p}} \right\}, \quad m+1 \le k \le q, \tag{1.36}$$

$$R_k := \max\left\{ \left\| \frac{\widehat{G_{k,\, n}}(p)}{p^2 + n^2} \right\|_{L^\infty_{n,p}}, \ \left\| \widehat{G_{k,\, n}}(p) \right\|_{L^\infty_{n,p}} \right\}, \quad q+1 \le k \le N_1, \tag{1.37}$$

$$R_k := \max\left\{ \left\| \frac{\widehat{G_{k,\, n}}(p)}{p^2 + n^2 + a_k} \right\|_{L^\infty_{n,p}}, \ \left\| \frac{(p^2+n^2)\widehat{G_{k,\, n}}(p)}{p^2 + n^2 + a_k} \right\|_{L^\infty_{n,p}} \right\}, \quad N_1+1 \le k \le N_2. \tag{1.38}$$

Assumption 1.6 along with Lemmas A4, A5 and A6 of [28] yield that the quantities given by (1.35), (1.36) and (1.37) are finite. It can be trivially checked that (1.38) are finite as well. Obviously, for $N_1 + 1 \le k \le N_2$ by virtue of (1.34) we have

$$\left| \frac{\widehat{G_{k,\, n}}(p)}{p^2 + n^2 + a_k} \right| \le \frac{|\widehat{G_{k,\, n}}(p)|}{a_k} \le \frac{1}{(2\pi)^{\frac{d+1}{2}} a_k} \|G_k\|_{L^1(\Omega)} < \infty$$

due to one of our assumptions. Using (1.34), we arrive at

$$\left| \frac{(p^2+n^2)\widehat{G_{k,\, n}}(p)}{p^2 + n^2 + a_k} \right| \le |\widehat{G_{k,\, n}}(p)| \le \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G_k\|_{L^1(\Omega)} < \infty,$$

so that $R_k < \infty$ for $N_1 + 1 \le k \le N_2$ as well. Thus, we can define

$$R := \max R_k, \quad 1 \le k \le N_2, \tag{1.39}$$

with R_k given in (1.35), (1.36), (1.37) and (1.38). Our final proposition is as follows.

Theorem 1.7 *Let $\Omega = I \times \mathbb{R}^d$, $d = 1, 2$, Assumptions 1.1 and 1.6 hold and $\sqrt{2}(2\pi)^{\frac{d+1}{2}} RL < 1$. Then the map $t_a v = u$ on $H^2(\Omega, \mathbb{R}^{N_2})$ defined by system (1.6), (1.7) has a unique fixed point $v_a(x) : \Omega \to \mathbb{R}^{N_2}$, which is the only stationary solution of the system of equations (1.2), (1.3) in $H^2(\Omega, \mathbb{R}^{N_2})$. This fixed point $v_a(x)$ is nontrivial provided that the intersection of supports of the Fourier transforms of the functions $supp\widehat{F_k(0, x)}_n(p) \cap supp\widehat{G_{k,\, n}}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for a certain $1 \le k \le N_2$ and some $n \in \mathbb{Z}$.*

Let us note that the maps considered in the theorems above are applied to the real valued vector functions by virtue of the conditions on $F_k(u, x)$ and $G_k(x)$, $1 \le k \le N_2$ involved in the nonlocal terms of (1.6), (1.7). The map t_a is an analog of the map T_a.

2 The Problem in the Whole Space

Proof of Theorem 1.3. Let us first suppose that in the case of $\Omega = \mathbb{R}^d$, $d = 1, 2, 3$ there exists $v(x) \in H^2(\mathbb{R}^d, \mathbb{R}^{N_2})$ such that system (1.6), (1.7) admits two solutions $u^{(1),(2)}(x) \in H^2(\mathbb{R}^d, \mathbb{R}^{N_2})$. The the difference vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^2(\mathbb{R}^d, \mathbb{R}^{N_2})$ satisfies the homogeneous system of equations

$$-\Delta w_k = a_k w_k, \quad a_k \ge 0, \quad 1 \le k \le N_1,$$
$$-\Delta w_k = -a_k w_k, \quad a_k > 0, \quad N_1 + 1 \le k \le N_2.$$

