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RESEARCH CONTRIBUTION

A Note on Complex-Hyperbolic Kleinian Groups

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Abstract
Let Γ be a discrete group of isometries acting on the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$. In this note, we prove that if Γ is convex-cocompact, torsion-free, and the critical exponent $\delta(\Gamma)$ is strictly lesser than 2, then the complex manifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein. We also discuss several related conjectures.

Keywords Discrete subgroups · Critical exponent · Stein manifold

The theory of complex hyperbolic manifolds and complex-hyperbolic Kleinian groups (i.e., discrete holomorphic isometry groups of complex hyperbolic spaces $\mathbb{H}^n_{\mathbb{C}}$) is a rich mixture of Riemannian and complex geometry, topology, dynamics, symplectic geometry and complex analysis. The purpose of this note is to discuss interactions of the theory of complex-hyperbolic Kleinian groups and the function theory of complex-hyperbolic manifolds. Let Γ be a discrete group of isometries acting on the complex-hyperbolic n-space, $\mathbb{H}^n_{\mathbb{C}}$, the unit ball $\mathbf{B}^n \subset \mathbb{C}^n$ equipped with the Bergman metric. A fundamental numerical invariant associated with Γ is the *critical exponent* $\delta(\Gamma)$ of Γ, defined by

$$\delta(\Gamma) = \inf\left\{ s : \sum_{\gamma\in\Gamma} e^{-s\cdot d(x,\gamma x)} < \infty \right\},$$

where $x \in \mathbb{H}^n_{\mathbb{C}}$ is any[1] point. The critical exponent measures the rate of exponential growth the Γ-orbit $\Gamma x \subset \mathbb{H}^n_{\mathbb{C}}$; it also equals the Hausdorff dimension of the conical limit set of Γ, see [7,8].

Our main result is:

[1] $\delta(\Gamma)$ does not depend on the choice of $x \in \mathbb{H}^n_{\mathbb{C}}$.

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Theorem 1 *Suppose that $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$ is a convex-cocompact, torsion-free discrete subgroup satisfying $\delta(\Gamma) < 2$. Then, $M_\Gamma = \mathbf{B}^n/\Gamma$ is Stein.*

The condition on the critical exponent in the above theorem is sharp, since for a complex Fuchsian subgroup $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$, $\delta(\Gamma) = 2$, but the quotient $M_\Gamma = \mathbf{B}^n/\Gamma$ is non-Stein, because the convex core of M_Γ is a complex curve, see Example 4. On the other hand, if Γ is a torsion-free real Fuchsian subgroup or a small deformation of such (see Example 3), then Γ satisfies the condition of the above theorem.

The main ingredients in the proof of Theorem 1 are Proposition 11 and Theorem 15. The condition "convex-cocompact" is only used in Proposition 11, whereas Theorem 15 holds for any torsion-free discrete subgroup $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$ satisfying $\delta(\Gamma) < 2$.

Conjecture 2 Theorem 1 holds if we omit the "convex-cocompact" assumption on Γ.

In Sect. 4, we discuss other conjectural generalizations of Theorem 1 and supporting results.

1 Preliminaries

In this section, we recall some definitions and basic facts about the n-dimensional complex hyperbolic space, we refer to [9,11] for details.

Consider the n-dimensional complex vector space $\mathbb{C}^{n+1}$ equipped with the pseudo-hermitian bilinear form:

$$\langle z, w \rangle = -z_0 \bar{w}_0 + \sum_{k=1}^{n} z_k \bar{w}_k \tag{1}$$

and define the quadratic form $q(z)$ of signature $(n, 1)$ by $q(z) := \langle z, z \rangle$. Then, q defines the *negative light cone* $V_- := \{z : q(z) < 0\} \subset \mathbb{C}^{n+1}$. The projection of V_- in the projectivization of $\mathbb{C}^{n+1}$, $\mathbb{P}^n$, is an open ball which we denote by $\mathbf{B}^n$.

