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RESEARCH CONTRIBUTION

The $\mathbb{F}_p$-Selberg Integral

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Abstract

We prove an $\mathbb{F}_p$-Selberg integral formula, in which the $\mathbb{F}_p$-Selberg integral is an element of the finite field $\mathbb{F}_p$ with odd prime number p of elements. The formula is motivated by the analogy between multidimensional hypergeometric solutions of the KZ equations and polynomial solutions of the same equations reduced modulo p.

Keywords Selberg integral · $\mathbb{F}_p$-integral · Morris' identity · Aomoto recursion · KZ equations · Reduction modulo p

Mathematics Subject Classification 13A35 · 33C60 · 32G20

Contents

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1 Introduction

In 1944 Atle Selberg proved the following integral formula:

$$\int_0^1 \cdots \int_0^1 \prod_{1\le i<j\le n} (x_i - x_j)^{2\gamma} \prod_{i=1}^{n} x_i^{\alpha-1}(1-x_i)^{\beta-1}\, \mathrm{d}x_1 \ldots \mathrm{d}x_n$$
$$= \prod_{j=1}^{n} \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)}, \tag{1.1}$$

see [2,22].[1] Hundreds of papers are devoted to the generalizations of the Selberg integral formula and its applications, see for example [2,11] and references therein. There are q-analysis versions of the formula, the generalizations associated with Lie algebras, elliptic versions, finite field versions, see some references in [1,2,4,5,7–9,11–13,18,19,24,28–31,39–41]. In the finite field versions, one considers additive and multiplicative characters of a finite field, which map the field to the field of complex numbers, and forms an analog of Eq. (1.1), in which both sides are complex numbers. The simplest of such formulas is the classical relation between Jacobi and Gauss sums, see [1,2,7].

In this paper we suggest another version of the Selberg integral formula, in which the $\mathbb{F}_p$-Selberg integral is an element of the finite field $\mathbb{F}_p$ with an odd prime number p of elements, see Theorem 4.1.

Our motivation comes from the theory of the Knizhnik–Zamolodchikov (KZ) equations, see [6,14]. These are the systems of linear differential equations, satisfied by conformal blocks on the sphere in the WZW model of conformal field theory. The KZ

[1] In [23] Selberg remarks: "This paper was published with some hesitation, and in Norwegian, since I was rather doubtful that the results were new. The journal is one which is read by mathematics-teachers in the gymnasium, and the proof was written out in some detail so it should be understandable to someone who knew a little about analytic functions and analytic continuation." See more in [11].

equations were solved in multidimensional hypergeometric integrals in [25], see also [32,33]. The following general principle was formulated in [17]: if an example of the KZ type equations has a one-dimensional space of solutions, then the corresponding multidimensional hypergeometric integral can be evaluated explicitly. As an illustration of that principle in [17], an example of KZ equations with a one-dimensional space of solutions was considered, the corresponding multidimensional hypergeometric integral was reduced to the Selberg integral and then evaluated by formula (1.1). Other illustrations see in [9,10,20,28–30,34].

Recently in [26] the KZ equations were considered modulo a prime number p and polynomial solutions of the reduced equations were constructed, see also [27, 35–38]. The construction is analogous to the construction of the multidimensional hypergeometric solutions, and the constructed polynomial solutions were called the $\mathbb{F}_p$-hypergeometric solutions.

In this paper we consider the reduction modulo p of the same example of the KZ equations, that led in [17] to the Selberg integral. The space of solutions of the reduced KZ equations is still one-dimensional and, according to the principle, we may expect that the corresponding $\mathbb{F}_p$-hypergeometric solution is related to a Selberg type formula. Indeed we have evaluated that $\mathbb{F}_p$-hypergeometric solution by analogy with the evaluation of the Selberg integral and obtained our $\mathbb{F}_p$-Selberg integral formula in Theorem 4.1.

The paper contains three proofs of our $\mathbb{F}_p$-Selberg integral formula. There might be more proofs. It would be interesting to see if our formula can be deduced from the known relations between the multidimensional Gauss and Jacobi sums, see for example [2, Section 8.11].

The paper is organized as follows. In Sect. 2 we collect useful facts. In Sect. 3 we introduce the notion of $\mathbb{F}_p$-integral and discuss the integral formula for the $\mathbb{F}_p$-beta integral. In Sect. 4 we formulate our main result, Theorem 4.1, and prove it by developing an $\mathbb{F}_p$-analog of Aomoto's recursion, defined in [3] for the Selberg integral. In Sect. 5 we give another proof of Theorem 4.1, based on Morris' identity, which is deduced from the classical Selberg integral formula (1.1) in [16]. In Sect. 6 we sketch a third proof of Theorem 4.1 based on a combinatorial identity, also deduced from the Selberg integral formula (1.1). In Sect. 7 we discuss in more detail how our $\mathbb{F}_p$-Selberg integral formula is related to the $\mathbb{F}_p$-hypergeometric solutions of KZ equations reduced modulo p.

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2 Preliminary Remarks

2.1 Lucas' Theorem

Theorem 2.1 [15] *For nonnegative integers m and n and a prime p, the following congruence relation holds*:

$$\binom{n}{m} \equiv \prod_{i=0}^{a} \binom{n_i}{m_i} \pmod{p}, \tag{2.1}$$

where $m = m_b p^b + m_{b-1} p^{b-1} + \cdots + m_1 p + m_0$ *and* $n = n_b p^b + n_{b-1} p^{b-1} + \cdots + n_1 p + n_0$ *are the base* p *expansions of* m *and* n *respectively. This uses the convention that* $\binom{n}{m} = 0$ *if* $n < m$. □

2.2 Binomial Lemma

Lemma 2.2 [38] *Let* a, b *be positive integers such that* $a < p$, $b < p$, $p \le a + b$. *Then we have an identity in* $\mathbb{F}_p$,

$$b \binom{b-1}{a+b-p} = b \binom{b-1}{p-a-1} = (-1)^{a+1} \frac{a!\, b!}{(a+b-p)!}. \tag{2.2}$$

□

2.3 Cancellation of Factorials

Lemma 2.3 *If* a, b *are nonnegative integers and* $a + b = p - 1$, *then in* $\mathbb{F}_p$ *we have*

$$a!\, b! = (-1)^{a+1}. \tag{2.3}$$

Proof We have $a! = (-1)^a (p-1) \cdots (p-a)$ and $p - a = b + 1$. Hence $a!\, b! = (-1)^a (p-1)! = (-1)^{a+1}$ by Wilson's Theorem. □

3 $\mathbb{F}_p$-Integrals

3.1 Definition

Let p be an odd prime number and M an $\mathbb{F}_p$-module. Let $P(x_1, \ldots, x_k)$ be a polynomial with coefficients in M,

$$P(x_1, \ldots, x_k) = \sum_d c_d\, x_1^{d_1} \ldots x_k^{d_k}. \tag{3.1}$$

Let $l = (l_1, \ldots, l_k) \in \mathbb{Z}^k_{>0}$. The coefficient $c_{l_1 p-1, \ldots, l_k p-1}$ is called the $\mathbb{F}_p$-*integral over the cycle* $[l_1, \ldots, l_k]_p$ and is denoted by $\int_{[l_1, \ldots, l_k]_p} P(x_1, \ldots, x_k)\, \mathrm{d}x_1 \ldots \mathrm{d}x_k$.