Since the operator $-\Delta : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ does not have any nontrivial eigenfunctions, we have that $w_k(x)$ vanishes in $\mathbb{R}^d$ for all $k = 1, \dots, N_2$.
We consider an arbitrary vector function $v(x) \in H^2(\mathbb{R}^d, \mathbb{R}^{N_2})$ and apply the standard Fourier transform (1.12) to both sides of system (1.6), (1.7). This gives us

$$\widehat{u_k}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k}(p)}{p^2 - a_k}, \quad a_k \ge 0, \quad 1 \le k \le N_1, \tag{2.1}$$

$$\widehat{u_k}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k}(p)}{p^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \le k \le N_2. \tag{2.2}$$

Here, $\widehat{f_k}(p)$ denotes the Fourier image of $F_k(v(x), x)$. We have the trivial estimates using expressions (1.14), (1.15) (1.16) and (1.17), namely

$$|\widehat{u_k}(p)| \le (2\pi)^{\frac{d}{2}} M |\widehat{f_k}(p)| \quad and \quad |p^2 \widehat{u_k}(p)| \le (2\pi)^{\frac{d}{2}} M |\widehat{f_k}(p)|, \ 1 \le k \le N_2.$$

This gives us the upper bound for the norm as

$$\|u\|^2_{H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})} = \sum_{k=1}^{N_2} \{\|\widehat{u_k}(p)\|^2_{L^2(\mathbb{R}^d)} + \|p^2\widehat{u_k}(p)\|^2_{L^2(\mathbb{R}^d)}\} \leq$$
$$\leq 2(2\pi)^d M^2 \sum_{k=1}^{N_2} \|F_k(v(x), x)\|^2_{L^2(\mathbb{R}^d)}. \tag{2.3}$$

Clearly, the right side of (2.3) is finite via inequality (1.4) of Assumption 1.1 above. Thus, for any $v(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})$ there exists a unique vector function $u(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})$, which satisfies system (1.6), (1.7) and its Fourier image is given by (2.1), (2.2). Hence, the map $T_a : H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2}) \to H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})$ is well defined.
This enables us to choose arbitrarily $v^{(1),(2)}(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})$ and obtain their images under this map $u^{(1),(2)} := T_a v^{(1),(2)} \in H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})$, such that by means of (1.6), (1.7)

$$-\Delta u_k^{(1)} - a_k u_k^{(1)} = \int_{\mathbb{R}^d} G_k(x-y) F_k(v_1^{(1)}(y), v_2^{(1)}(y), \dots, v_{N_2}^{(1)}(y), y)dy, \quad a_k \geq 0 \tag{2.4}$$

for $1 \leq k \leq N_1$,

$$-\Delta u_k^{(1)} + a_k u_k^{(1)} = \int_{\mathbb{R}^d} G_k(x-y) F_k(v_1^{(1)}(y), v_2^{(1)}(y), \dots, v_{N_2}^{(1)}(y), y)dy, \quad a_k > 0 \tag{2.5}$$

for $N_1 + 1 \leq k \leq N_2$. Similarly,

$$-\Delta u_k^{(2)} - a_k u_k^{(2)} = \int_{\mathbb{R}^d} G_k(x-y) F_k(v_1^{(2)}(y), v_2^{(2)}(y), \dots, v_{N_2}^{(2)}(y), y)dy, \quad a_k \geq 0 \tag{2.6}$$

if $1 \leq k \leq N_1$,

$$-\Delta u_k^{(2)} + a_k u_k^{(2)} = \int_{\mathbb{R}^d} G_k(x-y) F_k(v_1^{(2)}(y), v_2^{(2)}(y), \dots, v_{N_2}^{(2)}(y), y)dy, \quad a_k > 0, \tag{2.7}$$

if $N_1 + 1 \leq k \leq N_2$. Let us apply the standard Fourier transform (1.12) to both sides of systems (2.4), (2.5) and (2.6), (2.7). This yields

$$\widehat{u_k^{(1)}}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k^{(1)}}(p)}{p^2 - a_k}, \quad a_k \geq 0, \quad 1 \leq k \leq N_1, \tag{2.8}$$

$$\widehat{u_k^{(1)}}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k^{(1)}}(p)}{p^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \leq k \leq N_2 \tag{2.9}$$

and

$$\widehat{u_k^{(2)}}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k^{(2)}}(p)}{p^2 - a_k}, \quad a_k \geq 0, \quad 1 \leq k \leq N_1, \tag{2.10}$$

$$\widehat{u_k^{(2)}}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k^{(2)}}(p)}{p^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \leq k \leq N_2. \tag{2.11}$$