The tangent space $T_{[z]}\mathbb{P}^n$ is naturally identified with $z^\perp$, the orthogonal complement of $\mathbb{C}z$ in V, taken with respect to $\langle \cdot, \cdot \rangle$. If $z \in V_-$, then the restriction of q to $z^\perp$ is positive-definite, hence, $\langle \cdot, \cdot \rangle$ project to a hermitian metric h (also denoted $\langle \cdot, \cdot \rangle_h$) on $\mathbf{B}^n$. The *complex hyperbolic n-space* $\mathbb{H}^n_{\mathbb{C}}$ is $\mathbf{B}^n$ equipped with the hermitian metric h. The boundary $\partial \mathbf{B}^n$ of $\mathbf{B}^n$ in $\mathbb{P}^n$ gives a natural compactification of $\mathbf{B}^n$.

In this note, we usually denote the complex hyperbolic n-space by $\mathbf{B}^n$. The real part of the hermitian metric h defines a Riemannian metric g on $\mathbf{B}^n$. The sectional curvature of g varies between -4 and -1. We denote the distance function on $\mathbf{B}^n$ by d. The distance function satisfies

$$\cosh^2(d(0, z)) = (1 - |z|^2)^{-1}. \tag{2}$$

A real linear subspace $W \subset \mathbb{C}^{n+1}$ is said to be *totally real* with respect to the form (1) if for any two vectors $z, w \in W$, $\langle z, w \rangle \in \mathbb{R}$. Such a subspace is automatically totally real in the usual sense: $JW \cap W = \{0\}$, where J is the almost complex structure on V. *(Real) geodesics* in $\mathbf{B}^n$ are projections of totally real indefinite (with respect to q) 2-planes in $\mathbb{C}^{n+1}$ (intersected with V_-). For instance, geodesics through the origin $0 \in \mathbf{B}^n$ are Euclidean line segments in $\mathbf{B}^n$. More generally, totally-geodesic real subspaces in $\mathbf{B}^n$ are projections of totally real indefinite subspaces in $\mathbb{C}^{n+1}$ (intersected with V_-). They are isometric to the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ of constant sectional curvature -1.

Complex geodesics in $\mathbf{B}^n$ are projections of indefinite complex 2-planes. Complex geodesics are isometric to the unit disk with the hermitian metric:

$$\frac{dzd\bar{z}}{(1-|z|^2)^2},$$

which has constant sectional curvature -4. More generally, k-dimensional complex hyperbolic subspaces $\mathbb{H}^k_{\mathbb{C}}$ in $\mathbf{B}^n$ are projections of indefinite complex $(k+1)$-dimensional subspaces (intersected with V_-).

All complete totally-geodesic submanifolds in $\mathbb{H}^n_{\mathbb{C}}$ are either real or complex hyperbolic subspaces.

The group $\mathrm{U}(n, 1) \cong \mathrm{U}(q)$ of (complex) automorphisms of the form q projects to the group $\mathrm{Aut}(\mathbf{B}^n) \cong \mathrm{PU}(n, 1)$ of complex (biholomorphic, isometric) automorphisms of $\mathbf{B}^n$. The group $\mathrm{Aut}(\mathbf{B}^n)$ is linear, its matrix representation is given, for instance, by the adjoint representation, which is faithful since $\mathrm{Aut}(\mathbf{B}^n)$ has trivial center.

A discrete subgroup Γ of $\mathrm{Aut}(\mathbf{B}^n)$ is called a *complex-hyperbolic Kleinian group*. The accumulation set of an(y) orbit Γx in $\partial\mathbf{B}^n$ is called the *limit set* of Γ and denoted by $\Lambda(\Gamma)$. The complement of $\Lambda(\Gamma)$ in $\partial\mathbf{B}^n$ is called the *domain of discontinuity* of Γ and denoted by $\Omega(\Gamma)$. The group Γ acts properly discontinuously on $\mathbf{B}^n \cup \Omega(\Gamma)$.

For a (torsion-free) complex-hyperbolic Kleinian group Γ, the quotient $\mathbf{B}^n/\Gamma$ is a Riemannian orbifold (manifold) equipped with push-forward of the Riemannian metric of $\mathbf{B}^n$. We reserve the notation M_Γ to denote this quotient. The *convex core* of M_Γ is the the projection of the closed convex hull of $\Lambda(\Gamma)$ in $\mathbf{B}^n$. The subgroup Γ is called *convex-cocompact* if the convex core of M_Γ is a nonempty compact subset. Equivalently (see [3]), $\overline{M}_\Gamma = (\mathbf{B}^n \cup \Omega(\Gamma))/\Gamma$ is compact.