Lemma 3.1 *For* $i = 1, \ldots, k-1$ *we have*

$$\int_{[l_1, \ldots, l_{i+1}, l_i, \ldots, l_k]_p} P(x_1, \ldots, x_{i+1}, x_i, \ldots, x_k) \mathrm{d}x_1 \ldots \mathrm{d}x_k$$

$$= \int_{[l_1,\dots,l_k]_p} P(x_1,\dots,x_k)\, dx_1 \dots dx_k. \tag{3.2}$$

□

Lemma 3.2 *For any $i = 1,\dots,k$, we have*

$$\int_{[l_1,\dots,l_k]_p} \frac{\partial P}{\partial x_i}(x_1,\dots,x_k) = 0.$$

□

3.2 $\mathbb{F}_p$-Beta Integral

For nonnegative integers a, b the classical beta integral formula says

$$\int_0^1 x^a(1-x)^b dx = \frac{a!\,b!}{(a+b+1)!}. \tag{3.3}$$

Theorem 3.3 [38] *Let $0 \le a < p$, $0 \le b < p$, $p-1 \le a+b$. Then in $\mathbb{F}_p$ we have*

$$\int_{[1]_p} x^a(1-x)^b dx = -\frac{a!\,b!}{(a+b-p+1)!}. \tag{3.4}$$

If $a+b < p-1$, then

$$\int_{[1]_p} x^a(1-x)^b dx = 0. \tag{3.5}$$

Proof We have $x^a(1-x)^b = \sum_{k=0}^{b}(-1)^k\binom{b}{k}x^k$, and need $a+k = p-1$. Hence $k = p-1-a$ and

$$\int_{[1]_p} x^a(1-x)^b dx = (-1)^{p-1-a}\binom{b}{p-1-a}.$$

Now Lemma 2.2 implies (3.4). Formula (3.5) is clear. □

4 *n*-Dimensional $\mathbb{F}_p$-Selberg and $\mathbb{F}_p$-Aomoto Integrals

4.1 *n*-Dimensional Integral Formulas

The n-dimensional Selberg integral formula for nonnegative integers a, b, c is

$$\int_0^1 \dots \int_0^1 \prod_{1\le i<j\le n} (x_i - x_j)^{2c} \prod_{i=1}^{n} x_i^a(1-x_i)^b \, dx_1 \dots dx_n$$

$$= \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(a+(j-1)c)!\,(b+(j-1)c)!}{(a+b+(n+j-2)c+1)!}, \tag{4.1}$$

and for $k = 1, \dots, n-1$, the n-dimensional Aomoto integral formula is

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} x_i \prod_{1\le i<j\le n} (x_i - x_j)^{2c} \prod_{i=1}^{n} x_i^a (1-x_i)^b \, \mathrm{d}x_1 \dots \mathrm{d}x_n$$
$$= \prod_{j=1}^{k} \frac{a+(n-j)c+1}{a+b+(2n-j-1)c+1} \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(a+(j-1)c)!\,(b+(j-1)c)!}{(a+b+(n+j-2)c+2)!}, \tag{4.2}$$

[2,3,22].

Theorem 4.1 *Assume that a, b, c are nonnegative integers such that*

$$p-1 \le a+b+(n-1)c, \quad a+b+(2n-2)c < 2p-1. \tag{4.3}$$

Then we have an integral formula in $\mathbb{F}_p$:

$$\int_{[1,\dots,1]_p} \prod_{1\le i<j\le n} (x_i - x_j)^{2c} \prod_{i=1}^{n} x_i^a (1-x_i)^b \, \mathrm{d}x_1 \dots \mathrm{d}x_n$$
$$= (-1)^n \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(a+(j-1)c)!\,(b+(j-1)c)!}{(a+b+(n+j-2)c+1-p)!}. \tag{4.4}$$

Also, if $k = 1, \dots, n-1$, and

$$p-1 \le a+b+(n-1)c, \quad a+b+(2n-2)c < 2p-2, \tag{4.5}$$

then

$$\int_{[1,\dots,1]_p} \prod_{i=1}^{k} x_i \prod_{1\le i<j\le n} (x_i - x_j)^{2c} \prod_{i=1}^{n} x_i^a (1-x_i)^b \, \mathrm{d}x_1 \dots \mathrm{d}x_n$$
$$= (-1)^n \prod_{j=1}^{k} \frac{a+(n-j)c+1}{a+b+(2n-j-1)c+2} \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(a+(j-1)c)!\,(b+(j-1)c)!}{(a+b+(n+j-2)c+1-p)!}. \tag{4.6}$$

The first proof of Theorem 4.1 is given in Sects. 4.2–4.4, the second in Sect. 5, and the third one is sketched in Sect. 6.

Remark Formula (4.4) can be rewritten as

$$\sum_{x_1,\dots,x_n\in\mathbb{F}_p}\prod_{1\le i<j\le n}(x_i-x_j)^{2c}\prod_{i=1}^n x_i^a(1-x_i)^b$$
$$=\prod_{j=1}^n\frac{(jc)!}{c!}\,\frac{(a+(j-1)c)!(b+(j-1)c)!}{(a+b+(n+j-2)c+1-p)!},\tag{4.7}$$

if additionally $a+b+(2n-2)c<2p-2$.

Remark The fact that the $\mathbb{F}_p$-Selberg integral on the left-hand side of (4.4) equals an explicit alternating product on the right-hand side of (4.4) is surprising. But even more surprising is the fact that the alternating product on the right-hand side of (4.4) is simply given by the alternating product on the right-hand of the classical formula (4.1) with just several of factorials shifted by p.

Remark The Selberg integral (4.1) is related to the $\mathfrak{sl}_2$ KZ differential equations, see Sect. 7, and is called the Selberg integral of type A_1. The Selberg integral of type A_n, related to the $\mathfrak{sl}_{n+1}$ KZ differential equations, is introduced in [30,39–41].