Here $\widehat{f_k^{(1),(2)}}(p)$ stand for the Fourier images of $F_k(v^{(1),(2)}(x), x)$. From formulas (2.8), (2.9), (2.10), (2.11) via (1.14), (1.15) (1.16) and (1.17) we deduce easily the upper bounds for $1 \leq k \leq N_2$ as

$$\left|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right| \leq (2\pi)^{\frac{d}{2}} M \left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|,$$

$$\left|p^2\widehat{u_k^{(1)}}(p) - p^2\widehat{u_k^{(2)}}(p)\right| \leq (2\pi)^{\frac{d}{2}} M \left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|.$$

This enables us to derive the estimate from above on the corresponding norm of the difference of vector functions

$$\|u^{(1)} - u^{(2)}\|^2_{H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})} = \sum_{k=1}^{N_2} \left\{ \left\|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right\|^2_{L^2(\mathbb{R}^d)} + \left\|p^2\left[\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right]\right\|^2_{L^2(\mathbb{R}^d)} \right\}$$

$$\leq 2(2\pi)^d M^2 \sum_{k=1}^{N_2} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R}^d)}. \tag{2.12}$$

By means of the Sobolev embedding theorem for $1 \leq k \leq N_2$ we have $v_k^{(1),(2)}(x) \in H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),\ 1 \leq d \leq 3$. Inequality (1.5) of Assumption 1.1 above trivially implies

$$\|T_a v^{(1)} - T_a v^{(2)}\|_{H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})} \leq \sqrt{2}(2\pi)^{\frac{d}{2}} ML \|v^{(1)} - v^{(2)}\|_{H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})}. \tag{2.13}$$

The constant in the right side of (2.13) is less than one as assumed. Thus, the Fixed Point Theorem implies the existence of a unique vector function $v_a(x) \in H^2(\mathbb{R}^d, \mathbb{R}^{N_2})$, so that $T_a v_a = v_a$. This is the only stationary solution of system (1.2), (1.3) in $H^2(\mathbb{R}^d,\ \mathbb{R}^{N_2})$. Finally, we suppose that $v_a(x)$ is trivial in $\mathbb{R}^d$. This will give us the contradiction to our assumption that for a certain $1 \leq k \leq N_2$ the Fourier transforms of $G_k(x)$ and $F_k(0, x)$ do not vanish simultaneously on some set of nonzero Lebesgue measure in $\mathbb{R}^d$. □

3 The Problem on the $[0, 2\pi]$ Interval

Proof of Theorem 1.5. First we suppose that for some $v(x) \in H_c^2(I,\ \mathbb{R}^{N_2})$ there exist two solutions $u^{(1),(2)}(x) \in H_c^2(I,\ \mathbb{R}^{N_2})$ of system (1.6), (1.7) with $\Omega = I$. Then, the difference vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H_c^2(I, \mathbb{R}^{N_2})$ will solve the homogeneous system of equations

$$-w_k'' = a_k w_k, \quad a_k \ge 0, \quad 1 \le k \le N_1, \tag{3.1}$$

$$-w_k'' = -a_k w_k, \quad a_k > 0, \quad N_1 + 1 \le k \le N_2. \tag{3.2}$$

By means of Assumption 1.4, we have $a_k > 0$, $a_k \ne n^2$, $n \in \mathbb{Z}$ for $1 \le k \le m$. Hence, they are not the eigenvalues of our operator

