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RESEARCH CONTRIBUTION

The Roots of Exceptional Modular Lie Superalgebras with Cartan Matrix

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Abstract

For each of the exceptional (not entering infinite series) finite-dimensional modular Lie superalgebras with indecomposable Cartan matrix, we give the explicit list of its roots, and the corresponding Chevalley basis, for one of its inequivalent Cartan matrices, namely the one corresponding to the greatest number of mutually orthogonal isotropic odd simple roots (this number, called the defect of the Lie superalgebra, is important in the representation theory). Our main tools: Grozman's Mathematica-based code SuperLie, Python, and A. Lebedev's help.

Keywords Modular Lie superalgebra · Cartan matrix

Mathematics Subject Classification Primary 17B50; Secondary 17B20

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1 Introduction

The paper Bouarroudj et al. (2009) contains classification of finite-dimensional Lie superalgebras $\mathfrak{g}(A)$ with indecomposable Cartan matrix A over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$ together with all inequivalent Cartan matrices and the corresponding Dynkin–Kac diagrams for each such Lie superalgebra (recall that every modular Lie algebra and every Lie superalgebra over any field can have several inequivalent Cartan matrices).

Here we consider the *exceptional* (not entering infinite series) finite-dimensional Lie superalgebras $\mathfrak{g}(A)$ with A indecomposable, and supplement Bouarroudj et al. (2009) with new results and clarifications. In particular, we clarify the notion of equivalent Cartan matrices.

In Sect. 2 we recall the definition of Cartan matrix and root system for modular Lie algebras and Lie superalgebras $\mathfrak{g}(A)$; the definition of roots over $\mathbb{K}$ differs from the one in characteristic 0; this was suggested in Kuznetsov and Chebochko (2000) and, following Lebedev, in Bouarroudj et al. (2009).

A posteriori we see that all indecomposable Cartan matrices of finite-dimensional Lie superalgebras $\mathfrak{g}(A)$ are symmetrizable. The classification (Bouarroudj et al. 2009) did not impose any a priori conditions on A except for indecomposability and requirement $\mathfrak{g}(A) < \infty$.

The paper Bouarroudj et al. (2009) contains superdimensions of each Lie superalgebra $\mathfrak{g}(A)$, a description of its even part $\mathfrak{g}(A)_{\bar 0}$, and of its odd part $\mathfrak{g}(A)_{\bar 1}$ as a $\mathfrak{g}(A)_{\bar 0}$-module. Recall that $A/a|B$ means that $A|B = \operatorname{sdim} \mathfrak{g}(A)$ and $a|B = \operatorname{sdim} \mathfrak{g}^{(1)}(A)/\mathfrak{c}$, where $\mathfrak{c}$ is the center and $\mathfrak{g}^{(1)}$ is the first derived of $\mathfrak{g}$ (for brevity, we write $\mathfrak{g}^{(1)}(A)$ instead of $(\mathfrak{g}(A))^{(1)}$ and the like); note that if $p = 2$ we define

$$\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}] + \operatorname{Span}\{g^2 \mid g \in \mathfrak{g}_{\bar 1}\}, \text{ see Eq. (26).}$$

For presentations of these Lie superalgebras obtained with the aid of *SuperLie* code, see Grozman (2013), Bouarroudj et al. (2010).

The tables in Sects. 4, 5 are analogs of descriptions of root systems in Bourbaki (2002). Recall that whereas each Lie algebra with indecomposable Cartan matrix over

$\mathbb{C}$ (finite-dimensional or of polynomial growth, or even hyperbolic, see Chapovalov et al. 2010) has just one Cartan matrix, each of the exceptional Lie **super**algebras considered has several Cartan matrices, see Bouarroudj et al. (2009) and Chapovalov et al. (2010). One of Lie superalgebra we consider here has 135 inequivalent Cartan matrices; dozens inequivalent Cartan matrices for one Lie superalgebra is usual. Therefore, to list all sets of simple roots for every Lie superalgebra is hardly reasonable, so we list *system of roots* for each exceptional modular Lie superalgebra $\mathfrak{g}(A)$, but list *simple roots* for just one Cartan matrices of $\mathfrak{g}(A)$.

This selected Cartan matrix $\mathcal{A}$ is the one with the maximal[1] number of pairwise orthogonal with respect to a non-degenerate invariant symmetric bilinear form (briefly: NIS) isotropic simple roots. This NIS exists for Cartan matrices of Lie (super)algebras we consider here, but not always if $p = 2$, see Bouarroudj et al. (2018).

This Cartan matrix $\mathcal{A}$ of $\mathfrak{g}$ and its Dynkin–Kac diagram D_m are useful to define the *defect* of $\mathfrak{g}$, an important invariant, and in computing the homology $\operatorname{Ker} \operatorname{ad}_x / \operatorname{Im} \operatorname{ad}_x$ for any $x \in \mathfrak{g}(A)_{\bar{1}}$ such that $x^2 = 0$, see Krutov et al. (2020). To find simple roots for the other Cartan matrices, one has to use isotropic reflections, see Sect. 2.

Observe that even some even roots can be isotropic, not only odd ones, e.g., for $\mathfrak{br}(3)$ for $p = 3$, and for all Lie (super)algebras $\mathfrak{g}(A)$, except the one with $A = (\bar{1})$.

The referee asked us if the Lie (super)algebras we consider are restricted. Restrictedness is a very important feature of Lie (super)algebras; besides, there are several subtleties related both with "super" and the case "$p = 2$". In particular, if $\mathfrak{g}(A)$ is restricted, its derived algebra might be not. Since, however, this is not directly related with our main result, we answer this question in Appendix.

Observe that our definitions imply that all root vectors of the Lie (super)algebras $\mathfrak{g}(A)$ we consider are of multiplicity 1. This might be not so if we apply the term "root" speaking of subalgebras and quotients of $\mathfrak{g}(A)$, as we do (somewhat carelessly) below. For some infinite-dimensional Lie algebras $\mathfrak{g}(A)$, even over $\mathbb{C}$, root multiplicities are > 1.

2 Cartan Matrices, Reflections, Chevalley Generators, Chevalley Basis (from Chapovalov et al. (2010); Bouarroudj et al. (2009)) with Lebedev's Clarifications

2.1 Chevalley Generators and Cartan Matrices

Let us start with the construction of Lie (super)algebras with Cartan matrix. Let $A = (A_{ij})$ be an $n \times n$-matrix whose entries lie in the ground field $\mathbb{K}$. Let $\operatorname{rk} A = n - l$. It means that there exists an $l \times n$-matrix $T = (T_{ij})$ such that

$$\begin{aligned} &\text{(a) the rows of } T \text{ are linearly independent;} \\ &\text{(b) } TA = 0 \text{ (or, more precisely, "zero } l \times n\text{-matrix").} \end{aligned} \tag{1}$$

[1] In other words, $\mathcal{A}$ is the matrix with the maximal number of not connected grey vertices in its Dynkin–Kac diagram D_m; in terms of $\mathcal{A}$, this means that $\mathcal{A}$ has the maximal number of zeros on its main diagonal and if $\mathcal{A}_{ii} = \mathcal{A}_{jj} = 0$, then $\mathcal{A}_{ij} = \mathcal{A}_{ji} = 0$.

Indeed, if rk A^T = rk $A = n - l$, then there exist l linearly independent vectors v_i such that $A^T v_i = 0$; set

$$T_{ij} = (v_i)_j.$$

Let the elements $e_i^\pm$ and h_i, where $i = 1, \dots, n$, generate a Lie superalgebra denoted $\tilde{\mathfrak{g}}(A, I)$, where $I = (p_1, \dots p_n) \in (\mathbb{Z}/2)^n$ is a collection of parities ($p(e_i^\pm) = p_i$), free except for the relations

$$[e_i^+, e_j^-] = \delta_{ij} h_i; \quad [h_i, e_j^\pm] = \pm A_{ij} e_j^\pm \quad \text{and } [h_i, h_j] = 0 \text{ for any } i, j. \tag{2}$$

Let Lie (super)algebra with Cartan matrix $\mathfrak{g}(A, I)$ be the quotient of $\tilde{\mathfrak{g}}(A, I)$ modulo the ideal explicitly described in Grozman and Leites (2001) and Bouarroudj et al. (2010).

By abuse of notation we denote by $e_j^\pm$ and h_j—the elements of $\tilde{\mathfrak{g}}(A, I)$—and also their images in $\mathfrak{g}(A, I)$ and $\mathfrak{g}^{(i)}(A, I)$. We call these images the *Chevalley generators* of $\mathfrak{g}(A, I)$, and $\mathfrak{g}^{(i)}(A, I)$, *provided* A is normalized, cf. Sect. 2.5.1. (There is no name for their pre-images in $\tilde{\mathfrak{g}}(A, I)$; we used to call them *Chevalley generators* as well, by abuse of notation.)

2.1.1 In Small Font

First, observe that the formula

$$\deg e_i^\pm := (0, \dots, 0, \pm 1, 0, \dots, 0) \text{ with } \pm \text{ in the } i\text{th slot} \tag{3}$$

determines a $\mathbb{Z}^n$-grading in $\tilde{\mathfrak{g}}(A, I)$. The additional to (2) relations that turn $\tilde{\mathfrak{g}}(A, I)$ into $\mathfrak{g}(A, I)$ are of the form $R_i = 0$ whose left sides are implicitly described, for the general Cartan matrix with entries in $\mathbb{K}$, as

the R_i that generate the $\mathbb{Z}^n$-graded ideal $\mathfrak{r}$ maximal among the ideals of $\tilde{\mathfrak{g}}(A, I)$ whose intersection with the span of the above h_i is zero. (4)

Set

$$c_i = \sum_{1 \le j \le n} T_{ij} h_j, \text{ where } i = 1, \dots, l. \tag{5}$$

Then, from the properties of the matrix T, we deduce that

(a) the elements c_i are linearly independent;

(b) the elements c_i are central, because

$$[c_i, e_j^\pm] = \pm \left(\sum_{1 \le k \le n} T_{ik} A_{kj} \right) e_j^\pm = \pm (TA)_{ij} e_j^\pm. \tag{6}$$

The existence of central elements means that the linear span of all the roots is of dimension $n - l$ only. (This can be explained even without central elements: The weights can be considered as column-vectors whose i-th coordinates are the corresponding eigenvalues of ad_{h_i}. The weight of e_i is, therefore, the i-th column of A. Since $\mathrm{rk}\, A = n - l$, the linear span of all columns of A is $(n - l)$-dimensional just by definition of the rank. Since any root is an (integer) linear combination of the weights of the e_i, the linear span of all roots is $(n - l)$-dimensional).

This means that some elements which we would like to see having different (even opposite if $p = 2$) weights, actually, have identical weights. To remedy this, we do the following: let B be an arbitrary $l \times n$-matrix such that

$$\text{the } (n + l) \times n\text{-matrix } \begin{pmatrix} A \\ B \end{pmatrix} \text{ has rank } n. \tag{7}$$

Let us add to the algebra $\mathfrak{g} = \tilde{\mathfrak{g}}(A, I)$, and hence $\mathfrak{g}(A, I)$, the grading elements d_i, where $i = 1, \ldots, l$, subject to the following relations:

$$[d_i, e_j^{\pm}] = \pm B_{ij} e_j; \quad [d_i, d_j] = 0; \quad [d_i, h_j] = 0 \tag{8}$$

(the last two relations mean that the d_i lie in the Cartan subalgebra, and even in the maximal torus which will be denoted by $\mathfrak{h}$).

Note that these d_i are *outer* derivations of $\mathfrak{g}(A, I)^{(1)}$, i.e., they can not be obtained as linear combinations of brackets of the elements of $\mathfrak{g}(A, I)$, i.e., the d_i do not lie in $\mathfrak{g}(A, I)^{(1)}$.

2.2 Roots and Weights

In this subsection, $\mathfrak{g}$ denotes one of the algebras $\mathfrak{g}(A, I)$ or $\tilde{\mathfrak{g}}(A, I)$.

Let $\mathfrak{h}$ be the span of the h_i and the d_j. The elements of $\mathfrak{h}^*$ are called *weights*. For a given weight α, the *weight subspace* of a given $\mathfrak{g}$-module V is defined as

$$V_\alpha = \{x \in V \mid \text{an integer } N > 0 \text{ exists such that } (\alpha(h) - \mathrm{ad}_h)^N x = 0 \text{ for any } h \in \mathfrak{h}\}.$$

Any non-zero element $x \in V_\alpha$ is said to be *of weight* α. For the roots, which are particular cases of weights if $p = 0$, the above definition is inconvenient for various reasons, e.g., see Kuznetsov and Chebochko (2000), Bouarroudj et al. (2019), Krutov et al. (2020).

2.2.1 Statement

(Root decomposition Kac 1995) *Over* $\mathbb{C}$, *the space of any Lie algebra* $\mathfrak{g}(A)$ *can be represented as a direct sum of subspaces*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha. \tag{9}$$

Note that if $p = 2$, it might happen that $\mathfrak{h} \subsetneq \mathfrak{g}_0$. (For example, all weights of the form 2α over $\mathbb{C}$ become 0 over $\mathbb{K}$.)

For the Lie (super)algebras $\mathfrak{g} := \mathfrak{g}(A)$ with Cartan matrix A, we assume that the elements $e_i^{\pm}$ with the same superscript (either $+$ or $-$) correspond to linearly independent *roots* α_i, called *simple roots*. Any non-zero element $\alpha \in \mathbb{R}^n$ is called *a root* if the corresponding subspace homogenous with respect to the $\mathbb{Z}^n$-grading[2] of grade α (we denote this subspace by $\mathfrak{g}_\alpha$) is non-zero. The set R of all roots is called *the root system* of $\mathfrak{g}$.

The terms "root" and "root space" are often applied to various "relatives" of $\mathfrak{g}(A)$, e.g., the ith derived algebra $\mathfrak{g}^{(i)}$ or quotients of $\mathfrak{g}$ or $\mathfrak{g}^{(i)}$ modulo center, such as $\mathfrak{psl}(a|a)$.

Thus, any root α such that $\mathfrak{g}_\alpha \neq 0$ lies in the $\mathbb{Z}$-span of $\{\alpha_1, \dots, \alpha_n\}$, i.e.,

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{Z}\{\alpha_1, \dots, \alpha_n\}} \mathfrak{g}_\alpha. \tag{10}$$

Thus, $\mathfrak{g}$ has a $\mathbb{R}^n$-grading such that $e_i^{\pm}$ has grade $(0, \dots, 0, \pm 1, 0, \dots, 0)$, where ± 1 stands in the i-th slot (this can also be considered as $\mathbb{Z}^n$-grading, but we use $\mathbb{R}^n$ to simplify formulations of various statements). If $p = 0$, this grading is equivalent to the weight grading of $\mathfrak{g}$. If $p > 0$, these gradings may be inequivalent; in particular, if $p = 2$, then the elements e_i^+ and e_i^- have the same weight. (This is one of the reasons why in what follows we consider roots as elements of $\mathbb{R}^n$, not as weights; for one more reason, see Krutov et al. 2020).

Clearly, the subspaces $\mathfrak{g}_\alpha$ are purely even or purely odd, and the corresponding roots are said to be *even* or *odd*.

2.3 Systems of Simple and Positive Roots

In this subsection, $\mathfrak{g} = \mathfrak{g}(A, I)$, and R is the root system of $\mathfrak{g}$.

For any subset $B = \{\sigma_1, \dots, \sigma_m\} \subset R$, we set (we denote by $\mathbb{Z}_+$ the set of non-negative integers):

$$R_B^{\pm} = \{\alpha \in R \mid \alpha = \pm \sum n_i \sigma_i, \text{ where } n_i \in \mathbb{Z}_+\}.$$

The set B is called a *system of simple roots* of R (or $\mathfrak{g}$) if $\sigma_1, \dots, \sigma_m$ are linearly independent and $R = R_B^+ \cup R_B^-$. Note that R contains basis coordinate vectors, and therefore spans $\mathbb{R}^n$; thus, any system of simple roots contains exactly n elements.

Let $(x, y) = \sum x_i y_i$ for any $x, y \in \mathbb{R}^n$ denote the standard Euclidean inner product in $\mathbb{R}^n$. A subset $R^+ \subset R$ is called a *system of positive roots* of R (or $\mathfrak{g}$) if there exists $x \in \mathbb{R}^n$ such that

$$\begin{aligned} &(\alpha, x) \in \mathbb{R}\backslash\{0\} \text{ for all } \alpha \in R, \\ &R^+ = \{\alpha \in R \mid (\alpha, x) > 0\}. \end{aligned} \tag{11}$$

[2] NOT the *eigenspace*! That's the whole point. Eigenspaces (which are determined by the action of $\mathfrak{h}$) correspond to weights, not to roots.

Since R is a finite set, then the set

$$\{y \in \mathbb{R}^n \mid \text{there exists } \alpha \in R \text{ such that } (\alpha, y) = 0\}$$

is a finite union of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$, so it has zero measure. So for almost every x, condition (11) holds.

By construction, any system B of simple roots is contained in exactly one system of positive roots, which is precisely R_B^+.

2.3.1 Conjecture

(Simple roots). (1) *Any system* R^+ *of positive roots of* $\mathfrak{g}(A)$ *contains exactly one system of simple roots. This system consists of all the positive roots* (i.e., elements of R^+) *that can not be represented as a sum of two positive roots.*

(2) *For any system of simple roots* $\{\sigma_1, \dots, \sigma_n\}$ *in* $\mathfrak{g}(A, I)$, *there is a pair* (A', I') *such that there is an isomorphism between* $\mathfrak{g}(A, I)$ *and* $\mathfrak{g}(A', I')$ *which maps* $\mathfrak{h}$ *to* $\mathfrak{h}$ *and* $\mathfrak{g}(A, I)_{\pm\sigma_i}$ *to* $\mathfrak{g}(A', I')_{(0,\dots,0,\pm 1,0,\dots,0)}$, *with* ± 1 *in* i*-th position, for all* $i = 1, \dots, n$.

We do not know an *a priori* proof of this Conjecture. Item 1) is, however, true for Lie algebras and Lie superalgebras of the form $\mathfrak{g}(A)$ with A indecomposable and $\dim \mathfrak{g}(A) < \infty$. About item 2), see Lebedev's comment 3.2.2.

2.4 Normalization Convention and Equivalent Cartan Matrices

Clearly,

$$\text{the rescaling } e_i^{\pm} \mapsto \sqrt{\lambda_i} e_i^{\pm}, \text{ sends } A \text{ to } A' := \operatorname{diag}(\lambda_1, \dots, \lambda_n) \cdot A. \tag{12}$$

Two pairs (A, I) and (A', I') are said to be *equivalent* if (A', I') is obtained from (A, I) by a composition of a permutation of parities and the corresponding permutation of the matrix's rows and columns with a rescaling $A' = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \cdot A$, where $\lambda_1, \dots, \lambda_n \neq 0$. Clearly, equivalent pairs determine isomorphic Lie superalgebras.

The rescaling affects only the matrix A_B, not the set of parities I_B. The Cartan matrix A is said to be *normalized* if

$$A_{jj} = 0 \text{ or } 1, \text{ or } 2. \tag{13}$$

We let $A_{jj} = 2$ only if $i_j = \bar{0}$; in order to eliminate possible confusion, we write $A_{jj} = \bar{0}$ or $\bar{1}$ if $i_j = \bar{0}$, whereas if $i_j = \bar{1}$, we write $A_{jj} = 0$ or 1.

Normalization conditions correspond to the "natural" Chevalley generators of the most usual[3] "building blocks" of finite-dimensional Lie (super)algebras with Cartan

[3] Together with the possibility to build any simple Lie algebra with Cartan matrix over $\mathbb{C}$ and many Lie (super)algebras with Cartan matrix over various fields by just two generators, see Grozman and Leites (1996), there are other types of "building blocks". For simple modular Lie (super)algebras, to describe relations between a pair of generators, as for "Lie algebra of matrices of complex size", see Grozman and Leites (1996), is probably impossible in general because of the lack of complete reducibility of exterior powers of

matrix: $\mathfrak{sl}(2)$ if $A_{jj} = 2$, or $\mathfrak{gl}(1|1)$ if $A_{jj} = 0$, or $\mathfrak{osp}(1|2)$ if $A_{jj} = 1$, respectively. (In this paper we do not need "less usual" building blocks—Lie superalgebras with $A_{jj} = \bar{0}$ or $\bar{1}$ because they do not contribute to the list of Lie algebras of the type we consider: finite-dimensional modular, see Bouarroudj et al. (2009), nor are they needed to construct affine Kac–Moody or hyperbolic (almost affine) Lie (super)algebras, see Chapovalov et al. (2010).)