Below are two interesting examples of convex-cocompact complex-hyperbolic Kleinian groups which will also serve as illustrations our results.

Example 3 (Real Fuchsian subgroups) Let $\mathbb{H}^2_{\mathbb{R}} \subset \mathbf{B}^n$ be a totally real hyperbolic plane. This inclusion is induced by an embedding $\rho : \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}}) = \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{Aut}(\mathbf{B}^n)$, whose image preserves $\mathbb{H}^2_{\mathbb{R}}$. Let $\Gamma' < \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}})$ be a uniform lattice. Then $\Gamma = \rho(\Gamma')$ preserves $\mathbb{H}^2_{\mathbb{R}}$ and acts on it cocompactly. Such subgroups $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$ will be called *real Fuchsian subgroups*. The compact surface-orbifold $\Sigma = \mathbb{H}^2_{\mathbb{R}}/\Gamma$ is the convex core of M_Γ. The critical exponent $\delta(\Gamma)$ is 1.

Let Γ_t, $t \geq 0$, be a continuous family of deformations of $\Gamma_0 = \Gamma$ in $\mathrm{Aut}(\mathbf{B}^n)$ such that Γ_t's, for $t > 0$, are convex-cocompact but not real Fuchsian. Such deformation exist as long as Γ_t is, say, torsion-free, see, e.g., [15]. The groups Γ_t, $t > 0$, are called *real quasi-Fuchsian subgroups*. The critical exponents of such subgroups are strictly greater than 1.

Example 4 (Complex Fuchsian subgroups) Let Γ' be a cocompact subgroup of $\mathrm{SU}(1,1)$, the identity component isometry group of the real-hyperbolic plane (modulo $\mathbb{Z}_2$) and let $\mathrm{SU}(1,1) \to \mathrm{SU}(n,1)$ be any embedding. Note that $\mathrm{SU}(n,1)$ modulo center (isomorphic to $\mathbb{Z}_{n+1}$) is isomorphic to $\mathrm{PU}(n,1)$. By taking compositions, we get a representation $\rho : \Gamma' \to \mathrm{PU}(n,1)$. Then $\Gamma := \rho(\Gamma')$ leaves a complex geodesic invariant in $\mathbf{B}^n$. Such subgroups Γ will be called *complex Fuchsian subgroups*. In this case, $\mathrm{core}(M_\Gamma) = \mathbb{H}^1_{\mathbb{C}}/\Gamma$ is a compact complex curve in M_Γ where $\mathbb{H}^1_{\mathbb{C}}$ is the Γ-invariant complex geodesic. The critical exponent $\delta(\Gamma)$ is 2.

2 Generalities on Complex Manifolds

By a *complex manifold with boundary* M, we mean a smooth manifold with (possibly empty) boundary ∂M such that $\mathrm{int}(M)$ is equipped with a complex structure and that there exists a smooth embedding $f : M \to X$ to an equidimensional complex manifold X, biholomorphic on $\mathrm{int}(M)$. A holomorphic function on M is a smooth function which admits a holomorphic extension to a neighborhood of M in X.

Let X be a complex manifold and $Y \subset X$ is a codimension 0 smooth submanifold with boundary in X. The submanifold Y is said to be *strictly Levi-convex* if every boundary point of Y admits a neighborhood U in X such that the submanifold with boundary $Y \cap U$ can be written as

$$\{\phi \le 0\},$$

for some smooth submersion $\phi : U \to \mathbb{R}$ satisfying $\mathrm{Hess}(\phi) > 0$, where $\mathrm{Hess}(\phi)$ is the holomorphic Hessian:

$$\left(\frac{\partial^2 \phi}{\partial \bar{z}_i \partial z_j}\right).$$

Definition 5 A *strongly pseudoconvex manifold M* is a complex manifold with boundary which admits a strictly Levi-convex holomorphic embedding in an equidimensional complex manifold.

Definition 6 An open complex manifold Z is called *holomorphically convex* if for every discrete closed subset $A \subset Z$ there exists a holomorphic function $Z \to \mathbb{C}$ which is proper on A.