We call the $\mathbb{F}_p$-integral (4.4) the $\mathbb{F}_p$-Selberg integral of type A_1. The $\mathbb{F}_p$-Selberg integral of type A_n, $n>1$, is introduced in [21]. The $\mathbb{F}_p$-Selberg integral formula of type A_n is deduced in [21] from the $\mathbb{F}_p$-Selberg integral formula (4.4) by induction on n.

Remark The integral analogous to (4.4) but with x_i-x_j factors raised to an odd power vanishes:

$$\int_{[1,\dots,1]_p}\prod_{1\le i<j\le n}(x_i-x_j)^{2c+1}\prod_{i=1}^n x_i^a(1-x_i)^b\,\mathrm{d}x_1\dots\mathrm{d}x_n=0.\tag{4.8}$$

Indeed, after expanding the $(x_1-x_2)^{2c+1}$ factor, the integral (4.8) equals

$$\sum_{m=0}^{2c+1}(-1)^{m+1}\binom{2c+1}{m}\int_{[1,\dots,1]_p}x_1^{a+m}x_2^{a+(2c+1-m)}f(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n=0,$$

with f symmetric in x_1 and x_2. The terms corresponding to m and $2c+1-m$ cancel each other, making the sum 0.

4.2 Auxiliary Lemmas

Denote

$$P_n(a,b,c)=(-1)^n\prod_{j=1}^n\frac{(jc)!}{c!}\,\frac{(a+(j-1)c)!\,(b+(j-1)c)!}{(a+b+(n+j-2)c+1-p)!}.\tag{4.9}$$

The polynomial

$$\Phi(x_1,\dots,x_n,a,b,c)=\prod_{1\le i<j\le n}(x_i-x_j)^{2c}\prod_{i=1}^n x_i^a(1-x_i)^b$$

is called the *master polynomial*. Denote

$$S_n(a,b,c)=\int_{[1,\dots,1]_p}\prod_{1\le i<j\le n}(x_i-x_j)^{2c}\prod_{i=1}^n x_i^a(1-x_i)^b\mathrm{d}x_1\dots\mathrm{d}x_n,$$

$$S_{k,n}(a,b,c)=\int_{[1,\dots,1]_p}\prod_{i=1}^k x_i\prod_{1\le i<j\le n}(x_i-x_j)^{2c}\prod_{i=1}^n x_i^a(1-x_i)^b\,\mathrm{d}x_1\dots\mathrm{d}x_n,$$

for $k=0,\dots,n$. Then $S_{0,n}(a,b,c)=S_n(a,b,c)$, $S_{n,n}(a,b,c)=S_n(a+1,b,c)$. By (3.2), we also have

$$S_{k,n}(a,b,c)=\int_{[1,\dots,1]_p}\prod_{i=1}^k x_{\sigma_i}\prod_{1\le i<j\le n}(x_i-x_j)^{2c}\prod_{i=1}^n x_i^a(1-x_i)^b\,\mathrm{d}x_1\dots\mathrm{d}x_n$$

for any $1\le\sigma_1<\dots<\sigma_k\le n$.

Lemma 4.2 *We have* $S_n(a,b+p,c)=S_n(a,b,c)$.

Proof We have $(1-x_i)^{b+p}=(1-x_i)^b(1-x_i)^p=(1-x_i)^b(1-x_i^p)$. Hence the factors $(1-x_i)^b$ and $(1-x_i)^{b+p}$ contribute to the coefficient of x_i^{p-1} in the same way. □

Lemma 4.3 *If* $a+b+(2n-2)c<2p-2$ *and* $c>0$, *then* $n<p$. □

Lemma 4.4 *If* $a+b+(n-1)c<p-1$, *then* $S_n(a,b,c)=0$.

Proof The coefficient of $x_1^{p-1}\dots x_n^{p-1}$ in the expansion of $\Phi(a,b,c)$ equals zero. □

Lemma 4.5 *If* $p\le a+(n-1)c$, *then* $S_n(a,b,c)=0$.

Proof Expand $\Phi(x,a,b,c)$ in monomials $x_1^{d_1}\dots x_n^{d_n}$. If $p\le a+(n-1)c$, then each monomial $x_1^{d_1}\dots x_n^{d_n}$ in the expansion has at least one of $d_1,\dots,d_n$ greater than $p-1$, Hence the coefficient of $x_1^{p-1}\dots x_n^{p-1}$ in the expansion equals zero. □

Lemma 4.6 *If* $a+b+(2n-2)c<2p-1$, *then* $S_n(a,b,c)=S_n(b,a,c)$.

Proof Expand $\Phi(x,a,b,c)$ in monomials $x_1^{d_1}\dots x_n^{d_n}$. If $a+b+(2n-2)c<2p-1$, then
(a) for each monomial $x_1^{d_1}\dots x_n^{d_n}$ in the expansion all of $d_1,\dots,d_n$ are less than $2p-1$.

We also have

$$\Phi(1-y_1,\dots,1-y_n,a,b,c)=\prod_{1\le i<j\le n}(y_i-y_j)^{2c}\prod_{i=1}^{n}y_i^a(1-y_i)^b.$$

This transformation does not change the $\mathbb{F}_p$-integral due to Lucas' Theorem and property (a), see a similar reasoning in the proof of [36, Lemma 5.2]. □

4.3 Case $a+b+(n-1)c=p-1$

Lemma 4.7 *If* $a+b+(n-1)c=p-1$, *then*

$$S_n(a,b,c)=(-1)^{bn+cn(n-1)/2}\frac{(cn)!}{(c!)^n}. \tag{4.10}$$

Proof If $a+b+(n-1)c=p-1$, then $S_n(a,b,c)$ equals $(-1)^{bn}$ multiplied by the coefficient of $(x_1\dots x_n)^c$ in $\prod_{1\le i<j\le n}(x_i-x_j)^{2c}$, which equals $(-1)^{cn(n-1)/2}\frac{(cn)!}{(c!)^n}$ by Dyson's formula

$$\text{C.T.}\prod_{1\le i<j\le n}(1-x_i/x_j)^c(1-x_j/x_i)^c=\frac{(cn)!}{(c!)^n}. \tag{4.11}$$

Here C.T. denotes the constant term. See the formula in [2, Section 8.8]. □

Lemma 4.8 *If* $a+b+(n-1)c=p-1$, *then*

$$P_n(a,b,c)=(-1)^{bn+cn(n-1)/2}\frac{(cn)!}{(c!)^n}. \tag{4.12}$$

Proof We have

$$\begin{aligned}P_n(a,b,c)&=(-1)^n\prod_{j=1}^{n}\frac{(jc)!}{c!}\,\frac{(a+(j-1)c)!\,(b+(j-1)c)!}{(a+b+(n+j-2)c+1-p)!}\\&=(-1)^n\prod_{j=1}^{n}\frac{(jc)!}{c!}\,\frac{(a+(j-1)c)!\,(b+(j-1)c)!}{c!\,(2c)!\dots((n-1)c)!}.\end{aligned}$$

By Lemma 2.2 we have $a!\,(b+(n-1)c)!=(-1)^{b+(n-1)c+1}$, $(a+c)!\,(b+(n-2)c)!=(-1)^{b+(n-2)c+1}$, and so on. This proves the lemma. □

Lemmas 4.7 and 4.8 prove formula (4.4) for $a+b+(n-1)c=p-1$.

