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RESEARCH CONTRIBUTION

Two Parameters bt-Algebra and Invariants for Links and Tied Links

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Abstract
We introduce a two-parameters bt-algebra which, by specialization, becomes the one-parameter bt-algebra, introduced by the authors, as well as another one-parameter presentation of it; the invariant for links and tied links, associated to this two-parameter algebra via Jones recipe, contains as specializations the invariants obtained from these two presentations of the bt-algebra and then is more powerful than each of them. Also, a new non Homflypt polynomial invariant is obtained for links, which is related to the linking matrix.

Keywords Links invariants · Tied links invariants · bt-Algebra

Mathematics Subject Classification 57M25 · 20C08 · 20F36

1 Introduction

The bt-algebra is a one-parameter finite dimensional algebra defined by generators and relations, see Aicardi and Juyumaya (2000) and Ryom-Hansen (2011). In Marin (2018) it is shown how to associate to each Coxeter group a certain algebra, and in the case of the Weyl group of type A this algebra coincides with the bt-algebra; this may open new perspectives for the study of the bt-algebra in knot theory, cf. Flores (2020). The representation theory of the bt-algebra has been studied in Ryom-Hansen (2011), Espinoza and Ryom-Hansen (2018), Jacon and Poulain d'Andecy (2017).

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For every positive integer n, we denote by $\mathcal{E}_n(\mathsf{u})$ the bt-algebra over $\mathbb{C}(\mathsf{u})$, with parameter u. The original definition of $\mathcal{E}_n(\mathsf{u})$ is by braid generators $T_1, \ldots, T_{n-1}$ and tie generators $E_1, \ldots, E_{n-1}$, satisfying the defining generators of the tied braid monoid defined in Aicardi and Juyumaya (2016a, Definition 3.1) together with the polynomial relation

$$T_i^2 = 1 + (\mathsf{u} - 1)E_i + (\mathsf{u} - 1)E_i T_i, \quad \text{for all } i.$$

It is known that the bt-algebra is a knot algebra: indeed, in Aicardi and Juyumaya (2016b) we have defined a three-variable polynomial invariant for classical links which is denoted by $\bar{\Delta}$; this invariant was constructed originally by using the method—also called Jones recipe—that Jones introduced to construct the Homflypt polynomial (Jones 1987).

In Chlouveraki et al. (2020) another presentation for the bt-algebra is considered. More precisely, denote by $\sqrt{\mathsf{u}}$ a variable such that $(\sqrt{\mathsf{u}})^2 = \mathsf{u}$: the new presentation of the bt-algebra is now over $\mathbb{C}(\sqrt{\mathsf{u}})$ and is presented by the same tie generators E_i's but the generators T_i's are replaced by braid generators V_i's, still satisfying all original defining relation of the T_i's with exception of the polynomial relation above which is replaced by

$$V_i^2 = 1 + (\sqrt{\mathsf{u}} - \sqrt{\mathsf{u}}^{-1})E_i V_i, \quad \text{for all } i.$$

We denote by $\mathcal{E}(\sqrt{\mathsf{u}})$ the bt-algebra with this new presentation. Now, again, by using the Jones recipe on the bt-algebra but with the presentation $\mathcal{E}(\sqrt{\mathsf{u}})$, a three-variable polynomial invariant for classical links is constructed in Chlouveraki et al. (2020); this invariant is denoted by Θ.

It was noted by the first author that $\bar{\Delta}$ and Θ are not topologically equivalent, see Aicardi (2016), cf. Chlouveraki et al. (2020); this is an amazing fact that shows the subtlety of the Jones recipe. In fact, the main motivation of this note is to understand the relation between the invariants $\bar{\Delta}$ and Θ. To do that we introduce a bt-algebra with two commuting parameters u and v, denoted by $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$, presented by the tie generators E_i's and braid generators R_i's, subject to the same monomial relations as the bt-algebra and the polynomial relations

$$R_i^2 = 1 + (\mathsf{u} - 1)E_i + (\mathsf{v} - 1)E_i R_i, \quad \text{for all } i.$$

Similarly to what happens for the two-parameters Hecke algebra (Kassel and Turaev 2008, Subsection 4.2), the bt-algebra with two parameters is isomorphic to the bt-algebra with one parameter, see Proposition 1; this fact allows to define a Markov trace on $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$ (Proposition 2). Consequently, we apply the Jones recipe to the bt-algebra with two parameters, obtaining a four-variable invariant polynomial, denoted by Υ, for classical links as well its extension $\widetilde{\Upsilon}$ to tied links (Aicardi and Juyumaya 2016a). As it will be observed in Remark 2, specializations of the parameters in $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$ yields $\mathcal{E}_n(\mathsf{u})$ and $\mathcal{E}_n(\sqrt{\mathsf{u}})$; therefore, the respective specializations of Υ yields the invariants $\bar{\Delta}$ and Θ; this gives an answer to the initial question that motivated this work.

In Sect. 5 we define $\widetilde{\Upsilon}$ by skein relations. We also give a close look to the specialization of $\widetilde{\Upsilon}$ at $\mathsf{v} = 1$, which is denoted by $\widetilde{\Omega}$. In Theorem 4 we show some properties of $\widetilde{\Omega}$ by introducing a generalization of the linking number to tied links. Finally, in Sect. 6 we give a table comparing the invariant Υ and its specializations considered here. Section 7 is a digression on the bt-algebra, at one and two parameters, in comparison with two different presentations of the Hecke algebra.

2 Preliminaries

Here, $\mathbb{K}$-algebra means an associative algebra, with unity 1, over the field $\mathbb{K}$.

2.1 The Monoid of Tied Braids

As usual we denote by B_n the braid group on n-strands. The Artin presentation of B_n is by the braids generators $\sigma_1, \ldots, \sigma_{n-1}$ and the relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i - j| > 1$ and $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$, for $|i - j| = 1$. An extension of the braid group B_n is the *monoid of tied braids* TB_n, which is a master piece in the study of tied links.