For the role of the "best" (first among equals) order of indices we propose the one that minimizes the value

$$\max_{i,j\in\{1,\dots,n\} \text{ such that } (A_B)_{ij}\neq 0} |i-j| \tag{14}$$

(i.e., gather the non-zero entries of A as close to the main diagonal as possible). Observe that this numbering differs from the one that N. Bourbaki uses for the $\mathfrak{e}$ type Lie algebras.

We will only consider normalized Cartan matrices; for them, we do not have to indicate the set of parities I.

2.4.1 Warnings

(1) The notion of Cartan matrix, see Kac (1995), is standard now. However, for reasons difficult to understand, neither for Lie *super*algebras, nor for *modular* Lie algebras the definitions of the Cartan matrix, nor analog of the Dynkin diagram were correctly formulated until Bouarroudj et al. (2009) (in the modular case) and Chapovalov et al. (2010) (for $p = 0$) were made available. And it is still possible to hear or read "consider the central extension of the loop algebra $\mathfrak{g}^{\ell(1)}$ with values in simple finite-dimensional Lie algebra $\mathfrak{g}$; this central extension has Cartan matrix and Dynkin diagram extending those of $\mathfrak{g}$", whereas it is the *double extension* of $\mathfrak{g}^{\ell(1)}$ that has a Cartan matrix. Likewise, simple Lie superalgebras $\mathfrak{psl}(a|a)$ in characteristic 0 as well as $\mathfrak{psl}(a|a+pk)$ in characteristic $p > 0$ (and their central extensions $\mathfrak{sl}(a|a)$ and $\mathfrak{sl}(a|a+pk)$) do not have a Cartan matrix, whereas the double extensions $\mathfrak{gl}(a|a)$ and $\mathfrak{gl}(a|a+pk)$ of the simple Lie superalgebras have Cartan matrices. For a definition of *double extension* and succinct review for any p, see Bouarroudj et al. (2019).

(2) Unlike the case of simple finite-dimensional Lie algebras over $\mathbb{C}$, where the normalized Cartan matrix A is defined uniquely (up to a permutation of rows and columns), generally this is not so: each row with a 0 or $\bar{0}$ on the main diagonal can be multiplied by any nonzero factor. For example, when interested in non-degenerate invariant symmetric bilinear forms on $\mathfrak{g}$, see Bouarroudj et al. (2018), we multiply the rows so as to make A_B symmetric, if possible, spoiling normalization.

Which version of the Cartan matrix should be considered as its "normal form"? The defining relations give the answer: The normalized Cartan matrix is used, for

Footnote 3 continued
the adjoint module and because it is unclear what are analogs of "principal embedding" of $\mathfrak{sl}(2)$. For $p = 0$ and having replaced $\mathfrak{sl}(2)$ by $\mathfrak{osp}(1|2)$, we can consider such analogs in the cases classified in Leites et al. (1986)—when there are "superprincipal embeddings".

example, to describe relations between the Chevalley generators of the same sign, see Grozman and Leites (2001, 2005), Bouarroudj et al. (2009, 2010).

2.5 Remark: Which Systems of Simple Roots is Distinguished?

Let $B = \{\alpha_1, \dots, \alpha_n\}$ be a system of simple roots. Choose non-zero elements $e_i^\pm$ in the 1-dimensional superspaces $\mathfrak{g}_{\pm\alpha_i}$; set $h_i = [e_i^+, e_i^-]$, let $A_B = (A_{ij})$, where the entries A_{ij} are recovered from relations (2), and let $I_B = \{p(e_1), \cdots, p(e_n)\}$.

It would be nice to find a convenient way to fix some distinguished pair (A_B, I_B) in the equivalence class.

2.5.1 Chevalley Generators and Chevalley Bases

We often denote the set of generators of $\mathfrak{g}(A, I)$ and $\mathfrak{g}^{(i)}(A, I)$ corresponding to a normalized Cartan matrix by $X_1^\pm, \dots, X_n^\pm$ instead of $e_1^\pm, \dots, e_n^\pm$; and call these generators, together with the elements $H_i := [X_i^+, X_i^-]$, and the derivations d_j, see (8), the *Chevalley generators*.

For $p = 0$ and normalized Cartan matrices of simple finite-dimensional Lie algebras, the set of Chevalley generators can be uniquely (up to signs) extended to a basis all whose elements are homogenous with respect to the grading and all structure constants are integer, cf. Strade (2004). A certain choice of signs gives what is called a *Chevalley* basis.

Observe that, having normalized the Cartan matrix of $\mathfrak{sp}(2n)$ so that $A_{ii} = 2$ for all $i \neq n$, but $A_{nn} = 1$, we get **another** basis with integer structure constants. We think that this basis also qualifies to be called *Chevalley basis*.

For any p and Lie superalgebras $\mathfrak{osp}(2m+1|2n)$ such normalization is a must. Summing up, a *Chevalley basis* of $\mathfrak{g}(A)$ with a normalized Cartan matrix A is a one with generators $X_i^\pm$ and H_i, where all elements are homogenous with respect to the grading induced by Eq. (3) and some additional requirements on structure constants, see Cohen and Roozemond (2009). Everything goes as for $p = 0$ if A has integer elements and all structure constants lie in $\mathbb{Z}/p\mathbb{Z}$; otherwise we do not know how to define "Chevalley basis". These exceptional cases are

$$\begin{gathered}\mathfrak{br}(2;\varepsilon) \text{ for } \varepsilon \neq 0 \text{ and } p = 3,\ \mathfrak{osp}_\alpha(4|2) \text{ for } \alpha \neq 0, -1 \text{ and } p \neq 2;\\ \mathfrak{wk}(3;\alpha) \text{ and } \mathfrak{bgl}(3;\alpha) \text{ as well as } \mathfrak{wk}(4;\alpha) \text{ and } \mathfrak{bgl}(4;\alpha) \text{ for } \alpha \neq 0, 1 \text{ and } p = 2.\end{gathered} \tag{15}$$

2.6 Reflections

Let R^+ be the system of positive roots of Lie superalgebra $\mathfrak{g}$ over a field $\mathbb{K}$ of characteristic $p > 0$, and let $B = \{\sigma_1, \dots, \sigma_n\}$ be the corresponding system of simple roots and the corresponding pair $(A = A_B, I = I_B)$. Then, for any $k \in \{1, \dots, n\}$, the set

$$(R^+ \setminus \{m\sigma_k \mid m \in \mathbb{Z}_+, m\sigma_k \in R^+\}) \coprod \{-m\sigma_k \mid m \in \mathbb{Z}_+, m\sigma_k \in R^+\} \tag{16}$$

is a system of positive roots. The *reflection in* σ_k changes the system of simple roots by the formulas

$$r_{\sigma_k}(\sigma_j) = \begin{cases} -\sigma_j & \text{if } k = j, \\ \sigma_j + B_{kj}\sigma_k & \text{if } k \neq j, \end{cases} \tag{17}$$

where B_{kj} is the maximal $m \in \mathbb{Z}$ such that $\alpha_j + m\alpha_k \in R$ provided we consider $\mathbb{Z}/p\mathbb{Z}$ as a subfield of $\mathbb{K}$.

The name "reflection" is used because in the case of simple finite-dimensional complex Lie algebras this action, extended on the whole R by linearity, is a map from R to R, and it does not depend on R^+, only on σ_k. This map is usually denoted by r_{σ_k} or just r_k. The map r_{σ_k} extended to the $\mathbb{R}$-span of R is reflection in the hyperplane orthogonal to σ_k relative the bilinear form dual to the nondegenerate invarient symmetric bilinear form.

The reflections in the even (odd) roots are referred to as *even* (*odd*) *reflections*. A simple root is called *isotropic*, if the corresponding row of the Cartan matrix has zero on the main diagonal, and *non-isotropic* otherwise. The reflections that correspond to isotropic or non-isotropic roots will be referred to accordingly.

2.6.1 On Weyl Groups and Groupoids

(A) If there are isotropic simple roots, the reflections r_α do not, as a rule, generate a version of the *Weyl group* because the product of two reflections in nodes not connected by a (perhaps, multiple) edge is not defined. These reflections just connect a pair of "neighboring" systems of simple roots, and there is no reason to expect that we can multiply such two distinct reflections. In the case of modular Lie algebras, and in the case of Lie superalgebras for any p, the action of a given isotropic reflection (17) can not, generally, be extended to a linear map $R \longrightarrow R$. For Lie superalgebras over $\mathbb{C}$, one can extend the action of reflections by linearity to the root lattice, but this extension preserves the root system only for $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(2m+1|2n)$, cf. Serganova (1996).

(B) At seminars of Manin and Leites in early 1980s in Moscow, discussions of the question

$$\text{For Lie superalgebras, what might versions of the "Weyl group generated by reflections" be?} \tag{18}$$

culminated in the following answers.

(0) In 1978–80, Bernstein and Leites classified the irreducible finite-dimensional representations of vectorial Lie superalgebras $\mathfrak{vect}(m|n)$, in particular, $\mathfrak{vect}(0|2) \simeq \mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$. This result and its generalization for $\mathfrak{vect}(0|m)$, $\mathfrak{sl}(1|n)$ and $\mathfrak{osp}(2|2n)$ yielded analogs of Weyl character formula, where the analog of the Weyl group of these $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g}$ played the Weyl group of $\mathfrak{g}_0$, see a review (Leites 1985, p. 2504). This answer to the question (18) did not, however, look satisfactory for many reasons.

(1) Manin and Voronov introduced the notion of a *super Weyl group* in relation to the geometry of homogeneous superspaces. They constructed Schubert supercells which were labeled by elements of a super Weyl group, see Manin and Voronov (1988).
(2) In Heckenberger and Yamane (2008), Heckenberger and Yamane answered the question (18) Serganova formulated in Serganova (1996): they introduced Weyl groupoids. Cuntz and Heckenberger reformulated the definition of Weyl groupoids in terms of Cartan schemes, see Cuntz and Heckenberger (2009).
(3) Sergeev and Veselov (2017) and references therein, gave a *non-equivalent*, as far as we understand, definition of the Weyl groupoid in the super case.
(4) It is interesting to extend the question (18) to modular Lie (super)algebras. We would like to draw attention of the reader to an under-appreciated paper Skryabin (1993), where the analog of Weyl group was considered in relation with $\mathfrak{br}(3)$. This analog seems to be most close to the groupoid defined by Sergeev and Veselov.
(5) Completely independent approach to analogs of "Weyl group" for modular Lie algebras without Cartan matrix is due to Premet, see Bois et al. (2014) and references therein. This approach is meaningful for vectorial and periplectic (preserving an odd non-degenerate symmetric bilinear form) Lie superalgebras over fields of any characteristic.

2.6.2 How Reflections Act on Chevalley Generators

If σ_i is an **odd** isotropic root, then the corresponding reflection r_i sends one set of Chevalley generators into a new one:

$$\tilde{X}_i^{\pm} = X_i^{\mp}; \quad \tilde{X}_j^{\pm} = \begin{cases} [X_i^{\pm}, X_j^{\pm}] & \text{if } A_{ij} \neq 0, \\ X_j^{\pm} & \text{otherwise.} \end{cases} \tag{19}$$

If σ_k is an **even** isotropic root (i.e., $i_k = \bar{0}$) and $p > 0$, then the corresponding reflection r_k acts as follows: for $j \neq k$, we have:

$$\tilde{X}_k^{\pm} = X_k^{\mp}; \quad \tilde{X}_j^{\pm} = \begin{cases} (\mathrm{ad}_{\tilde{X}_k^{\pm}})^{p-1}\tilde{X}_j^{\pm} & \text{if } A_{kj} \neq 0, \\ X_j^{\pm} & \text{otherwise.} \end{cases} \tag{20}$$

The Cartan matrix $r_i(A)$ corresponding to the Chevalley generators (19) should be obtained as described above: set

$$\tilde{H}_i := [\tilde{X}_i^+, \tilde{X}_i^-]$$

and compute

$$[\tilde{H}_i, \tilde{X}_j^+] = \tilde{B}_{ij}\tilde{X}_j^+.$$

Normalize the matrix $\tilde{B}$ as we agreed, see Sect. 2.4; let B be the normalized matrix. Then, $r_k(A) := B = (B_{ij})$.

2.6.3 How Reflections Act on Cartan Matrices ([Lebedev 2008, Chapovalov et al. 2010])

Let A be a Cartan matrix of size n and $I = (p_1, \ldots, p_n)$ the vector of parities. If $p_k = \bar{1}$ and $A_{kk} = 0$, then the reflection in the kth simple **odd** root sends A to $r_k(A)$, and I to $r_k(I)$, where

$$(r_k(A))_{ij} = A_{ij} + c_i A_{kj} + b_j A_{ik}, \quad r_k(p_j) \equiv p_j + b_j \pmod 2 \tag{21}$$

and where

$$c_i = \begin{cases} -2 & \text{if } i = k, \\ 0 & \text{if } i \neq k \text{ and } A_{ik} = 0, \\ \frac{A_{ik}}{A_{ki}} & \text{if } i \neq k \text{ and } A_{ik} \neq 0; \end{cases} \quad \text{and} \quad b_j = \begin{cases} -2 & \text{if } j = k, \\ 0 & \text{if } j \neq k \text{ and } A_{jk} = 0, \\ 1 & \text{if } j \neq k \text{ and } A_{jk} \neq 0. \end{cases}$$

This can be expressed in terms of matrices as

$$r_k(A) = (E + \mathcal{C})A(E + \mathcal{B}),$$

where all columns of the matrix $\mathcal{C}$, except the kth one, are zero, whereas the ith coordinate of the kth column-vector is c_i, the ith coordinate of the kth row-vector of $\mathcal{B}$ is b_i, the other rows of $\mathcal{B}$ being zero; E is the unit matrix.

3 Lebedev's Comments

Serganova (1984) proved (for $p = 0$) that for two systems of simple roots B_1 and B_2, there is always a chain of reflections connecting a system of simple roots B_1 with either some system of simple roots B_2' equivalent to a system of simple roots B_2 in the sense of definition in Sect. 2.4 or with $-B_2'$. Here is a version of Serganova's statement suitable for any p.

3.1 Lemma

(Any two systems of simple roots are connected by a chain of reflections) *For any two systems of simple roots B_1 and B_2 of any finite-dimensional Lie superalgebra with Cartan matrix, there is always a chain of reflections connecting B_1 with B_2.*

Let me start with a statement I am certain of. I will formulate it in terms which I will define without any relation to Lie superalgebras with Cartan matrices (CM LSA), so that the proof wouldn't rely on any properties of CM LSA I am not sure about. But I will use terms which are also used for concepts related to CM LSA.

Let R be a non-empty finite subset of the vector space $V = \mathbb{R}^n$ such that $0 \notin R$ and $-r \in R$ for all $r \in R$. We will call such a set R a *system of roots*; compare this definition with that in Serganova (1996, p. 4298).

A subset $P \subset R$ will be called a *system of positive roots* if there is an $h \in V^*$ such that $h(r) \neq 0$ for all $r \in R$ and $P = \{r \in R \mid h(r) > 0\}$.

A non-empty subset $L \subset R$ will be called a *root ray* if it is a maximal subset of R such that all of its elements are positive multiples of each other, or, equivalently, if there is an $r_0 \in R$ such that $L = \{r \in R \mid r = cr_0$ for some $c \in \mathbb{R}_{>0}\}$; for an element $r \in R$, we will denote the root ray it belongs to by L_r.

Clearly, if P is a system of positive roots and $r \in P$, then $L_r \subset P$. If P is a system of positive roots, we will call a root ray $L \subset P$ *simple in* P if there is an $h \in V^*$ such that $h(r) = 0$ for all $r \in L$ and $h(r) > 0$ for all $r \in P \setminus L$.

3.2 Statement

(On chains of positive root systems) *Let $P', P'' \subset R$ be two system of positive roots. Then, there is a finite sequence of positive systems of roots*

$$P_1 = P', P_2, \dots, P_m = P''$$

such that each next one is obtained from the previous one by removing a simple root ray and adding the opposite ray, i.e., for every $k = 1, \dots, m-1$, there is a root ray L simple in P_k such that $P_{k+1} = (P \setminus L) \sqcup -L$, where $-L = \{-x \mid x \in L\}$.

Proof We will prove it by induction on $|P' \setminus P''|$. If it is equal to 0, i.e., $P' \subset P''$, then $P' = P''$, and the fact is trivial.

Now, let $|P' \setminus P''| > 0$. Let $h', h'' \in V^*$ be two elements which define P' and P''. Consider the convex envelope of $P' \cup \{0\}$. It is a bounded convex polytope, and 0 is one of its vertices, since the $(n-1)$-dimensional plane $h'(x) = 0$ passes through 0 and the rest of the polytope lies on one side of the plane. At least one of the vertices connected to 0 by an edge must lie outside of the half-space $h''(x) > 0$, because otherwise the whole polytope lies within that half-space, which would mean that $P' \subset P''$; let us denote this vertex r_0.

Then, the root ray L_{r_0} is simple in P', since there is a half-plane containing the edge connecting r_0 and 0 such that the rest of the polytope lies on one side of it, i.e., there is $h_1 \in V^*$ such that $h_1(r) = 0$ for all $r \in L_{r_0}$ and $h_1(r) > 0$ for all $r \in P \setminus L_{r_0}$. Then, for a sufficiently small $\varepsilon > 0$, the element $h_1 - \varepsilon h'$ determines positive system $P_2 := (P' \setminus L_{r_0}) \sqcup -L_{r_0}$. Since $L_{r_0} \subset P'$, it follows that $(-L_{r_0}) \cap P' = \emptyset$, so

$$|P_2 \setminus P''| = |P' \setminus P''| - |L_{r_0}| < |P' \setminus P''|,$$

and by the induction hypothesis, there is a sequence of systems of positive roots of the required form connecting P_2 and P'', and by prepending P' to it, we get a sequence of systems of positive roots connecting P' and P''. □

3.2.1 Proof of Lemma 3.1

The system of roots of any finite-dimensional Lie superalgebra $\mathfrak{g}(A)$ is a system of roots in the sense of the definition in Sect. 3, and its subset is a system of positive roots if and only if it is a system of positive roots in the above sense. Assuming that Conjecture 2.3.1.1) is true, there is a one-to-one correspondence between systems of positive roots and systems of simple roots, and a simple root ray has to contain a simple root. So if we define reflections as "removing a root ray with a simple root (i.e., all positive multiples of a given simple root) and adding the opposite root ray", then the above argument proves Lemma 3.1 in view of Conjecture 2.3.1.1). □

3.2.2 On Conjecture 2.3.1.2

When we construct the Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A, I)$, we start with elements $e_i^{\pm}$. Let me call the roots which correspond to e_i^+ *basic roots*. The basic roots form a system of simple roots in the sense of definition in Subsect. 2.3; but we can choose some other system of simple roots, let us denote them $\sigma_1, \dots, \sigma_n$. Conjecture 2.3.1.2 claims that all systems of simple roots are equally suitable for constructing $\mathfrak{g}$ as a Lie superalgebra with Cartan matrix, i.e., that we could construct $\mathfrak{g}$ so that $\sigma_1, \dots, \sigma_n$ would be basic roots, with some other pair (A', I').

3.2.3 On the Reviewer's Question: "Aren't All Root Spaces 1-dimensional?"

Assume definitions of Sect. 3; then, since in the above proof (Sect. 3.2.1) we change one system of positive roots into another by replacing root rays with their opposites one by one, and at each step, the root ray we replace is simple in the current system of positive roots, it means that every root of $\mathfrak{g}(A, I)$ is a multiple of a simple one. If we assume that Conjecture 2.3.1.2 holds, then every root space is 1-dimensional.

4 Roots and Root Vectors

Let the π_i be the fundamental weights relative to a fixed system of simple roots. For any simple Lie algebra with Cartan matrix, let $R(\sum a_i\pi_i)$ denote both the irreducible representation with highest weight $(a_1, a_2, \dots)$ and the respective module. The tautological module over the Lie algebra of series $\mathfrak{sl}$, $\mathfrak{o}$ or $\mathfrak{sp}$ is denoted by $\mathrm{id} := R(\pi_1)$.

