Arnold Mathematical Journal (2023) 9:381–391
https://doi.org/10.1007/s40598-022-00220-3

RESEARCH CONTRIBUTION

A Generalized Problem Associated to the Kummer–Vandiver Conjecture

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Received: 18 May 2022 / Revised: 22 August 2022 / Accepted: 17 October 2022 /
Published online: 7 November 2022

Abstract
To discuss the validity of the Kummer–Vandiver conjecture, we consider a generalized problem associated to the conjecture. Let p be an odd prime number and ζ_p a primitive p-th root of unity. Using new programs, we compute the Iwasawa invariants of $\mathbf{Q}(\sqrt{d}, \zeta_p)$ in the range $|d| < 200$ and $200 < p < 1{,}000{,}000$. From our data, the actual numbers of exceptional cases seem to be near the expected numbers for $p < 1{,}000{,}000$. Moreover, we find a few rare exceptional cases for $|d| < 10$ and $p > 1{,}000{,}000$. We give two partial reasons why it is difficult to find exceptional cases for $d = 1$ including counter-examples to the Kummer–Vandiver conjecture.

Keywords Iwasawa invariants · Kummer–Vandiver conjecture · Ideal class group

Mathematics Subject Classification Primary 11R23; Secondary 11R18 · 11R29 · 11R70

1 Introduction

Let p be an odd prime number and K a finite extension of $\mathbf{Q}$. K_∞ denotes the cyclotomic $\mathbf{Z}_p$-extension of K. Let K_n be its n-th layer and $A_n = A_n(K)$ the p-part of the ideal class group of K_n.

First, let K be the p-cyclotomic field $\mathbf{Q}(\zeta_p)$, then $K_n = \mathbf{Q}(\zeta_{p^{n+1}})$. Let $\omega = \omega_p$ be the Teichmüller character $(\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p$ such that $\omega(a) \equiv a \bmod p$. We identify $\Delta = \mathrm{Gal}(K_\infty/\mathbf{Q}_\infty)$ with $(\mathbf{Z}/p\mathbf{Z})^\times$. Put $e_{\omega^k} = \frac{1}{\sharp\Delta}\sum_{\delta\in\Delta}\omega^k(\delta)\delta^{-1}$ the idempotent of the group ring $\mathbf{Q}_p[\Delta]$. Then we have

The author was partially supported by JSPS KAKENHI Grant Number JP17K05176 and JP20H00115.

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$$A_n = \bigoplus_{k:even} e_{\omega^k} A_n \oplus \bigoplus_{p-k:odd} e_{\omega^{p-k}} A_n,$$

where k is an even integer with $2 \leq k \leq p-1$. Let A_n^+ (resp. A_n^-) be the even part (resp. odd part). Let r_p be the irregularity index, i.e., the number of irregular pairs (p,k). Irregular pairs have been computed by Kummer, Vandiver, D.H. Lehmer, E. Lehmer, Selfridge, Nicol, Pollack, Johnson, Wada, Wagstaff, Tanner, Ernvall, Metsänkylä, Buhler, Crandall, Sompolski, Shokrollahi, Hart, Harvey and Ong. These computations had been connected with verification of Fermat's last theorem. However, even after the proof was completed by Wiles, they are still interesting because they give us concrete knowledge of the ideal class group of cyclotomic fields. In [1, 2, 5] etc., for any prime number $p < 2^{31} = 2,147,483,648$, it has been verified that

$$A_n^+ = \{0\} \text{ and } A_n^- \simeq (\mathbf{Z}/p^{n+1}\mathbf{Z})^{r_p} \text{ for all } n \geq 0.$$

The former statement is called the Kummer–Vandiver conjecture. We have a naive explanation of the fact that we have not been able to find any counter-example. If we follow the heuristic argument of [15, pp.158–159], we can expect that the number of exceptions to the Kummer–Vandiver conjecture for $x_0 \leq p \leq x_1$ is approximately $(\log\log x_1 - \log\log x_0)/2$. Then, $(\log\log 2^{31} - \log\log 37)/2 = 0.891756\cdots$ is probably too small to find one counter-example, where 37 is the smallest irregular prime number. Furthermore, the expected number would not be exact, because there are some effects on ideal class groups from an upper bound for the numerator of the Bernoulli number or the K-groups (cf. [10]). If there are another strong effects, the actual number could be much less than the above number. To study the heuristic, we consider the following generalized problem.