Definition 1 (Aicardi and Juyumaya 2016a, *Definition 3.1*) TB_n is the monoid presented by the usual braids generators $\sigma_1, \ldots, \sigma_{n-1}$ together with the tied generators $\eta_1, \ldots, \eta_{n-1}$ and the relations:

$$\eta_i\eta_j = \eta_j\eta_i \qquad \text{for all } i, j, \tag{1}$$
$$\eta_i\eta_i = \eta_i \qquad \text{for all } i, \tag{2}$$
$$\eta_i\sigma_i = \sigma_i\eta_i \qquad \text{for all } i, \tag{3}$$
$$\eta_i\sigma_j = \sigma_j\eta_i \qquad \text{for } |i - j| > 1, \tag{4}$$
$$\eta_i\sigma_j\sigma_i = \sigma_j\sigma_i\eta_j \qquad \text{for } |i - j| = 1, \tag{5}$$
$$\eta_i\eta_j\sigma_i = \eta_j\sigma_i\eta_j \quad = \quad \sigma_i\eta_i\eta_j \qquad \text{for } |i - j| = 1, \tag{6}$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{for } |i - j| > 1, \tag{7}$$
$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad \text{for } |i - j| = 1, \tag{8}$$
$$\eta_i\sigma_j\sigma_i^{-1} = \sigma_j\sigma_i^{-1}\eta_j \qquad \text{for } |i - j| = 1. \tag{9}$$

2.2 The One Parameter bt-Algebra

Set u a variable: the bt-algebra $\mathcal{E}_n(\mathsf{u})$ (Aicardi and Juyumaya 2000; Ryom-Hansen 2011; Aicardi and Juyumaya 2016b) can be conceived as the quotient algebra of the monoid algebra of TB_n over $\mathbb{C}(\mathsf{u})$, by the two-sided ideal generated by

$$\sigma_i^2 - 1 - (\mathsf{u} - 1)\eta_i(1 + \sigma_i), \quad \text{for all} \quad i.$$

See Aicardi and Juyumaya (2016a, Remark 4.3). In other words, $\mathcal{E}_n(\mathsf{u})$ is the $\mathbb{C}(\mathsf{u})$-algebra generated by $T_1, \ldots, T_{n-1}, E_1, \ldots, E_{n-1}$ satisfying the relations (1)–(8), where σ_i is replaced by T_i and η_i by E_i, together with the relations

$$T_i^2 = 1 + (\mathsf{u}-1)E_i + (\mathsf{u}-1)E_iT_i, \quad \text{for all} \quad i. \tag{10}$$

We consider now another presentation of the bt-algebra, used in Chlouveraki et al. (2020), Marin (2018). Let $\sqrt{\mathsf{u}}$ be a variable s.t. $\sqrt{\mathsf{u}}^2 = \mathsf{u}$. We denote by $\mathcal{E}_n(\sqrt{\mathsf{u}})$ the bt-algebra presented by the generators $V_1, \ldots, V_{n-1}$ and $E_1, \ldots, E_{n-1}$, where

$$V_i := T_i + \left(\frac{1}{\sqrt{\mathsf{u}}} - 1\right) E_iT_i.$$

The V_i's still satisfy the defining relations (1)–(8), substituting σ_i with V_i, η_i with E_i, but equation (10) becomes

$$V_i^2 = 1 + \left(\sqrt{\mathsf{u}} - \frac{1}{\sqrt{\mathsf{u}}}\right) E_iV_i, \quad \text{for all} \quad i. \tag{11}$$

2.3 Tied Links

Tied links were introduced in Aicardi and Juyumaya (2016a) and roughly correspond to links which may have ties connecting pairs of points of two components or of the same component. The ties in the diagrams of the tied links are drawn as springs, to outline the fact that they can be contracted and extended, letting their extremes to slide along the components. The ties define a partition of the set of components in this way: two components connected by a tie belong to the same part of the partition. Every classical link can be considered as a tied link without ties; in this case each component form a distinct part of the partition. Alternatively, a classical link can be considered as a tied link in which all components form a sole part.

We denote by $\mathfrak{L}$ the set of classical links in $\mathbb{R}^3$ and by $\widetilde{\mathfrak{L}}$ the set of tied links. As we have just recalled, $\mathfrak{L} \subset \widetilde{\mathfrak{L}}$, but the set $\mathfrak{L}$ can be identified also with the subset $\widetilde{\mathfrak{L}}^*$ of $\widetilde{\mathfrak{L}}$, formed by the tied links whose components are all tied. In terms of braids, the situation is as follows. Recall that the tied links are in bijection with the equivalence classes of TB_∞ under the t-Markov moves (Aicardi and Juyumaya 2016a, Theorem 3.7). Now, observe that B_n can be naturally considered as a submonoid of TB_n and the t-Markov moves at level of B_n are the classical Markov moves: this implies the inclusion $\mathfrak{L} \subset \widetilde{\mathfrak{L}}$. On the other hand, the group B_n is isomorphic, as group, to the submonoid EB_n of TB_n,

$$EB_n := \{\eta^n\sigma \,;\, \sigma \in B_n\}, \qquad \eta^n := \eta_1 \cdots \eta_{n-1},$$

where the group isomorphism from EB_n to B_n, denoted by $\mathtt{f}$, is given by $\mathtt{f}(\eta^n\sigma) = \sigma$. Moreover, two tied braids of EB_n are t-Markov equivalent if and only if their images by $\mathtt{f}$ are Markov equivalent. This explains, in terms of braids, the identification between $\mathfrak{L}$ and $\widetilde{\mathfrak{L}}^*$ mentioned above. For more details see Aicardi and Juyumaya (2018, Subsection 2.3).

2.4 Invariants

Invariants for classical and tied links were constructed by using the bt-algebra in the Jones recipe (Jones 1987). We recall some facts and introduce some notations for these invariants:

(1) Δ and $\widetilde{\Delta}$ denote respectively the three-variable invariant for classical links and tied links, defined through the original bt-algebra. The invariant Δ, called $\bar{\Delta}$ in Aicardi and Juyumaya (2016b), is the restriction of $\widetilde{\Delta}$ to $\mathfrak{L}$; the invariant $\widetilde{\Delta}$ was defined in Aicardi and Juyumaya (2016a), where was denoted $\mathcal{F}$.
(2) Θ and $\widetilde{\Theta}$ denote respectively the three-variable invariant for classical links and tied links, defined in Chlouveraki et al. (2020); the original notation for $\widetilde{\Theta}$ was $\overline{\Theta}$. Notice that the invariant Θ is the restriction of $\widetilde{\Theta}$ to $\mathfrak{L}$.
(3) The invariants $\widetilde{\Delta}$ and $\widetilde{\Theta}$, restricted to $\widetilde{\mathfrak{L}}^*$, coincide with the Homplypt polynomial, which is denoted by $\mathrm{P} = \mathrm{P}(t, x)$; we keep the defining skein relation of P as in Jones (1987, Proposition 6.2).
(4) The invariants Δ and Θ coincide with the Homplypt polynomial, whenever they are evaluated on knots; however they distinguish pairs of links that are not distinguished by P. See Chlouveraki et al. (2020, Theorem 8.3) and Aicardi (2016, Proposition 2).
(5) It is intriguing to note that despite the only difference in the construction Δ and Θ is the presentation used for the bt-algebra, these invariants are not topologically equivalent, see Aicardi (2016), Chlouveraki et al. (2020).

3 The Two-Parameters bt-Algebra

3.1 Generators and Relations

Let v be a variable commuting with u, and set $\mathbb{K} = \mathbb{C}(\mathsf{u}, \mathsf{v})$.