$$-\frac{d^2}{dx^2} : H^2(I) \to L^2(I). \tag{3.3}$$

Thus, $w_k(x)$ are trivial in I if $1 \le k \le m$. When $m + 1 \le k \le q$ the values of a_k coincide with the eigenvalues of (3.3) as assumed. But w_k belong to the constrained subspaces $H_k^2(I)$. Therefore, w_k vanish identically in I if $m + 1 \le k \le q$ due to their orthogonality to the eigenfunctions $\frac{e^{\pm i n_k x}}{\sqrt{2\pi}}$ of (3.3). By virtue of Assumption 1.4, the constants $a_k = 0$ for $q + 1 \le k \le N_1$. But $w_k \in H_0^2(I)$, so that they are orthogonal to the zero mode of our operator (3.3). Therefore, $w_k(x)$ are trivial in I as well if $q + 1 \le k \le N_1$. Finally, let us consider the situation when $N_1 + 1 \le k \le N_2$. But operator (3.3) cannot have any negative eigenvalues. Therefore, by means of (3.2) we have that $w_k(x)$ vanish identically in I for $N_1 + 1 \le k \le N_2$.
We choose arbitrarily $v(x) \in H_c^2(I, \mathbb{R}^{N_2})$. Let us apply the Fourier transform (1.20) to both sides of the system of equations (1.6), (1.7) considered on the interval $[0, 2\pi]$. This yields

$$u_{k,\,n} = \sqrt{2\pi}\frac{G_{k,\,n} f_{k,\,n}}{n^2 - a_k}, \quad a_k \ge 0, \quad 1 \le k \le N_1, \quad n \in \mathbb{Z}, \tag{3.4}$$

$$u_{k,\,n} = \sqrt{2\pi}\frac{G_{k,\,n} f_{k,\,n}}{n^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \le k \le N_2, \quad n \in \mathbb{Z}, \tag{3.5}$$

with $f_{k,\,n} := F_k(v(x), x)_n$. Clearly, the Fourier coefficients of the second derivatives are equal to

$$(-u_k'')_n = \sqrt{2\pi}\frac{n^2 G_{k,\,n} f_{k,\,n}}{n^2 - a_k}, \quad a_k \ge 0, \quad 1 \le k \le N_1, \quad n \in \mathbb{Z},$$

$$(-u_k'')_n = \sqrt{2\pi}\frac{n^2 G_{k,\,n} f_{k,\,n}}{n^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \le k \le N_2, \quad n \in \mathbb{Z}.$$

We easily obtain the estimate from above

$$\|u\|^2_{H_c^2(I,\,\mathbb{R}^{N_2})} = \sum_{k=1}^{N_2} \left\{ \sum_{n=-\infty}^{\infty} |u_{k,\,n}|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_{k,\,n}|^2 \right\}$$

$$\leq 4\pi \sum_{k=1}^{N_2} P_k^2 \|F_k(v(x), x)\|_{L^2(I)}^2. \tag{3.6}$$

Let us recall inequality (1.4) of Assumption 1.1. Hence, the right side of (3.6) is finite. Thus, for an arbitrarily chosen vector function $v(x) \in H_c^2(I, \mathbb{R}^{N_2})$ there exists a unique $u(x) \in H_c^2(I, \mathbb{R}^{N_2})$, which satisfies the system of equations (1.6), (1.7) and its Fourier coefficients are given by formulas (3.4), (3.5), so that the map τ_a : $H_c^2(I, \mathbb{R}^{N_2}) \to H_c^2(I, \mathbb{R}^{N_2})$ is well defined. Clearly, orthogonality relations (1.18) and (1.19) along with (3.4) yield that for $m+1 \leq k \leq q$ the components $u_k(x)$ are orthogonal to the Fourier harmonics $\frac{e^{\pm i n_k x}}{\sqrt{2\pi}}$ in $L^2(I)$ and for $q+1 \leq k \leq N_1$ the functions $u_k(x)$ are orthogonal to 1 in $L^2(I)$, because the corresponding Fourier coefficients are equal to zero.