Problem 1.1 *Let F be an abelian extension of $\mathbf{Q}$. Let $N_F(x)$ be the number of prime numbers p such that $A_0(F(\zeta_p)^+) \neq \{0\}$ for $p \leq x$, where $F(\zeta_p)^+$ is the maximal real subfield of $F(\zeta_p)$. Is $N_F(x)$ bounded as $x \to \infty$? If it is not so, give an approximate function for $N_F(x)$.*

The Kummer–Vandiver conjecture claims that $N_{\mathbf{Q}}(x) = 0$ for all x, which is much stronger than its boundedness.

In this paper, following [11–14], we study the above problem when F is $\mathbf{Q}$ or a quadratic field, because they are easy to be compared. Let χ be the Dirichlet character associated to F and f_χ its conductor. The main purpose of the paper is to find exceptional cases associated to the $\chi\omega^k$-part to argue about the expected number. We actually computed the Iwasawa invariants of $\mathbf{Q}(\sqrt{d}, \zeta_p)$ in the range $|d| < 200$ and $200 < p < 1{,}000{,}000$ by using new programs, where $d = d_\chi = \chi(-1) f_\chi$. From our data, the actual number seems to be near the expected number in the range. Moreover, we found a few rare exceptional cases for $|d| < 10$ and $1{,}000{,}000 < p < 20{,}000{,}000$.

Our main computations are executed in $O((f_\chi p)^{1+\epsilon})$ bit operations. See [12, 14] on the relation between these Iwasawa invariants and the higher K-groups of the integer ring of $\mathbf{Q}(\sqrt{d})$.

2 Iwasawa Invariants of $\mathbf{Q}(\sqrt{d}, \zeta_p)$

Let χ be the trivial character or a quadratic Dirichlet character conductor $f = f_\chi$ and p an odd prime number such that p does not divide f. Put $d = d_\chi = \chi(-1)f_\chi$, $K = \mathbf{Q}(\sqrt{d_\chi}, \zeta_p)$, then $K_n = \mathbf{Q}(\sqrt{d_\chi}, \zeta_{p^{n+1}})$. Let A_n be the p-part of the ideal class group of K_n.

Put $\Gamma = \mathrm{Gal}(K_\infty/K)$, $\Delta = \mathrm{Gal}(K_\infty/\mathbf{Q}_\infty) \simeq \mathrm{Gal}(K_0/\mathbf{Q})$ and $e_\psi = \frac{1}{\sharp\Delta}\sum_{\delta\in\Delta}\psi(\delta)\delta^{-1}$ for a character ψ of Δ. We put $f_0 = fp$ and identify Δ with a subquotient of $(\mathbf{Z}/f_0\mathbf{Z})^\times$ in the ordinary way. For a $\mathbf{Z}_p[\Delta]$-module A, A^ψ denotes $e_\psi A$. Let $\lambda_p(\psi)$, $\mu_p(\psi)$ and $\nu_p(\psi)$ be the Iwasawa invariants associated to A_n^ψ, i.e.,

$$\sharp A_n^\psi = p^{\lambda_p(\psi)n+\mu_p(\psi)p^n+\nu_p(\psi)}$$

for sufficiently large n. By Ferrero-Washington's theorem, we have $\mu_p(\psi) = 0$ for all p and ψ.

Assume that ψ is even. The Iwasawa polynomial $g_\psi(T) \in \mathbf{Z}_p[T]$ for the p-adic L-function is defined as follows. Let $L_p(s, \psi)$ be the p-adic L-function constructed by [8]. By [7, §6], there uniquely exists $G_\psi(T) \in \mathbf{Z}_p[[T]]$ satisfying $G_\psi((1+f_0)^{1-s} - 1) = L_p(s, \psi)$ for all $s \in \mathbf{Z}_p$ if $\psi \neq \chi^0$. By [3], p does not divide $G_\psi(T)$. By the p-adic Weierstrass preparation theorem, we can uniquely write $G_\psi(T) = g_\psi(T)u_\psi(T)$, where $g_\psi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\psi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. Similarly we can define $g_\psi^*(T) \in \mathbf{Z}_p[T]$ from $G_\psi^*(T) \in \mathbf{Z}_p[[T]]$ satisfying $G_\psi^*((1+f_0)^s - 1) = L_p(s, \psi)$. Put