Alternatively,[2] one can define holomorphically convex manifolds as follows: For a compact K in a complex manifold M, the *holomorphic convex hull* $\hat{K}_M$ of K in M is

$$\hat{K}_M = \{z \in M : |f(z)| \le \sup_{w \in K} |f(w)|, \forall f \in \mathcal{O}_M\}.$$

In the above, $\mathcal{O}_M$ denotes the ring of holomorphic functions on M. Then M is holomorphically convex iff for every compact $K \subset M$, the hull $\hat{K}_M$ is also compact.

[2] And this is the standard definition.

Theorem 7 (Grauert [10]) *The interior of every compact strongly pseudoconvex manifold M is holomorphically convex.*

Definition 8 A complex manifold M is called *Stein* if it admits a proper holomorphic embedding in $\mathbb{C}^n$ for some n.

Equivalently, M is Stein iff it is holomorphically convex and *holomorphically separable*: That is, for every distinct points $x, y \in M$, there exists a holomorphic function $f : M \to \mathbb{C}$ such that $f(x) \neq f(y)$. We will use:

Theorem 9 (Rossi [13], Corollary on page 20) *If a compact complex manifold M is strongly pseudoconvex and contains no compact complex subvarieties of positive dimension, then* $\mathrm{int}(M)$ *is Stein.*

We now discuss strong quasiconvexity and Stein property in the context of complex-hyperbolic manifolds. A classical example of a complex submanifold with Levi-convex boundary is a closed round ball $\overline{\mathbf{B}}^n$ in $\mathbb{C}^n$. Suppose that $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$ is a discrete torsion-free subgroup of the group of holomorphic automorphisms of $\mathbf{B}^n$ with (nonempty) domain of discontinuity $\Omega = \Omega(\Gamma) \subset \partial\mathbf{B}^n$. The quotient

$$\overline{M}_\Gamma = (\mathbf{B}^n \cup \Omega)/\Gamma$$

is a smooth manifold with boundary.

Lemma 10 *$\overline{M}_\Gamma$ is strongly pseudoconvex.*

Proof We let T_Λ denote the union of all projective hyperplanes in $P^n_{\mathbb{C}}$ tangent to $\partial\mathbf{B}^n$ at points of Λ, the limit set of Γ. Let $\widehat{\Omega}$ denote the connected component of $P^n_{\mathbb{C}} \setminus T_\Lambda$ containing $\mathbf{B}^n$. It is clear that $\mathbf{B}^n \cup \Omega \subset \widehat{\Omega}$ is strictly Levi-convex. By the construction, Γ preserves $\widehat{\Omega}$. It is proven in [5, Thm. 7.5.3] that the action of Γ on $\widehat{\Omega}$ is properly discontinuous. Hence, $X := \widehat{\Omega}/\Gamma$ is a complex manifold containing $\overline{M}_\Gamma$ as a strictly Levi-convex submanifold with boundary. □

Specializing to the case when $\overline{M}_\Gamma$ is compact, i.e. Γ is convex-cocompact, we obtain:

Proposition 11 *Suppose that Γ is torsion-free, convex-cocompact and $n > 1$. Then:*

1. *$\partial\overline{M}_\Gamma$ is connected.*
2. *If $\mathrm{int}(\overline{M}_\Gamma) = M_\Gamma$ contains no compact complex subvarieties of positive dimension, then M_Γ is Stein.*

For example, as it was observed in [4], the quotient-manifold $\mathbf{B}^2/\Gamma$ of a real-Fuchsian subgroup $\Gamma < \mathrm{Aut}(\mathbf{B}^2)$ is Stein while the quotient-manifold of a complex-Fuchsian subgroup $\Gamma < \mathrm{Aut}(\mathbf{B}^2)$ is non-Stein.

3 Proof of Theorem 1

In this section, we construct certain plurisubharmonic functions on M_Γ, for each finitely generated, discrete subgroup $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$ satisfying $\delta(\Gamma) < 2$. We use these functions to show that M_Γ has no compact subvarieties of positive dimension. At the end of this section, we prove the main result of this paper.

Let X be a complex manifold. Recall that a continuous function $f : X \to \mathbb{R}$ is called *plurisubharmonic*[3] if for any homomorphic map $\phi : V(\subset \mathbb{C}) \to X$, the composition $f \circ \phi$ is subharmonic. Plurisubharmonic functions f satisfy the maximum principle; in particular, if f restricts to a nonconstant function on a connected complex subvariety $Y \subset X$, then Y is noncompact.