4.4 Aomoto Recursion

We follow the paper [3], where recurrence relations were developed for the classical Selberg integral. See also [2, Section 8.2].

Using Lemma 3.2, for $k = 1, \dots, n$ we have

$$\begin{aligned}
0 &= \int_{[1,\dots,1]_p} \frac{\partial}{\partial x_1}\left[(1-x_1)\prod_{i=1}^{k} x_i \Phi(x,a,b,c)\right] \mathrm{d}x_1 \dots \mathrm{d}x_n \\
&= (a+1)\int_{[1,\dots,1]_p} (1-x_1)\prod_{i=2}^{k} x_i \Phi(x,a,b,c) \mathrm{d}x_1 \dots \mathrm{d}x_n \\
&\quad - (b+1)\int_{[1,\dots,1]_p} \prod_{i=1}^{k} x_i \Phi(x,a,b,c) \mathrm{d}x_1 \dots \mathrm{d}x_n \\
&\quad + 2c\int_{[1,\dots,1]_p} \sum_{j=2}^{n} \frac{1-x_1}{x_1-x_j} \prod_{i=1}^{k} x_i \Phi(x,a,b,c) \mathrm{d}x_1 \dots \mathrm{d}x_n.
\end{aligned} \tag{4.13}$$

Lemma 4.9 *The $\mathbb{F}_p$-integral*

$$\int_{[1,\dots,1]_p} \frac{1}{x_1-x_j} \prod_{i=1}^{k} x_i \, \Phi(x,a,b,c) \, \mathrm{d}x_1 \dots \mathrm{d}x_n \tag{4.14}$$

equals 0 *if* $2 \le j \le k$ *and equals* $S_{k-1,n}/2$ *if* $k < j \le n$. *The* $\mathbb{F}_p$*-integral*

$$\int_{[1,\dots,1]_p} \frac{x_1}{x_1-x_j} \prod_{i=1}^{k} x_i \, \Phi(x,a,b,c) \, \mathrm{d}x_1 \dots \mathrm{d}x_n \tag{4.15}$$

equals $S_{k,n}/2$ *if* $2 \le j \le k$ *and equals* $S_{k,n}$ *if* $k < j \le n$.

Proof By Lemma 3.1 each of these integrals does not change if x_1, x_j are permuted. The four statements of the lemma hold since $\frac{x_1x_j}{x_1-x_j} + \frac{x_1x_j}{x_j-x_1} = 0$, $\frac{x_1}{x_1-x_j} + \frac{x_j}{x_j-x_1} = 1$, $\frac{x_1^2x_j}{x_1-x_j} + \frac{x_1x_j^2}{x_j-x_1} = x_1x_j$, $\frac{x_1^2}{x_1-x_j} + \frac{x_j^2}{x_j-x_1} = x_1 + x_j$, respectively. □

Lemma 4.10 *For* $k = 1, \dots, n$ *we have*

$$S_{k,n} = \frac{a+(n-k)c+1}{a+b+(2n-k-1)c+2} S_{k-1,n}. \tag{4.16}$$

Proof Using Lemma 4.9 we rewrite (4.13) as

$$0 = (a+1)S_{k-1,n} - (a+b+2)S_{k,n} + c(n-k)S_{k-1,n} - c(2n-k-1)S_{k,n}.$$

□

4.5 Proof of Theorem 4.1

Theorem 4.1 is proved by induction on a and b. The base induction step $a + b + (n - 1)c = p - 1$ is proved in Sect. 4.3.

Lemma 4.10 gives

$$S_n(a+1,b,c) = S_n(a,b,c) \prod_{j=1}^{n} \frac{a+(n-j)c+1}{a+b+(2n-j-1)c+1}.$$

Together with the symmetry $S_n(a,b,c) = S_n(b,a,c)$ this gives formula (4.4). Then formula (4.16) gives formula (4.6). Theorem 4.1 is proved.

4.6 Relation to Jacobi Polynomials

The statements (4.6) for different values of k can be captured in a single equation, which involves a Jacobi polynomial—like it was done by K. Aomoto in [3] for the classical Selberg integral. Recall that the degree n Jacobi polynomial is

$$P^{(n)}_{\alpha,\beta}(x) = \frac{1}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} \prod_{i=1}^{\nu}(n+\alpha+\beta+i) \prod_{i=\nu+1}^{n} (\alpha+i) \left(\frac{x-1}{2}\right)^{\nu}.$$

Proposition 4.11 *Assuming inequalities* (4.5) *let* $\alpha = (a+1)/c-1$, $\beta = (b+1)/c-1$. *Then*

$$\int_{[1,\dots,1]_p} \prod_{i=1}^{n}(x_i - t) \cdot \Phi(x,a,b,c)\mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{n!c^n \cdot S_n(a,b,c)}{\prod_{i=n-1}^{2n-2}(a+b+ic+2)} \cdot P_n^{(\alpha,\beta)}(1-2t). \tag{4.17}$$

The proof is the same is in [3]: After expanding $\prod_{i=1}^{n}(x_i - t)$ we have the sum of integrals of the type

$$\int_{[1,\dots,1]_p} x_{\sigma_1}x_{\sigma_2}\dots x_{\sigma_k}\Phi(x,a,b,c)\mathrm{d}x_1\dots\mathrm{d}x_n,$$

which—by symmetry (3.2)—are equal to $S_{k,n}(a,b,c)$. Substituting

$$S_{k,n}(a,b,c) = S_n(a,b,c) \cdot \prod_{j=1}^{k} \frac{a+(n-j)c+1}{a+b+(2n-j-1)c+2}$$

from (4.4) and (4.6) yields (4.17).

5 $\mathbb{F}_p$-Selberg Integral from Morris' Identity

5.1 Morris' Identity

In this section we work out the integral formula (4.4) for the $\mathbb{F}_p$-Selberg integral from Morris' identity. Suppose that α, β, γ are nonnegative integers. Then

$$\begin{aligned} \text{C. T.} \prod_{i=1}^{n} (1-x_i)^{\alpha}(1-1/x_i)^{\beta} \prod_{1\le j\ne k\le n} (1-x_j/x_k)^{\gamma} \\ = \prod_{j=1}^{n} \frac{(j\gamma)!}{\gamma!} \frac{(\alpha+\beta+(j-1)\gamma)!}{(\alpha+(j-1)\gamma)!\,(\beta+(j-1)\gamma)!}. \end{aligned} \tag{5.1}$$

Morris identity was deduced in [16] from the integral formula (4.1) for the classical Selberg integral, see [2, Section 8.8].