Let us consider arbitrary vector functions $v^{(1),(2)}(x) \in H_c^2(I, \mathbb{R}^{N_2})$, so that their images under the map discussed above are $u^{(1),(2)} = \tau_a v^{(1),(2)} \in H_c^2(I, \mathbb{R}^{N_2})$. By virtue of (1.6), (1.7), we have

$$-\frac{d^2}{dx^2}u_k^{(1)} - a_k u_k^{(1)} = \int_0^{2\pi} G_k(x-y)F_k(v_1^{(1)}(y), v_2^{(1)}(y), \ldots, v_{N_2}^{(1)}(y), y)dy, \quad a_k \geq 0 \tag{3.7}$$

if $1 \leq k \leq N_1$,

$$-\frac{d^2}{dx^2}u_k^{(1)} + a_k u_k^{(1)} = \int_0^{2\pi} G_k(x-y)F_k(v_1^{(1)}(y), v_2^{(1)}(y), \ldots, v_{N_2}^{(1)}(y), y)dy, \quad a_k > 0 \tag{3.8}$$

if $N_1 + 1 \leq k \leq N_2$. Analogously,

$$-\frac{d^2}{dx^2}u_k^{(2)} - a_k u_k^{(2)} = \int_0^{2\pi} G_k(x-y)F_k(v_1^{(2)}(y), v_2^{(2)}(y), \ldots, v_{N_2}^{(2)}(y), y)dy, \quad a_k \geq 0 \tag{3.9}$$

for $1 \leq k \leq N_1$,

$$-\frac{d^2}{dx^2}u_k^{(2)} + a_k u_k^{(2)} = \int_0^{2\pi} G_k(x-y)F_k(v_1^{(2)}(y), v_2^{(2)}(y), \ldots, v_{N_2}^{(2)}(y), y)dy, \quad a_k > 0 \tag{3.10}$$

for $N_1 + 1 \leq k \leq N_2$. We apply the Fourier transform (1.20) to both sides of systems (3.7), (3.8) and (3.9), (3.10). This gives us

$$u_{k,\,n}^{(1)} = \sqrt{2\pi}\frac{G_{k,\,n} f_{k,\,n}^{(1)}}{n^2 - a_k}, \quad a_k \geq 0, \quad 1 \leq k \leq N_1, \quad n \in \mathbb{Z}, \tag{3.11}$$

$$u_{k,\, n}^{(1)} = \sqrt{2\pi}\frac{G_{k,\, n} f_{k,\, n}^{(1)}}{n^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \le k \le N_2, \quad n \in \mathbb{Z} \tag{3.12}$$

and

$$u_{k,\, n}^{(2)} = \sqrt{2\pi}\frac{G_{k,\, n} f_{k,\, n}^{(2)}}{n^2 - a_k}, \quad a_k \ge 0, \quad 1 \le k \le N_1, \quad n \in \mathbb{Z}, \tag{3.13}$$

$$u_{k,\, n}^{(2)} = \sqrt{2\pi}\frac{G_{k,\, n} f_{k,\, n}^{(2)}}{n^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \le k \le N_2, \quad n \in \mathbb{Z}. \tag{3.14}$$

Here $f_{k,\, n}^{(1),(2)}$ denote the Fourier images of $F_k(v^{(1),(2)}(x), x)$ under transform (1.20). Let us recall (1.22), (1.23), (1.24), (1.25) and (1.26). By means of formulas (3.11), (3.12), (3.13) and (3.14), we obtain for $1 \le k \le N_2,\ n \in \mathbb{Z}$ that

$$|u_{k,\, n}^{(1)} - u_{k,\, n}^{(2)}| \le \sqrt{2\pi} P |f_{k,\, n}^{(1)} - f_{k,\, n}^{(2)}|,$$

$$|n^2 u_{k,\, n}^{(1)} - n^2 u_{k,\, n}^{(2)}| \le \sqrt{2\pi} P |f_{k,\, n}^{(1)} - f_{k,\, n}^{(2)}|.$$

Therefore,

$$\|u^{(1)} - u^{(2)}\|^2_{H^2_c(I,\, \mathbb{R}^{N_2})} = \sum_{k=1}^{N_2} \left\{ \sum_{n=-\infty}^{\infty} |u_{k,\, n}^{(1)} - u_{k,\, n}^{(2)}|^2 + \sum_{n=-\infty}^{\infty} |n^2 (u_{k,\, n}^{(1)} - u_{k,\, n}^{(2)})|^2 \right\}$$

$$\le 4\pi P^2 \sum_{k=1}^{N_2} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(I)}.$$