Definition 2 (Cf. Aicardi and Juyumaya 2000; Ryom-Hansen 2011; Aicardi and Juyumaya 2016b) The two-parameter bt-algebra, denoted by $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$, is defined by $\mathcal{E}_1(\mathsf{u}, \mathsf{v}) := \mathbb{K}$ and, for $n > 1$, as the unital associative $\mathbb{K}$-algebra, with unity 1, presented by the braid generators $R_1, \ldots, R_{n-1}$ and the tie generators $E_1, \ldots, E_{n-1}$, subject to the following relations:

$$E_i E_j = E_j E_i \qquad \text{for all } i, j, \tag{12}$$
$$E_i^2 = E_i \qquad \text{for all } i, \tag{13}$$
$$E_i R_j = R_j E_i \qquad \text{for } |i - j| > 1, \tag{14}$$
$$E_i R_i = R_i E_i \qquad \text{for all } i, \tag{15}$$
$$E_i R_j R_i = R_j R_i E_j \qquad \text{for } |i - j| = 1, \tag{16}$$
$$E_i E_j R_i = E_j R_i E_j = R_i E_i E_j \qquad \text{for } |i - j| = 1, \tag{17}$$
$$R_i R_j = R_j R_i \qquad \text{for } |i - j| > 1, \tag{18}$$
$$R_i R_j R_i = R_j R_i R_j \qquad \text{for } |i - j| = 1, \tag{19}$$

$$R_i^2 = 1 + (\mathsf{u} - 1)E_i + (\mathsf{v} - 1)E_i R_i \qquad \text{for all } i. \tag{20}$$

Notice that every R_i is invertible, and

$$R_i^{-1} = R_i + (1 - \mathsf{v})\mathsf{u}^{-1} E_i + (\mathsf{u}^{-1} - 1)E_i R_i. \tag{21}$$

Remark 1 The algebra $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$ can be conceived as the quotient of the monoid algebra of TB_n, over $\mathbb{K}$, by the two-sided ideal generated by all expressions of the form $\sigma_i^2 - 1 - (\mathsf{u} - 1)\eta_i - (\mathsf{v} - 1)\eta_i\sigma_i$, for all i.

Remark 2 Observe that the original bt-algebra $\mathcal{E}_n(\mathsf{u})$ is obtained as $\mathcal{E}_n(\mathsf{u}, \mathsf{u})$, while the presentation $\mathcal{E}_n(\sqrt{\mathsf{u}})$ corresponds to $\mathcal{E}_n(1, \mathsf{v})$, with $\mathsf{v} = \sqrt{\mathsf{u}} - \sqrt{\mathsf{u}}^{\,-1} + 1$.

3.2 Isomorphism of bt-Algebras with One and Two Parameters

We show here that the new two-parameters algebra is isomorphic to the original bt-algebra.

Let δ be a root of the quadratic polynomial

$$\mathsf{u}(\mathsf{z} + 1)^2 - (\mathsf{v} - 1)(\mathsf{z} + 1) - 1. \tag{22}$$

Define the elements T_i's by

$$T_i := R_i + \delta E_i R_i, \quad \text{for all} \quad i. \tag{23}$$

Proposition 1 *The $\mathbb{L}$-algebras $\mathcal{E}_n(\mathsf{u}, \mathsf{v}) \otimes_{\mathbb{K}} \mathbb{L}$ and $\mathcal{E}_n(\mathsf{u}(\delta + 1)^2)$, are isomorphic through the mappings $R_i \mapsto T_i$, $E_i \mapsto E_i$, where $\mathbb{L}$ is the smaller field containing $\mathbb{K}$ and δ.*

Proof The T_i's satisfy the relations (12)–(19) and we have, using relation (20),

$$T_i^2 = R_i^2 + (\delta^2 + 2\delta)E_i R_i^2 = 1 + (\mathsf{u}(\delta + 1)^2 - 1)E_i + (\mathsf{v} - 1)(\delta + 1)^2 E_i R_i.$$

Now, since

$$R_i = T_i - \frac{\delta}{\delta + 1} E_i T_i,$$

we have $E_i R_i = (\delta + 1)^{-1} E_i T_i$, and substituting we get

$$T_i^2 = 1 + (\mathsf{u}(\delta + 1)^2 - 1)E_i + (\mathsf{v} - 1)(\delta + 1)E_i T_i. \tag{24}$$

Therefore, the coefficients of E_i and $E_i T_i$ are equal since δ is a root of the polynomial (22). □

Remark 3 Notice that the roots of (22) are: $\mathsf{z}_{\pm} = (\mathsf{v} - 1 - 2\mathsf{u} \pm \sqrt{(\mathsf{v}-1)^2 + 4\mathsf{u}})/2\mathsf{u}$, so

$$T_i^2 = 1 + \frac{(\mathsf{v}-1)\left(\mathsf{v} - 1 \pm \sqrt{(\mathsf{v}-1)^2 + 4\mathsf{u}}\right)}{2\mathsf{u}}(E_i + E_iT_i). \quad (25)$$

Thus, for $\mathsf{v} = \mathsf{u}$, we have: $\mathsf{z}_+ = 0$ and $\mathsf{z}_- = -\mathsf{u}^{-1}(\mathsf{u}+1)$ with the corresponding quadratic relations:

$$T_i^2 = 1 + (\mathsf{u}-1)(E_i + E_iT_i), \quad T_i^2 = 1 + \frac{1-\mathsf{u}}{\mathsf{u}}(E_i + E_iT_i).$$

The first solution gives trivially $\mathcal{E}_n(\mathsf{u})$, while the second one gives another presentation of $\mathcal{E}_n(\mathsf{u})$, obtained by keeping as parameter u^{-1}; note that $\mathcal{E}_n(\mathsf{u}) = \mathcal{E}_n(\mathsf{u}^{-1})$.

On the other hand, for $\mathsf{u} = 1$, we get $\mathsf{z}_{\pm} = (\mathsf{v} - 3 \pm \sqrt{\mathsf{v}^2 - 2\mathsf{v} + 5})/2$ giving

$$T_i^2 = 1 + \frac{(\mathsf{v}-1)\left(\mathsf{v} - 1 \pm \sqrt{\mathsf{v}^2 - 2\mathsf{v} + 5}\right)}{2}(E_i + E_iT_i). \quad (26)$$

These two solutions determine isomorphisms between $\mathcal{E}_n(\sqrt{\mathsf{u}})$ and $\mathcal{E}_n(\mathsf{u})$.