Now, we turn to our construction of plurisubharmonic functions. Let $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$ be a discrete subgroup. Consider the Poincaré series

$$\sum_{\gamma \in \Gamma} (1 - |\gamma(z)|^2), \quad z \in \mathbf{B}^n. \tag{3}$$

Lemma 12 *Suppose that $\delta(\Gamma) < 2$. Then* (3) uniformly converges on compact sets.

Proof Since $\delta(\Gamma) < 2$, the Poincaré series

$$\sum_{\gamma \in \Gamma} e^{-2d(0,\gamma(z))}$$

uniformly converges on compact subsets in $\mathbf{B}^n$. By (2), we get

$$e^{-2d(0,\gamma(z))} \le (1 - |\gamma(z)|^2) \le 4e^{-2d(0,\gamma(z))}. \tag{4}$$

Then, the result follows from the upper inequality. □

Remark 13 Note that when $\delta(\Gamma) > 2$, or when Γ is of *divergent type* (e.g., convex-cocompact) and $\delta(\Gamma) = 2$, then (3) does not converge. This follows from the lower inequality of (4).

Assume that $\delta(\Gamma) < 2$. Define $F : \mathbf{B}^n \to \mathbb{R}$,

$$F(z) = \sum_{\gamma \in \Gamma} (|\gamma(z)|^2 - 1).$$

Since F is Γ-invariant, i.e., $F(\gamma z) = F(z)$, for all $\gamma \in \Gamma$ and all $z \in \mathbf{B}^n$, F descends to a function

$$f : M_\Gamma \to \mathbb{R}.$$

[3] There is a more general notion of *plurisubharmonic functions*; for our purpose, we only consider this restrictive definition.

Lemma 14 *The function $f : M_\Gamma \to \mathbb{R}$ is plurisubharmonic.*

Proof Enumerate Γ as $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$. Consider the sequence of partial sums of the series F:

$$S_k(z) = \sum_{j \leq k} (|\gamma_j(z)|^2 - 1).$$

Since each summand in the above is plurisubharmonic[4], S_k is plurisubharmonic for each $k \geq 1$. Moreover, the sequence of functions S_k is monotonically decreasing. Thus, the limit $F = \lim_{k\to\infty} S_k$ is also plurisubharmonic, and hence so is f. □

Note, however, that at this point, we do not yet know that the function f is nonconstant.

Now, we prove the main result of this section.

Theorem 15 *Let Γ be a torsion-free discrete subgroup of* $\mathrm{Aut}(\mathbf{B}^n)$. *If $\delta(\Gamma) < 2$, then M_Γ contains no compact complex subvarieties of positive dimension.*

Proof Suppose that Y is a compact connected subvariety of positive dimension in M_Γ. Since $\pi_1(Y)$ is finitely generated, so is its image Γ' in $\Gamma = \pi_1(M_\Gamma)$. Since $\delta(\Gamma') \leq \delta(\Gamma)$, by passing to the subgroup Γ' we can (and will) assume that the group Γ is finitely generated.

We construct a sequence of functions $F_k : \mathbf{B}^n \to \mathbb{R}$ as follows. For $k \in \mathbb{N}$, let $\Sigma_k \subset \Gamma - \{1\}$ denote the subset consisting of $\gamma \in \Gamma$ satisfying $d(0, \gamma(0)) \leq k$. Since Γ is a finitely generated linear group, it is residually finite and, hence, there exists a finite index subgroup $\Gamma_k < \Gamma$ disjoint from Σ_k. For each $k \in \mathbb{N}$, define $F_k : \mathbf{B}^n \to \mathbb{R}$ as the sum:

$$F_k(z) = \sum_{\gamma \in \Gamma_k} (|\gamma(z)|^2 - 1).$$

Since

$$\bigcap_{k \in \mathbb{N}} \Gamma_k = \{1\},$$

the sequence of functions F_k converges to $(|z|^2 - 1)$ uniformly on compact subsets of $\mathbf{B}^n$. As before, each F_k is plurisubharmonic (cf. Lemmata 12, 14).