Evidently, due to the Sobolev embedding theorem we have $v_k^{(1),(2)}(x) \in H^2(I) \subset L^\infty(I)$ with $1 \le k \le N_2$. Let us recall inequality (1.5). Hence,

$$\|\tau_a v^{(1)} - \tau_a v^{(2)}\|_{H^2_c(I,\, \mathbb{R}^{N_2})} \le 2\sqrt{\pi} P L \|v^{(1)} - v^{(2)}\|_{H^2_c(I,\, \mathbb{R}^{N_2})}. \tag{3.15}$$

Note that the constant in the right side of bound (3.15) is less than one as assumed. Thus, the Fixed Point Theorem implies the existence and uniqueness of a vector function $v_a(x) \in H^2_c(I,\ \mathbb{R}^{N_2})$ satisfying $\tau_a v_a = v_a$. This is the only stationary solution of the system of equations (1.2), (1.3) in $H^2_c(I,\ \mathbb{R}^{N_2})$. Finally, we suppose that $v_a(x)$ is trivial in the interval I. This will contradict to the stated assumption that the Fourier coefficients $G_{k,\, n} F_k(0, x)_n \neq 0$ for a certain $1 \le k \le N_2$ and some $n \in \mathbb{Z}$. □

4 The Problem in the Layer Domain

Proof of Theorem 1.7. First, we suppose that there exists $v(x) \in H^2(\Omega,\ \mathbb{R}^{N_2})$, which generates $u^{(1),(2)}(x) \in H^2(\Omega,\ \mathbb{R}^{N_2})$ satisfying the system of equations (1.6), (1.7). Then the difference of these vector functions $w(x) := u^{(1)}(x) - u^{(2)}(x) \in$

$H^2(\Omega, \mathbb{R}^{N_2})$ will solve the homogeneous system

$$-\Delta w_k = a_k w_k, \quad a_k \geq 0, \quad 1 \leq k \leq N_1, \tag{4.1}$$

$$-\Delta w_k = -a_k w_k, \quad a_k > 0, \quad N_1 + 1 \leq k \leq N_2. \tag{4.2}$$

We apply the partial Fourier transform to (4.1) and derive

$$-\Delta_\perp w_{k,\ n}(x_\perp) = (a_k - n^2) w_{k,\ n}(x_\perp), \quad 1 \leq k \leq N_1, \quad n \in \mathbb{Z}.$$

Here, $w_{k,\ n}(x_\perp) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} w_k(x_1, x_\perp) e^{-inx_1} dx_1$ and $\Delta_\perp$ is the Laplace operator with respect to $x_\perp$. Clearly,

$$\|w_k\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|w_{k,\ n}\|^2_{L^2(\mathbb{R}^d)}, \quad 1 \leq k \leq N_1.$$

Thus,

$$w_{k,\ n}(x_\perp) \in L^2(\mathbb{R}^d), \quad 1 \leq k \leq N_1, \quad n \in \mathbb{Z}.$$

But the operator $-\Delta_\perp$ considered in the whole $\mathbb{R}^d$ does not have any nontrivial square integrable eigenfunctions. Therefore, $w_k(x)$ vanish in Ω for $1 \leq k \leq N_1$. The analogous assertion is true for $w_k(x)$ with $N_1 + 1 \leq k \leq N_2$ by virtue of (4.2), since the operator $-\Delta : H^2(\Omega) \to L^2(\Omega)$ cannot possess any negative eigenvalues.
Let us consider an arbitrary vector function $v(x) \in H^2(\Omega, \mathbb{R}^{N_2})$ and apply the Fourier transform (1.33) to both sides of system (1.6), (1.7). This yields

$$\widehat{u_{k,\ n}}(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G_{k,\ n}}(p)\widehat{f_{k,\ n}}(p)}{p^2 + n^2 - a_k}, \quad a_k \geq 0, \quad 1 \leq k \leq N_1, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d, \tag{4.3}$$

$$\widehat{u_{k,\ n}}(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G_{k,\ n}}(p)\widehat{f_{k,\ n}}(p)}{p^2 + n^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \leq k \leq N_2, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d. \tag{4.4}$$