At this point we have to note that there is another interesting specialization of $\mathcal{E}_n(\mathsf{u},\mathsf{v})$, namely when $\mathsf{v} = 1$. In fact, $\mathcal{E}_n(\mathsf{u}, 1)$ deserves a deeper investigation. Here we gives some relations holding only in this specialization. More precisely, we have:

$$R_i^2 = 1 + (\mathsf{u}-1)E_i \quad \text{and} \quad R_i^{-1} = R_i + (\mathsf{u}^{-1} - 1)E_iR_i \quad \text{for all } i. \quad (27)$$

Then we deduce

$$(\mathsf{u}+1)R_i - \mathsf{u}R_i^{-1} = R_i^3 \quad (28)$$

since $E_iR_i^{-1} = \mathsf{u}^{-1}E_iR_i$. So,

$$R_i^4 - (\mathsf{u}+1)R_i^2 + \mathsf{u} = 0, \quad \text{or equivalently} \quad (R_i^2 - 1)(R_i^2 - \mathsf{u}) = 0. \quad (29)$$

3.3 Markov Trace on $\mathcal{E}_n(\mathsf{u},\mathsf{v})$

Proposition 2 *Let a and b two mutually commuting variables. There exists a unique Markov trace $\rho = \{\rho_n\}_{n\in\mathbb{N}}$ on $\mathcal{E}_n(u, v)$, where the ρ_n's are linear maps from $\mathcal{E}_n(u, v)$ to $\mathbb{L}(a, b)$, satisfying $\rho_n(1) = 1$, and defined inductively by the rules:*

(1) $\rho_n(XY) = \rho_n(YX)$,
(2) $\rho_{n+1}(XR_n) = \rho_{n+1}(XR_nE_n) = a\rho_n(X)$,
(3) $\rho_{n+1}(XE_n) = b\rho_n(X)$,

where $X, Y \in \mathcal{E}_n(u, v)$.

Proof The proof follows from Proposition 1, since is obtained by carrying the Markov trace on the bt-algebra (Aicardi and Juyumaya 2016b, Theorem 3) to $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$. More precisely, if we denote by ρ' the Markov trace on the bt-algebra, then ρ is defined by $\rho' \circ \phi$, where ϕ denote the isomorphism of Proposition 1; moreover denoting by a' and b' the parameters trace of ρ', we have $\mathsf{a} = (\delta + 1)^{-1}\mathsf{a}'$ and $\mathsf{b} = \mathsf{b}'$. □

4 Invariants

In this section we define, via Jones recipe, the invariants of classical and tied links associated to the algebra $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$.

4.1 An Invariant for Classical Links

Define the homomorphism π_{c} from B_n to $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$ by taking

$$\pi_{\mathsf{c}}(\sigma_i) = \sqrt{\mathsf{c}} R_i, \tag{30}$$

where the scaling factor c is obtained by imposing, due to the second Markov move, that $(\rho \circ \pi_{\mathsf{c}})(\sigma_i) = (\rho \circ \pi_{\mathsf{c}})(\sigma_i^{-1})$; thus

$$\mathsf{c} := \frac{\mathsf{a} + (1 - \mathsf{v})\mathsf{u}^{-1}\mathsf{b} + (\mathsf{u}^{-1} - 1)\mathsf{a}}{\mathsf{a}} = \frac{\mathsf{a} + \mathsf{b}(1 - \mathsf{v})}{\mathsf{a}\mathsf{u}}. \tag{31}$$

Theorem 1 *The function $\Upsilon : \mathfrak{L} \longrightarrow \mathbb{C}(u, v, a, \sqrt{c})$, defines an invariant for classical links,*

$$\Upsilon(L) := \left(\frac{1}{a\sqrt{c}}\right)^{n-1} (\rho \circ \pi_c)(\sigma),$$

where $L = \widehat{\sigma}$, $\sigma \in B_n$.

Proof The proof follows step by step the proof done for the invariant $\bar{\Delta}$ in Aicardi and Juyumaya (2016b), replacing the elements T_i by R_i. Observe that the only differences consist in the expressions of L (see (46), Aicardi and Juyumaya 2016b), that must be replaced by c, and of the inverse element, that contains now two parameters. However, it is a routine to check that the proof is not affected by the presence of two parameters instead of one. □

Remark 4 From Remark 2 it follows that, respectively, the invariants Δ and Θ correspond to the specializations $\mathsf{u} = \mathsf{v}$ and $\mathsf{u} = 1$ with $\mathsf{v} = \sqrt{\mathsf{u}} - \sqrt{\mathsf{u}}^{-1}$ of Υ.

4.2 An Invariant for Tied Links

The invariant Υ can be extended to an invariant of tied links, denoted by $\widetilde{\Upsilon}$, simply extending π_{c} to TB_n by mapping η_i to E_i. We denote this extension by $\widetilde{\pi}_{\mathsf{c}}$.

Theorem 2 *The function* $\widetilde{\Upsilon} : \widetilde{\mathfrak{L}} \longrightarrow \mathbb{C}(\mathsf{u}, \mathsf{v}, \mathsf{a}, \sqrt{\mathsf{c}})$, *defines an invariant for tied links, where*

$$\widetilde{\Upsilon}(L) := \left(\frac{1}{\mathsf{a}\sqrt{\mathsf{c}}}\right)^{n-1} (\rho \circ \widetilde{\pi}_{\mathsf{c}})(\eta),$$

L being the closure of the n-tied braid η.

This theorem will be proved together with Theorem 3 of the next section.

5 The Invariant $\widetilde{\Upsilon}$ via Skein Relation

This section is two parts: the first one describes $\widetilde{\Upsilon}$ by skein relation and the second is devoted to analyze a specialization $\widetilde{\Omega}$ of $\widetilde{\Upsilon}$.

In the sequel, if there is no risk of confusion, we indicate by L both the oriented tied link and its diagram and we denote by L_+, L_-, $L_\sim$, $L_{+,\sim}$ and $L_{-,\sim}$ the diagrams of tied links, that are identical outside a small disc into which enter two strands, whereas inside the disc the two strands look as shown in Fig. 1.

The following theorem is the counterpart of Aicardi and Juyumaya (2016a, Theorem 2.1).

Theorem 3 *The function* $\widetilde{\Upsilon}$ *is defined uniquely by the following three rules:*

I *The value of* $\widetilde{\Upsilon}$ *is equal to* 1 *on the unknot.*
II *Let L be a tied link. By $L \sqcup \bigcirc$ we denote the tied link consisting of L and the unknot, unlinked to L. Then*

$$\widetilde{\Upsilon}(L \sqcup \bigcirc) = \frac{1}{\mathsf{a}\sqrt{\mathsf{c}}}\widetilde{\Upsilon}(L).$$

III *Skein rule:*

$$\frac{1}{\sqrt{\mathsf{c}}}\widetilde{\Upsilon}(L_+) - \sqrt{\mathsf{c}}\widetilde{\Upsilon}(L_-) = \frac{\mathsf{v}-1}{\mathsf{u}}\widetilde{\Upsilon}(L_\sim) + \frac{1}{\sqrt{\mathsf{c}}}(1-\mathsf{u}^{-1})\widetilde{\Upsilon}(L_{+,\sim}).$$

Proof (of Theorems 2 and 3) See the proof done for the invariant $\mathcal{F}$ in Aicardi and Juyumaya (2016a, Theorem 2.1) replacing the variables z and w respectively by a and

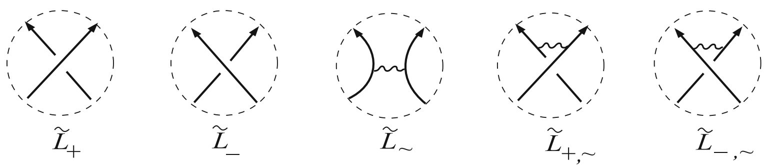

Fig. 1 The discs where the corresponding diagrams differ

c. The definition of t must be replaced by that of b given by

$$\mathsf{b} = \mathsf{a}(\mathsf{uc} - 1)/(1 - \mathsf{v}), \tag{32}$$

according to (31). All steps of the proofs are still holding for the new skein rules involving the new parameter v.