$$\tilde{\lambda}_p(\psi) = \deg g_\psi(T) = \deg g_\psi^*(T).$$

Put $f_n = f_0 p^n$ and let $\gamma \in \Gamma \simeq \mathrm{Gal}(\cup_{n\geq 0}\mathbf{Q}(\zeta_{f_n})/\mathbf{Q}(\zeta_{f_0}))$ be the generator of Γ such that $\zeta_{f_n}^{\tilde{\gamma}} = \zeta_{f_n}^{1+f_0}$ for all $n \geq 0$. As usual, we can identify the complete group ring $\mathbf{Z}_p[[\Gamma]]$ with the formal power series ring $\Lambda = \mathbf{Z}_p[[T]]$ by $\gamma = 1 + T$. By this identification, we can consider a $\mathbf{Z}_p[[\Gamma]]$-module as a Λ-module. For a finitely generated torsion Λ-module A, we define the Iwasawa polynomial $\mathrm{char}_\Lambda(A)$ to be the characteristic polynomial of the action T on $A \otimes \mathbf{Q}_p$ (cf. [15, §13]). Let L_n be the maximal unramified abelian extension of K_n and M_n the maximal abelian extension of K_n unramified outside p. By the class field theory, we have $A_n \simeq \mathrm{Gal}(L_n/K_n)$. Set $L_\infty = \cup_{n\geq 0}L_n$, $M_\infty = \cup_{n\geq 0}M_n$, $X_\infty = \mathrm{Gal}(L_\infty/K_\infty)$ and $Y_\infty = \mathrm{Gal}(M_\infty/K_\infty)$. By the Iwasawa main conjecture proved by [4, 9], $\mathrm{char}_\Lambda(X_\infty^{\psi^{-1}\omega}) = g_\psi^*(T)$ and $\mathrm{char}_\Lambda(Y_\infty^\psi) = g_\psi(T)$.

In the following, we assume that

$$\psi = \chi\omega^k \text{ is even, and } \psi^* = \psi^{-1}\omega = \chi\omega^{p-k} \text{ is odd}$$

with $2 \leq k \leq p-2$. Since p does not divide f,

$$\text{(C)} \qquad \psi(p) \neq 1 \text{ and } \psi^*(p) \neq 1.$$

Table 1 Exceptional pairs for $|d| < 200$ and $200{,}000 < p < 1{,}000{,}000$

$[\nu]$			$[a_0]$		
p	k	d	p	k	d
240,571	146,919	−43	241,817	134,764	53
289,897	186,889	−131	290,627	50,599	−151
384,487	13,724	161	292,801	242,013	−104
384,847	226,771	−143	333,581	180,787	−71
386,119	263,582	149	399,181	1683	−4
401,321	205,162	185	788,687	186,548	141
937,943	11,057	−167			
$[b_0]$			[lmd]		
p	k	d	p	k	d
292,157	48,631	−111	245,177	59,489	−20
434,389	402,352	93	312,089	21,817	−159
512,891	91,273	−120	372,871	329,947	−104
516,323	63,368	136	429,427	61,972	92
541,759	285,435	−71	483,773	271,222	33
570,781	405,689	−52	509,581	402,749	−195
785,303	359,267	−67	667,727	487,990	113
800,447	136,068	161	768,013	754,145	−111
			794,141	494,244	165
			911,831	821,980	165