By $N\mathfrak{g}(A)$ we denote the realization of $\mathfrak{g}(A)$ corresponding to the Nth Cartan matrix A as listed in Bouarroudj et al. (2009). The odd root vectors are $\boxed{\text{boxed}}$ and isotropic roots are <u>underlined</u>.

4.1 $\mathfrak{osp}(4|2; a)$, $\mathfrak{ag}(2)$, and $\mathfrak{ab}(3)$ for $p \geq 5$

The answer is the same as is well-known for $p = 0$, namely (here $\mathfrak{sl}_i(2)$ is the ith copy of $\mathfrak{sl}(2)$ with the tautological module id_i):

$\mathfrak{osp}(4\vert 2;a)_{\bar 0} = \mathfrak{sl}_1(2)\oplus\mathfrak{sl}_2(2)\oplus\mathfrak{sl}_3(2)$	$\mathfrak{osp}(4\vert 2;a)_{\bar 1} = \mathrm{id}_1\boxtimes\mathrm{id}_2\boxtimes\mathrm{id}_3$
$\mathfrak{ag}(2)_{\bar 0} = \mathfrak{sl}(2)\oplus\mathfrak{g}(2)$	$\mathfrak{ag}(2)_{\bar 1} = \mathrm{id}\boxtimes R(\pi_1)$
$\mathfrak{ab}(3)_{\bar 0} = \mathfrak{sl}(2)\oplus\mathfrak{o}(7)$	$\mathfrak{ab}(3)_{\bar 1} = \mathrm{id}\boxtimes R(\pi_1)$

Each of the other exceptional Lie superalgebras $\mathfrak{g}(A)$ with indecomposable Cartan matrix A exists only in characteristics 2, 3 and 5. The Lie superalgebras $3\mathfrak{g}(2,3)$ and $1\mathfrak{g}(3,3)$ (indigenous to $p=3$) resemble $3\mathfrak{ag}(2)$ and $6\mathfrak{ab}(3)$ (existing for $p=0$ and any $p>3$), respectively, other exceptional Lie superalgebras $\mathfrak{g}(A)$ have no analogs except for two pairs $\mathfrak{brj}(2;5)\leftrightarrow\mathfrak{brj}(2;3)$ and $\mathfrak{el}(5;5)\leftrightarrow\mathfrak{el}(5;3)$ (existing for $p=5\leftrightarrow p=3$) which we consider one after the other for clarity.

Every Cartan matrix is considered only once, for p declared. More precisely, certain Lie (super)algebras have incarnations in several characteristics their elements being integers of parameter a evaluated in $\mathbb{K}$. For example, $\mathfrak{osp}(4|2;a)$, $\mathfrak{ag}(2)$, and $\mathfrak{ab}(3)$ have incarnations for $p\geq 5$ and 0 and two pairs of exceptions ($\mathfrak{brj}(2;5)\leftrightarrow\mathfrak{brj}(2;3)$ and $\mathfrak{el}(5;5)\leftrightarrow\mathfrak{el}(5;3)$) have incarnations for $p=3$ and 5. A version of $\mathfrak{osp}(4|2;a)$ for $p=2$ is called $\mathfrak{bgl}(3;a)$; its desuperization—$\mathfrak{wk}(3;a)$—has the "same" Cartan matrix, but with different diagonal elements ($\bar 0$ instead of 0). One can not consider any of the Cartan matrices for values of p different from those declared (if one does, one gets an infinite-dimensional algebra: thanks to the classification).

Recall that the Cartan matrix of the Brown algebra $\mathfrak{br}(2;\varepsilon)$ is $\begin{pmatrix}2 & -1\\ -2 & 1-\varepsilon\end{pmatrix}$, where $\varepsilon\neq 0$.

4.2 $1\mathfrak{brj}(2;5)$ of sdim $10|12$, $p=5$

For $\mathfrak{g}=\mathfrak{brj}(2;5)$, we have $\mathfrak{g}_{\bar 0}=\mathfrak{sp}(4)=\mathfrak{br}(2;-1)$ and $\mathfrak{g}_{\bar 1}=R(\pi_1+\pi_2)$ is irreducible $\mathfrak{g}_{\bar 0}$-module. We consider the Cartan matrix and basis elements

$$\begin{pmatrix}0 & -1\\ -2 & 1\end{pmatrix}$$

the root vectors	the roots
$\boxed{x_1}$, $\boxed{x_2}$	$\underline{\alpha_1}$, α_2
$x_3=[x_1,\ x_2],\ x_4=[x_2,\ x_2]$	$\alpha_1+\alpha_2,\ \ 2\alpha_2$
$\boxed{x_5}=[x_2,\ [x_1,\ x_2]]$	$\alpha_1+2\alpha_2$
$x_6=[[x_1,\ x_2],\ [x_2,\ x_2]]$	$\alpha_1+3\alpha_2$
$\boxed{x_7}=[[x_1,\ x_2],[x_2,\ [x_1,\ x_2]]]$,	$\underline{2\alpha_1+3\alpha_2}$,
$\boxed{x_8}=[[x_2,\ x_2],[x_2,\ [x_1,\ x_2]]]$	$\underline{\alpha_1+4\alpha_2}$
$x_9=[[x_1,\ x_2],[[x_1,\ x_2],\ [x_2,\ x_2]]]$	$2\alpha_1+4\alpha_2$
$\boxed{x_{10}}=[[x_2,\ [x_1,\ x_2]],\ [[x_1,\ x_2],[x_2,\ x_2]]]$	$\underline{2\alpha_1+5\alpha_2}$

4.3 $1\mathfrak{brj}(2;3)$ of sdim $10|8$, $p=3$

For $\mathfrak{g}=\mathfrak{brj}(2;3)$, we have $\mathfrak{g}_{\bar 0}=\mathfrak{br}(2;1)$ and $\mathfrak{g}_{\bar 1}=R(2\pi_2)$ as $\mathfrak{g}_{\bar 0}$-module. We consider the Cartan matrix and basis elements

$$\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$$

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}$	$\underline{\alpha_1},\ \alpha_2$
$x_3 = [x_1,\ x_2],\ x_4 = [x_2,\ x_2]$	$\alpha_1 + \alpha_2,\ 2\alpha_2$
$\boxed{x_5} = [x_2,\ [x_1,\ x_2]]$	$\alpha_1 + 2\alpha_2$
$x_6 = [[x_1,\ x_2],[x_2,\ x_2]]$	$\alpha_1 + 3\alpha_2$
$\boxed{x_7} = [[x_2,\ x_2],[x_2,[x_1,\ x_2]]]$	$\underline{\alpha_1 + 4\alpha_2}$
$x_8 = [[x_1,\ x_2],[[x_1,\ x_2],[x_2,\ x_2]]]$	$2\alpha_1 + 4\alpha_2$

4.4 1$\mathfrak{el}(5; 5)$ of sdim $= 55|32, p = 5$

For $\mathfrak{g} = \mathfrak{el}(5; 5)$, we have $\mathfrak{g}_{\bar 0} = \mathfrak{o}(11)$ and $\mathfrak{g}_{\bar 1} = R(\pi_5)$ as $\mathfrak{g}_{\bar 0}$-module. We consider the Cartan matrix

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & -4 & -4 \\ 0 & 0 & -4 & 0 & -2 \\ 0 & -1 & -4 & -2 & 0 \end{pmatrix}$$

the Chevalley basis and the root system are:

the root vectors	the roots
$x_1, x_2, \boxed{x_3}, \boxed{x_4}, \boxed{x_5}$	$\alpha_1, \alpha_2, \underline{\alpha_3}, \underline{\alpha_4}, \underline{\alpha_5}$
$\boxed{x_6} = [x_1, x_3], \boxed{x_7} = [x_2, x_5], x_8 = [x_3, x_4],$ $x_9 = [x_3, x_5], x_{10} = [x_4, x_5]$	$\underline{\alpha_1 + \alpha_3}, \underline{\alpha_2 + \alpha_5}, \alpha_3 + \alpha_4,$ $\alpha_3 + \alpha_5, \alpha_4 + \alpha_5$
$x_{11} = [x_3, [x_2, x_5]], x_{12} = [x_4, [x_1, x_3]],$ $x_{13} = [x_4, [x_2, x_5]], x_{14} = [x_5, [x_1, x_3]],$ $\boxed{x_{15}} = [x_5, [x_3, x_4]]$	$\alpha_2 + \alpha_3 + \alpha_5, \alpha_1 + \alpha_3 + \alpha_4,$ $\alpha_2 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_3 + \alpha_5,$ $\underline{\alpha_3 + \alpha_4 + \alpha_5}$
$x_{16} = [[x_1, x_3], [x_2, x_5]],$ $\boxed{x_{17}} = [[x_1, x_3], [x_4, x_5]],$ $\boxed{x_{18}} = [[x_2, x_5], [x_3, x_4]],$ $\boxed{x_{19}} = [[x_2, x_5], [x_4, x_5]]$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5,$ $\underline{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5},$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $\underline{\alpha_2 + \alpha_4 + 2\alpha_5}$
$\boxed{x_{20}} = [[x_2, x_5], [x_4, [x_1, x_3]]],$ $x_{21} = [[x_3, x_5], [x_4, [x_1, x_3]]],$ $x_{22} = [[x_4, x_5], [x_3, [x_2, x_5]]],$ $x_{23} = [[x_4, x_5], [x_4, [x_2, x_5]]]$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $\alpha_1 + 2\alpha_3 + \alpha_4 + \alpha_5,$ $\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5,$ $\alpha_2 + 2\alpha_4 + 2\alpha_5$
$x_{24} = [[x_3, [x_2, x_5]], [x_4, [x_1, x_3]]],$ $x_{25} = [[x_4, [x_2, x_5]], [x_5, [x_1, x_3]]],$ $\boxed{x_{26}} = [[x_4, [x_2, x_5]], [x_5, [x_3, x_4]]]$	$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5,$ $\underline{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5}$

$\boxed{x_{27}} = [[x_4, [x_2, x_5]], [[x_1, x_3], [x_4, x_5]]]$, $\boxed{x_{28}} = [[x_5, [x_3, x_4]], [[x_1, x_3], [x_2, x_5]]]$, $x_{29} = [[x_5, [x_3, x_4]], [[x_2, x_5], [x_4, x_5]]]$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5}$, $\underline{\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5}$, $\alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5$
$x_{30} = [[[x_1, x_3], [x_4, x_5]], [[x_2, x_5], [x_3, x_4]]]$, $x_{31} = [[[x_1, x_3], [x_4, x_5]], [[x_2, x_5], [x_4, x_5]]]$, $x_{32} = [[[x_2, x_5], [x_3, x_4]], [[x_2, x_5], [x_4, x_5]]]$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5$, $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5$, $2\alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5$
$x_{33} = [[[x_2, x_5], [x_4, x_5]], [[x_2, x_5], [x_4, [x_1, x_3]]]]$, $\boxed{x_{34}} = [[[x_2, x_5], [x_4, x_5]], [[x_3, x_5], [x_4, [x_1, x_3]]]]$	$\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5$, $\underline{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 3\alpha_5}$
$\boxed{x_{35}} = [[[x_2, x_5], [x_4, [x_1, x_3]]], [[x_4, x_5], [x_3, [x_2, x_5]]]]$, $x_{36} = [[[x_3, x_5], [x_4, [x_1, x_3]]], [[x_4, x_5], [x_4, [x_2, x_5]]]]$	$\underline{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 3\alpha_5}$, $\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5$
$x_{37} = [[[x_4, x_5], [x_3, [x_2, x_5]]], [[x_4, [x_2, x_5]], [x_5, [x_1, x_3]]]]$, $x_{38} = [[[x_4, x_5], [x_4, [x_2, x_5]]], [[x_3, [x_2, x_5]], [x_4, [x_1, x_3]]]]$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 4\alpha_5$, $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5$
$\boxed{x_{39}} = [[[x_4, [x_2, x_5]], [x_5, [x_1, x_3]]], [[x_4, [x_2, x_5]], [x_5, [x_3, x_4]]]]$	$\underline{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5}$
$x_{40} = [[[x_4, [x_2, x_5]], [x_5, [x_3, x_4]]]$, $[[x_5, [x_3, x_4]], [[x_1, x_3], [x_2, x_5]]]]$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 4\alpha_5$
$x_{41} = [[[x_4, [x_2, x_5]], [[x_1, x_3], [x_4, x_5]]]$, $[[x_5, [x_3, x_4]], [[x_1, x_3], [x_2, x_5]]]]$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 4\alpha_5$

4.5 $7\mathfrak{el}(5;3)$ of sdim $= 39|32$, $p = 3$

For $\mathfrak{g} = \mathfrak{el}(5;3)$, we have $\mathfrak{g}_{\bar 0} = \mathfrak{o}(9) \oplus \mathfrak{sl}(2)$ and $\mathfrak{g}_{\bar 1} = R(\pi_4) \boxtimes \mathrm{id}$ as $\mathfrak{g}_{\bar 0}$-module. We consider the Cartan matrix and Chevalley basis elements

$$\begin{pmatrix} 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -2 \\ -2 & -1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix}$$

the root vectors	the roots
$\boxed{x_1}, x_2, \boxed{x_3}, \boxed{x_4}, \boxed{x_5}$,	$\underline{\alpha_1}, \alpha_2, \underline{\alpha_3}, \underline{\alpha_4}, \underline{\alpha_5}$,
$x_6 = [x_1, x_4]$, $\boxed{x_7} = [x_2, x_4]$, $x_8 = [x_3, x_4]$, $x_9 = [x_3, x_5]$,	$\alpha_1 + \alpha_4$, $\underline{\alpha_2 + \alpha_4}$, $\alpha_3 + \alpha_4$, $\alpha_3 + \alpha_5$,
$x_{10} = [x_2, [x_1, x_4]]$, $\boxed{x_{11}} = [x_3, [x_1, x_4]]$, $x_{12} = [x_3, [x_2, x_4]]$, $\boxed{x_{13}} = [x_5, [x_3, x_4]]$,	$\alpha_1 + \alpha_2 + \alpha_4$, $\underline{\alpha_1 + \alpha_3 + \alpha_4}$, $\alpha_2 + \alpha_3 + \alpha_4$, $\underline{\alpha_3 + \alpha_4 + \alpha_5}$,

$\boxed{x_{14}} = [x_3, [x_2, [x_1, x_4]]]$, $x_{15} = [[x_1, x_4], [x_3, x_5]]$, $\boxed{x_{16}} = [[x_2, x_4], [x_3, x_4]]$, $\boxed{x_{17}} = [[x_2, x_4], [x_3, x_5]]$,	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}$, $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$, $\underline{2\alpha_4 + \alpha_2 + \alpha_3}$, $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$,
$x_{18} = [[x_2, x_4], [x_5, [x_3, x_4]]]$, $x_{19} = [[x_3, x_4], [x_2, [x_1, x_4]]]$, $x_{20} = [[x_3, x_4], [x_3, [x_2, x_4]]]$, $x_{21} = [[x_3, x_5], [x_2, [x_1, x_4]]]$,	$2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5$, $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3$, $2\alpha_3 + 2\alpha_4 + \alpha_2$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$,
$\boxed{x_{22}} = [[x_2, [x_1, x_4]], [x_5, [x_3, x_4]]]$, $\boxed{x_{23}} = [[x_3, [x_1, x_4]], [x_3, [x_2, x_4]]]$, $\boxed{x_{24}} = [[x_3, [x_2, x_4]], [x_5, [x_3, x_4]]]$,	$\underline{2\alpha_4 + \alpha_1 + \alpha_2}$ $\underline{+\alpha_3 + \alpha_5}$, $\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2}$, $\underline{2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5}$,
$x_{25} = [[x_3, [x_1, x_4]], [[x_2, x_4], [x_3, x_4]]]$, $x_{26} = [[x_5, [x_3, x_4]], [x_3, [x_2, [x_1, x_4]]]]$,	$2\alpha_3 + 3\alpha_4 + \alpha_1 + \alpha_2$, $2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5$,
$x_{27} = [[x_3, [x_2, [x_1, x_4]]], [[x_2, x_4], [x_3, x_4]]]$, $\boxed{x_{28}} = [[[x_1, x_4], [x_3, x_5]], [[x_2, x_4], [x_3, x_4]]]$,	$2\alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_1$, $\underline{2\alpha_3 + 3\alpha_4 + \alpha_1}$ $\underline{+\alpha_2 + \alpha_5}$,
$x_{29} = [[[x_1, x_4], [x_3, x_5]], [[x_3, x_4], [x_3, [x_2, x_4]]]]$, $\boxed{x_{30}} = [[[x_2, x_4], [x_3, x_5]], [[x_3, x_4], [x_2, [x_1, x_4]]]]$,	$3\alpha_3 + 3\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5$, $\underline{2\alpha_2 + 2\alpha_3 + 3\alpha_4}$ $\underline{+\alpha_1 + \alpha_5}$,
$x_{31} = [[[x_3, x_4], [x_3, [x_2, x_4]]]$, $[[x_3, x_5], [x_2, [x_1, x_4]]]]$,	$2\alpha_2 + 3\alpha_3 + 3\alpha_4 +$ $\alpha_1 + \alpha_5$,
$\boxed{x_{32}} = [[[x_3, x_4], [x_3, [x_2, x_4]]]$, $[[x_2, [x_1, x_4]], [x_5, [x_3, x_4]]]]$,	$\underline{2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \alpha_1 + \alpha_5}$,
$x_{33} = [[[x_2, [x_1, x_4]], [x_5, [x_3, x_4]]]$, $[[x_3, [x_1, x_4]], [x_3, [x_2, x_4]]]]$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \alpha_5$

4.6 $1\mathfrak{g}(1, 6)$ of sdim $21|14$, $p = 3$

For $\mathfrak{g} = \mathfrak{g}(1, 6)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(6)$ and $\mathfrak{g}_{\bar{1}} = R(\pi_3)$ as $\mathfrak{g}_{\bar{0}}$-module. We consider the Cartan matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ and the Chevalley basis elements

the root vectors	the roots
x_1, $\boxed{x_2}$, $\boxed{x_3}$,	$\alpha_1, \alpha_2, \underline{\alpha_3}$,
$\boxed{x_4} = [x_1, x_2]$, $x_5 = [x_2, x_2]$, $x_6 = [x_2, x_3]$,	$\alpha_1 + \alpha_2, 2\alpha_2, \alpha_2 + \alpha_3$,
$x_7 = [x_2, [x_1, x_2]]$, $x_8 = [x_3, [x_1, x_2]]$, $\boxed{x_9} = [x_3, [x_2, x_2]]$,	$2\alpha_2 + \alpha_1, \alpha_1 + \alpha_2 + \alpha_3$, $\underline{2\alpha_2 + \alpha_3}$,
$x_{10} = [[x_1, x_2], [x_1, x_2]]$, $\boxed{x_{11}} = [[x_1, x_2], [x_2, x_3]]$,	$2\alpha_1 + 2\alpha_2, 2\alpha_2 + \alpha_1 + \alpha_3$,
$\boxed{x_{12}} = [[x_1, x_2], [x_3, [x_1, x_2]]]$, $x_{13} = [[x_2, x_3], [x_2, [x_1, x_2]]]$,	$\underline{2\alpha_1 + 2\alpha_2 + \alpha_3}$, $3\alpha_2 + \alpha_1 + \alpha_3$,
$x_{14} = [[x_2, [x_1, x_2]], [x_3, [x_1, x_2]]]$,	$2\alpha_1 + 3\alpha_2 + \alpha_3$,
$\boxed{x_{15}} = [[x_3, [x_2, x_2]], [[x_1, x_2], [x_1, x_2]]]$,	$\underline{2\alpha_1 + 4\alpha_2 + \alpha_3}$,
$x_{16} = [[[x_1, x_2], [x_2, x_3]], [[x_1, x_2], [x_2, x_3]]]$	$2\alpha_1 + 2\alpha_3 + 4\alpha_2$