The left-hand side of (5.1) can be written as

$$\text{C. T.}\, (-1)^{\binom{n}{2}\gamma+n\beta} \prod_{1\le i<j\le n} (x_i-x_j)^{2\gamma} \prod_{i=1}^{n} x_i^{-\beta-(n-1)\gamma} (1-x_i)^{\alpha+\beta}, \tag{5.2}$$

while

$$S_n(a,b,c) = \text{C. T.} \prod_{1\le i<j\le n} (x_i-x_j)^{2c} \prod_{i=1}^{n} x_i^{a+1-p}(1-x_i)^b, \tag{5.3}$$

where the constant term is projected to $\mathbb{F}_p$.

Putting $a+1-p = -\beta-(n-1)\gamma,\ b=\alpha+\beta, c=\gamma$, or

$$\alpha = a+b+(n-1)c+1-p, \quad \beta = p-a-(n-1)c-1, \quad \gamma = c. \tag{5.4}$$

we obtain the following theorem.

Theorem 5.1 *If the nonnegative integers a, b, c satisfy the inequalities*

$$p-1 \le a+b+(n-1)c, \quad a+(n-1)c \le p-1, \tag{5.5}$$

then the $\mathbb{F}_p$-Selberg integral is given by the formula:

$$\begin{aligned} S_n(a,b,c) = (-1)^{\binom{n}{2}c+na} \\ \times \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(b+(j-1)c)!}{(p-a-(n-j)c-1)!\,(a+b+(n+j-2)c+1-p)!}, \end{aligned} \tag{5.6}$$

where the integer on the right-hand side is projected to $\mathbb{F}_p$.

Lemma 5.2 *If both inequalities* (4.3) *and* (5.5) *hold, that is, if*

$$p-1 \le a+b+(n-1)c, \quad a+b+(2n-2)c < 2p-1, \tag{5.7}$$
$$a+(n-1)c \le p-1, \tag{5.8}$$

then in $\mathbb{F}_p$ *we have*

$$(-1)^{\binom{n}{2}c+na} \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(b+(j-1)c)!}{(p-a-(n-j)c-1)!\,(a+b+(n+j-2)c)!},$$
$$= (-1)^n \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(a+(j-1)c)!\,(b+(j-1)c)!}{(a+b+(n+j-2)c+1-p)!}, \tag{5.9}$$

and hence (5.6)

$$S_n(a,b,c) = (-1)^n \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(a+(j-1)c)!\,(b+(j-1)c)!}{(a+b+(n+j-2)c+1-p)!}. \tag{5.10}$$

Notice that by Lemma 4.5 we have $S_n(a,b,c) = 0$ if inequality (5.8) does not hold.

Proof We have

$$\prod_{j=1}^{n} \frac{1}{(p-a-(n-j)c-1)!} = \prod_{j=1}^{n} \frac{(a+(n-j)c)!}{(p-a-(n-j)c-1)!\,(a+(n-j)c)!}$$
$$= \prod_{j=1}^{n} (-1)^{a+(n-j)c+1}(a+(n-j)c)!,$$

by Lemma 2.3. This implies the Lemma 5.2. □

5.2 More on Values of $S_n(a,b,c)$

Theorem 5.3 *If inequalities* (5.5) *hold and* $a = p-1-(n-1)c-k$, *then*

$$S_n(p-1-(n-1)c-k,b,c) = (-1)^{\binom{n}{2}c+na} \frac{(nc)!}{(c!)^n} \prod_{j=1}^{n} \frac{\binom{b+(j-1)c}{k}}{\binom{(j-1)c+k}{k}}, \tag{5.11}$$

where the integer in the right-hand side is projected to $\mathbb{F}_p$. □

Notice that the projections to $\mathbb{F}_p$ of the binomial coefficients $\binom{b+(j-1)c}{k}$ can be calculated by Lucas's Theorem and both integers in the binomial coefficients $\binom{(j-1)c+k}{k}$ are nonnegative and less than p.

Proof We have

$$\frac{(\alpha+\beta+(j-1)\gamma)!}{(\alpha+(j-1)\gamma)!\,(\beta+(j-1)\gamma)!} = \binom{\alpha+\beta+(j-1)\gamma}{\beta}\prod_{i=1}^{(j-1)\gamma}\frac{1}{\beta+i}$$
$$= \binom{b+(j-1)c}{p-a-(n-1)c-1}\prod_{i=1}^{(j-1)c}\frac{1}{p-a-(n-1)c-1+i}.$$

If $a = p-1-(n-1)c-k$, then this equals

$$\binom{b+(j-1)c}{k}\prod_{i=1}^{(j-1)c}\frac{1}{k+i} = \binom{b+(j-1)c}{k}\frac{k!}{((j-1)c)!\,\prod_{i=1}^{k}((j-1)c+i)}$$
$$= \frac{1}{((j-1)c)!}\frac{\binom{b+(j-1)c}{k}}{\binom{(j-1)c+k}{k}}.$$

Substituting this to (5.6) we obtain (5.11). □

Example Formula (5.11) gives

$$S_2(p-c-1,b,c) = (-1)^c\binom{2c}{c}, \quad S_2(p-c-2,b,c) = (-1)^c\binom{2c}{c}\frac{b(b+c)}{c+1},$$

and so on. Notice that these values are not given by Theorem 4.1. See more examples in Fig. 1.