By (C), we have that $A_n^{\psi} \simeq X_{\infty}^{\psi}/\omega_n X_{\infty}^{\psi}$ and $A_n^{\psi^*} \simeq X_{\infty}^{\psi^*}/\omega_n X_{\infty}^{\psi^*}$, where $\omega_n = (1+T)^{p^n} - 1$ (cf. [6, Lemma 3 and Remark 4]). Moreover, if A_0^{ψ} is trivial, we have $\lambda(\psi) = \nu(\psi) = 0$, $X_{\infty}^{\psi} = \{0\}$, $Y_{\infty}^{\psi} \simeq \Lambda/(g_{\psi}(T))$ and $X_{\infty}^{\psi^*} \simeq \Lambda/(g_{\psi}^*(T))$. Put $a_0 = a_0(\psi) = L_p(1,\psi) = G_{\psi}(0)$ and $b_0 = b_0(\psi) = L_p(0,\psi) = G_{\psi}^*(0)$. Note that $v_p(a_0) = v_p(\sharp \mathrm{Gal}(M_0/K_0)^{\psi})$ and $v_p(b_0) = v_p(\sharp \mathrm{Gal}(L_0/K_0)^{\psi^*})$.

We call $(p, \chi\omega^k)$ exceptional pairs when one of the following conditions holds: $[\nu]: \nu(\chi\omega^k) > 0$, $[a_0]: v_p(a_0) > 1$, $[b_0]: v_p(b_0) > 1$ or [lmd]: $\tilde{\lambda}(\chi\omega^k) > 1$. In [11–14], we computed exceptional pairs for $|d| < 200$ and $p < 200{,}000$. By further computation, we obtain the following.

Proposition 2.1 *For $|d| < 200$ and $200{,}000 < p < 1{,}000{,}000$, all exceptional pairs $(p, \chi\omega^k)$ are given in Table 1.*

To study Problem 1.1 efficiently, we consider the following problem.

Problem 2.1 *Let X be a set of primitive Dirichlet characters. Let $N_{X,x_0}^{[\nu]}(x)$ be the number of pairs $(p, \chi\omega^k)$ such that $\nu(\chi\omega^k) > 0$ for $\chi \in X$ in the range $x_0 \le p \le x$. We similarly define $N_{X,x_0}^{[a_0]}(x)$, $N_{X,x_0}^{[b_0]}(x)$ and $N_{X,x_0}^{[lmd]}(x)$. Are they bounded as $x \to \infty$? If they are not so, give approximate functions for them.*

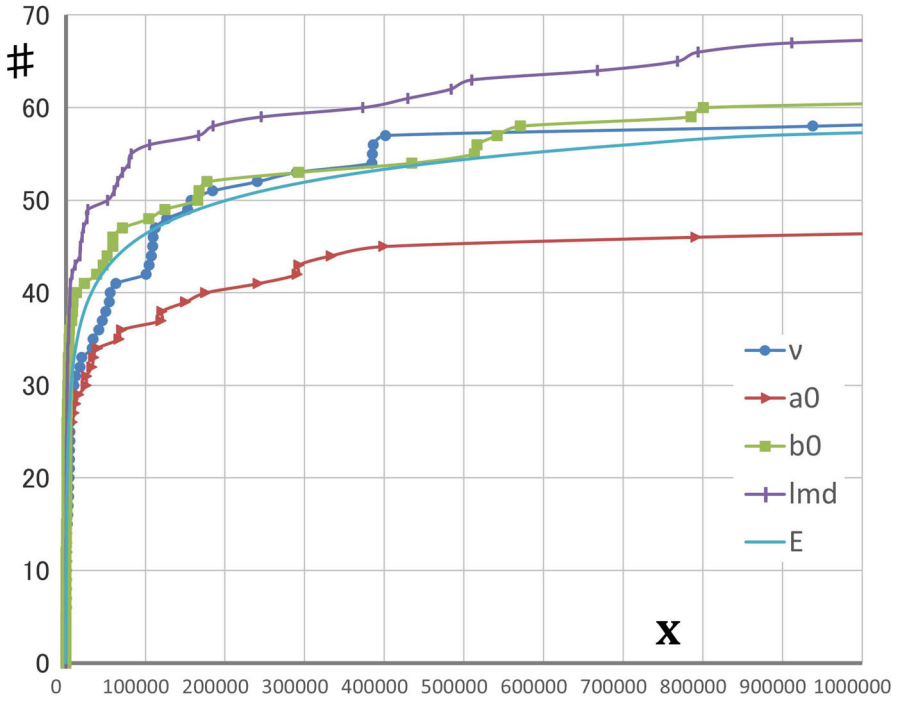

Fig. 1 Actual numbers and the expected number ($200 < p < 1{,}000{,}000$)

In [13], when X is a set of the trivial character or quadratic characters, we give a candidate of approximate functions:

$$E(x) = \sharp X \sum_{x_0 < p \le x} \frac{p-3}{2} \left(\frac{1}{p}\right)^2$$

which comes from a heuristic argument similar to that of [15, pp.158–159]. In Fig. 1, we compare actual numbers of exceptional pairs in the range $f_\chi < 200$ and $200 < p \le x$ with the expected number $E(x)$ ($x_0 = 200$ and $\sharp X = 123$). From our data, actual numbers still seem to be near the expected number.