4.7 $2\mathfrak{g}(2, 3)$ of sdim $12/10|14$, $p = 3$

For $\mathfrak{g} = \mathfrak{g}(2, 3)$, we have $\mathfrak{g}_{\bar 0} = \mathfrak{gl}(3) \oplus \mathfrak{sl}(2)$ and $\mathfrak{g}_{\bar 1} = \mathfrak{psl}(3) \boxtimes \mathrm{id}$ as $\mathfrak{g}_{\bar 0}$-module. Clearly, $(\mathfrak{g}^{(1)}(2, 3)/\mathfrak{c})_{\bar 0} = \mathfrak{psl}(3) \oplus \mathfrak{sl}(2)$. We consider the Cartan matrix $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ and the Chevalley basis elements

the root vectors	the roots
$\boxed{x_1}$, $\boxed{x_2}$, $\boxed{x_3}$,	$\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3}$,
$x_4 = [x_1, x_3]$, $x_5 = [x_2, x_3]$,	$\alpha_1 + \alpha_3, \alpha_2 + \alpha_3$,
$\boxed{x_6} = [x_2, [x_1, x_3]]$,	$\alpha_1 + \alpha_2 + \alpha_3$,
$x_7 = [[x_1, x_3], [x_2, x_3]]$,	$2\alpha_3 + \alpha_1 + \alpha_2$,
$\boxed{x_8} = [[x_1, x_3], [x_2, [x_1, x_3]]]$, $\boxed{x_9} = [[x_2, x_3], [x_2, [x_1, x_3]]]$,	$\underline{2\alpha_1 + 2\alpha_3 + \alpha_2}$, $\underline{2\alpha_2 + 2\alpha_3 + \alpha_1}$,
$x_{10} = [[x_2, [x_1, x_3]], [x_2, [x_1, x_3]]]$,	$2\alpha_1 + 2\alpha_2 + 2\alpha_3$,
$\boxed{x_{11}} = [[x_2, [x_1, x_3]], [[x_1, x_3], [x_2, x_3]]]$	$\underline{2\alpha_1 + 2\alpha_2 + 3\alpha_3}$

4.8 $2\mathfrak{g}(2,6)$ of sdim $36/34|20$, $p=3$

We have $\mathfrak{g}(2,6)_{\bar 0} = \mathfrak{gl}(6)$ and $\mathfrak{g}(2,6)_{\bar 1} = R(\pi_3)$ as $\mathfrak{g}(2,6)_{\bar 0}$-module. Clearly, $(\mathfrak{g}^{(1)}(2,6)/\mathfrak{c})_{\bar 0} = \mathfrak{psl}(6)$. We consider the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

and the Chevalley basis elements

the root vectors	the roots
$x_1, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, x_5,$	$\alpha_1, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \alpha_5,$
$\boxed{x_6} = [x_1, x_2], x_7 = [x_2, x_3], x_8 = [x_2, x_4],$ $x_9 = [x_3, x_4], \boxed{x_{10}} = [x_3, x_5],$	$\underline{\alpha_1 + \alpha_2}, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4,$ $\alpha_3 + \alpha_4, \underline{\alpha_3 + \alpha_5},$
$x_{11} = [x_3, [x_1, x_2]], x_{12} = [x_4, [x_1, x_2]],$ $\boxed{x_{13}} = [x_4, [x_2, x_3]], x_{14} = [x_5, [x_2, x_3]],$ $x_{15} = [x_5, [x_3, x_4]],$	$\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4}, \alpha_2 + \alpha_3 + \alpha_5,$ $\alpha_3 + \alpha_4 + \alpha_5,$
$\boxed{x_{16}} = [[x_1, x_2], [x_3, x_4]],$ $x_{17} = [[x_1, x_2], [x_3, x_5]],$ $\boxed{x_{18}} = [[x_2, x_4], [x_3, x_5]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4},$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$
$x_{19} = [[x_2, x_4], [x_3, [x_1, x_2]]],$ $\boxed{x_{20}} = [[x_3, x_5], [x_4, [x_1, x_2]]],$ $x_{21} = [[x_3, x_5], [x_4, [x_2, x_3]]],$	$2\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4,$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_5,$
$x_{22} = [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]],$ $x_{23} = [[x_4, [x_1, x_2]], [x_5, [x_2, x_3]]],$	$2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5,$ $2\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5,$
$\boxed{x_{24}} = [[x_5, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]],$	$\underline{2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_5},$
$x_{25} = [[[x_1, x_2], [x_3, x_4]], [[x_2, x_4], [x_3, x_5]]]$	$2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5$

4.9 $7\mathfrak{g}(3,3)$ of sdim $23/21|16$, $p=3$

Let $\mathrm{spin}_7 := R(\pi_3)$. For $\mathfrak{g} = \mathfrak{g}(3,3)$, we have $\mathfrak{g}_{\bar 0} = (\mathfrak{o}(7) \oplus \mathbb{K}z) \oplus \mathbb{K}d$ and $\mathfrak{g}_{\bar 1} = (\mathrm{spin}_7)_+ \oplus (\mathrm{spin}_7)_-$ as $\mathfrak{g}_{\bar 0}$-module. The action of d—the outer derivative of $\mathfrak{g}^{(1)}$—separates the identical $\mathfrak{o}(7)$-modules spin_7 by acting on these modules as the scalar multiplication by ± 1, as indicated by subscripts, z spans the center of $\mathfrak{g}(3,3)$.

We consider the Cartan matrix $\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ and the Chevalley basis elements

the root vectors	the roots
$\boxed{x_1}$, $\boxed{x_2}$, x_3, $\boxed{x_4}$,	$\underline{\alpha_1}, \underline{\alpha_2}, \alpha_3, \underline{\alpha_4}$,
$x_5 = [x_1, x_2]$, $\boxed{x_6} = [x_2, x_3]$, $x_7 = [x_2, x_4]$,	$\alpha_1 + \alpha_2$, $\underline{\alpha_2 + \alpha_3}$, $\alpha_2 + \alpha_4$,
$x_8 = [x_3, [x_1, x_2]]$, $\boxed{x_9} = [x_4, [x_1, x_2]]$, $x_{10} = [x_4, [x_2, x_3]]$,	$\alpha_1 + \alpha_2 + \alpha_3$, $\underline{\alpha_1 + \alpha_2 + \alpha_4}$, $\alpha_2 + \alpha_3 + \alpha_4$,
$\boxed{x_{11}} = [x_4, [x_3, [x_1, x_2]]]$, $\boxed{x_{12}} = [[x_1, x_2], [x_2, x_3]]$,	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}$, $\underline{2\alpha_2 + \alpha_1 + \alpha_3}$,
$x_{13} = [[x_1, x_2], [x_3, [x_1, x_2]]]$, $x_{14} = [[x_2, x_4], [x_3, [x_1, x_2]]]$,	$2\alpha_1 + 2\alpha_2 + \alpha_3$, $2\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4$,
$\boxed{x_{15}} = [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]]$,	$\underline{2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4}$,
$x_{16} = [[x_4, [x_1, x_2]], [[x_1, x_2], [x_2, x_3]]]$,	$2\alpha_1 + 3\alpha_2 + \alpha_3 + \alpha_4$,
$x_{17} = [[x_4, [x_3, [x_1, x_2]]], [[x_1, x_2], [x_2, x_3]]]$	$2\alpha_1 + 2\alpha_3 + 3\alpha_2 + \alpha_4$

4.10 $2\mathfrak{g}(3, 6)$ of sdim $36|40$, $p = 3$

For $\mathfrak{g} = \mathfrak{g}(3, 6)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(8)$ and $\mathfrak{g}_{\bar{1}} = R(\pi_3)$ as $\mathfrak{g}_{\bar{0}}$-module. We consider the Cartan matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and the Chevalley basis elements

the root vectors	the roots
$\boxed{x_1}$, $\boxed{x_2}$, $\boxed{x_3}$, $\boxed{x_4}$,	$\underline{\alpha_1}, \underline{\alpha_2}, \alpha_3, \underline{\alpha_4}$,
$x_5 = [x_1, x_2]$, $x_6 = [x_2, x_3]$, $x_7 = [x_3, x_3]$, $x_8 = [x_3, x_4]$,	$\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $2\alpha_3$, $\alpha_3 + \alpha_4$,
$\boxed{x_9} = [x_3, [x_1, x_2]]$, $\boxed{x_{10}} = [x_3, [x_2, x_3]]$, $\boxed{x_{11}} = [x_4, [x_2, x_3]]$, $\boxed{x_{12}} = [x_4, [x_3, x_3]]$,	$\alpha_1 + \alpha_2 + \alpha_3$, $\underline{2\alpha_3 + \alpha_2}$, $\underline{\alpha_2 + \alpha_3 + \alpha_4}$, $\underline{2\alpha_3 + \alpha_4}$,
$x_{13} = [[x_1, x_2], [x_3, x_3]]$, $x_{14} = [[x_1, x_2], [x_3, x_4]]$, $x_{15} = [[x_2, x_3], [x_3, x_4]]$,	$2\alpha_3 + \alpha_1 + \alpha_2$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $2\alpha_3 + \alpha_2 + \alpha_4$,

$\boxed{x_{16}} = [[x_2, x_3], [x_3, [x_1, x_2]]],$ $\boxed{x_{17}} = [[x_3, x_4], [x_3, [x_1, x_2]]],$ $\boxed{x_{18}} = [[x_3, x_4], [x_3, [x_2, x_3]]],$	$\underline{2\alpha_2 + 2\alpha_3 + \alpha_1},$ $\underline{2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4},$ $\underline{3\alpha_3 + \alpha_2 + \alpha_4},$
$x_{19} = [[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]],$ $x_{20} = [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]],$ $x_{21} = [[x_3, [x_1, x_2]], [x_4, [x_3, x_3]]],$	$2\alpha_1 + 2\alpha_2 + 2\alpha_3,$ $2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4,$ $3\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4,$
$\boxed{x_{22}} = [[x_3, [x_1, x_2]], [[x_1, x_2], [x_3, x_4]]],$ $\boxed{x_{23}} = [[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_3]]],$	$\underline{2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4},$ $\underline{2\alpha_2 + 3\alpha_3 + \alpha_1 + \alpha_4},$
$x_{24} = [[[x_1, x_2], [x_3, x_3]], [[x_1, x_2], [x_3, x_4]]],$ $x_{25} = [[[x_1, x_2], [x_3, x_3]], [[x_2, x_3], [x_3, x_4]]],$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4,$ $2\alpha_2 + 4\alpha_3 + \alpha_1 + \alpha_4,$
$\boxed{x_{26}} = [[[x_1, x_2], [x_3, x_3]], [[x_3, x_4], [x_3, [x_1, x_2]]]],$ $\boxed{x_{27}} = [[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_3, [x_1, x_2]]]],$ $\boxed{x_{28}} = [[[x_2, x_3], [x_3, x_4]], [[x_3, x_4], [x_3, [x_1, x_2]]]],$	$\underline{2\alpha_1 + 2\alpha_2 + 4\alpha_3 + \alpha_4},$ $\underline{2\alpha_1 + 3\alpha_2 + 3\alpha_3 + \alpha_4},$ $\underline{2\alpha_2 + 2\alpha_4 + 4\alpha_3 + \alpha_1},$
$x_{29} = [[[x_2, x_3], [x_3, [x_1, x_2]]], [[x_3, x_4], [x_3, [x_1, x_2]]]],$ $x_{30} = [[[x_3, x_4], [x_3, [x_1, x_2]]], [[x_3, x_4], [x_3, [x_1, x_2]]]],$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + \alpha_4,$ $2\alpha_1 + 2\alpha_2 + 2\alpha_4 + 4\alpha_3,$
$\boxed{x_{31}} = [[[x_3, x_4], [x_3, [x_1, x_2]]], [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]],$ $\boxed{x_{32}} = [[[x_3, x_4], [x_3, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]]],$	$\underline{2\alpha_1 + 2\alpha_4 + 3\alpha_2 + 4\alpha_3},$ $\underline{2\alpha_1 + 3\alpha_2 + 5\alpha_3 + \alpha_4},$
$x_{33} = [[[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_4, [x_3, x_3]]]],$	$2\alpha_1 + 2\alpha_4 + 3\alpha_2 + 5\alpha_3,$
$\boxed{x_{34}} = [[[x_3, [x_1, x_2]], [x_4, [x_3, x_3]]], [[x_4, [x_2, x_3]],$ $[[x_1, x_2], [x_3, x_3]]]],$	$\underline{2\alpha_1 + 2\alpha_4 + 3\alpha_2 + 6\alpha_3},$
$x_{35} = [[[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_3]]], [[x_4, [x_2, x_3]],$ $[[x_1, x_2], [x_3, x_3]]]],$	$2\alpha_1 + 2\alpha_4 + 4\alpha_2 + 6\alpha_3,$
$\boxed{x_{36}} = [[[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_3]]], [[[x_1, x_2],$ $[x_3, x_3]], [[x_1, x_2], [x_3, x_4]]]]$	$\underline{2\alpha_4 + 3\alpha_1 + 4\alpha_2 + 6\alpha_3}$

4.11 $6\mathfrak{g}(4, 3)$ of sdim $24|26$, $p = 3$

For $\mathfrak{g} = \mathfrak{g}(4, 3)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(6) \oplus \mathfrak{sl}(2)$ and $\mathfrak{g}_{\bar{1}} = R(\pi_2) \boxtimes \mathrm{id}$ as $\mathfrak{g}_{\bar{0}}$-module. We consider the Cartan matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and the Chevalley basis elements

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}, \boxed{x_3}, \boxed{x_4},$	$\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4},$
$x_5 = [x_1, x_2],$ $x_6 = [x_2, x_3],$ $x_7 = [x_3, x_4],$	$\alpha_1 + \alpha_2,$ $\alpha_2 + \alpha_3,$ $\alpha_3 + \alpha_4,$
$\boxed{x_8} = [x_3, [x_1, x_2]],$ $x_9 = [x_4, [x_2, x_3]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3},$ $\alpha_2 + \alpha_3 + \alpha_4,$
$x_{10} = [[x_1, x_2], [x_3, x_4]],$ $x_{11} = [[x_2, x_3], [x_3, x_4]],$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$ $2\alpha_3 + \alpha_2 + \alpha_4,$
$\boxed{x_{12}} = [[x_2, x_3], [x_4, [x_2, x_3]]],$ $x_{13} = [[x_3, x_4], [x_3, [x_1, x_2]]],$ $x_{14} = [[x_3, x_4], [x_4, [x_2, x_3]]],$	$\underline{2\alpha_2 + 2\alpha_3 + \alpha_4},$ $2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4,$ $2\alpha_3 + 2\alpha_4 + \alpha_2,$
$x_{15} = [[x_3, x_4], [[x_1, x_2], [x_3, x_4]]],$ $x_{16} = [[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]],$ $\boxed{x_{17}} = [[x_4, [x_2, x_3]], [x_4, [x_2, x_3]]],$	$2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2,$ $2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4,$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4},$
$x_{18} = [[x_4, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]],$ $x_{19} = [[x_4, [x_2, x_3]], [[x_2, x_3], [x_3, x_4]]],$	$2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1,$ $2\alpha_2 + 2\alpha_4 + 3\alpha_3,$
$\boxed{x_{20}} = [[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_3, x_4]]],$	$\underline{2\alpha_2 + 2\alpha_4 + 3\alpha_3 + \alpha_1},$
$x_{21} = [[[x_1, x_2], [x_3, x_4]], [[x_2, x_3], [x_4, [x_2, x_3]]]],$	$2\alpha_4 + 3\alpha_2 + 3\alpha_3 + \alpha_1,$
$\boxed{x_{22}} = [[[x_2, x_3], [x_4, [x_2, x_3]]], [[x_3, x_4],$ $[x_3, [x_1, x_2]]]],$	$\underline{2\alpha_4 + 3\alpha_2 + 4\alpha_3 + \alpha_1},$
$x_{23} = [[[x_3, x_4], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]],$ $[x_4, [x_2, x_3]]]],$	$3\alpha_2 + 3\alpha_4 + 4\alpha_3 + \alpha_1,$

4.12 13$\mathfrak{g}(8, 3)$ of sdim 55|50, $p = 3$

For $\mathfrak{g} = \mathfrak{g}(8, 3)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{f}(4) \oplus \mathfrak{sl}(2)$ and $\mathfrak{g}_{\bar{1}} = R(\pi_4) \boxtimes \text{id}$ as $\mathfrak{g}_{\bar{0}}$-module. We consider the Cartan matrix

$$\begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & -2 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

and the Chevalley basis elements

the root vectors	the roots
$x_1, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, \boxed{x_5},$	$\alpha_1, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \underline{\alpha_5},$
$\boxed{x_6} = [x_1, x_2], \boxed{x_7} = [x_1, x_3], x_8 = [x_2, x_3],$ $x_9 = [x_2, x_4], x_{10} = [x_4, x_5],$	$\underline{\alpha_1 + \alpha_2}, \underline{\alpha_1 + \alpha_3}, \alpha_2 + \alpha_3,$ $\alpha_2 + \alpha_4, \alpha_4 + \alpha_5,$
$x_{11} = [x_3, [x_1, x_2]], x_{12} = [x_4, [x_1, x_2]],$ $\boxed{x_{13}} = [x_4, [x_2, x_3]], \boxed{x_{14}} = [x_5, [x_2, x_4]],$	$\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4},$ $\underline{\alpha_2 + \alpha_4 + \alpha_5},$
$x_{15} = [[x_1, x_2], [x_1, x_3]],$ $\boxed{x_{16}} = [[x_1, x_2], [x_4, x_5]],$ $\boxed{x_{17}} = [[x_1, x_3], [x_2, x_4]],$ $x_{18} = [[x_2, x_3], [x_4, x_5]],$	$2\alpha_1 + \alpha_2 + \alpha_3,$ $\underline{\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5},$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4},$ $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$
$\boxed{x_{19}} = [[x_1, x_3], [x_4, [x_1, x_2]]],$ $x_{20} = [[x_2, x_4], [x_3, [x_1, x_2]]],$ $x_{21} = [[x_4, x_5], [x_3, [x_1, x_2]]],$	$\underline{2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4},$ $2\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$
$x_{22} = [[x_4, x_5], [[x_1, x_2], [x_1, x_3]]],$ $x_{23} = [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]],$ $\boxed{x_{24}} = [[x_3, [x_1, x_2]], [x_5, [x_2, x_4]]],$ $\boxed{x_{25}} = [[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]],$	$2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$ $2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4,$ $\underline{2\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5},$ $\underline{2\alpha_2 + 2\alpha_4 + \alpha_1 + \alpha_3},$
$\boxed{x_{26}} = [[x_4, [x_1, x_2]], [[x_1, x_3], [x_2, x_4]]],$ $\boxed{x_{27}} = [[x_4, [x_2, x_3]], [[x_1, x_2], [x_1, x_3]]],$ $\boxed{x_{28}} = [[x_5, [x_2, x_4]], [[x_1, x_2], [x_1, x_3]]],$ $x_{29} = [[x_5, [x_2, x_4]], [[x_1, x_3], [x_2, x_4]]],$	$\underline{2\alpha_1 + 2\alpha_2 + 2\alpha_4 + \alpha_3},$ $\underline{2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4},$ $\underline{2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $2\alpha_2 + 2\alpha_4 + \alpha_1 + \alpha_3 + \alpha_5,$
$x_{30} = [[x_4, [x_1, x_2]], [[x_2, x_4], [x_3, [x_1, x_2]]]],$ $x_{31} = [[[x_1, x_2], [x_1, x_3]], [[x_2, x_3], [x_4, x_5]]],$ $x_{32} = [[[x_1, x_2], [x_4, x_5]], [[x_1, x_3], [x_2, x_4]]],$ $x_{33} = [[[x_1, x_3], [x_2, x_4]], [[x_1, x_3], [x_2, x_4]]],$	$2\alpha_1 + 2\alpha_4 + 3\alpha_2 + \alpha_3,$ $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5,$ $2\alpha_1 + 2\alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5,$ $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4,$
$\boxed{x_{34}} = [[[x_1, x_2], [x_4, x_5]], [[x_2, x_4], [x_3, [x_1, x_2]]]],$ $\boxed{x_{35}} = [[[x_1, x_3], [x_2, x_4]], [[x_2, x_4], [x_3, [x_1, x_2]]]],$ $\boxed{x_{36}} = [[[x_2, x_3], [x_4, x_5]], [[x_1, x_3], [x_4, [x_1, x_2]]]],$	$\underline{2\alpha_1 + 2\alpha_4 + 3\alpha_2 + \alpha_3 + \alpha_5},$ $\underline{2\alpha_1 + 2\alpha_3 + 2\alpha_4 + 3\alpha_2},$ $\underline{2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5},$
$x_{37} = [[[x_1, x_2], [x_4, x_5]], [[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]]],$ $\boxed{x_{38}} = [[[x_1, x_3], [x_4, [x_1, x_2]]], [[x_2, x_4], [x_3, [x_1, x_2]]]],$ $x_{39} = [[[x_2, x_4], [x_3, [x_1, x_2]]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$2\alpha_1 + 3\alpha_2 + 3\alpha_4 + \alpha_3 + \alpha_5,$ $\underline{2\alpha_3 + 2\alpha_4 + 3\alpha_1 + 3\alpha_2},$ $2\alpha_1 + 2\alpha_3 + 2\alpha_4 + 3\alpha_2 + \alpha_5,$
$x_{40} = [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_3, [x_1, x_2]], [x_4, [x_1, x_2]]]],$ $\boxed{x_{41}} = [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]]],$	$2\alpha_3 + 2\alpha_4 + 3\alpha_1 + 3\alpha_2 + \alpha_5,$ $\underline{2\alpha_1 + 2\alpha_3 + 3\alpha_2 + 3\alpha_4 + \alpha_5},$
$\boxed{x_{42}} = [[[x_4, x_5], [[x_1, x_2], [x_1, x_3]]],$ $[[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]]],$ $x_{43} = [[[x_3, [x_1, x_2]], [x_5, [x_2, x_4]]], [[x_4, [x_1, x_2]],$ $[x_4, [x_2, x_3]]]],$	$\underline{2\alpha_3 + 3\alpha_1 + 3\alpha_2 + 3\alpha_4 + \alpha_5},$ $2\alpha_1 + 2\alpha_3 + 3\alpha_4 + 4\alpha_2 + \alpha_5,$