At this point we have an invariant for tied links $\widetilde{\Upsilon}$, uniquely defined by the rules I–III. It remains to prove that it coincides with that obtained via Jones recipe: the proof now proceeds exactly as that of Aicardi and Juyumaya (2016a, Theorem 4.5). In this way we have proven also Theorem 2. □

Remark 5 Rules I and II imply that the value of the invariant on a collection of n unlinked circles is $(\mathsf{a}\sqrt{\mathsf{c}})^{1-n}$.

Remark 6 The following skein rule IV is obtained from rule III, adding a tie between the two strands inside the disc. Rules Va and Vb are equivalent to the skein rule III, by using rule IV.

IV

$$\frac{1}{\mathsf{u}\sqrt{\mathsf{c}}}\widetilde{\Upsilon}(L_{+,\sim}) - \sqrt{\mathsf{c}}\widetilde{\Upsilon}(L_{-,\sim}) = \frac{\mathsf{v}-1}{\mathsf{u}}\widetilde{\Upsilon}(L_{\sim}).$$

Va

$$\frac{1}{\sqrt{\mathsf{c}}}\widetilde{\Upsilon}(L_{+}) = \sqrt{\mathsf{c}}\left[\widetilde{\Upsilon}(L_{-}) + (\mathsf{u}-1)\,\widetilde{\Upsilon}(L_{-,\sim})\right] + (\mathsf{v}-1)\,\widetilde{\Upsilon}(L_{\sim}).$$

Vb

$$\sqrt{\mathsf{c}}\widetilde{\Upsilon}(L_{-}) = \frac{1}{\sqrt{\mathsf{c}}}\left[\widetilde{\Upsilon}(L_{+}) + \frac{1-\mathsf{u}}{\mathsf{u}}\widetilde{\Upsilon}(L_{+,\sim})\right] + \frac{1-\mathsf{v}}{\mathsf{u}}\widetilde{\Upsilon}(L_{\sim}).$$

Remark 7 The value of the invariant $\widetilde{\Upsilon}(\mathsf{u},\mathsf{v})$ on a tied link made by n unlinked circles all tied, is obtained by rule IV (cf. Aicardi and Juyumaya 2016a, Remark 2.3), and it is

$$\left(\frac{\mathsf{uc}-1}{\sqrt{\mathsf{c}}(1-\mathsf{v})}\right)^{n-1} = \left(\frac{\mathsf{b}}{\mathsf{a}\sqrt{\mathsf{c}}}\right)^{n-1}. \tag{33}$$

The last equality comes from (32).

Remark 8 For tied links in $\widetilde{\mathfrak{L}}^{*}$, the invariant $\widetilde{\Upsilon}$ is uniquely defined by rules I and IV. Observe that, by multiplying skein rule IV by $\sqrt{\mathsf{u}}$, we get that $\widetilde{\Upsilon}$ coincides with the Homflypt polynomial in the variables $t = \sqrt{\mathsf{uc}}$ and $x = (\mathsf{v}-1)/\sqrt{\mathsf{c}}$; that is, if L is the tied link in $\widetilde{\mathfrak{L}}^{*}$, associated to the classical links L, then $\widetilde{\Upsilon}(L) = \mathrm{P}(L)$.

Remark 9 The invariants of tied links $\widetilde{\Delta}$ and $\widetilde{\Theta}$ are, respectively, the specializations $\widetilde{\Upsilon}(\mathsf{u},\mathsf{u})$ and $\widetilde{\Upsilon}(1,\mathsf{v})$.

5.1 The Invariant $\widetilde{\Omega}$

We shall denote by $\widetilde{\Omega}$ the specialization $\widetilde{\Upsilon}_{\mathsf{u,v}}$ at $\mathsf{v} = 1$.

We observe firstly that if $\mathsf{v} = 1$ then $\mathsf{c} = \mathsf{u}^{-1}$, so the invariant $\widetilde{\Omega}$ takes in fact values in $\mathbb{C}(\sqrt{\mathsf{u}}, \mathsf{a}, \mathsf{b})$. The next lemma describes $\widetilde{\Omega}$ by skein relations and is the key to show its main properties.

Lemma 1 *The invariant $\widetilde{\Omega}$ is uniquely defined by the following rules:*

I $\widetilde{\Omega}(\bigcirc) = 1$.
II $\widetilde{\Omega}(L \sqcup \bigcirc) = a^{-1}\sqrt{u}\,\widetilde{\Omega}(L)$.
III *By $L\,\widetilde{\sqcup}\,\bigcirc$ we denote the tied link consisting of the tied link L and the unknot, unlinked to L, but tied to one component of L. Then*

$$\widetilde{\Omega}(L\,\widetilde{\sqcup}\,\bigcirc) = \frac{b\sqrt{u}}{a}\,\widetilde{\Omega}(L).$$

IV *Skein rule:*

$$\sqrt{u}\,\widetilde{\Omega}(L_+) - \frac{1}{\sqrt{u}}\,\widetilde{\Omega}(L_-) + \sqrt{u}(u^{-1} - 1)\,\widetilde{\Omega}(L_{+,\sim}) = 0.$$

Proof By comparing the rules of the lemma with those of Theorem 3, we observe that: rule I coincides with rule I for $\widetilde{\Upsilon}$, rules II and IV are obtained by setting $\mathsf{v} = 1$ in the corresponding rules II and III. Notice that, when the two components of the considered crossing are tied, rule IV becomes

$$\widetilde{\Omega}(L_{+,\sim}) - \widetilde{\Omega}(L_{-,\sim}) = 0. \tag{34}$$

Observe now that the necessity of rule III for defining $\widetilde{\Omega}$, depends on the fact that the skein rule IV does not involve the diagram $L_\sim$, so that the value of $\widetilde{\Omega}$ on two unlinked circles tied together cannot be deduced, using skein rules; note this value is chosen by imposing that it coincide with the value obtained through the Jones recipe, and indeed matches with (33). Rule III is in fact the unique point that makes the case $\mathsf{v} = 1$ to be considered separately from Theorem 3. □

To present the next result we need to highlight some facts and to introduce some notations.