For $|d| < 10$, we obtain the following by further computation.

Proposition 2.2 *For $|d| < 10$ and $10 < p < 20{,}000{,}000$, all exceptional pairs $(p, \chi\omega^k)$ are given in Table 2.*

Remark 2.1 In [14, Proposition 2], we reported that there is only one exceptional pair $(399181, \chi_{-4}\omega^{1683})$ in the range $|d| < 10$ and $200{,}000 < p < 1{,}000{,}000$, which is included in [lmd] by mistake.

Table 2 Exceptional pairs for $|d| < 10$ and $10 < p < 20{,}000{,}000$

[ν]			[a_0]		
p	k	d	p	k	d
379	317	−4	59	36	8
34,301	114	8	1381	609	−4
157,229	140,434	8	399,181	1683	−4
			5,911,877	1,629,992	5
[b_0]			[lmd]		
p	k	d	p	k	d
173	97	−7	23	11	−8
257	101	−3	1151	842	8
2221	1600	8	3613	1147	−7
4,953,979	1,174,520	5	27,791	11,840	8
			1,744,817	928,867	−3

Table 3 Expected numbers of exceptional pairs up to $x = 10^n$

$x\backslash d$	1	5	8	−3	−4	−7	−8
10^2	0.06880	0.19020	0.28139	0.12712	0.19738	0.17111	0.15889
10^3	0.24468	0.37505	0.42811	0.25481	0.33259	0.38703	0.33137
10^4	0.38967	0.51498	0.56373	0.39079	0.47707	0.52158	0.47146
10^5	0.49996	0.62089	0.67539	0.50485	0.58772	0.63171	0.58758
10^6	0.59054	0.71359	0.76666	0.59531	0.67887	0.72346	0.67885
10^7	0.66785	0.79089	0.84353	0.67206	0.75614	0.80081	0.75607

3 A Conjecture on the Number of Exceptional Pairs

We give two partial reasons why it is difficult to find exceptional pairs for $\chi = \chi^0$, i.e., $d = 1$. The first partial reason is the fact that the expected number for χ^0 is smaller than those for the other characters. Let $r_{p,\chi}$ be the number of pairs $(p, \chi\omega^k)$ such that $\tilde{\lambda}_p(\chi\omega^k) > 0$. Then we have $0 \le r_{p,\chi} \le (p-3)/2$. The distribution of the number of p such that $r_{p,\chi} = r$ for each χ in the range $200 < p < 20{,}000{,}000$ is similar to that for χ^0. However, the distribution for χ in the range $10 < p < 200$ is not always similar to that for χ^0, which affects expected numbers. For each χ, put $E_\chi(x) = \sum_{10<p\le x,\ p:\text{prime}} \frac{r_{p,\chi}}{p}$. The following table and figures show differences among $E_\chi(x)$s.

In Figs. 2 and 3, we compare $E_\chi(x)$s with $E'(x) = \sum_{37\le p\le x} \frac{p-3}{2p^2}$ and $E''(x) = \sum_{37\le p\le x} \frac{p-17}{2p^2}$, where $E''(x)$ is defined by considering trivialities of $K_4(\mathbf{Z})$ and numerators of Bernoulli numbers B_{2i} for i =1–5 and 7 (see [11, §4.2]).