$x_{44} = [[[x_4, [x_1, x_2]], [x_4, [x_2, x_3]]], [[x_5, [x_2, x_4]], [[x_1, x_2], [x_1, x_3]]]]$,	$2\alpha_3+3\alpha_1+3\alpha_4+4\alpha_2+\alpha_5$,
$x_{45} = [[[x_4, [x_1, x_2]], [[x_1, x_3], [x_2, x_4]]], [[x_5, [x_2, x_4]], [[x_1, x_2], [x_1, x_3]]]]$, $\boxed{x_{46}} = [[[x_4, [x_2, x_3]], [[x_1, x_2], [x_1, x_3]]], [[x_5, [x_2, x_4]], [[x_1, x_3], [x_2, x_4]]]]$,	$2\alpha_3+3\alpha_4+4\alpha_1+4\alpha_2+\alpha_5$, $\underline{3\alpha_1+3\alpha_3+3\alpha_4+4\alpha_2+\alpha_5}$,
$\boxed{x_{47}} = [[[x_5, [x_2, x_4]], [[x_1, x_2], [x_1, x_3]]], [[[x_1, x_3], [x_2, x_4]], [[x_1, x_3], [x_2, x_4]]]]$,	$\underline{3\alpha_3+3\alpha_4+4\alpha_1+4\alpha_2+\alpha_5}$,
$x_{48} = [[[x_4, [x_1, x_2]], [[x_2, x_4], [x_3, [x_1, x_2]]]], [[[x_1, x_2], [x_1, x_3]], [[x_2, x_3], [x_4, x_5]]]]$,	$3\alpha_3+3\alpha_4+4\alpha_1+5\alpha_2+\alpha_5$,
$\boxed{x_{49}} = [[[[x_1, x_3], [x_2, x_4]], [[x_1, x_3], [x_2, x_4]]], [[[x_1, x_2], [x_4, x_5]], [[x_2, x_4], [x_3, [x_1, x_2]]]]]$,	$\underline{3\alpha_3+4\alpha_1+4\alpha_4+5\alpha_2+\alpha_5}$,
$x_{50} = [[[[x_1, x_2], [x_4, x_5]], [[x_2, x_4], [x_3, [x_1, x_2]]]], [[[x_2, x_3], [x_4, x_5]], [[x_1, x_3], [x_4, [x_1, x_2]]]]]$	$2\alpha_5+3\alpha_3+4\alpha_1+4\alpha_4+5\alpha_2$

4.13 $2\mathfrak{g}(4, 6)$ of sdim $66|32$, $p = 3$

For $\mathfrak{g} = \mathfrak{g}(4, 6)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{o}(12)$ and $\mathfrak{g}_{\bar{1}} = R(\pi_5)$ as $\mathfrak{g}_{\bar{0}}$-module. We consider the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

and the Chevalley basis elements

the root vectors	the roots
$x_1, x_2, \boxed{x_3}, \boxed{x_4}, \boxed{x_5}, x_6$,	$\alpha_1, \alpha_2, \underline{\alpha_3}, \underline{\alpha_4}, \underline{\alpha_5}, \alpha_6$,
$x_7 = [x_1, x_2]$, $\boxed{x_8} = [x_2, x_3]$, $x_9 = [x_3, x_4]$, $x_{10} = [x_3, x_5]$, $x_{11} = [x_4, x_5]$, $\boxed{x_{12}} = [x_4, x_6]$,	$\alpha_1+\alpha_2$, $\underline{\alpha_2+\alpha_3}$, $\alpha_3+\alpha_4$, $\alpha_3+\alpha_5$, $\alpha_4+\alpha_5$, $\underline{\alpha_4+\alpha_6}$,
$\boxed{x_{13}} = [x_3, [x_1, x_2]]$, $x_{14} = [x_4, [x_2, x_3]]$, $x_{15} = [x_5, [x_2, x_3]]$, $\boxed{x_{16}} = [x_5, [x_3, x_4]]$, $x_{17} = [x_6, [x_3, x_4]]$, $x_{18} = [x_6, [x_4, x_5]]$,	$\underline{\alpha_1+\alpha_2+\alpha_3}$, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_2+\alpha_3+\alpha_5$, $\underline{\alpha_3+\alpha_4+\alpha_5}$, $\alpha_3+\alpha_4+\alpha_6$, $\alpha_4+\alpha_5+\alpha_6$,
$x_{19} = [[x_1, x_2], [x_3, x_4]]$, $x_{20} = [[x_1, x_2], [x_3, x_5]]$, $\boxed{x_{21}} = [[x_2, x_3], [x_4, x_5]]$, $x_{22} = [[x_2, x_3], [x_4, x_6]]$, $\boxed{x_{23}} = [[x_3, x_5], [x_4, x_6]]$,	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3+\alpha_5$, $\underline{\alpha_2+\alpha_3+\alpha_4+\alpha_5}$, $\alpha_2+\alpha_3+\alpha_4+\alpha_6$, $\underline{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$,

$x_{24} = [[x_3, x_5], [x_4, [x_2, x_3]]]$, $\boxed{x_{25}} = [[x_4, x_5], [x_3, [x_1, x_2]]]$, $x_{26} = [[x_4, x_6], [x_3, [x_1, x_2]]]$, $\boxed{x_{27}} = [[x_4, x_6], [x_5, [x_2, x_3]]]$, $x_{28} = [[x_4, x_6], [x_5, [x_3, x_4]]]$,	$2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_5$, $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6$, $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}$, $2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_6$,
$x_{29} = [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]]$, $\boxed{x_{30}} = [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]$, $x_{31} = [[x_4, [x_2, x_3]], [x_6, [x_4, x_5]]]$, $x_{32} = [[x_5, [x_2, x_3]], [x_6, [x_3, x_4]]]$,	$2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5$, $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}$, $2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6$, $2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$,
$x_{33} = [[x_5, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]]$, $x_{34} = [[x_6, [x_3, x_4]], [[x_1, x_2], [x_3, x_5]]]$, $\boxed{x_{35}} = [[x_6, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]$, $x_{36} = [[x_6, [x_4, x_5]], [[x_1, x_2], [x_3, x_4]]]$,	$2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_5$, $2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$, $\underline{2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_6}$, $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6$,
$\boxed{x_{37}} = [[[x_1, x_2], [x_3, x_4]], [[x_3, x_5], [x_4, x_6]]]$, $x_{38} = [[[x_1, x_2], [x_3, x_5]], [[x_2, x_3], [x_4, x_6]]]$, $x_{39} = [[[x_2, x_3], [x_4, x_5]], [[x_3, x_5], [x_4, x_6]]]$,	$\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6}$, $2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_5 + \alpha_6$, $2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6$,
$\boxed{x_{40}} = [[[x_2, x_3], [x_4, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]]$, $x_{41} = [[[x_3, x_5], [x_4, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]]$,	$\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_6}$, $2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6$,
$x_{42} = [[[x_3, x_5], [x_4, [x_2, x_3]]], [[x_4, x_6], [x_3, [x_1, x_2]]]]$, $x_{43} = [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_4, x_6], [x_5, [x_2, x_3]]]]$,	$2\alpha_2 + 2\alpha_4 + 3\alpha_3 + \alpha_1 + \alpha_5 + \alpha_6$, $2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_6$,
$\boxed{x_{44}} = [[[x_4, x_6], [x_5, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]]]$,	$\underline{2\alpha_2 + 2\alpha_4 + 2\alpha_5 + 3\alpha_3 + \alpha_1 + \alpha_6}$,
$x_{45} = [[[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]], [[x_4, [x_2, x_3]],$ $[x_6, [x_4, x_5]]]]$,	$2\alpha_2 + 2\alpha_5 + 3\alpha_3 + 3\alpha_4 + \alpha_1 + \alpha_6$,
$x_{46} = [[[x_5, [x_2, x_3]], [x_6, [x_3, x_4]]], [[x_6, [x_4, x_5]],$ $[[x_1, x_2], [x_3, x_4]]]]$	$2\alpha_2 + 2\alpha_5 + 2\alpha_6 + 3\alpha_3 +$ $3\alpha_4 + \alpha_1$

4.14 $4\mathfrak{g}(6, 6)$ of sdim $78\|64$, $p = 3$

For $\mathfrak{g} = \mathfrak{g}(6, 6)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{o}(13)$ and $\mathfrak{g}_{\bar{1}} = \mathrm{spin}_{13} := R(\pi_6)$ as $\mathfrak{g}_{\bar{0}}$-module. We consider the Cartan matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

and the Chevalley basis elements

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, \boxed{x_5}, x_6,$	$\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \underline{\alpha_5}, \alpha_6,$
$x_7 = [x_1, x_2], x_8 = [x_2, x_3], x_9 = [x_3, x_4],$ $x_{10} = [x_3, x_5], x_{11} = [x_4, x_5], \boxed{x_{12}} = [x_4, x_6],$	$\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4,$ $\alpha_3 + \alpha_5, \alpha_4 + \alpha_5, \underline{\alpha_4 + \alpha_6},$
$\boxed{x_{13}} = [x_3, [x_1, x_2]], \boxed{x_{14}} = [x_4, [x_2, x_3]],$ $\boxed{x_{15}} = [x_5, [x_2, x_3]], \boxed{x_{16}} = [x_5, [x_3, x_4]],$ $x_{17} = [x_6, [x_3, x_4]], x_{18} = [x_6, [x_4, x_5]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3}, \underline{\alpha_2 + \alpha_3 + \alpha_4},$ $\underline{\alpha_2 + \alpha_3 + \alpha_5}, \underline{\alpha_3 + \alpha_4 + \alpha_5},$ $\alpha_3 + \alpha_4 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6,$
$x_{19} = [[x_1, x_2], [x_3, x_4]],$ $x_{20} = [[x_1, x_2], [x_3, x_5]],$ $x_{21} = [[x_2, x_3], [x_4, x_5]],$ $\boxed{x_{22}} = [[x_2, x_3], [x_4, x_6]],$ $\boxed{x_{23}} = [[x_3, x_5], [x_4, x_6]],$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5,$ $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6},$ $\underline{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$
$\boxed{x_{24}} = [[x_3, x_5], [x_4, [x_2, x_3]]],$ $\boxed{x_{25}} = [[x_4, x_5], [x_3, [x_1, x_2]]],$ $x_{26} = [[x_4, x_6], [x_3, [x_1, x_2]]],$ $x_{27} = [[x_4, x_6], [x_5, [x_2, x_3]]],$ $x_{28} = [[x_4, x_6], [x_5, [x_3, x_4]]],$	$\underline{2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_5},$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6,$ $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_6,$
$x_{29} = [[x_3, [x_1, x_2]], [x_5, [x_3, x_4]]],$ $\boxed{x_{30}} = [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$ $x_{31} = [[x_4, [x_2, x_3]], [x_5, [x_2, x_3]]],$ $\boxed{x_{32}} = [[x_4, [x_2, x_3]], [x_6, [x_4, x_5]]],$ $\boxed{x_{33}} = [[x_5, [x_2, x_3]], [x_6, [x_3, x_4]]],$	$2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5,$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$ $2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5,$ $\underline{2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6},$ $\underline{2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6},$
$\boxed{x_{34}} = [[x_5, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]],$ $x_{35} = [[x_5, [x_2, x_3]], [[x_2, x_3], [x_4, x_6]]],$ $x_{36} = [[x_6, [x_3, x_4]], [[x_1, x_2], [x_3, x_5]]],$ $x_{37} = [[x_6, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]],$ $x_{38} = [[x_6, [x_4, x_5]], [[x_1, x_2], [x_3, x_4]]],$	$\underline{2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_5},$ $\underline{2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$ $2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6,$ $2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_6,$ $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6,$
$\boxed{x_{39}} = [[[x_1, x_2], [x_3, x_4]], [[x_3, x_5], [x_4, x_6]]],$ $\boxed{x_{40}} = [[[x_1, x_2], [x_3, x_5]], [[x_2, x_3], [x_4, x_6]]],$ $\boxed{x_{41}} = [[[x_2, x_3], [x_4, x_5]], [[x_2, x_3], [x_4, x_6]]],$ $\boxed{x_{42}} = [[[x_2, x_3], [x_4, x_5]], [[x_3, x_5], [x_4, x_6]]],$	$\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6},$ $\underline{2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_5 + \alpha_6},$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6},$ $\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6},$
$x_{43} = [[[x_2, x_3], [x_4, x_5]], [[x_4, x_6], [x_5, [x_2, x_3]]]],$ $x_{44} = [[[x_2, x_3], [x_4, x_6]], [[x_3, x_5], [x_4, [x_2, x_3]]]],$ $x_{45} = [[[x_2, x_3], [x_4, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$ $x_{46} = [[[x_3, x_5], [x_4, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6,$ $2\alpha_2 + 2\alpha_4 + 3\alpha_3 + \alpha_5 + \alpha_6,$ $2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_6,$ $2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6,$
$\boxed{x_{47}} = [[[x_3, x_5], [x_4, [x_2, x_3]]], [[x_4, x_6], [x_3, [x_1, x_2]]]],$ $\boxed{x_{48}} = [[[x_3, x_5], [x_4, [x_2, x_3]]], [[x_4, x_6], [x_5, [x_2, x_3]]]],$ $\boxed{x_{49}} = [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_4, x_6], [x_5, [x_2, x_3]]]],$	$\underline{2\alpha_2 + 2\alpha_4 + 3\alpha_3 + \alpha_1 + \alpha_5 + \alpha_6},$ $\underline{2\alpha_2 + 2\alpha_4 + 2\alpha_5 + 3\alpha_3 + \alpha_6},$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_6},$
$x_{50} = [[[x_4, x_6], [x_3, [x_1, x_2]]], [[x_4, [x_2, x_3]],$ $[x_5, [x_2, x_3]]]],$ $x_{51} = [[[x_4, x_6], [x_5, [x_2, x_3]]], [[x_3, [x_1, x_2]],$ $[x_5, [x_3, x_4]]]],$ $x_{52} = [[[x_4, x_6], [x_5, [x_3, x_4]]], [[x_4, [x_2, x_3]],$ $[x_5, [x_2, x_3]]]],$	$2\alpha_4 + 3\alpha_2 + 3\alpha_3 + \alpha_1 + \alpha_5 + \alpha_6,$ $2\alpha_2 + 2\alpha_4 + 2\alpha_5 + 3\alpha_3 + \alpha_1 + \alpha_6,$ $2\alpha_2 + 2\alpha_5 + 3\alpha_3 + 3\alpha_4 + \alpha_6,$

$\boxed{x_{53}} = [[[x_3,[x_1,x_2]],[x_5,[x_3,x_4]]],[[x_4,[x_2,x_3]],[x_6,[x_4,x_5]]]]$,	$\underline{2\alpha_2+2\alpha_5+3\alpha_3+3\alpha_4+\alpha_1+\alpha_6}$,
$\boxed{x_{54}} = [[[x_3,[x_1,x_2]],[x_6,[x_4,x_5]]],[[x_4,[x_2,x_3]],[x_5,[x_2,x_3]]]]$,	$\underline{2\alpha_4+2\alpha_5+3\alpha_2+3\alpha_3+\alpha_1+\alpha_6}$,
$x_{55} = [[[x_4,[x_2,x_3]],[x_6,[x_4,x_5]]],[[x_5,[x_2,x_3]],[x_6,[x_3,x_4]]]]$,	$2\alpha_2+2\alpha_5+2\alpha_6+3\alpha_3+3\alpha_4$,
$x_{56} = [[[x_4,[x_2,x_3]],[x_6,[x_4,x_5]]],[[x_5,[x_2,x_3]],[[x_1,x_2],[x_3,x_4]]]]$,	$2\alpha_5+3\alpha_2+3\alpha_3+3\alpha_4+\alpha_1+\alpha_6$,
$x_{57} = [[[x_5,[x_2,x_3]],[x_6,[x_3,x_4]]],[[x_5,[x_2,x_3]],[[x_1,x_2],[x_3,x_4]]]]$,	$2\alpha_4+2\alpha_5+3\alpha_2+4\alpha_3+\alpha_1+\alpha_6$,
$\boxed{x_{58}} = [[[x_5,[x_2,x_3]],[x_6,[x_3,x_4]]],[[x_6,[x_4,x_5]],[[x_1,x_2],[x_3,x_4]]]]$,	$\underline{2\alpha_2+2\alpha_5+2\alpha_6+3\alpha_3+3\alpha_4+\alpha_1}$,
$\boxed{x_{59}} = [[[x_5,[x_2,x_3]],[[x_1,x_2],[x_3,x_4]]],[[x_6,[x_3,x_4]],[[x_2,x_3],[x_4,x_5]]]]$,	$\underline{2\alpha_5+3\alpha_2+3\alpha_4+4\alpha_3+\alpha_1+\alpha_6}$,
$x_{60} = [[[x_5,[x_2,x_3]],[[x_2,x_3],[x_4,x_6]]],[[x_6,[x_4,x_5]],[[x_1,x_2],[x_3,x_4]]]]$,	$2\alpha_5+2\alpha_6+3\alpha_2+3\alpha_3+3\alpha_4+\alpha_1$,
$x_{61} = [[[x_5,[x_2,x_3]],[[x_1,x_2],[x_3,x_4]]],[[[x_2,x_3],[x_4,x_5]],[[x_3,x_5],[x_4,x_6]]]]$,	$3\alpha_2+3\alpha_4+3\alpha_5+4\alpha_3+\alpha_1+\alpha_6$,
$\boxed{x_{62}} = [[[x_6,[x_3,x_4]],[[x_2,x_3],[x_4,x_5]]],[[[x_1,x_2],[x_3,x_5]],[[x_2,x_3],[x_4,x_6]]]]$,	$\underline{2\alpha_5+2\alpha_6+3\alpha_2+3\alpha_4+4\alpha_3+\alpha_1}$,
$x_{63} = [[[[x_1,x_2],[x_3,x_4]],[[x_3,x_5],[x_4,x_6]]],[[[x_2,x_3],[x_4,x_5]],[[x_2,x_3],[x_4,x_6]]]]$,	$2\alpha_5+2\alpha_6+3\alpha_2+4\alpha_3+4\alpha_4+\alpha_1$,
$x_{64} = [[[[x_1,x_2],[x_3,x_5]],[[x_2,x_3],[x_4,x_6]]],[[[x_2,x_3],[x_4,x_5]],[[x_3,x_5],[x_4,x_6]]]]$,	$2\alpha_6+3\alpha_2+3\alpha_4+3\alpha_5+4\alpha_3+\alpha_1$,
$\boxed{x_{65}} = [[[[x_2,x_3],[x_4,x_5]],[[x_3,x_5],[x_4,x_6]]],[[[x_2,x_3],[x_4,x_6]],[[x_4,x_5],[x_3,[x_1,x_2]]]]]$,	$\underline{2\alpha_6+3\alpha_2+3\alpha_5+4\alpha_3+4\alpha_4+\alpha_1}$,
$x_{66} = [[[[x_2,x_3],[x_4,x_6]],[[x_3,x_5],[x_4,[x_2,x_3]]]],[[[x_3,x_5],[x_4,x_6]],[[x_4,x_5],[x_3,[x_1,x_2]]]]]$,	$2\alpha_6+3\alpha_2+3\alpha_5+4\alpha_4+5\alpha_3+\alpha_1$,
$\boxed{x_{67}} = [[[[x_2,x_3],[x_4,x_6]],[[x_4,x_5],[x_3,[x_1,x_2]]]],[[[x_3,x_5],[x_4,[x_2,x_3]]],[[x_4,x_6],[x_5,[x_2,x_3]]]]]$,	$\underline{2\alpha_6+3\alpha_5+4\alpha_2+4\alpha_4+5\alpha_3+\alpha_1}$,
$x_{68} = [[[[x_3,x_5],[x_4,[x_2,x_3]]],[[x_4,x_6],[x_3,[x_1,x_2]]]],[[[x_4,x_5],[x_3,[x_1,x_2]]],[[x_4,x_6],[x_5,[x_2,x_3]]]]]$	$2\alpha_1+2\alpha_6+3\alpha_5+4\alpha_2+4\alpha_4+5\alpha_3$