Here $\widehat{f_{k,\ n}}(p)$ designates the Fourier image of $F_k(v(x), x)$ under transform (1.33). Obviously, we have the estimates from above in terms of the expressions given by (1.35), (1.36), (1.37) and (1.38), namely

$$|\widehat{u_{k,\ n}}(p)| \leq (2\pi)^{\frac{d+1}{2}} R_k |\widehat{f_{k,\ n}}(p)|, \quad |(p^2 + n^2)\widehat{u_{k,\ n}}(p)| \leq (2\pi)^{\frac{d+1}{2}} R_k |\widehat{f_{k,\ n}}(p)|,$$

with $1 \leq k \leq N_2,\ n \in \mathbb{Z},\ p \in \mathbb{R}^d$. We arrive at

$$\|u\|^2_{H^2(\Omega,\ \mathbb{R}^{N_2})} = \sum_{k=1}^{N_2} \left\{ \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |\widehat{u_{k,\ n}}(p)|^2 dp + \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2 + n^2)\widehat{u_{k,\ n}}(p)|^2 dp \right\}$$

$$\leq 2(2\pi)^{d+1}\sum_{k=1}^{N_2} R_k{}^2\|F_k(v(x),x)\|^2_{L^2(\Omega)}. \tag{4.5}$$

Let us recall inequality (1.4) of Assumption 1.1. Hence, the right side of (4.5) is finite. Thus, for any vector function $v(x) \in H^2(\Omega,\ \mathbb{R}^{N_2})$ there exists a unique $u(x) \in H^2(\Omega,\ \mathbb{R}^{N_2})$ satisfying system (1.6), (1.7). Its Fourier transform is given by (4.3), (4.4). Therefore, the map $t_a : H^2(\Omega,\ \mathbb{R}^{N_2}) \to H^2(\Omega,\ \mathbb{R}^{N_2})$ is well defined.
We choose arbitrarily two vector functions $v^{(1),(2)} \in H^2(\Omega,\ \mathbb{R}^{N_2})$, so that their images under the map discussed above are $u^{(1),(2)} := t_a v^{(1),(2)} \in H^2(\Omega,\ \mathbb{R}^{N_2})$. By means of (1.6), (1.7), we have precisely

$$-\Delta u_k^{(1)} - a_k u_k^{(1)} = \int_\Omega G_k(x-y)F_k(v_1^{(1)}(y), v_2^{(1)}(y), \dots, v_{N_2}^{(1)}(y), y)dy, \quad a_k \geq 0 \tag{4.6}$$

with $1 \leq k \leq N_1$,

$$-\Delta u_k^{(1)} + a_k u_k^{(1)} = \int_\Omega G_k(x-y)F_k(v_1^{(1)}(y), v_2^{(1)}(y), \dots, v_{N_2}^{(1)}(y), y)dy, \quad a_k > 0 \tag{4.7}$$

with $N_1 + 1 \leq k \leq N_2$. Similarly,

$$-\Delta u_k^{(2)} - a_k u_k^{(2)} = \int_\Omega G_k(x-y)F_k(v_1^{(2)}(y), v_2^{(2)}(y), \dots, v_{N_2}^{(2)}(y), y)dy, \quad a_k \geq 0 \tag{4.8}$$

if $1 \leq k \leq N_1$,

$$-\Delta u_k^{(2)} + a_k u_k^{(2)} = \int_\Omega G_k(x-y)F_k(v_1^{(2)}(y), v_2^{(2)}(y), \dots, v_{N_2}^{(2)}(y), y)dy, \quad a_k > 0 \tag{4.9}$$

if $N_1+1 \leq k \leq N_2$. Let us apply the Fourier transform (1.33) to both sides of systems (4.6), (4.7) and (4.8), (4.9). This yields

$$\widehat{u_{k,\,n}^{(1)}}(p) = (2\pi)^{\frac{d+1}{2}}\frac{\widehat{G_{k,\,n}}(p)\widehat{f_{k,\,n}^{(1)}}(p)}{p^2+n^2-a_k}, \quad a_k \geq 0, \quad 1 \leq k \leq N_1, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d, \tag{4.10}$$