Table 4 shows the total number N_d of exceptional pairs in $200 < p < 1{,}000{,}000$ for each d, the distribution (%) and a Poisson distribution with $\lambda = 4$

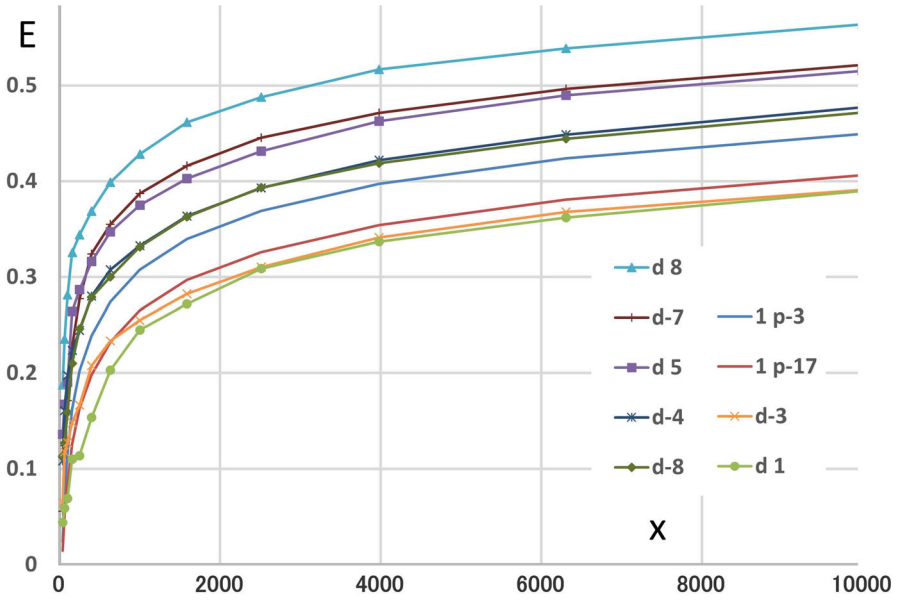

Fig. 2 Expected numbers of exceptional pairs up to 10,000

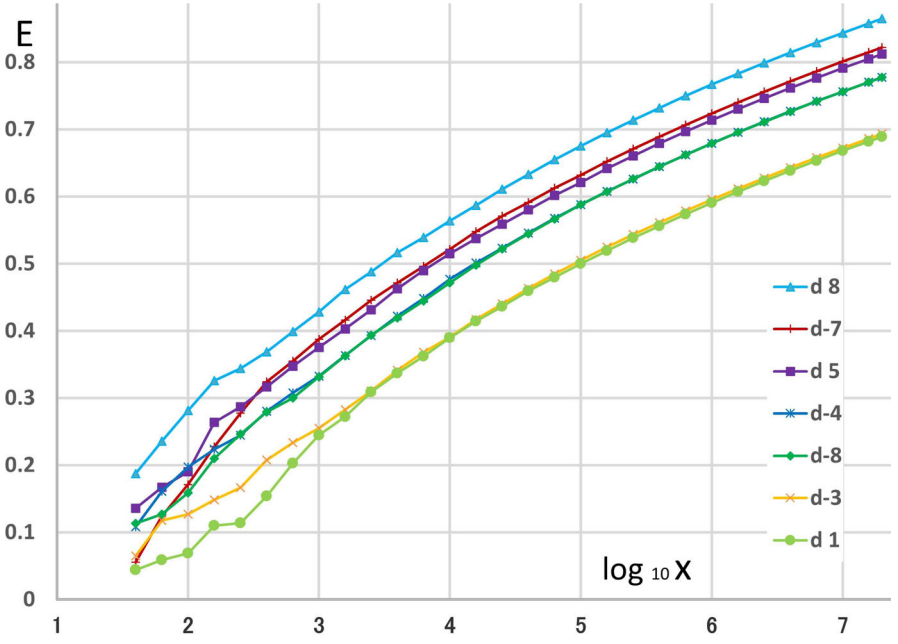

Fig. 3 Expected numbers of exceptional pairs up to 20,000,000

$\sum_{200<p<1,000,000} \frac{p-3}{2p^2} = 1.8721\cdots$. Since there are 23 d's with $N_d = 0$ among 123 d's, it is not very unusual that $N_1 = 0$. This is the second partial reason.

From our data and computational results on p-adic L-functions, it is natural to consider the following conjecture.