5.3 Factorization Properties

By Lemmas 4.2 and 4.5 we have $S_n(a,b+p,c) = S_n(a,b,c)$ and $S_n(a,b,c) = 0$ if $a \ge p-(n-1)c$. Thus, for given c, it is enough to analyze $S_n(a,b,c)$ in the rectangle $\Omega = \{(a,b) \mid a \in [0, p-1-(n-1)c], b \in [0, p-1]\}$. This rectangle is partitioned into n smaller rectangles :

$$\Omega_0(n,c) = \{(a,b) \mid a \in [0,\ p-1-(n-1)c],\quad b \in [0,\ p-1-(n-1)c]\},$$
$$\Omega_i(n,c) = \{(a,b) \mid a \in [0,\ p-1-(n-1)c],$$
$$b \in [p-1-(n-i)c+1,\ p-1-(n-i-1)c]\},\quad i=1,\dots,n-1,$$

see the tables in Fig. 1. The values of $S_n(a,b,c)$ in $\Omega_0(n,c)$ are given by Theorem 4.1 and Lemma 4.4. The values of $S_n(a,b,c)$ in a rectangle $\Omega_i(n,c)$ are given by Theorem 4.1 and Lemma 4.4 also, but applied to $\mathbb{F}_p$-Selberg integrals of smaller dimensions with the same value of c and suitable choices of values for a and b. Namely, we have the following factorization property.

	b=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
a=0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
a=1	0	0	0	0	0	0	0	0	0	10	1	0	0	0	0	0	0	0	0	0	10	1	0	0	0	0
a=2	0	0	0	0	0	0	0	0	1	9	1	0	0	0	0	0	0	0	0	1	9	1	0	0	0	0
a=3	0	0	0	0	0	0	0	10	3	8	1	0	0	0	0	0	0	0	10	3	8	1	0	0	0	0
a=4	0	0	0	0	0	0	1	7	6	7	1	0	0	0	0	0	0	1	7	6	7	1	0	0	0	0
a=5	0	0	0	0	0	10	5	1	10	6	1	0	0	0	0	0	10	5	1	10	6	1	0	0	0	0
a=6	0	0	0	0	1	5	4	2	4	5	1	0	0	0	0	1	5	4	2	4	5	1	0	0	0	0
a=7	0	0	0	10	7	1	2	9	10	4	1	0	0	0	10	7	1	2	9	10	4	1	0	0	0	10
a=8	0	0	1	3	6	10	4	10	6	3	1	0	0	1	3	6	10	4	10	6	3	1	0	0	1	3
a=9	0	10	9	8	7	6	5	4	3	2	1	0	10	9	8	7	6	5	4	3	2	1	0	10	9	8
a=10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a=11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a=12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	b=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
a=0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0
a=1	0	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	2	2	0	0	0	0	0	0	0
a=2	0	0	0	0	0	2	5	2	0	0	0	0	0	0	0	0	2	5	2	0	0	0	0	0	0	0
a=3	0	0	0	0	2	9	9	2	0	0	0	0	0	0	0	2	9	9	2	0	0	0	0	0	0	0
a=4	0	0	0	2	3	1	3	2	0	0	0	0	0	0	2	3	1	3	2	0	0	0	0	0	0	2
a=5	0	0	2	9	1	1	9	2	0	0	9	0	0	2	9	1	1	9	2	0	0	9	0	0	2	9
a=6	0	2	5	9	3	9	5	2	0	10	10	0	2	5	9	3	9	5	2	0	10	10	0	2	5	9
a=7	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
a=8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a=9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	b=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
a=0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0
a=1	0	0	0	8	3	0	0	0	0	0	0	0	0	0	8	3	0	0	0	0	0	0	0	0	0	8
a=2	0	0	3	4	3	0	0	6	0	0	6	0	0	3	4	3	0	0	6	0	0	6	0	0	3	4
a=3	0	8	4	7	3	0	6	7	0	4	5	0	8	4	7	3	0	6	7	0	4	5	0	8	4	7
a=4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
a=5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a=6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a=7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a=8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 1 Tables of $S_1(a, b, -)$, $S_2(a, b, 3)$, $S_3(a, b, 3)$ values for $p = 11$ and small integers a, b. Yellow shading indicates the range covered by Theorem 4.1, and the dotted lines enclose the region covered by Theorem 5.1. The structure of the gray shading is discussed in Sect. 5.3

Theorem 5.4 *For* $(a, b) \in \Omega_i(n, c)$ *with* $i > 0$, *we have*

$$S_n(a, b, c) = (-1)^{(n-i)ic} \binom{nc}{ic} \times \frac{\prod_{j=1}^{n-i} \binom{p-1-(n-j)c-a}{(j-1)c} \prod_{j=1}^{i} \binom{p-1-(n-j)c-a}{(j-1)c}}{\prod_{j=1}^{n} \binom{p-1-(n-j)c-a}{(j-1)c}} \times S_{n-i}(a + ic, b, c)\, S_i(a + (n-i)c, b + (n-i)c - p, c). \tag{5.12}$$

Notice that all binomials $\binom{\alpha}{\beta}$ in the second line of (5.12) have $p > \alpha \geq \beta \geq 0$. Notice also that $(a+ic, b) \in \Omega_0(n-i, c)$ and $(a+(n-i)c, b+(n-i)c-p) \in \Omega_0(i, c)$, and hence Theorem 4.1 and Lemma 4.4 can be applied to $S_{n-i}(a + ic, b, c)$ and $S_i(a + (n-i)c, b + (n-i)c - p, c)$.

Proof The theorem follows from formula (5.11) and Lucas' Theorem. □

6 A Remarkable Combinatorial Identity

In this section we sketch another proof of Theorem 4.1. We do this because at the heart of this proof there is a remarkable identity (Theorem 6.1) for polynomials in two variables.

Notation. Let c, n be positive integers. For $1 \leq i < j \leq n$ we will consider non-negative integers $0 \leq m_{ij} \leq 2c$ and we set $\overline{m}_{ij} = 2c - m_{ij}$. For $1 \leq k \leq n$ define

$$r_k = \sum_{1 \leq i < k} \overline{m}_{ik} + \sum_{k < i \leq n} m_{ki}, \quad s_k = \sum_{1 \leq i < k} m_{ik} + \sum_{k < i \leq n} \overline{m}_{ki}.$$

We will use the Pochhammer symbol $(x)_m = x(x+1)(x+2)\cdots(x+m-1)$.

Theorem 6.1 *Let* $n \geq 2,\ c \geq 1$ *be positive integers. In* $\mathbb{Z}[x, y]$ *we have the identity*

$$\sum_{\mathbf{m}} \left((-1)^{\sum_{i<j} m_{ij}} \prod_{i<j} \binom{2c}{m_{ij}} \cdot \prod_{k=1}^{n} (x)_{r_k} (y)_{s_k} \right) = \prod_{k=1}^{n-1} \frac{((k+1)c)!}{c!} (x)_{kc} (y)_{kc} (x + y + (2n - k - 2)c)_{kc},$$

where by $\sum_{\mathbf{m}}$ *we mean the* $\binom{n}{2}$*-fold summation* $\sum_{m_{12}=0}^{2c} \sum_{m_{13}=0}^{2c} \sum_{m_{14}=0}^{2c} \cdots \sum_{m_{n-1,n}=0}^{2c}$.