4.15 $5\mathfrak{g}(8,6)$ of sdim $133|56$, $p=3$

For $\mathfrak{g}=\mathfrak{g}(8,6)$, we have $\mathfrak{g}_{\bar{0}}=\mathfrak{e}(7)$ and $\mathfrak{g}_{\bar{1}}=R(\pi_1)$ as $\mathfrak{g}_{\bar{0}}$-module. We consider the following Cartan matrix and the Chevalley basis elements

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

the root vectors	the roots
$x_1, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, x_5, x_6, x_7,$	$\alpha_1, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \alpha_5, \alpha_6, \alpha_7,$
$\boxed{x_8} = [x_1, x_3], x_9 = [x_2, x_3], x_{10} = [x_2, x_4],$ $x_{11} = [x_3, x_4], \boxed{x_{12}} = [x_4, x_5], x_{13} = [x_5, x_6],$ $x_{14} = [x_6, x_7],$	$\underline{\alpha_1 + \alpha_3}, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4,$ $\alpha_3 + \alpha_4, \underline{\alpha_4 + \alpha_5}, \alpha_5 + \alpha_6,$ $\alpha_6 + \alpha_7,$
$x_{15} = [x_2, [x_1, x_3]], x_{16} = [x_4, [x_1, x_3]],$ $\boxed{x_{17}} = [x_4, [x_2, x_3]], x_{18} = [x_5, [x_2, x_4]],$ $x_{19} = [x_5, [x_3, x_4]], \boxed{x_{20}} = [x_6, [x_4, x_5]],$ $x_{21} = [x_7, [x_5, x_6]],$	$\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4}, \alpha_2 + \alpha_4 + \alpha_5,$ $\alpha_3 + \alpha_4 + \alpha_5, \underline{\alpha_4 + \alpha_5 + \alpha_6},$ $\alpha_5 + \alpha_6 + \alpha_7,$
$\boxed{x_{22}} = [[x_1, x_3], [x_2, x_4]],$ $x_{23} = [[x_1, x_3], [x_4, x_5]],$ $\boxed{x_{24}} = [[x_2, x_3], [x_4, x_5]],$ $x_{25} = [[x_2, x_4], [x_5, x_6]],$ $x_{26} = [[x_3, x_4], [x_5, x_6]],$ $\boxed{x_{27}} = [[x_4, x_5], [x_6, x_7]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4},$ $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6,$ $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $\underline{\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$
$x_{28} = [[x_3, x_4], [x_2, [x_1, x_3]]],$ $\boxed{x_{29}} = [[x_4, x_5], [x_2, [x_1, x_3]]],$ $x_{30} = [[x_4, x_5], [x_4, [x_2, x_3]]],$ $x_{31} = [[x_5, x_6], [x_4, [x_1, x_3]]],$ $\boxed{x_{32}} = [[x_5, x_6], [x_4, [x_2, x_3]]],$ $x_{33} = [[x_6, x_7], [x_5, [x_2, x_4]]],$ $x_{34} = [[x_6, x_7], [x_5, [x_3, x_4]]],$	$2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4,$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5,$ $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$ $\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$ $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$

$x_{35} = [[x_2, [x_1, x_3]], [x_5, [x_3, x_4]]],$ $\boxed{x_{36}} = [[x_2, [x_1, x_3]], [x_6, [x_4, x_5]]],$ $x_{37} = [[x_4, [x_1, x_3]], [x_5, [x_2, x_4]]],$ $x_{38} = [[x_4, [x_1, x_3]], [x_7, [x_5, x_6]]],$ $x_{39} = [[x_4, [x_2, x_3]], [x_6, [x_4, x_5]]],$ $\boxed{x_{40}} = [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]],$	$2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5,$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$ $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5,$ $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$ $2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$
$x_{41} = [[x_2, [x_1, x_3]], [[x_3, x_4], [x_5, x_6]]],$ $x_{42} = [[x_4, [x_2, x_3]], [[x_4, x_5], [x_6, x_7]]],$ $\boxed{x_{43}} = [[x_5, [x_3, x_4]], [[x_1, x_3], [x_2, x_4]]],$ $x_{44} = [[x_6, [x_4, x_5]], [[x_1, x_3], [x_2, x_4]]],$ $x_{45} = [[x_6, [x_4, x_5]], [[x_2, x_3], [x_4, x_5]]],$ $\boxed{x_{46}} = [[x_7, [x_5, x_6]], [[x_1, x_3], [x_2, x_4]]],$	$2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6,$ $2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7,$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5},$ $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6,$ $2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6,$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$ $\underline{+\alpha_6 + \alpha_7},$
$x_{47} = [[x_7, [x_5, x_6]], [[x_3, x_4], [x_2, [x_1, x_3]]]],$ $x_{48} = [[[x_1, x_3], [x_2, x_4]], [[x_2, x_3], [x_4, x_5]]],$ $\boxed{x_{49}} = [[[x_1, x_3], [x_2, x_4]], [[x_3, x_4], [x_5, x_6]]],$ $x_{50} = [[[x_1, x_3], [x_2, x_4]], [[x_4, x_5], [x_6, x_7]]],$ $x_{51} = [[[x_1, x_3], [x_4, x_5]], [[x_2, x_4], [x_5, x_6]]],$ $x_{52} = [[[x_2, x_3], [x_4, x_5]], [[x_4, x_5], [x_6, x_7]]],$	$2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$ $2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5,$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6},$ $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7,$ $2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6,$ $2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7,$
$x_{53} = [[[x_2, x_4], [x_5, x_6]], [[x_3, x_4], [x_2, [x_1, x_3]]]],$ $\boxed{x_{54}} = [[[x_3, x_4], [x_5, x_6]], [[x_4, x_5], [x_2, [x_1, x_3]]]],$ $\boxed{x_{55}} = [[[x_4, x_5], [x_6, x_7]], [[x_3, x_4], [x_2, [x_1, x_3]]]],$ $x_{56} = [[[x_4, x_5], [x_6, x_7]], [[x_4, x_5], [x_2, [x_1, x_3]]]],$ $x_{57} = [[[x_4, x_5], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$	$2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_6,$ $\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6},$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2}$ $\underline{+\alpha_5 + \alpha_6 + \alpha_7},$ $2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7,$ $2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_2 + \alpha_3 + \alpha_7,$

$x_{58} = [[[x_3, x_4], [x_2, [x_1, x_3]]], [[x_6, x_7], [x_5, [x_2, x_4]]]]$, $x_{59} = [[[x_4, x_5], [x_2, [x_1, x_3]]], [[x_5, x_6], [x_4, [x_2, x_3]]]]$, $\boxed{x_{60}} = [[[x_4, x_5], [x_2, [x_1, x_3]]], [[x_6, x_7], [x_5, [x_3, x_4]]]]$, $x_{61} = [[[x_4, x_5], [x_4, [x_2, x_3]]], [[x_5, x_6], [x_4, [x_1, x_3]]]]$, $x_{62} = [[[x_5, x_6], [x_4, [x_1, x_3]]], [[x_6, x_7], [x_5, [x_2, x_4]]]]$,	$2\alpha_2+2\alpha_3+2\alpha_4+\alpha_1+\alpha_5+\alpha_6+\alpha_7$, $2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_1+\alpha_6$, $\underline{2\alpha_3+2\alpha_4+2\alpha_5+\alpha_1+\alpha_2+\alpha_6+\alpha_7}$, $2\alpha_3+2\alpha_5+3\alpha_4+\alpha_1+\alpha_2+\alpha_6$, $2\alpha_4+2\alpha_5+2\alpha_6+\alpha_1+\alpha_2+\alpha_3+\alpha_7$,
$\boxed{x_{63}} = [[[x_5, x_6], [x_4, [x_2, x_3]]], [[x_4, [x_1, x_3]], [x_5, [x_2, x_4]]]]$, $x_{64} = [[[x_6, x_7], [x_5, [x_2, x_4]]], [[x_2, [x_1, x_3]], [x_5, [x_3, x_4]]]]$, $\boxed{x_{65}} = [[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_2, [x_1, x_3]], [x_6, [x_4, x_5]]]]$, $x_{66} = [[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_4, [x_1, x_3]], [x_5, [x_2, x_4]]]]$,	$\underline{2\alpha_2+2\alpha_3+2\alpha_5+3\alpha_4+\alpha_1+\alpha_6}$, $2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_1+\alpha_6+\alpha_7$, $\underline{2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_1+\alpha_2+\alpha_7}$, $2\alpha_3+2\alpha_5+3\alpha_4+\alpha_1+\alpha_2+\alpha_6+\alpha_7$,

$x_{67} = [[[x_2, [x_1, x_3]], [x_5, [x_3, x_4]]], [[x_4, [x_2, x_3]], [x_6, [x_4, x_5]]]]$, $x_{68} = [[[x_2, [x_1, x_3]], [x_6, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]]$, $\boxed{x_{69}} = [[[x_4, [x_1, x_3]], [x_5, [x_2, x_4]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]]$, $x_{70} = [[[x_4, [x_1, x_3]], [x_7, [x_5, x_6]]], [[x_4, [x_2, x_3]], [x_6, [x_4, x_5]]]]$,	$2\alpha_2+2\alpha_5+3\alpha_3+3\alpha_4+\alpha_1+\alpha_6$, $2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_1+\alpha_7$, $\underline{2\alpha_2+2\alpha_3+2\alpha_5+3\alpha_4+\alpha_1+\alpha_6+\alpha_7}$, $2\alpha_3+2\alpha_5+2\alpha_6+3\alpha_4+\alpha_1+\alpha_2+\alpha_7$,
$x_{71} = [[[x_4, [x_1, x_3]], [x_5, [x_2, x_4]]], [[x_2, [x_1, x_3]], [[x_3, x_4], [x_5, x_6]]]]$, $x_{72} = [[[x_4, [x_1, x_3]], [x_7, [x_5, x_6]]], [[x_6, [x_4, x_5]], [[x_2, x_3], [x_4, x_5]]]]$, $x_{73} = [[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_5, [x_3, x_4]], [[x_1, x_3], [x_2, x_4]]]]$, $\boxed{x_{74}} = [[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_6, [x_4, x_5]], [[x_1, x_3], [x_2, x_4]]]]$,	$2\alpha_1+2\alpha_2+2\alpha_5+3\alpha_3+3\alpha_4+\alpha_6$, $2\alpha_3+2\alpha_6+3\alpha_4+3\alpha_5+\alpha_1+\alpha_2+\alpha_7$, $2\alpha_2+2\alpha_5+3\alpha_3+3\alpha_4+\alpha_1+\alpha_6+\alpha_7$, $\underline{2\alpha_2+2\alpha_3+2\alpha_5+2\alpha_6+3\alpha_4+\alpha_1+\alpha_7}$,
$x_{75} = [[[x_2, [x_1, x_3]], [[x_3, x_4], [x_5, x_6]]], [[x_4, [x_2, x_3]], [[x_4, x_5], [x_6, x_7]]]]$, $x_{76} = [[[x_5, [x_3, x_4]], [[x_1, x_3], [x_2, x_4]]], [[x_7, [x_5, x_6]], [[x_1, x_3], [x_2, x_4]]]]$, $\boxed{x_{77}} = [[[x_6, [x_4, x_5]], [[x_2, x_3], [x_4, x_5]]], [[x_7, [x_5, x_6]], [[x_1, x_3], [x_2, x_4]]]]$,	$2\alpha_2+2\alpha_5+2\alpha_6+3\alpha_3+3\alpha_4+\alpha_1+\alpha_7$, $2\alpha_1+2\alpha_2+2\alpha_5+3\alpha_3+3\alpha_4+\alpha_6+\alpha_7$, $\underline{2\alpha_2+2\alpha_3+2\alpha_6+3\alpha_4+3\alpha_5+\alpha_1+\alpha_7}$,
$x_{78} = [[[x_6, [x_4, x_5]], [[x_2, x_3], [x_4, x_5]]], [[x_7, [x_5, x_6]], [[x_3, x_4], [x_2, [x_1, x_3]]]]]$, $x_{79} = [[[x_6, [x_4, x_5]], [[x_2, x_3], [x_4, x_5]]], [[[x_1, x_3], [x_2, x_4]], [[x_4, x_5], [x_6, x_7]]]]$, $x_{80} = [[[x_7, [x_5, x_6]], [[x_1, x_3], [x_2, x_4]]], [[[x_1, x_3], [x_2, x_4]], [[x_3, x_4], [x_5, x_6]]]]$,	$2\alpha_2+2\alpha_6+3\alpha_3+3\alpha_4+3\alpha_5+\alpha_1+\alpha_7$, $2\alpha_2+2\alpha_3+2\alpha_6+3\alpha_5+4\alpha_4+\alpha_1+\alpha_7$, $2\alpha_1+2\alpha_2+2\alpha_5+2\alpha_6+3\alpha_3+3\alpha_4+\alpha_7$,
$x_{81} = [[[x_7, [x_5, x_6]], [[x_3, x_4], [x_2, [x_1, x_3]]]], [[[x_1, x_3], [x_4, x_5]], [[x_2, x_4], [x_5, x_6]]]]$, $\boxed{x_{82}} = [[[[x_1, x_3], [x_2, x_4]], [[x_3, x_4], [x_5, x_6]]], [[[x_2, x_3], [x_4, x_5]], [[x_4, x_5], [x_6, x_7]]]]$,	$2\alpha_1+2\alpha_2+2\alpha_6+3\alpha_3+3\alpha_4+3\alpha_5+\alpha_7$, $\underline{2\alpha_2+2\alpha_6+3\alpha_3+3\alpha_5+4\alpha_4+\alpha_1+\alpha_7}$,
$\boxed{x_{83}} = [[[[x_1, x_3], [x_4, x_5]], [[x_2, x_4], [x_5, x_6]]], [[[x_4, x_5], [x_6, x_7]], [[x_3, x_4], [x_2, [x_1, x_3]]]]]$, $x_{84} = [[[[x_2, x_3], [x_4, x_5]], [[x_4, x_5], [x_6, x_7]]], [[[x_2, x_4], [x_5, x_6]], [[x_3, x_4], [x_2, [x_1, x_3]]]]]$,	$\underline{2\alpha_1+2\alpha_2+2\alpha_6+3\alpha_3+3\alpha_5+4\alpha_4+\alpha_7}$, $2\alpha_6+3\alpha_2+3\alpha_3+3\alpha_5+4\alpha_4+\alpha_1+\alpha_7$,

$x_{85} = [[[[x_2, x_4], [x_5, x_6]], [[x_3, x_4], [x_2, [x_1, x_3]]]],$ $[[[x_4, x_5], [x_6, x_7]], [[x_4, x_5], [x_2, [x_1, x_3]]]]],$ $x_{86} = [[[[x_3, x_4], [x_5, x_6]], [[x_4, x_5], [x_2, [x_1, x_3]]]],$ $[[[x_4, x_5], [x_6, x_7]], [[x_3, x_4], [x_2, [x_1, x_3]]]]],$	$2\alpha_1+2\alpha_6+3\alpha_2+3\alpha_3+3\alpha_5+4\alpha_4+\alpha_7,$ $2\alpha_1+2\alpha_2+2\alpha_6+3\alpha_5+4\alpha_3+4\alpha_4+\alpha_7,$
$\boxed{x_{87}} = [[[[x_4, x_5], [x_6, x_7]], [[x_3, x_4], [x_2, [x_1, x_3]]]],$ $[[[x_4, x_5], [x_2, [x_1, x_3]]], [[x_5, x_6],$ $[x_4, [x_2, x_3]]]]],$	$\underline{2\alpha_1 + 2\alpha_6 + 3\alpha_2 + 3\alpha_5 + 4\alpha_3}$ $\underline{+4\alpha_4 + \alpha_7},$
$x_{88} = [[[[x_3, x_4], [x_2, [x_1, x_3]]], [[x_6, x_7],$ $[x_5, [x_2, x_4]]]], [[[x_4, x_5], [x_4, [x_2, x_3]]],$ $[[x_5, x_6], [x_4, [x_1, x_3]]]]],$	$2\alpha_1+2\alpha_6+3\alpha_2+3\alpha_5+4\alpha_3+5\alpha_4+\alpha_7,$
$x_{89} = [[[[x_4, x_5], [x_4, [x_2, x_3]]], [[x_5, x_6],$ $[x_4, [x_1, x_3]]]], [[[x_6, x_7], [x_5, [x_2, x_4]]],$ $[[x_2, [x_1, x_3]], [x_5, [x_3, x_4]]]]],$	$2\alpha_1+2\alpha_6+3\alpha_2+4\alpha_3+4\alpha_5+5\alpha_4+\alpha_7,$
$x_{90} = [[[[x_5, x_6], [x_4, [x_2, x_3]]], [[x_4, [x_1, x_3]],$ $[x_5, [x_2, x_4]]]], [[[x_6, x_7], [x_5, [x_3, x_4]]],$ $[[x_2, [x_1, x_3]], [x_6, [x_4, x_5]]]]],$	$2\alpha_1+3\alpha_2+3\alpha_6+4\alpha_3+4\alpha_5+5\alpha_4+\alpha_7,$
$x_{91} = [[[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_4, [x_1, x_3]],$ $[x_5, [x_2, x_4]]]], [[[x_2, [x_1, x_3]], [x_6, [x_4, x_5]]],$ $[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]]]$	$2\alpha_1+2\alpha_7+3\alpha_2+3\alpha_6+4\alpha_3+4\alpha_5+5\alpha_4$

4.16 $p = 2$

4.16.1 Notation $\mathfrak{A} \oplus_c \mathfrak{B}$ needed to describe $\mathfrak{bgl}(4; \alpha)$, $\mathfrak{e}(6, 6)$, $\mathfrak{e}(7, 6)$, and $\mathfrak{e}(8, 1)$

This notation describes the case where $\mathfrak{A}$ and $\mathfrak{B}$ are nontrivial central extensions of the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$, respectively, and $\mathfrak{A} \oplus_c \mathfrak{B}$—a nontrivial central extension of $\mathfrak{a} \oplus \mathfrak{b}$ (or, perhaps, a more complicated $\mathfrak{a} \subplus \mathfrak{b}$) with 1-dimensional center spanned by c—is such that the restriction of the extension of $\mathfrak{a} \oplus \mathfrak{b}$ to $\mathfrak{a}$ gives $\mathfrak{A}$ and that to $\mathfrak{b}$ gives $\mathfrak{B}$.