Let $\widetilde{Y}$ be a connected component of the preimage of Y under the projection map $\mathbf{B}^n \to M_\Gamma$. Since $\widetilde{Y}$ is a closed, noncompact subset of $\mathbf{B}^n$, the function $(|z|^2 - 1)$ is nonconstant on $\widetilde{Y}$. As the sequence (F_k) converges to $(|z|^2 - 1)$ uniformly on compacts, there exists $k \in \mathbb{N}$, such that F_k is nonconstant on $\widetilde{Y}$. Let $f_k : M_k = M_{\Gamma_k} \to \mathbb{R}$ denote the function obtained by projecting F_k to M_k, and Y_k be the image of $\widetilde{Y}$ under the projection map $\mathbf{B}^n \to M_k$. Since M_k is a finite covering of M_Γ, the subvariety $Y_k \subset M_k$ is compact. Moreover, f_k is a nonconstant plurisubharmonic function on Y_k since F_k is such a function on $\widetilde{Y}$. This contradicts the maximum principle. □

[4] This follows from the fact that the function $|z|^2$ is plurisubharmonic.

Remark 16 Regarding Remark 13: The failure of convergence of the series (3) as pointed out in Remark 13 is not so surprising. In fact, if Γ is a complex Fuchsian group, then $\delta(\Gamma) = 2$ and the convex core of M_Γ is a compact Riemann surface, see Example 4. Thus, our construction of F must fail in this case.

We conclude this section with a proof of the main result of this paper.

Proof of Theorem 1 By Theorem 15, M_Γ does not have compact complex subvarieties of positive dimensions. Then, by the second part of Proposition 11, M_Γ is Stein. □

4 Further Remarks

In relation to Theorem 1, it is also interesting to understand the case when $\delta(\Gamma) = 2$, that is: For which convex-cocompact, torsion-free subgroups Γ of $\mathrm{Aut}(\mathbf{B}^n)$ satisfying $\delta(\Gamma) = 2$, is the manifold M_Γ Stein? It has been pointed out before that a complex Fuchsian subgroup $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$ satisfies $\delta(\Gamma) = 2$, but the manifold M_Γ is not Stein. In fact, the convex core of M_Γ is a complex curve, see Remark 16. We conjecture that complex Fuchsian subgroups are the only such non-Stein examples.

Conjecture 17 Let $\Gamma < \mathrm{Aut}(\mathbf{B}^n)$ be a convex-cocompact, torsion-free subgroup such that $\delta(\Gamma) = 2$. Then, M_Γ is non-Stein if and only if Γ is a complex Fuchsian subgroup.

We illustrate this conjecture in the following very special case: Let $\phi : \pi_1(\Sigma) \to \mathrm{Aut}(\mathbf{B}^n)$ be a faithful convex-cocompact representation where Σ is a compact Riemann surface of genus $g \geq 2$. Then ϕ induces a (unique) equivariant harmonic map:

$$F : \widetilde{\Sigma} \to \mathbf{B}^n.$$

which descends to a harmonic map $f : \Sigma \to M_\Gamma$.

Proposition 18 *Suppose that F is a holomorphic immersion. Then $\Gamma = \phi(\pi_1(\Sigma))$ satisfies $\delta(\Gamma) \geq 2$. Moreover, if $\delta(\Gamma) = 2$, then Γ preserves a complex line. In particular, Γ is a complex Fuchsian subgroup of* $\mathrm{Aut}(\mathbf{B}^n)$.

Proof Noting that M_Γ contains a compact complex curve, namely, $f(\Sigma)$, the first part follows directly from Theorem 1.

For the second part, we let Y denote the surface $\tilde{\Sigma}$ equipped with the Riemannian metric obtained via pull-back of the Riemannian metric g on $\mathbf{B}^n$. The entropy[5] $h(Y)$ of Y is bounded above by $\delta(\Gamma)$, that is

$$h(Y) \leq 2. \tag{5}$$

This can be seen as follows: The distance function d_Y on Y satisfies

$$d_Y(y_1, y_2) \geq d(F(y_1), F(y_2)).$$

[5] The *volume entropy* of a simply connected Riemannian manifold (X, g) is defined as $\lim_{r\to\infty} \log \mathrm{Vol}(B(r, x))/r$, where $x \in X$ is a chosen base-point and $B(r, x)$ denotes the ball of radius r centered at x. This limit exists and is independent of x, see [12].