The summands of the left-hand side are of degree $4c\binom{n}{2}$ polynomials, and according to the theorem, their sum is the right-hand side, which is the product of degree $3c\binom{n}{2}$, with linear factors. The reader is invited to verify that for $n = 2$ the theorem reduces to a hypergeometric identity, namely Dixon's Theorem [2, Theorem 3.4.1] on the factorization of ${}_3F_2$ with certain parameters. For instance the $n = 2, c = 2$ case of Theorem 6.1 states that the sum of the terms

$$(x+2)(x+3)(y+2)(y+3), \quad -4x(x+2)y(y+2), \quad 6x(x+1)y(y+1),$$
$$-4x(x+2)y(y+2), \quad (x+2)(x+3)(y+2)(y+3)$$

is $12(x+y+2)(x+y+3)$ (here we canceled the factor $xy(x+1)(y+1)$, which appears in each term and on the right-hand side as well). The explicit form of the identity for $n = 3$ is

$$\begin{aligned}
&\sum_{m_{12},m_{23},m_{13}=0}^{2c} (-1)^{m_{12}+m_{13}+m_{23}} \binom{2c}{m_{12}}\binom{2c}{m_{23}}\binom{2c}{m_{13}} \\
&\quad \times \prod_{k=0}^{m_{12}+m_{13}-1} (x+k) \prod_{k=0}^{2c-m_{12}+m_{23}-1} (x+k) \prod_{k=0}^{4c-m_{13}-m_{23}-1} (x+k) \\
&\quad \times \prod_{k=0}^{4c-m_{12}-m_{13}-1} (y+k) \prod_{k=0}^{2c-m_{23}+m_{12}-1} (y+k) \prod_{k=0}^{m_{13}+m_{23}-1} (y+j) \\
&= \frac{(2c)!}{c!}\,\frac{(3c)!}{c!} \prod_{k=1}^{c} (x+k-1)(y+k-1)(x+y+4c-k) \\
&\quad \times \prod_{k=1}^{2c} (x+k-1)(y+k-1)(x+y+4c-k).
\end{aligned}$$

Sketch of the proof of Theorem 6.1. Consider Eq. (4.1) for a positive integer c, that is, the classical Selberg integral formula in n dimensions. On the left-hand side we *decouple* the variables, i.e. we substitute $(x_i - x_j)^{2c} = \sum_{m_{ij}=0}^{2c} \binom{2c}{m_{ij}} x_i^{m_{ij}} (-x_j)^{\overline{m}_{ij}}$. We obtain

$$\begin{aligned}
&\sum_{\mathbf{m}} \left((-1)^{\sum_{i<j} m_{ij}} \prod_{i<j} \binom{2c}{m_{ij}} \cdot \prod_{k=1}^{n} \left(\int_0^1 x_k^{a+r_k} (1-x_k)^b \mathrm{d}x_k \right) \right) \\
&\quad = \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(a+(j-1)c)!(b+(j-1)c)!}{(a+b+(n+j-2)c+1)!}.
\end{aligned}$$

Now writing $\Gamma(a+r_k+1)\Gamma(b+1)/\Gamma(a+r_k+b+2)$ for the one-dimensional Selberg integrals on the left-hand side, and substituting

$$x = a+1, \quad y = -(a+2(n-1)c+b+1),$$

the obtained identity rearranges to the statement in the theorem. □

We believe that the identity in Theorem 6.1 is interesting in its own right, but here is a sketch of how to use the identity to prove Theorem 4.1.

Consider the left-hand side of (4.4), and carry out the same *decoupling* of variables as we did in the proof of Theorem 6.1. We obtain a sum, parameterized by choices of m_{ij}, and in each summand we get a product of one-dimensional $\mathbb{F}_p$-Selberg integrals of the form $\int_{[1]_p} x_k^{A_k}(1-x_k)^b \mathrm{d}x_k$ for some A_k. Substituting the value $-A_k!b!/(A_k+b+1-p)!$ for such a one-dimensional integral (formula (3.4)), we obtain an explicit formula (no integrals anymore!) for the left-hand side of (4.4). The summation Theorem 6.1 brings that sum to a product form, and one obtains exactly the right-hand side of (4.4).

In this proof one has to pay additional attention to the case $a+b < p-1$, when some integrals $\int_{[1]_p} x_k^{A_k}(1-x_k)^b \mathrm{d}x_k$ have $A_k+b < p-1$ and are equal to zero by formula (3.5). Still in this case the sum of nonzero terms is transformed to the desired product by the identity of Theorem 6.1 with parameter c replaced by $d := a+b+(n-1)c+1-p$. □

7 KZ Equations

7.1 Special Case of $\mathfrak{sl}_2$ KZ Equations Over $\mathbb{C}$

Let e, f, h be the standard basis of the complex Lie algebra $\mathfrak{sl}_2$ with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. The element

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \ \in \ \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \tag{7.1}$$

is called the Casimir element. For $i \in \mathbb{Z}_{\geq 0}$ let V_i be the irreducible $i+1$-dimensional $\mathfrak{sl}_2$-module with basis $v_i, fv_i, \dots, f^i v_i$ such that $ev_i = 0$, $hv_i = iv_i$.

Let $u(z_1, z_2)$ be a function taking values in $V_{m_1} \otimes V_{m_2}$ and solving the KZ equations

$$\kappa \frac{\partial u}{\partial z_1} = \frac{\Omega}{z_1 - z_2} u, \quad \kappa \frac{\partial u}{\partial z_2} = \frac{\Omega}{z_2 - z_1} u, \tag{7.2}$$

where $\kappa \in \mathbb{C}^\times$ is a parameter of the equations. Let $\mathrm{Sing}[m_1 + m_2 - 2n]$ denote the space of singular vectors of weight $m_1 + m_2 - 2n$ in $V_{m_1} \otimes V_{m_2}$,

$$\mathrm{Sing}[m_1 + m_2 - 2n] = \{v \in V_{m_1} \otimes V_{m_2} \mid hv = (m_1 + m_2 - 2n)v, \ ev = 0\}.$$

This space is one-dimensional if the integer n satisfies $0 \leq n \leq \min(m_1, m_2)$ and is zero-dimensional otherwise. According to [25], solutions u with values in $\mathrm{Sing}[m_1 + m_2 - 2n]$ are expressible in terms of n-dimensional hypergeometric integrals

$$u(z_1, z_2) = \sum_r u_r(z_1, z_2) \, f^r v_{m_1} \otimes f^{n-r} v_{m_2}$$

with

$$u_r(z_1, z_2) = (z_1 - z_2)^{m_1 m_2/2\kappa} \int_C W_r(z_1, z_2, t)\Psi(z_1, z_2, t)\, \mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