We start by recalling that the ties of a tied link define a partition of the set of components: if there is a tie between two components, then these components belong to the same class, see Aicardi and Juyumaya (2018, Section 2.1).

Definition 3 We call linking graph of a link, the m-graph whose vertices represent the m components and where two vertices are connected by an edge if the corresponding components have a nonzero linking number. Each edge is labeled by the corresponding linking number.

We generalize the linking number to tied links.

Definition 4 We call class linking number or c-linking number, between two classes of components, the sum of linking numbers of the components of one class with the components of the other class.

Definition 5 We call c-linking graph of a tied link L, the k-graph whose vertices represent the k classes of the L components and where two vertices are connected by an edge if the corresponding classes have a nonzero c-linking number. Each edge is labeled by the corresponding c-linking number.

Example 1 The links in Fig. 2 have three components: 1, 2 and 3, and two classes, $A = \{1, 3\}$ and $B = \{2\}$. All crossings have positive sign. The c-linking number between the classes A and B is in both cases equal to 2. The corresponding c-linking graph is shown at right.

Remark 10 For tied links in $\mathfrak{L}$, the c-linking graph coincides with the linking graph.

Theorem 4 *The invariant $\widetilde{\Omega}$ has the following properties:*

(1) *The value of $\widetilde{\Omega}$ is equal to* 1 *on knots.*
(2) *$\widetilde{\Omega}$ takes the same value on links with the same number of components all tied together. The value depends only on the number of components m, namely it is equal to $(\mathsf{b}\sqrt{\mathsf{u}}/\mathsf{a})^{m-1}$.*
(3) *$\widetilde{\Omega}$ takes the same value on tied links having the same number of components and the same c-linking graph.*

Proof Rule (34) implies that, given any knot diagram, $\widetilde{\Omega}$ takes the same value on any other diagram obtained by changing any crossing from positive to negative or viceversa. Thus, it takes the same value also on the diagram corresponding to the unknot: by rule I this value is equal to 1. This proves claim (1).

Claim (2) is a consequence of rule (34) together with rule III of Lemma 1.

Suppose the tied link $\widetilde{L}$ has m components, partitioned into k classes. We order arbitrarily the classes, and inside each class, using rule (34), we change the signs of some crossings in order to unlink the components and transform each component into the unknot. Then we start from the first class c_1 and consider in their order all the other classes c_i linked with it: we mark all the undercrossing of c_1 with c_i as *deciding* crossings. Then we pass to the class c_2, we select all classes c_j linked with it and having indices greater than 2, and mark the undercrossings of c_2 with c_j, so increasing the list of deciding crossings. We proceed this way till the last class. At the end we have obtained an ordered sequence of q pairs of classes characterized by the corresponding

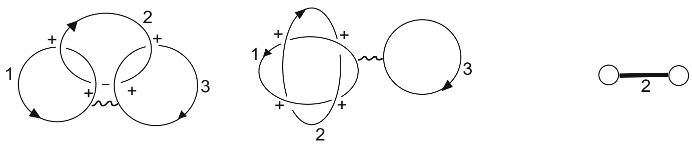

Fig. 2 Two tied links with the same c-linking graph

c-linking numbers. So, we construct a graph with k vertices, and q ordered edges, labeled with the c-linking numbers.

Consider now the first pair of classes (i, j) in the sequence. We apply the skein rule IV of Lemma 1, to each one of the n *deciding* crossings between the components of this pair. These points have signs $s_1, \ldots, s_n$. By using rule (34), rule IV becomes, respectively for positive and negative crossings,

$$\widetilde{\Omega}(L_+) = \frac{1}{\mathsf{u}}\widetilde{\Omega}(L_-) + \left(1 - \frac{1}{\mathsf{u}}\right)\widetilde{\Omega}(L_{-,\sim})$$
$$\text{and} \quad \widetilde{\Omega}(L_-) = \mathsf{u}\widetilde{\Omega}(L_+) + (1-\mathsf{u})\widetilde{\Omega}(L_{+,\sim}).$$

So, consider the first deciding point with signs s_1. We have

$$\widetilde{\Omega}(L_{s_1}) = \mathsf{u}^{-s_1}\widetilde{\Omega}(L_{-s_1}) + (1 - \mathsf{u}^{-s_1})\widetilde{\Omega}(L_{-s_1,\sim}).$$

The two diagrams at the right member are identical, but in the second one there is a tie between the classes i and j. We denote this diagram by $L^{i\sim j}$; observe that in this diagram the classes i and j merge in a sole class.

To calculate the first term $\mathsf{u}^{-s_1}\widetilde{\Omega}(L_{-s_1})$, we pass to the second deciding point, so obtaining a first term $\mathsf{u}^{-(s_1+s_2)}\widetilde{\Omega}(L_{-s_2})$, and a second term $\mathsf{u}^{-s_1}(1-\mathsf{u}^{-s_2})\widetilde{\Omega}(L^{i\sim j})$. At the n-th deciding point, we obtain

$$\widetilde{\Omega}(L) = \mathsf{u}^{-(s_1+s_2+\cdots+s_n)}\widetilde{\Omega}(L_{-s_n}) + \sum_{i=1}^{n} \mathsf{u}^{-(s_0+\cdots+s_{i-1})}(1-\mathsf{u}^{-s_i})\widetilde{\Omega}(L^{i\sim j}),$$

where $s_0 = 0$. Now, L_{-s_n} is the link obtained by L by unlinking the classes i and j, that we shall denote by $L^{i\|j}$. By expanding the sum we obtain

$$\sum_{i=1}^{n} \mathsf{u}^{-(s_0+\cdots+s_{i-1})}(1-\mathsf{u}^{-s_i}) = 1 - \mathsf{u}^{-(s_1+s_2+\cdots+s_n)}.$$

The sum $s_1 + \cdots + s_n$ is the sum of the signs of all undercrossings, and therefore equals the c-linking number of the two classes, that we denote by $\ell(i, j)$. Therefore we get

$$\widetilde{\Omega}(L) = \mathsf{u}^{-\ell(i,j)}\widetilde{\Omega}(L^{i\|j}) + (1-\mathsf{u}^{-\ell(i,j)})\widetilde{\Omega}(L^{i\sim j}). \tag{35}$$

Observe now that Eq. (35) is a *generalized* skein relation, that is used to unlink two classes of components (or two components, when the classes contain a sole component), see Fig. 3. The independence of the calculation by skein of $\widetilde{\Upsilon}$ from the order of the deciding points, implies here the independence of the calculation of $\widetilde{\Omega}$ by the generalized skein equation (35) from the order of the pairs of classes.