Therefore, the exponential growth-rate δ_Y of $\pi_1(\Sigma)$-orbits in Y satisfies $\delta_Y \leq \delta(\Gamma)$. On the other hand, the quantity $\delta_Y = h(Y)$ since $\pi_1(\Sigma)$ acts cocompactly on Y.

Assume that $\widetilde{\Sigma}$ is endowed with a conformal Riemannian metric of constant -4 sectional curvature. Since $\tilde{\Sigma}$ is a symmetric space, we have

$$h^2(Y)\text{Area}(Y/\Gamma) \geq h^2(\widetilde{\Sigma})\text{Area}(\Sigma),$$

see [1, p. 624]. The inequality (5) together with the above implies that

$$\text{Area}(Y/\Gamma) \geq \text{Area}(\Sigma).$$

On the other hand, since $f : Y/\Gamma \to M_\Gamma$ is holomorphic, $4 \cdot \text{Area}(Y/\Gamma)$ equals to the Toledo invariant $c(\phi)$ (see [14]) of the representation ϕ. Since $c(\phi) \leq 4\pi(g-1)$, the inequality $\text{Area}(Y/\Gamma) \geq \text{Area}(\Sigma) = \pi(g-1)$ shows that $\text{Area}(Y/\Gamma) = \pi(g-1)$ or, equivalently, $c(\phi) = 4\pi(g-1)$. By the main result of [14], Γ preserves a complex-hyperbolic line in $\mathbf{B}^n$. □

Remark 19 The assumption that F is an immersion can be eliminated: Instead of working with a Riemannian metric, one can work with a Riemannian metric with finitely many singularities.

Motivated by Theorem 15, we also make the following conjecture.

Conjecture 20 If $\Gamma < \text{Aut}(\mathbf{B}^n)$ is discrete, torsion-free, and $\delta(\Gamma) < 2k$, then M_Γ does not contain compact complex subvarieties of dimension $\geq k$.

We conclude this section with a verification of this conjecture under a stronger hypothesis.

Proposition 21 *If $\Gamma < \text{Aut}(\mathbf{B}^n)$ is discrete, torsion-free, and $\delta(\Gamma) < 2k-1$, then M_Γ does not contain compact complex subvarieties of dimension $\geq k$.*

Proof Note that if Γ is elementary (i.e., virtually abelian), then $\delta(\Gamma) = 0$. In this case, the result follows from Theorem 15. For the rest, we assume that Γ is nonelementary.

By [2, Sec. 4], there is a natural map $f : M_\Gamma \to M_\Gamma$ homotopic to the identity map $\text{id}_{M_\Gamma} : M_\Gamma \to M_\Gamma$ and satisfying

$$|\text{Jac}_p(f)| \leq \left(\frac{\delta(\Gamma)+1}{p}\right)^p, \quad 2 \leq p \leq 2n,$$

where $\text{Jac}_p(f)$ denotes the p-Jacobian of f. When $\delta(\Gamma) < 2k-1$, we have $|\text{Jac}_p(f)| < 1$, for $p \in [2k, 2n]$. This means that f strictly contracts the volume form on each p-dimensional tangent space at every point $x \in M_\Gamma$, for $p \in [2k, 2n]$.

Let $Y \subset M_\Gamma$ be a compact complex subvariety of dimension $\geq k$ (real dimension $\geq 2k$). Then, Y is also a volume minimizer in its homology class. Since f strictly contracts volume on Y, $f(Y)$ has volume strictly lesser than that of Y. However, f being homotopic to id_{M_Γ}, $f(Y)$ belongs to the homology class of Y. This is a contradiction to the fact that Y minimizes volume its homology class. □

Remark 22 Note that Proposition 21 gives an alternative proof of Theorem 15 (hence Theorem 1) under a stronger hypothesis, namely $\delta(\Gamma) \in (0, 1)$. However, this method fails to verify Theorem 15 in the case when $\delta(\Gamma) \in [1, 2)$.

Finally, we note that the papers [6,16] contain other interesting results and conjectures on Stein properties of complex-hyperbolic manifolds.

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