Here the domain of integration is the simplex $C = \{t \in \mathbb{R}^n \mid z_1 \leq t_n \leq \cdots \leq t_1 \leq z_2\}$. The function $\Psi(z_1, z_2, t)$ is called the *master function*,

$$\Psi(z_1, z_2, t) = \prod_{1\leq i<j\leq n} (t_i - t_j)^{2/\kappa} \prod_{i=1}^{n} (t_i - z_1)^{-m_1/\kappa} (t_i - z_2)^{-m_2/\kappa},$$

the rational functions $W_r(z_1, z_2, t)$ are called the *weight functions*,

$$W_r(z_1, z_2, t) = \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=r}} \prod_{j\in J} \frac{1}{t_j - z_1} \prod_{j\notin J} \frac{1}{t_i - z_2}.$$

The fact that u is a solution in $\mathrm{Sing}[m_1 + m_2 - 2n]$ implies that

$$(n - r)(m_2 - n + r + 1)u_r + (r + 1)(m_1 - r)u_{r+1} = 0, \quad r = 1, \ldots, n - 1. \tag{7.3}$$

The coordinate functions u_r are generalizations of the Selberg integral. In fact, u_0 and u_n are exactly the Selberg integrals. For example,

$$u_0(z_1, z_2) = (z_1 - z_2)^{m_1 m_2/2\kappa} \int_C \prod_{1\leq i<j\leq n} (t_i - t_j)^{2/\kappa} \times \prod_{i=1}^{n} (t_i - z_1)^{-m_1/\kappa} (t_i - z_2)^{-m_2/\kappa - 1} \mathrm{d}t_1 \ldots \mathrm{d}t_n.$$

The change of variables $t_i = (z_2 - z_1)s_i + z_1$ for $i = 1, \ldots, n$ gives

$$u_0(z_1, z_2) = \frac{(-1)^A (z_1 - z_2)^B}{n!} \tilde{S}_n\left(1 - \frac{m_1}{\kappa}, -\frac{m_2}{\kappa}, \frac{1}{\kappa}\right),$$

where $\tilde{S}_n(\alpha, \beta, \gamma)$ denotes the Selberg integral (1.1), $A = \frac{n(n-1-m_1)}{\kappa} + n$, $B = \frac{m_1 m_2 - 2n(m_1+m_2)+2n(n-1)}{2\kappa}$. By formula (7.3), we obtain

$$u(z_1, z_2) = \kappa^n \frac{(-1)^A (z_1 - z_2)^B}{n!} \prod_{j=1}^{n} \frac{\Gamma(1 + \frac{j}{\kappa})}{\Gamma(1 + \frac{1}{\kappa})} \frac{\Gamma(1 - \frac{m_1-j+1}{\kappa})\,\Gamma(1 - \frac{m_2-j+1}{\kappa})}{\Gamma(1 - \frac{m_1+m_2-n-j+2}{\kappa})}, \times \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{f^r v_1 \otimes f^{n-r} v_2}{\prod_{j=1}^{r}(m_1 - j + 1) \prod_{j=1}^{n-r}(m_2 - j + 1)}. \tag{7.4}$$

7.2 Special Case of $\mathfrak{sl}_2$ KZ Equations Over $\mathbb{F}_p$

Let p be an odd prime number. Let κ be a ratio of two integers not divisible by p. Let m_1, m_2 be positive integers such that $m_1, m_2 < p$. Consider the Lie algebra $\mathfrak{sl}_2$ over the field $\mathbb{F}_p$. Let $V^p_{m_1}$, $V^p_{m_2}$ be the $\mathfrak{sl}_2$-modules over $\mathbb{F}_p$, corresponding to the complex representations V_{m_1}, V_{m_2}. Then the KZ differential equations (7.2) with values in $V^p_{m_1} \otimes V^p_{m_2}$ are well-defined, and we may discuss their polynomial solutions in variables z_1, z_2. Let

$$\mathrm{Sing}[m_1 + m_2 - 2n]_p = \{v \in V^p_{m_1} \otimes V^p_{m_2} \mid hv = (m_1 + m_2 - 2n)v,\ ev = 0\}.$$

This space is one-dimensional, if the integer n satisfies $0 \le n \le \min(m_1, m_2)$ and is zero-dimensional otherwise.

Choose the least positive integers M_1, M_2, M_{12}, c such that

$$M_i \equiv -\frac{m_i}{\kappa}, \quad M_{12} \equiv \frac{m_1 m_2}{2\kappa}, \quad c \equiv \frac{1}{\kappa} \pmod{p}. \tag{7.5}$$

According to [26], solutions u with values in $\mathrm{Sing}[m_1 + m_2 - 2n]_p$ are expressible in terms of n-dimensional $\mathbb{F}_p$-hypergeometric integrals

$$u(z_1, z_2) = \sum_r u_r(z_1, z_2)\, f^r v_{m_1} \otimes f^{n-r} v_{m_2} \tag{7.6}$$

with

$$u_r(z_1, z_2) = (z_1 - z_2)^{M_{12}} \int_{[1,\dots,1]_p} W_r(z_1, z_2, t)\Psi_p(z_1, z_2, t)\, \mathrm{d}t_1 \dots \mathrm{d}t_n,$$

where $\Psi_p(z_1, z_2, t)$ is the *master polynomial*,

$$\Psi_p(z_1, z_2, t) = \prod_{1 \le i < j \le n} (t_i - t_j)^{2c} \prod_{i=1}^{n} (t_i - z_1)^{M_1} (t_i - z_2)^{M_2}.$$

Theorem 7.1 *Assume that M_1, M_2, M_{12}, c, n are positive integers such that*

$$\begin{aligned} &M_1 + (n-1)c < p, \quad M_2 + (n-1)c < p, \\ &p \le M_1 + M_2 + (n-1)c, \quad M_1 + M_2 + (2n-2)c < 2p - 1. \end{aligned} \tag{7.7}$$

Then the function $u(z_1, z_2)$, defined by (7.6), is given by the formula

$$\begin{aligned} u(z_1, z_2) = (-1)^A (z_1 - z_2)^B \prod_{j=1}^{n} \frac{(jc)!}{c!} \frac{(M_1 + (j-1)c)!\,(M_2 + (j-1)c)!}{(M_1 + M_2 + (n+j-2)c - p)!}, \\ \times \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{f^r v_1 \otimes f^{n-r} v_2}{\prod_{j=1}^{r} (M_1 + (j-1)c) \prod_{j=1}^{n-r} (M_2 + (j-1)c)}, \end{aligned} \tag{7.8}$$

where

$$A = n(M_1 + (n-1)c + 1), \quad B = M_{12} + n(M_1 + M_2 + (n-1)c - p).$$

For $n = 1$ this is [38, Theorem 4.3].

Proof The proof follows from the $\mathbb{F}_p$-Selberg integral formula of Theorem 4.1 and formula (7.3), cf. Sect. 7.1. □

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