However, Eq. (35) becomes $\widetilde{\Omega}(L) = \widetilde{\Omega}(L^{i\|j})$ when the c-linking number is zero. Therefore, the invariant $\widetilde{\Omega}$ does not distinguish two linked classes with c-linking number zero from the same classes unlinked. So, we obtain the c-linking graph by deleting

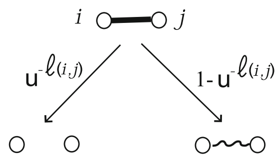

Fig. 3 Generalized skein rule

the edges labeled by zero in the graph before constructed. Observe that, if there remain p edges with non zero label, the generalized skein relation (35) defines a tree terminating in 2^p diagrams L_j, all having the classes unlinked. These diagrams differ only for a certain number of ties, and each one of them can be represented by a graph obtained from the c-linking graph where each edge is either deleted or substituted by a tie. The value of $\widetilde{\Omega}(L)$ is then the sum

$$\sum_{j=1}^{2^p} \alpha_j \widetilde{\Omega}(L_j). \tag{36}$$

Notice that each vertex of the tree is labeled by a pair (x, y) of classes, that is, the classes that are unlinked by the skein rule at that vertex. To calculate the coefficient α_j, consider all the p vertices of the path in the skein tree, going from L_j to L. For each one of these vertices, say with label (x, y), choose the factor $\mathsf{u}^{-\ell(x,y)}$ if it is reached from left, otherwise the factor $(1-\mathsf{u}^{-\ell(x,y)})$. The coefficient α_j is the product of these p factors.

The value $\widetilde{\Omega}(L_j)$ depends only on the number m of components, and on the number of classes h, $h \leq k$, of L_j; indeed, by rules II and III of Lemma 1 we have:

$$\widetilde{\Omega}(L_j) = \left(\frac{\sqrt{\mathsf{u}}}{\mathsf{a}}\right)^{m-1} \mathsf{b}^{m-h}. \tag{37}$$

To calculate h for the diagram L_j, we start from the c-linking graph of L, and use again the p vertices of the considered path in the skein tree: if the path reaches a vertex labeled (x, y) from left, then the edge (x, y) is eliminated from the graph, otherwise the edge is substituted by a tie. The number of connected components of the graph so obtained, having ties as edges, is the resulting number h of classes, e.g. see Fig. 5.

To conclude the proof, it is now sufficient to observe that the calculation of $\widetilde{\Omega}(L)$ depends only on the c-linking graph and on the total number of components of L. □

Corollary 1 *Let L be a tied link with m components and k classes. Let r be the exponent of a and s the minimal exponent of b in $\widetilde{\Omega}(L)$. Then $m = 1 - r$ and $k = 1 - r - s$.*

Proof It follows from Eqs. (36) and (37), noting that the coefficients α_i depend only on the variable u. □

We shall denote by Ω the specialization of Υ at $\mathsf{v} = 1$, that is, Ω is the restriction of $\widetilde{\Omega}$ to $\mathfrak{L}$. We have the following results for classical links.

Corollary 2 *The invariant* Ω *has the following properties:* Ω *takes the same value on links having the same linking graph. If* L *has* m *components, the exponent of* a *in* Ω *is* $1-m$ *and there is a term in* Ω *non containing* b*.*

Proof It follows from Theorem 4 and Corollary 1. □

Example 2 Consider the link L in Fig. 4. Here $m=3$, $c=3$ and $p=3$. All linking numbers $\ell(i,j)$ are equal to 1.

The value $\Omega(L)$ is obtained by adding the value of Ω on the 2^3 graphs shown in Fig. 5, where they are subdivided in four groups, according to the value of Ω, i.e., to the number of classes, indicated at bottom. The coefficients, here written for the four groups, are:

$$\mathsf{u}^{-3},\quad \mathsf{u}^{-2}(1-\mathsf{u}^{-1}),\quad \mathsf{u}^{-1}(1-\mathsf{u}^{-1})^2 \quad\text{and}\quad (1-\mathsf{u}^{-1})^3,$$

whereas the corresponding values of Ω are

$$\mathsf{u}/\mathsf{a}^2,\quad \mathsf{bu}/\mathsf{a}^2,\quad \mathsf{b}^2\mathsf{u}/\mathsf{a}^2 \quad\text{and}\quad \mathsf{b}^2\mathsf{u}/\mathsf{a}^2.$$

Then, $\Omega(L)=\mathsf{ua}^{-2}(\mathsf{u}^{-3}+3\mathsf{b}(\mathsf{u}^{-2}(1-\mathsf{u}^{-1}))+3\mathsf{b}^2(\mathsf{u}^{-1}(1-\mathsf{u}^{-1})^2)+\mathsf{b}^2(1-\mathsf{u}^{-1})^3)$, so

$$\Omega(L)=\mathsf{a}^{-2}\mathsf{u}^{-2}(1+3\mathsf{bu}-3\mathsf{b}-3\mathsf{b}^2\mathsf{u}+2\mathsf{b}^2+\mathsf{b}^2\mathsf{u}^3).$$

Finally, observe that $r=-2$ and $s=0$; indeed, L has 3 components and 3 classes.

6 Results of Calculations

Here the notations of the links with ten or eleven crossings are taken from Cha and Livingston (2018).

The following table shows eight pairs of non isotopic links with three components, distinguished by $\Upsilon(\mathsf{u},\mathsf{v})$, but non distinguished by the Homflypt polynomial. A star indicates when they are distinguished also by a specialization of $\Upsilon(\mathsf{u},\mathsf{v})$.

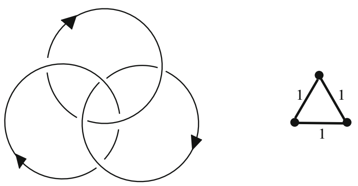

Fig. 4 A link and its linking graph

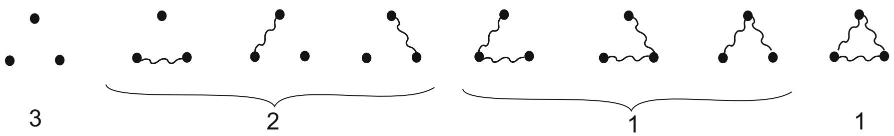

Fig. 5 The eight graph obtained from the c-linking graph of Fig. 4

Link	*l.graph*	*Link*	*l.graph*	$\Upsilon(\mathsf{u},\mathsf{v})$	$\Upsilon(1,\mathsf{v})$	$\Upsilon(\mathsf{u},\mathsf{u})$	$\Upsilon(\mathsf{u},1)$
$L11n358\{0,1\}$	-2, -1	$L11n418\{0,0\}$	-2, -1	$\star$	$\star$		
$L11n358\{1,1\}$	2, 1	$L11n418\{1,0\}$	2, 1	$\star$		$\star$	
$L11n356\{1,0\}$	2, -1	$L11n434\{0,0\}$	2, -1	$\star$		$\star$	
$L11n325\{1,1\}$	-1, •	$L11n424\{0,0\}$	-1, •	$\star$	$\star$	$\star$	
$L10n79\{1,1\}$	-3, •	$L10n95\{1,0\}$	1, -2, -2	$\star$	$\star$	$\star$	$\star$
$L11a404\{1,1\}$	1, 1, -2	$L11a428\{1,0\}$	1, 1, 2	$\star$	$\star$	$\star$	$\star$
$L11a467\{0,1\}$	-2, -1	$L11a527\{0,0\}$	-2, -1	$\star$	$\star$		
$L10n76\{1,1\}$	-3, •	$L11n425\{1,0\}$	1, -2, -2	$\star$	$\star$	$\star$	$\star$