Conjecture 3.1 *Let X be a set of primitive Dirichlet characters. For $\chi \in X$, let n_χ be the order of χ, $\mathcal{O}_{p,\chi}$ the integer ring of $\mathbf{Q}_p(\zeta_{n_\chi})$, and $\mathfrak{m}_{p,\chi}$ the maximal ideal of $\mathcal{O}_{p,\chi}$. For a sufficiently large number x_0, an approximate function of $N^{[\nu]}_{X,x_0}(x)$, $N^{[a_0]}_{X,x_0}(x)$, $N^{[b_0]}_{X,x_0}(x)$ and $N^{[lmd]}_{X,x_0}(x)$ is given by the sum of*

$$\begin{aligned}\sum_{x_0\le p\le x} \frac{p-3}{2}\left(\frac{1}{\sharp(\mathcal{O}_{p,\chi}/\mathfrak{m}_{p,\chi})}\right)^2 &= \sum_{\substack{x_0\le p\le x\\ p\equiv 1 \bmod n_\chi}} \frac{p-3}{2}\left(\frac{1}{p}\right)^2 + \sum_{\substack{x_0\le p\le x\\ p\not\equiv 1 \bmod n_\chi}} \frac{p-3}{2}\left(\frac{1}{p^{f_{p,\chi}}}\right)^2\\ &\approx \frac{1}{\varphi(n_\chi)} \sum_{x_0\le p\le x} \frac{p-3}{2p^2} + O(1)\\ &\approx (\log\log x - \log\log x_0)/(2\varphi(n_\chi)) + O(1),\end{aligned}$$

over $\chi \in X$, where $p^{f_{p,\chi}} = \sharp(\mathcal{O}_{p,\chi}/\mathfrak{m}_{p,\chi}) \ge p^2$.

Remark 3.1 By replacing $\mathbf{Z}_p$ by $\mathcal{O}_{p,\chi}$, we can define $N^{[\nu]}_{X,x_0}(x)$, …and $N^{[lmd]}_{X,x_0}(x)$. If χ is not the trivial character nor a quadratic character, then $\varphi(n_\chi) \ge 2$ and the expected number is clearly smaller than $\sum_{x_0\le p\le x} \frac{p-3}{2p^2}$. This is a reason why we first study characters with $n_\chi \le 2$.

4 Computations of Arithmetic Elements

To study Iwasawa invariants, we compute the following arithmetic elements (see [12, §5]):

(I) the generalized Bernoulli numbers modulo p, i.e., $\sum_{k=0}^{p-3} B_{k,\chi} t^k/k! \bmod p$,
(II)$_n$ the Iwasawa polynomial $g_{\chi\omega^k}(T) \mod p^{n+1}$,
(III)$_n$ the special cyclotomic unit $(c_n^{\chi\omega^k})^{Y_n(T)}$ modulo a prime ideal $\mathfrak{L}_n$,
(IV)$_n$ the Gauss sum $g_0(N_{K_n/K_0}\mathfrak{L}_n)^{\chi\omega^{p-k}}$ modulo a prime ideal $\mathfrak{L}_0^*$, where $\mathfrak{L}_n$ (resp. $\mathfrak{L}_n^*$) is a prime ideal above $l = 1 + \kappa f_n$ (resp. $l^* = 1 + \kappa^*(2 f_n l)$) of K_n.

From 2002 to 2007, we used 32-bit programs (bcn.c for even χ's and bcm.c for odd ones) for computations of (I), (II)$_1$ and (III)$_0$ in [11–14]. These programs work when f_0 and $8(p-3)\log_{16}(2p^3)$ are smaller than 2^{31}. Therefore, for $f = 1$ (resp. 199), they do not work when $p > 15{,}000{,}000$ (resp. 11,000,000). Since 2015, inspired by [2], we have used 64-bit programs (bcn64.c and bcm64.c), which work when p is smaller than 162 million. Further, the program is available to check Greenberg's conjecture by computing (III)$_1$. However, since it needs $O(f_1^{1+\epsilon})$ bit operations, we did not check it for $p > 100{,}000$ except for $(p,k,d) = (157229, 140434, 8)$.