In these four cases, $\mathfrak{g}(A)_{\bar{0}}$ is of the form

$$\mathfrak{g}(B) \oplus_c \mathfrak{hei}(2) \simeq \mathfrak{g}(B) \oplus \mathrm{Span}(X^+, X^-),$$

where the matrix B is not invertible (so $\mathfrak{g}(B)$ has a grading element d and a central element c), and where X^+, X^- and c span the Heisenberg Lie algebra $\mathfrak{hei}(2)$. The brackets are:

$$\begin{aligned} &[\mathfrak{g}^{(1)}(B), X^\pm] = 0; \\ &[d, X^\pm] = X^\pm; \qquad ([d, X^\pm] = \alpha X^\pm \text{ for } \mathfrak{bgl}(4; \alpha)) \\ &[X^+, X^-] = c. \end{aligned} \tag{22}$$

The odd part of $\mathfrak{g}(A)$ (at least in two of the four cases) consists of two copies of the same $\mathfrak{g}(B)$-module N, the operators $\mathrm{ad}_{X^\pm}$ permute these copies, and $\mathrm{ad}^2_{X^\pm} = 0$, so each of the operators maps one of the copies to the other, and this other copy to zero.

4.16.2 $\mathfrak{bgl}(3; \alpha)$, where $\alpha \neq 0, 1$; sdim $= 10/8|8$

The roots of $\mathfrak{g} = \mathfrak{bgl}(3; \alpha)$ are the same as those of $\mathfrak{osp}(4|2; \alpha)$ (or, more correctly, of $\mathfrak{wk}(3; \alpha)$), with the same division into even and odd ones; $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{gl}(3) \oplus \mathbb{K}Z$ and the

$\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is the sum of two irreducibles whose highest weight vectors are x_7 and y_1, where the roots corresponding to x_i and y_i are opposite.

We consider the Cartan matrix and the Chevalley basis elements

$$\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

the root vectors	the roots
$\boxed{x_1}$, $\boxed{x_2}$, $\boxed{x_3}$,	$\underline{\alpha_1}$, $\underline{\alpha_2}$, $\underline{\alpha_3}$
$x_4 = [x_1, x_2]$, $x_5 = [x_2, x_3]$,	$\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$,
$\boxed{x_6} = [x_3, [x_1, x_2]]$	$\underline{\alpha_1 + \alpha_2 + \alpha_3}$
$x_7 = [[x_1, x_2], [x_2, x_3]]$	$\alpha_1 + 2\alpha_2 + \alpha_3$

4.16.3 $\mathfrak{bgl}(4; \alpha)$, where $\alpha \neq 0, 1$; sdim $= 18|16$

The roots of $\mathfrak{bgl}(4; \alpha)$ are the same as those of $\mathfrak{wk}(4; \alpha)$, but divided into even and odd ones: $\mathfrak{bgl}(4; \alpha)_{\bar{0}} = \mathfrak{gl}(4) \oplus_c \mathfrak{hei}(2)$ and $\mathfrak{bgl}(4; \alpha)_{\bar{1}} = N \boxtimes \mathrm{id}$, where N is an 8-dimensional $\mathfrak{gl}(4)$ module, and id is the irreducible 2-dimensional $\mathfrak{hei}(2)$-module. We consider the Cartan matrix and the Chevalley basis elements

$$\begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

root vectors	roots
$\boxed{x_1}$, $\boxed{x_2}$ $\boxed{x_3}$, $\boxed{x_4}$,	$\underline{\alpha_1}$, $\underline{\alpha_2}$, $\underline{\alpha_3}$, $\underline{\alpha_4}$
$x_5 = [x_1, x_2], x_6 = [x_1, x_3]$,	$\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_3$,
$x_7 = [x_3, x_4]$,	$\alpha_3 + \alpha_4$
$\boxed{x_8} = [x_3, [x_1, x_2]]$,	$\underline{\alpha_1 + \alpha_2 + \alpha_3}$,
$\boxed{x_9} = [x_4, [x_1, x_3]]$,	$\underline{\alpha_1 + \alpha_3 + \alpha_4}$
$x_{10} = [[x_1, x_2], [x_1, x_3]]$,	$2\alpha_1 + \alpha_2 + \alpha_3$,
$x_{11} = [[x_1, x_2], [x_3, x_4]]$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$
$\boxed{x_{12}} = [[x_1, x_2], [x_4, [x_1, x_3]]]$,	$\underline{2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}$
$x_{13} = [[x_3, [x_1, x_2]], [x_4, [x_1, x_3]]]$,	$2\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$
$\boxed{x_{14}} = [[x_4, [x_1, x_3]], [[x_1, x_2], [x_1, x_3]]]$	$\underline{3\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4}$
$x_{15} = [[[x_1, x_2], [x_1, x_3]], [[x_1, x_2], [x_3, x_4]]]$	$3\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$

4.16.4 $\mathfrak{e}(6, 1)$ of sdim $= 46|32$

We have $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{oc}(2; 10) \oplus \mathbb{K}z$ and $\mathfrak{g}_{\bar{1}}$ is a reducible module of the form $R(\pi_4) \oplus R(\pi_5)$ with the two highest weight vectors

$$x_{36} = [[[x_4, x_5], [x_6, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_6, [x_3, x_4]]]]$$

and y_5. We consider the Cartan matrix of $\mathfrak{e}(6)$ with parities of simple roots 111100. The Chevalley basis elements are

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, x_5, x_6,$	$\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \alpha_5, \alpha_6,$
$x_7 = [x_1, x_2], x_8 = [x_2, x_3], x_9 = [x_3, x_4],$ $\boxed{x_{10}} = [x_3, x_6], \boxed{x_{11}} = [x_4, x_5],$	$\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4,$ $\underline{\alpha_3 + \alpha_6}, \underline{\alpha_4 + \alpha_5},$
$\boxed{x_{12}} = [x_3, [x_1, x_2]], \boxed{x_{13}} = [x_4, [x_2, x_3]],$ $x_{14} = [x_5, [x_3, x_4]], x_{15} = [x_6, [x_2, x_3]],$ $x_{16} = [x_6, [x_3, x_4]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3}, \underline{\alpha_2 + \alpha_3 + \alpha_4},$ $\alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_6,$ $\alpha_3 + \alpha_4 + \alpha_6,$
$\boxed{x_{17}} = [x_6, [x_4, [x_2, x_3]]],$ $x_{18} = [[x_1, x_2], [x_3, x_4]],$ $\boxed{x_{19}} = [[x_1, x_2], [x_3, x_6]],$ $\boxed{x_{20}} = [[x_2, x_3], [x_4, x_5]],$ $x_{21} = [[x_3, x_6], [x_4, x_5]],$	$\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6},$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6},$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$
$x_{22} = [[x_1, x_2], [x_6, [x_3, x_4]]],$ $x_{23} = [[x_3, x_6], [x_4, [x_2, x_3]]],$ $x_{24} = [[x_4, x_5], [x_3, [x_1, x_2]]],$ $\boxed{x_{25}} = [[x_4, x_5], [x_6, [x_2, x_3]]],$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6,$ $2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_6,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$
$x_{26} = [[x_4, x_5], [[x_1, x_2], [x_3, x_6]]],$ $\boxed{x_{27}} = [[x_3, [x_1, x_2]], [x_6, [x_3, x_4]]],$ $x_{28} = [[x_5, [x_3, x_4]], [x_6, [x_2, x_3]]],$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $\underline{2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_6},$ $2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6,$
$\boxed{x_{29}} = [[x_5, [x_3, x_4]], [[x_1, x_2], [x_3, x_6]]],$ $x_{30} = [[x_6, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]],$ $\boxed{x_{31}} = [[x_6, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]],$	$\underline{2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6},$ $2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_6,$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_6},$
$x_{32} = [[[x_1, x_2], [x_3, x_4]], [[x_3, x_6], [x_4, x_5]]],$ $x_{33} = [[[x_1, x_2], [x_3, x_6]], [[x_2, x_3], [x_4, x_5]]],$	$2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6,$ $2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_5 + \alpha_6,$
$\boxed{x_{34}} = [[[x_2, x_3], [x_4, x_5]], [[x_1, x_2], [x_6, [x_3, x_4]]]],$	$\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_6},$
$x_{35} = [[[x_3, x_6], [x_4, [x_2, x_3]]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$2\alpha_2 + 2\alpha_4 + 3\alpha_3 + \alpha_1 + \alpha_5 + \alpha_6,$
$x_{36} = [[[x_4, x_5], [x_6, [x_2, x_3]]], [[x_3, [x_1, x_2]],$ $[x_6, [x_3, x_4]]]]$	$2\alpha_2 + 2\alpha_4 + 2\alpha_6 + 3\alpha_3 + \alpha_1 + \alpha_5$

4.16.5 $\mathfrak{e}(6, 6)$ of sdim $= 38|40$

In this case, $\mathfrak{g}(B) \simeq \mathfrak{gl}(6)$, see (4.16.1). The $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is irreducible with the highest weight vector

$$x_{35} = [[[x_3, x_6], [x_4, [x_2, x_3]]], [[x_4, x_5], [x_3, [x_1, x_2]]]] \text{ of weight } (0, 0, 1, 0, 0, 1).$$

We consider the Cartan matrix of $\mathfrak{e}(6)$ with parities of simple roots 111111. The Chevalley basis elements are

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, \boxed{x_5}, \boxed{x_6},$	$\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \underline{\alpha_5}, \underline{\alpha_6},$
$x_7 = [x_1, x_2], x_8 = [x_2, x_3], x_9 = [x_3, x_4], x_{10} = [x_3, x_6],$ $x_{11} = [x_4, x_5],$	$\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4,$ $\alpha_3 + \alpha_6, \alpha_4 + \alpha_5,$
$\boxed{x_{12}} = [x_3, [x_1, x_2]], \boxed{x_{13}} = [x_4, [x_2, x_3]],$ $\boxed{x_{14}} = [x_5, [x_3, x_4]], \boxed{x_{15}} = [x_6, [x_2, x_3]],$ $\boxed{x_{16}} = [x_6, [x_3, x_4]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3}, \underline{\alpha_2 + \alpha_3 + \alpha_4},$ $\underline{\alpha_3 + \alpha_4 + \alpha_5}, \underline{\alpha_2 + \alpha_3 + \alpha_6},$ $\underline{\alpha_3 + \alpha_4 + \alpha_6},$
$x_{17} = [x_6, [x_4, [x_2, x_3]]], x_{18} = [[x_1, x_2], [x_3, x_4]],$ $x_{19} = [[x_1, x_2], [x_3, x_6]], x_{20} = [[x_2, x_3], [x_4, x_5]],$ $x_{21} = [[x_3, x_6], [x_4, x_5]],$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6,$ $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$ $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$
$\boxed{x_{22}} = [[x_1, x_2], [x_6, [x_3, x_4]]],$ $\boxed{x_{23}} = [[x_3, x_6], [x_4, [x_2, x_3]]],$ $\boxed{x_{24}} = [[x_4, x_5], [x_3, [x_1, x_2]]],$ $\boxed{x_{25}} = [[x_4, x_5], [x_6, [x_2, x_3]]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6},$ $\underline{2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_6},$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$
$x_{26} = [[x_4, x_5], [[x_1, x_2], [x_3, x_6]]],$ $x_{27} = [[x_3, [x_1, x_2]], [x_6, [x_3, x_4]]],$ $x_{28} = [[x_5, [x_3, x_4]], [x_6, [x_2, x_3]]],$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_6,$ $2\alpha_3 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6,$
$\boxed{x_{29}} = [[x_5, [x_3, x_4]], [[x_1, x_2], [x_3, x_6]]],$ $\boxed{x_{30}} = [[x_6, [x_2, x_3]], [[x_1, x_2], [x_3, x_4]]],$ $\boxed{x_{31}} = [[x_6, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]],$	$\underline{2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6},$ $\underline{2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_6},$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_6},$
$x_{32} = [[[x_1, x_2], [x_3, x_4]], [[x_3, x_6], [x_4, x_5]]],$ $x_{33} = [[[x_1, x_2], [x_3, x_6]], [[x_2, x_3], [x_4, x_5]]],$	$2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6,$ $2\alpha_2 + 2\alpha_3 + \alpha_1 + \alpha_4 + \alpha_5 + \alpha_6,$
$\boxed{x_{34}} = [[[x_2, x_3], [x_4, x_5]], [[x_1, x_2], [x_6, [x_3, x_4]]]],$	$\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_6},$
$x_{35} = [[[x_3, x_6], [x_4, [x_2, x_3]]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$2\alpha_2 + 2\alpha_4 + 3\alpha_3 + \alpha_1 + \alpha_5 + \alpha_6,$
$\boxed{x_{36}} = [[[x_4, x_5], [x_6, [x_2, x_3]]], [[x_3, [x_1, x_2]],$ $[x_6, [x_3, x_4]]]]$	$\underline{2\alpha_2 + 2\alpha_4 + 2\alpha_6 + 3\alpha_3 + \alpha_1 + \alpha_5}$

4.16.6 $\mathfrak{e}(7, 1)$ of sdim $= 80/78|54$

We consider the Cartan matrix of $\mathfrak{e}(7)$ with the parities of simple roots 1111001. The Chevalley basis elements are

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, x_5, x_6, \boxed{x_7},$	$\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \alpha_5, \alpha_6, \underline{\alpha_7},$
$x_8 = [x_1, x_2], x_9 = [x_2, x_3], x_{10} = [x_3, x_4],$ $\boxed{x_{11}} = [x_4, x_5], x_{12} = [x_4, x_7], x_{13} = [x_5, x_6],$	$\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \underline{\alpha_4 + \alpha_5},$ $\alpha_4 + \alpha_7, \alpha_5 + \alpha_6,$
$\boxed{x_{14}} = [x_3, [x_1, x_2]], \boxed{x_{15}} = [x_4, [x_2, x_3]],$ $x_{16} = [x_5, [x_3, x_4]], \boxed{x_{17}} = [x_6, [x_4, x_5]],$ $\boxed{x_{18}} = [x_7, [x_3, x_4]], x_{19} = [x_7, [x_4, x_5]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3}, \underline{\alpha_2 + \alpha_3 + \alpha_4},$ $\alpha_3 + \alpha_4 + \alpha_5, \underline{\alpha_4 + \alpha_5 + \alpha_6},$ $\underline{\alpha_3 + \alpha_4 + \alpha_7}, \alpha_4 + \alpha_5 + \alpha_7,$
$\boxed{x_{20}} = [x_7, [x_5, [x_3, x_4]]], x_{21} = [[x_1, x_2], [x_3, x_4]],$ $\boxed{x_{22}} = [[x_2, x_3], [x_4, x_5]], x_{23} = [[x_2, x_3], [x_4, x_7]],$ $x_{24} = [[x_3, x_4], [x_5, x_6]], x_{25} = [[x_4, x_7], [x_5, x_6]],$	$\underline{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_7},$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_7,$ $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$
$x_{26} = [[x_2, x_3], [x_7, [x_4, x_5]]],$ $x_{27} = [[x_4, x_5], [x_3, [x_1, x_2]]],$ $\boxed{x_{28}} = [[x_4, x_7], [x_3, [x_1, x_2]]],$ $x_{29} = [[x_4, x_7], [x_5, [x_3, x_4]]],$ $\boxed{x_{30}} = [[x_5, x_6], [x_4, [x_2, x_3]]],$ $\boxed{x_{31}} = [[x_5, x_6], [x_7, [x_3, x_4]]],$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7},$ $2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_7,$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$ $\underline{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$
$x_{32} = [[x_5, x_6], [[x_2, x_3], [x_4, x_7]]],$ $x_{33} = [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$ $\boxed{x_{34}} = [[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]],$ $\boxed{x_{35}} = [[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]],$ $x_{36} = [[x_6, [x_4, x_5]], [x_7, [x_3, x_4]]],$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7},$ $\underline{2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_7},$ $2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7,$
$\boxed{x_{37}} = [[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]],$ $\boxed{x_{38}} = [[x_6, [x_4, x_5]], [[x_2, x_3], [x_4, x_7]]],$ $x_{39} = [[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]],$ $x_{40} = [[x_7, [x_4, x_5]], [[x_1, x_2], [x_3, x_4]]],$ $x_{41} = [[x_7, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7},$ $2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_7,$ $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_7,$ $2\alpha_4 + 2\alpha_5 + \alpha_3 + \alpha_6 + \alpha_7,$
$\boxed{x_{42}} = [[x_7, [x_5, [x_3, x_4]]], [[x_1, x_2], [x_3, x_4]]],$ $x_{43} = [[[x_1, x_2], [x_3, x_4]], [[x_4, x_7], [x_5, x_6]]],$ $\boxed{x_{44}} = [[[x_2, x_3], [x_4, x_5]], [[x_4, x_7], [x_5, x_6]]],$ $x_{45} = [[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]],$	$\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_7},$ $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7,$ $\underline{2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7},$
$x_{46} = [[[x_2, x_3], [x_4, x_7]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$ $x_{47} = [[[x_3, x_4], [x_5, x_6]], [[x_2, x_3], [x_7, [x_4, x_5]]]],$ $\boxed{x_{48}} = [[[x_3, x_4], [x_5, x_6]], [[x_4, x_7], [x_3, [x_1, x_2]]]],$ $x_{49} = [[[x_4, x_7], [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_7,$ $2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6 + \alpha_7,$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7},$ $2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7,$
$\boxed{x_{50}} =$ $[[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_5, x_6], [x_7, [x_3, x_4]]]],$ $x_{51} = [[[x_4, x_7], [x_3, [x_1, x_2]]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$ $\boxed{x_{52}} =$ $[[[x_4, x_7], [x_5, [x_3, x_4]]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$	$\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_5 + 3\alpha_4 + \alpha_2 + \alpha_6 + \alpha_7},$

$x_{53} = [[[x_4, x_7], [x_5, [x_3, x_4]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $x_{54} = [[[x_5, x_6], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]]],$ $x_{55} = [[[x_5, x_6], [x_7, [x_3, x_4]]], [[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]]],$	$2\alpha_3+2\alpha_5+3\alpha_4+\alpha_1+\alpha_2+\alpha_6+\alpha_7,$ $2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_1+\alpha_6+\alpha_7,$ $2\alpha_3+2\alpha_5+2\alpha_7+3\alpha_4+\alpha_2+\alpha_6,$
$\boxed{x_{56}} = [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]]],$ $\boxed{x_{57}} = [[[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]], [[x_6, [x_4, x_5]], [x_7, [x_3, x_4]]]],$	$\underline{2\alpha_2+2\alpha_3+2\alpha_5+3\alpha_4+\alpha_1+\alpha_6+\alpha_7},$ $\underline{2\alpha_3+2\alpha_5+2\alpha_7+3\alpha_4+\alpha_1+\alpha_2+\alpha_6},$
$x_{58} = [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]],$ $x_{59} = [[[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]], [[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]]],$	$2\alpha_2+2\alpha_5+3\alpha_3+3\alpha_4+\alpha_1+\alpha_6+\alpha_7,$ $2\alpha_2+2\alpha_3+2\alpha_5+2\alpha_7+3\alpha_4+\alpha_1+\alpha_6,$
$\boxed{x_{60}} = [[[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]], [[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]],$	$\underline{2\alpha_2+2\alpha_5+2\alpha_7+3\alpha_3+3\alpha_4+\alpha_1+\alpha_6},$
$x_{61} = [[[x_7, [x_4, x_5]], [[x_1, x_2], [x_3, x_4]]], [[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]]],$	$2\alpha_2+2\alpha_5+2\alpha_7+3\alpha_3+4\alpha_4+\alpha_1+\alpha_6,$
$x_{62} = [[[x_7, [x_5, [x_3, x_4]]], [[x_1, x_2], [x_3, x_4]]], [[[x_2, x_3], [x_4, x_5]], [[x_4, x_7], [x_5, x_6]]]],$	$2\alpha_2+2\alpha_7+3\alpha_3+3\alpha_5+4\alpha_4+\alpha_1+\alpha_6,$
$x_{63} = [[[[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]], [[[x_4, x_7], [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]]]]$	$2\alpha_2+2\alpha_6+2\alpha_7+3\alpha_3+3\alpha_5+4\alpha_4+\alpha_1$