Observe that, among the eight pairs distinguished by $\Upsilon(\mathsf{u},\mathsf{v})$, six are distinguished by $\Upsilon(\mathsf{u},\mathsf{u})$, six by $\Upsilon(1,\mathsf{v})$; the pair distinguished by both $\Upsilon(\mathsf{u},\mathsf{u})$ and $\Upsilon(1,\mathsf{v})$ are four, three of which are distinguished also by $\Upsilon(\mathsf{u},1)$. We don't know whether it is necessary, for being distinguished by $\Upsilon(\mathsf{u},1)$, to be distinguished by all other specializations.

7 Final Remarks

7.1 Other Similar Recent Results

Recently, Jacon and Poulain d'Andecy have constructed an explicit isomorphism between the Yokonuma–Hecke algebra and a direct sum of matrix algebras over tensor products of Iwahori–Hecke algebras, also they have classified the Markov traces on the Yokonuma–Hecke algebra, see Jacon and Poulain d'Andecy (2016, Theorems 3.1, 5.3). In the same paper they defined an invariant of three parameters and have shown that the invariants Θ and Δ can be obtained from it, see Jacon and Poulain d'Andecy (2016, Subsection 6.5). In Chlouveraki et al. (2020, Appendix) Lickorish found a formula for Θ which allows to compute $\Theta(L)$ through the linking numbers and the Homflypt polynomials of the sublinks of the oriented link L, cf. Poulain d'Andecy and Wagner (2018). On the other hand, in Espinoza and Ryom-Hansen (2018, Theorem 14) Espinoza and Ryom–Hansen proved that the bt-algebra can be considered as a subalgebra of the Yokonuma–Hecke algebra; cf. Jacon and Poulain d'Andecy (2017). This result together with the results of Jacon and Poulain d'Andecy and the formula for Θ induce to think that some of the results of this paper, at level of classical links, could be recovered by a combination of the results mentioned before. However, this combination do not imply the results proved here for tied links.

An open problem yet is to know how strong is the four variable invariant Υ (respectively, $\widetilde{\Upsilon}$) with respect to Θ and Δ (respectively, $\widetilde{\Theta}$ and $\widetilde{\Delta}$).

7.2 Different Hecke Algebra Presentations

Denote by $\mathrm{H}_n(\mathsf{u})$ the Hecke algebra, that is, the $\mathbb{C}(\mathsf{u})$-algebra generated by $h_1,\ldots,h_{n-1}$ subject to the braid relations of type A, together with the quadratic relation

$$h_i^2 = \mathsf{u} + (\mathsf{u} - 1)h_i, \quad \text{for all } i.$$

Now, there exits another presentation used to describe the Hecke algebra, which is obtained by rescaling h_i by $\sqrt{\mathsf{u}}^{-1}$; more precisely, taking $f_i := \sqrt{\mathsf{u}}^{-1} h_i$. In this case the f_i's satisfy the braid relations and the quadratic relation

$$f_i^2 = 1 + (\sqrt{\mathsf{u}} - \sqrt{\mathsf{u}}^{-1}) f_i.$$

Denote by $\mathrm{H}_n(\sqrt{\mathsf{u}})$ the presentation of the Hecke algebra through the f_i's. The construction of the Homflypt polynomial can be made indistinctly from any of the above presentations for the Hecke algebra.

The bt-algebra can be regarded as a generalization of the Hecke algebra, in the sense that, by taking $E_i = 1$ in the presentation of the bt-algebra, we get the Hecke algebra; indeed, under $E_i = 1$ the presentations, respectively, of $\mathcal{E}_n(\mathsf{u})$ and $\mathcal{E}_n(\sqrt{\mathsf{u}})$ becomes $\mathrm{H}_n(\mathsf{u})$ and $\mathrm{H}_n(\sqrt{\mathsf{u}})$. Now we recall that, as we noted in observation 5 of Subsection 2.4, these two presentations of the bt-algebra yield different invariants. The authors don't know other situations where different presentations of the same algebra produce different invariants.

7.3 Relation with Two Parameters Hecke Algebra

Also the Hecke algebra with two parameters can be considered; that is, by taking two commuting parameters u_1 and u_2, and imposing that the generators h_i's satisfy $h_i^2 = \mathsf{u}_1 + \mathsf{u}_2 h_i$, for all i; however, the Hecke algebras with one and two parameters are isomorphic, see Kassel and Turaev (2008, Subsection 4.2); hence, from the algebraic point of view these algebras are the same. Now, regarding the behavior of the Hecke algebra with two parameters $\mathrm{H}_n(\mathsf{u}_1, \mathsf{u}_2)$, in the construction of polynomial invariants, we have that, after suitable rescaling, $\mathrm{H}_n(\mathsf{u}_1, \mathsf{u}_1)$ becomes of the type $\mathrm{H}_n(\sqrt{\mathsf{u}})$ and $\mathrm{H}_n(\mathsf{u}_1, 0)$ becomes the group algebra of the symmetric group. For $\mathrm{H}_n(0, \mathsf{u}_2)$, we obtain the so-called 0-Hecke algebra.

We examine now the bt-algebra with one more parameter. Taking u_0, u_1, u_2 and u_3 commuting variables, it is natural to keep generators R_i's instead the T_i's, satisfying $R_i^2 = \mathsf{u}_0 + \mathsf{u}_1 E_i + \mathsf{u}_2 E_i R_i + \mathsf{u}_3 R_i$, for all i; notice that a simple rescaling shows that we can take $\mathsf{u}_0 = 1$. Now, we need that these R_i's, together with the E_i's, satisfy all defining relations of the bt-algebra with the only exception of relation (20); it is straightforward to see that these defining relations hold if and only if we take $\mathsf{u}_3 = 0$. This is the motivation for defining the bt-algebra $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$ with two parameters in this paper. Observe that we have a homomorphism from $\mathcal{E}_n(\mathsf{u}, \mathsf{v})$ onto $\mathrm{H}_n(\mathsf{u}, \mathsf{v} - 1)$, defined by sending E_i to 1 and R_i to h_i; so, the 0-Hecke algebra is the homomorphic image of $\mathcal{E}_n(0, \mathsf{v})$.

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