Table 4 The total number of exceptional pairs for d in $200 < p < 1{,}000{,}000$

N_d	♯	%	Poisson	d
0	23	18.699	15.380	1,5, −11, 13, −15, 24, −24, −35, −40, −56, −68, 69, −87, −95 −103, −107, −123, 129, −132, −148, −164, 173, −191
1	35	28.455	28.793	−3, −7, 12, 17, −23, 29, −31, −55, −59, 65, 73, 77, −83, 85, 88 89, 92, 93, 101, 104, −119, 120, 137, −143, 145, −155, 168 172, −179, −187, 188, −195, 197, −199
2	30	24.390	26.952	−8, −20, 21, 33, 37, 40, 41, −43, 44, −47, 57, 60, 61, 76, −84 −88, 109, −184, −111, 113, −115, 136, 140, 152, −159, −163 177, −183, 184, 185, 193
3	16	13.008	16.819	−4, 28, −51, −52, 56, 97, 105, −127, 133, −136, −151, −152 157, −167, −168, 181
4	11	8.9431	7.8716	−39, 53, −67, −91, −116, −120, −139, 141, 156, 161, 165
5	4	3.2520	2.9472	8, −71, −104, 124
6	3	2.4390	0.9196	−19, −79, 149
7	1	0.8130	0.2459	−131

Table 5 The ratios of execution times

p	157,229	937,943	999,983	999,983	19,999,873	19,999,999
d	8	−167	1	−199	1	−7
(I)	1	41	5.4	47	280	320
$(\mathrm{II})_1$	0.5	190	1.3	240	48	300
$(\mathrm{III})_0$	0.4	74	1.1	94	38	83
$(\mathrm{IV})_0$	10	$*$	–	–	–	–
$(\mathrm{IV})_{0d}$	44	9000	–	–	–	–

Computation of $(\mathrm{IV})_0$ rigorously proves that the cyclotomic unit $c_0^{\chi\omega^k}$ is a p-th power element in K_0, i.e., $\nu_p(\chi\omega^k) > 0$. From 2002 to 2007, we used 32-bit programs (gauss.c for small f_0's and gaussd.c for large ones) for computation of $(\mathrm{IV})_0$, which work when $2l$ is smaller that 2^{31}. Therefore, for $f = 1$ (resp. 199), they do not work when $p > 11{,}000{,}000/\kappa$ (resp. $550{,}000/\kappa$) with κ =2–24. To reduce memory usage, we use HDD and several auxiliary primes $l_i \approx 1000$ in gaussd.c, which slows down the computation. Since 2020, we have been used 64-bit programs (gauss64.c and gaussd64.c), which work when $2l$ and $8f_0 \log_{16}(2l^{*2} f_0)$ are smaller than 2^{63}. To reduce memory usage, we use HDD and several auxiliary primes $l_i \approx 100{,}000$ in gaussd64.c.

Computations of (I), $(\mathrm{II})_1$ and $(\mathrm{III})_0$ (resp. $(\mathrm{IV})_0$) are executed in $O(f_0^{1+\epsilon})$ (resp. $O((\kappa f_0)^{1+\epsilon})$) bit operations. In Table 5, we give the ratios of execution times for (I)–(IV) by a single thread program on a Linux PC, where $*$ means that we did not compute it for lack of RAM.

For computations of (I) and $(\mathrm{IV})_0$, we need large RAM for the FFT algorithm. Various methods in [5] will be useful for speed-up and reduction of memory usage in computations of (I), $(\mathrm{II})_1$ and $(\mathrm{III})_0$. It will be able to speed up computations of $(\mathrm{II})_1$ and $(\mathrm{III})_0$ by parallel computing with GPU.

The above programs and further data have been available in our web page: https://math0.pm.tokushima-u.ac.jp/~hiroki/major/galois1-e.html. These data were obtained by six personal computers for several years. They agree with previous computations by several authors when p is small. Though temporary hardware errors are unusual, they could occur in long-term computation. We computed twice for $|d| < 200$ and $p < 1{,}000{,}000$, and only once for $|d| < 10$ and $1{,}000{,}000 < p < 20{,}000{,}000$. As we fixed the data for some temporary hardware errors in the former computation, we are afraid that they occur in the latter computation. However, irregular pairs are double-checked by (I) and $(\mathrm{II})_1$, and the number of pairs is near the expected number. Therefore these unusual errors would not affect Proposition 2.2.

Acknowledgements The author is grateful to the referee for carefully reading the original manuscript and for valuable comments.

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