$$\widehat{u_{k,\,n}^{(1)}}(p) = (2\pi)^{\frac{d+1}{2}}\frac{\widehat{G_{k,\,n}}(p)\widehat{f_{k,\,n}^{(1)}}(p)}{p^2+n^2+a_k}, \quad a_k > 0, \quad N_1+1 \leq k \leq N_2, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d. \tag{4.11}$$

and

$$\widehat{u_{k,\,n}^{(2)}}(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G_{k,\,n}}(p)\widehat{f_{k,\,n}^{(2)}}(p)}{p^2 + n^2 - a_k}, \quad a_k \geq 0, \quad 1 \leq k \leq N_1, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d, \tag{4.12}$$

$$\widehat{u_{k,\,n}^{(2)}}(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G_{k,\,n}}(p)\widehat{f_{k,\,n}^{(2)}}(p)}{p^2 + n^2 + a_k}, \quad a_k > 0, \quad N_1 + 1 \leq k \leq N_2, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d. \tag{4.13}$$

Here, $\widehat{f_{k,\,n}^{(1),(2)}}(p)$ stand for the Fourier images of $F_k(v^{(1),(2)}(x), x)$ under transform (1.33). We recall (1.35), (1.36), (1.37), (1.38) and (1.39). By virtue of formulas (4.10), (4.11), (4.12) and (4.13) we derive for $1 \leq k \leq N_2,\ n \in \mathbb{Z},\ p \in \mathbb{R}^d$ that

$$\left|\widehat{u_{k,\,n}^{(1)}}(p) - \widehat{u_{k,\,n}^{(2)}}(p)\right| \leq (2\pi)^{\frac{d+1}{2}} R \left|\widehat{f_{k,\,n}^{(1)}}(p) - \widehat{f_{k,\,n}^{(2)}}(p)\right|,$$

$$\left|(p^2 + n^2)\left[\widehat{u_{k,\,n}^{(1)}}(p) - \widehat{u_{k,\,n}^{(2)}}(p)\right]\right| \leq (2\pi)^{\frac{d+1}{2}} R \left|\widehat{f_{k,\,n}^{(1)}}(p) - \widehat{f_{k,\,n}^{(2)}}(p)\right|.$$

Hence,

$$\|u^{(1)} - u^{(2)}\|^2_{H^2(\Omega,\ \mathbb{R}^{N_2})} = \sum_{k=1}^{N_2} \left\{ \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} \left\{ |\widehat{u^{(1)}}_{k,\,n}(p) - \widehat{u^{(2)}}_{k,\,n}(p)|^2 dp \right.\right.$$
$$\left.+ \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} |(p^2 + n^2)(\widehat{u^{(1)}}_{k,\,n}(p) - \widehat{u^{(2)}}_{k,\,n}(p))|^2 dp \right\}$$
$$\leq 2(2\pi)^{d+1} R^2 \sum_{k=1}^{N_2} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\Omega)}.$$

By means of the Sobolev embedding theorem, $v_k^{(1),(2)}(x) \in H^2(\Omega) \subset L^\infty(\Omega)$ with $1 \leq k \leq N_2$. Let us recall inequality (1.5). We arrive at

$$\|t_a v^{(1)} - t_a v^{(2)}\|_{H^2(\Omega,\ \mathbb{R}^{N_2})} \leq \sqrt{2}(2\pi)^{\frac{d+1}{2}} R L \|v^{(1)} - v^{(2)}\|_{H^2(\Omega,\ \mathbb{R}^{N_2})}. \tag{4.14}$$

The constant in its right side of (4.14) is less than one via one of our assumptions. The Fixed Point Theorem yields the existence and uniqueness of a vector function $v_a(x) \in H^2(\Omega,\ \mathbb{R}^{N_2})$, for which $t_a v_a = v_a$ is valid. This is the only stationary solution of system (1.2), (1.3) in $H^2(\Omega,\ \mathbb{R}^{N_2})$. Let us suppose that $v_a(x)$ is trivial in Ω. This will give us the contradiction to the imposed condition that there exist $1 \leq k \leq N_2$ and $n \in \mathbb{Z}$, so that supp$\widehat{F_k(0, x)}_n(p) \cap$ supp$\widehat{G_{k,\,n}}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. □

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Declarations

Conflict of Interest The authors declare that they have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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