4.16.7 $\mathfrak{e}(7, 6)$ of sdim $= 70/68|64$

We consider the Cartan matrix of $\mathfrak{e}(7)$ with the parities of simple roots 0101010 and the Chevalley basis elements

the root vectors	the roots
$x_1, \boxed{x_2}, x_3, \boxed{x_4}, x_5, \boxed{x_6}, x_7,$	$\alpha_1, \underline{\alpha_2}, \alpha_3, \underline{\alpha_4}, \alpha_5, \underline{\alpha_6}, \alpha_7,$
$\boxed{x_8} = [x_1, x_2], \boxed{x_9} = [x_2, x_3], \boxed{x_{10}} = [x_3, x_4],$ $\boxed{x_{11}} = [x_4, x_5], \boxed{x_{12}} = [x_4, x_7], \boxed{x_{13}} = [x_5, x_6],$	$\underline{\alpha_1+\alpha_2}, \underline{\alpha_2+\alpha_3}, \underline{\alpha_3+\alpha_4},$ $\underline{\alpha_4+\alpha_5}, \underline{\alpha_4+\alpha_7}, \underline{\alpha_5+\alpha_6},$
$\boxed{x_{14}} = [x_3, [x_1, x_2]], x_{15} = [x_4, [x_2, x_3]],$ $\boxed{x_{16}} = [x_5, [x_3, x_4]], x_{17} = [x_6, [x_4, x_5]],$ $\boxed{x_{18}} = [x_7, [x_3, x_4]], \boxed{x_{19}} = [x_7, [x_4, x_5]],$	$\underline{\alpha_1+\alpha_2+\alpha_3}, \alpha_2+\alpha_3+\alpha_4,$ $\underline{\alpha_3+\alpha_4+\alpha_5}, \alpha_4+\alpha_5+\alpha_6,$ $\underline{\alpha_3+\alpha_4+\alpha_7}, \underline{\alpha_4+\alpha_5+\alpha_7},$
$\boxed{x_{20}} = [x_7, [x_5, [x_3, x_4]]], x_{21} = [[x_1, x_2], [x_3, x_4]],$ $x_{22} = [[x_2, x_3], [x_4, x_5]], x_{23} = [[x_2, x_3], [x_4, x_7]],$ $x_{24} = [[x_3, x_4], [x_5, x_6]], x_{25} = [[x_4, x_7], [x_5, x_6]],$	$\underline{\alpha_3+\alpha_4+\alpha_5+\alpha_7}, \alpha_1+\alpha_2+\alpha_3+\alpha_4,$ $\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_2+\alpha_3+\alpha_4+\alpha_7,$ $\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6+\alpha_7,$
$x_{26} = [[x_2, x_3], [x_7, [x_4, x_5]]],$ $x_{27} = [[x_4, x_5], [x_3, [x_1, x_2]]],$ $x_{28} = [[x_4, x_7], [x_3, [x_1, x_2]]],$ $x_{29} = [[x_4, x_7], [x_5, [x_3, x_4]]],$ $\boxed{x_{30}} = [[x_5, x_6], [x_4, [x_2, x_3]]],$ $x_{31} = [[x_5, x_6], [x_7, [x_3, x_4]]],$	$\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_7,$ $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,$ $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_7,$ $2\alpha_4+\alpha_3+\alpha_5+\alpha_7,$ $\underline{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6},$ $\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7,$

$\boxed{x_{32}} = [[x_5, x_6], [[x_2, x_3], [x_4, x_7]]],$ $\boxed{x_{33}} = [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$ $x_{34} = [[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]],$ $\boxed{x_{35}} = [[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]],$ $\boxed{x_{36}} = [[x_6, [x_4, x_5]], [x_7, [x_3, x_4]]],$	$\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7,$ $\underline{2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_7},$ $\underline{2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7},$
$\boxed{x_{37}} = [[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]],$ $x_{38} = [[x_6, [x_4, x_5]], [[x_2, x_3], [x_4, x_7]]],$ $\boxed{x_{39}} = [[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]],$ $\boxed{x_{40}} = [[x_7, [x_4, x_5]], [[x_1, x_2], [x_3, x_4]]],$ $\boxed{x_{41}} = [[x_7, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$ $2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7,$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_7},$ $\underline{2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_7},$ $\underline{2\alpha_4 + 2\alpha_5 + \alpha_3 + \alpha_6 + \alpha_7},$
$\boxed{x_{42}} = [[x_7, [x_5, [x_3, x_4]]], [[x_1, x_2], [x_3, x_4]]],$ $x_{43} = [[[x_1, x_2], [x_3, x_4]], [[x_4, x_7], [x_5, x_6]]],$ $x_{44} = [[[x_2, x_3], [x_4, x_5]], [[x_4, x_7], [x_5, x_6]]],$ $x_{45} = [[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]],$	$\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_7},$ $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7,$ $2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7,$ $2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7,$
$x_{46} = [[[x_2, x_3], [x_4, x_7]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$ $x_{47} = [[[x_3, x_4], [x_5, x_6]], [[x_2, x_3], [x_7, [x_4, x_5]]]],$ $x_{48} = [[[x_3, x_4], [x_5, x_6]], [[x_4, x_7], [x_3, [x_1, x_2]]]],$ $x_{49} = [[[x_4, x_7], [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_7,$ $2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6 + \alpha_7,$ $2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7,$ $2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7,$
$x_{50} =$ $[[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_5, x_6], [x_7, [x_3, x_4]]]],$ $\boxed{x_{51}} =$ $[[[x_4, x_7], [x_3, [x_1, x_2]]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$ $\boxed{x_{52}} =$ $[[[x_4, x_7], [x_5, [x_3, x_4]]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$	$2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6 + \alpha_7,$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_5 + 3\alpha_4 + \alpha_2 + \alpha_6 + \alpha_7},$
$\boxed{x_{53}} = [[[x_4, x_7], [x_5, [x_3, x_4]]],$ $[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $\boxed{x_{54}} = [[[x_5, x_6], [x_4, [x_2, x_3]]],$ $[[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]]],$ $\boxed{x_{55}} = [[[x_5, x_6], [x_7, [x_3, x_4]]],$ $[[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]]],$	$\underline{2\alpha_3 + 2\alpha_5 + 3\alpha_4 + \alpha_1 + \alpha_2 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_5 + 2\alpha_7 + 3\alpha_4 + \alpha_2 + \alpha_6},$
$x_{56} = [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$ $[[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]]],$ $\boxed{x_{57}} = [[[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]],$ $[[x_6, [x_4, x_5]], [x_7, [x_3, x_4]]]],$	$2\alpha_2 + 2\alpha_3 + 2\alpha_5 + 3\alpha_4 + \alpha_1 + \alpha_6 + \alpha_7,$ $\underline{2\alpha_3 + 2\alpha_5 + 2\alpha_7 + 3\alpha_4 + \alpha_1 + \alpha_2 + \alpha_6},$
$x_{58} = [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$ $[[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]],$ $x_{59} = [[[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]],$ $[[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]]],$	$2\alpha_2 + 2\alpha_5 + 3\alpha_3 + 3\alpha_4 + \alpha_1 + \alpha_6 + \alpha_7,$ $2\alpha_2 + 2\alpha_3 + 2\alpha_5 + 2\alpha_7 + 3\alpha_4 + \alpha_1 + \alpha_6,$
$x_{60} = [[[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]],$ $[[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]],$	$2\alpha_2 + 2\alpha_5 + 2\alpha_7 + 3\alpha_3 + 3\alpha_4 + \alpha_1 + \alpha_6,$

$\boxed{x_{61}} = [[[x_7, [x_4, x_5]], [[x_1, x_2], [x_3, x_4]]],$ $[[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]]],$	$\underline{2\alpha_2 + 2\alpha_5 + 2\alpha_7 + 3\alpha_3 + 4\alpha_4 + \alpha_1 + \alpha_6},$
$\boxed{x_{62}} = [[[x_7, [x_5, [x_3, x_4]]], [[x_1, x_2], [x_3, x_4]]],$ $[[[x_2, x_3], [x_4, x_5]], [[x_4, x_7], [x_5, x_6]]]],$	$\underline{2\alpha_2 + 2\alpha_7 + 3\alpha_3 + 3\alpha_5 + 4\alpha_4 + \alpha_1 + \alpha_6},$
$x_{63} = [[[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]],$ $[[[x_4, x_7], [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]]]$	$2\alpha_2 + 2\alpha_6 + 2\alpha_7 + 3\alpha_3 + 3\alpha_5 + 4\alpha_4 + \alpha_1$

4.16.8 $\mathfrak{e}(7, 7)$ of sdim $= 64/62|70$

We consider the Cartan matrix of $\mathfrak{e}(7)$ with the parities of simple roots 1111111 and the Chevalley basis elements

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, \boxed{x_5}, \boxed{x_6}, \boxed{x_7},$	$\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \underline{\alpha_5}, \underline{\alpha_6}, \underline{\alpha_7},$
$x_8 = [x_1, x_2], x_9 = [x_2, x_3], x_{10} = [x_3, x_4],$ $x_{11} = [x_4, x_5], x_{12} = [x_4, x_7], x_{13} = [x_5, x_6],$	$\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5,$ $\alpha_4 + \alpha_7, \alpha_5 + \alpha_6,$
$\boxed{x_{14}} = [x_3, [x_1, x_2]], \boxed{x_{15}} = [x_4, [x_2, x_3]],$ $\boxed{x_{16}} = [x_5, [x_3, x_4]], \boxed{x_{17}} = [x_6, [x_4, x_5]],$ $\boxed{x_{18}} = [x_7, [x_3, x_4]], \boxed{x_{19}} = [x_7, [x_4, x_5]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3}, \underline{\alpha_2 + \alpha_3 + \alpha_4},$ $\underline{\alpha_3 + \alpha_4 + \alpha_5}, \underline{\alpha_4 + \alpha_5 + \alpha_6},$ $\underline{\alpha_3 + \alpha_4 + \alpha_7}, \underline{\alpha_4 + \alpha_5 + \alpha_7},$
$x_{20} = [x_7, [x_5, [x_3, x_4]]], x_{21} = [[x_1, x_2], [x_3, x_4]],$ $x_{22} = [[x_2, x_3], [x_4, x_5]], x_{23} = [[x_2, x_3], [x_4, x_7]],$ $x_{24} = [[x_3, x_4], [x_5, x_6]], x_{25} = [[x_4, x_7], [x_5, x_6]],$	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$ $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7,$ $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$
$\boxed{x_{26}} = [[x_2, x_3], [x_7, [x_4, x_5]]],$ $\boxed{x_{27}} = [[x_4, x_5], [x_3, [x_1, x_2]]],$ $\boxed{x_{28}} = [[x_4, x_7], [x_3, [x_1, x_2]]],$ $\boxed{x_{29}} = [[x_4, x_7], [x_5, [x_3, x_4]]],$ $\boxed{x_{30}} = [[x_5, x_6], [x_4, [x_2, x_3]]],$ $\boxed{x_{31}} = [[x_5, x_6], [x_7, [x_3, x_4]]],$	$\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7},$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$ $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7},$ $\underline{2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_7},$ $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$ $\underline{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$
$x_{32} = [[x_5, x_6], [[x_2, x_3], [x_4, x_7]]],$ $x_{33} = [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$ $x_{34} = [[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]],$ $x_{35} = [[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]],$ $x_{36} = [[x_6, [x_4, x_5]], [x_7, [x_3, x_4]]],$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$ $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7,$ $2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_7,$ $2\alpha_4 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7,$
$\boxed{x_{37}} = [[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]],$ $\boxed{x_{38}} = [[x_6, [x_4, x_5]], [[x_2, x_3], [x_4, x_7]]],$ $\boxed{x_{39}} = [[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]],$ $\boxed{x_{40}} = [[x_7, [x_4, x_5]], [[x_1, x_2], [x_3, x_4]]],$ $\boxed{x_{41}} = [[x_7, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_7},$ $\underline{2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_7},$ $\underline{2\alpha_4 + 2\alpha_5 + \alpha_3 + \alpha_6 + \alpha_7},$
$x_{42} = [[x_7, [x_5, [x_3, x_4]]], [[x_1, x_2], [x_3, x_4]]],$ $x_{43} = [[[x_1, x_2], [x_3, x_4]], [[x_4, x_7], [x_5, x_6]]],$ $x_{44} = [[[x_2, x_3], [x_4, x_5]], [[x_4, x_7], [x_5, x_6]]],$ $x_{45} = [[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]],$	$2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_7,$ $2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7,$ $2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7,$ $2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7,$
$\boxed{x_{46}} = [[[x_2, x_3], [x_4, x_7]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$ $\boxed{x_{47}} = [[[x_3, x_4], [x_5, x_6]], [[x_2, x_3], [x_7, [x_4, x_5]]]],$ $\boxed{x_{48}} = [[[x_3, x_4], [x_5, x_6]], [[x_4, x_7], [x_3, [x_1, x_2]]]],$ $\boxed{x_{49}} = [[[x_4, x_7], [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7},$

$x_{50} =$ $[[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_5, x_6], [x_7, [x_3, x_4]]]],$ $x_{51} =$ $[[[x_4, x_7], [x_3, [x_1, x_2]]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$ $x_{52} =$ $[[[x_4, x_7], [x_5, [x_3, x_4]]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$	$2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6 + \alpha_7,$ $2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_1 + \alpha_5 + \alpha_6 + \alpha_7,$ $2\alpha_3 + 2\alpha_5 + 3\alpha_4 + \alpha_2 + \alpha_6 + \alpha_7,$
$\boxed{x_{53}} = [[[x_4, x_7], [x_5, [x_3, x_4]]],$ $[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $\boxed{x_{54}} = [[[x_5, x_6], [x_4, [x_2, x_3]]],$ $[[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]]],$ $\boxed{x_{55}} = [[[x_5, x_6], [x_7, [x_3, x_4]]],$ $[[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]]],$	$\underline{2\alpha_3 + 2\alpha_5 + 3\alpha_4 + \alpha_1 + \alpha_2 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_5 + 2\alpha_7 + 3\alpha_4 + \alpha_2 + \alpha_6},$
$x_{56} = [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$ $[[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]]],$ $x_{57} = [[[x_3, [x_1, x_2]], [x_7, [x_4, x_5]]],$ $[[x_6, [x_4, x_5]], [x_7, [x_3, x_4]]]]$	$2\alpha_2 + 2\alpha_3 + 2\alpha_5 + 3\alpha_4 + \alpha_1 + \alpha_6 + \alpha_7,$ $2\alpha_3 + 2\alpha_5 + 2\alpha_7 + 3\alpha_4 + \alpha_1 + \alpha_2 + \alpha_6,$
$\boxed{x_{58}} = [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$ $[[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]],$ $\boxed{x_{59}} = [[[x_4, [x_2, x_3]], [x_7, [x_4, x_5]]],$ $[[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]]],$	$\underline{2\alpha_2 + 2\alpha_5 + 3\alpha_3 + 3\alpha_4 + \alpha_1 + \alpha_6 + \alpha_7},$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_5 + 2\alpha_7 + 3\alpha_4 + \alpha_1 + \alpha_6},$
$x_{60} = [[[x_3, [x_1, x_2]], [[x_4, x_7], [x_5, x_6]]],$ $[[x_7, [x_3, x_4]], [[x_2, x_3], [x_4, x_5]]]],$	$2\alpha_2 + 2\alpha_5 + 2\alpha_7 + 3\alpha_3 + 3\alpha_4 + \alpha_1 + \alpha_6,$
$\boxed{x_{61}} = [[[x_7, [x_4, x_5]], [[x_1, x_2], [x_3, x_4]]],$ $[[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]]],$	$\underline{2\alpha_2 + 2\alpha_5 + 2\alpha_7 + 3\alpha_3 + 4\alpha_4 + \alpha_1 + \alpha_6},$
$x_{62} = [[[x_7, [x_5, [x_3, x_4]]], [[x_1, x_2], [x_3, x_4]]],$ $[[[x_2, x_3], [x_4, x_5]], [[x_4, x_7], [x_5, x_6]]]],$	$2\alpha_2 + 2\alpha_7 + 3\alpha_3 + 3\alpha_5 + 4\alpha_4 + \alpha_1 + \alpha_6,$
$\boxed{x_{63}} = [[[[x_2, x_3], [x_4, x_7]], [[x_3, x_4], [x_5, x_6]]],$ $[[[x_4, x_7], [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]]]$	$\underline{2\alpha_2 + 2\alpha_6 + 2\alpha_7 + 3\alpha_3 + 3\alpha_5 + 4\alpha_4 + \alpha_1}$

4.16.9 $\mathfrak{e}(8, 1)$ of sdim $= 136|112$

We have (cf. Sect. 4.16.1) $\mathfrak{g}(B) \simeq \mathfrak{e}(7)$. We consider the Cartan Matrix of $\mathfrak{e}(8)$ with the parities of simple roots 11001111 and the Chevalley basis elements

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}, x_3, x_4, \boxed{x_5}, \boxed{x_6}, \boxed{x_7}, \boxed{x_8},$	$\underline{\alpha_1}, \underline{\alpha_2}, \alpha_3, \alpha_4, \underline{\alpha_5}, \underline{\alpha_6}, \underline{\alpha_7}, \underline{\alpha_8},$
$x_9 = [x_1, x_2], \boxed{x_{10}} = [x_2, x_3], x_{11} = [x_3, x_4],$ $\boxed{x_{12}} = [x_4, x_5], x_{13} = [x_5, x_6], x_{14} = [x_5, x_8],$ $x_{15} = [x_6, x_7],$	$\alpha_1 + \alpha_2, \underline{\alpha_2 + \alpha_3}, \alpha_3 + \alpha_4,$ $\underline{\alpha_4 + \alpha_5}, \alpha_5 + \alpha_6, \alpha_5 + \alpha_8,$ $\alpha_6 + \alpha_7,$
$x_{16} = [x_3, [x_1, x_2]], \boxed{x_{17}} = [x_4, [x_2, x_3]],$ $\boxed{x_{18}} = [x_5, [x_3, x_4]], x_{19} = [x_6, [x_4, x_5]],$ $\boxed{x_{20}} = [x_7, [x_5, x_6]], x_{21} = [x_8, [x_4, x_5]],$ $\boxed{x_{22}} = [x_8, [x_5, x_6]],$	$\alpha_1 + \alpha_2 + \alpha_3, \underline{\alpha_2 + \alpha_3 + \alpha_4},$ $\underline{\alpha_3 + \alpha_4 + \alpha_5}, \alpha_4 + \alpha_5 + \alpha_6,$ $\underline{\alpha_5 + \alpha_6 + \alpha_7}, \alpha_4 + \alpha_5 + \alpha_8,$ $\underline{\alpha_5 + \alpha_6 + \alpha_8},$

$\boxed{x_{23}} = [x_8, [x_6, [x_4, x_5]]]$, $x_{24} = [[x_1, x_2], [x_3, x_4]]$, $x_{25} = [[x_2, x_3], [x_4, x_5]]$, $x_{26} = [[x_3, x_4], [x_5, x_6]]$, $x_{27} = [[x_3, x_4], [x_5, x_8]]$, $\boxed{x_{28}} = [[x_4, x_5], [x_6, x_7]]$, $x_{29} = [[x_5, x_8], [x_6, x_7]]$,	$\underline{\alpha_4 + \alpha_5 + \alpha_6 + \alpha_8}, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_8, \underline{\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7}$, $\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$,
$\boxed{x_{30}} = [[x_3, x_4], [x_8, [x_5, x_6]]]$, $\boxed{x_{31}} = [[x_4, x_5], [x_3, [x_1, x_2]]]$, $\boxed{x_{32}} = [[x_5, x_6], [x_4, [x_2, x_3]]]$, $\boxed{x_{33}} = [[x_5, x_8], [x_4, [x_2, x_3]]]$, $x_{34} = [[x_5, x_8], [x_6, [x_4, x_5]]]$, $\boxed{x_{35}} = [[x_6, x_7], [x_5, [x_3, x_4]]]$, $x_{36} = [[x_6, x_7], [x_8, [x_4, x_5]]]$,	$\underline{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_8}$, $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}$, $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}$, $\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8}$, $2\alpha_5 + \alpha_4 + \alpha_6 + \alpha_8$, $\underline{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7}$, $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$,
$x_{37} = [[x_6, x_7], [[x_3, x_4], [x_5, x_8]]]$, $x_{38} = [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]$, $x_{39} = [[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]]$, $x_{40} = [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]$, $x_{41} = [[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]]$, $x_{42} = [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]$, $\boxed{x_{43}} = [[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]]$,	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8$, $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$, $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_8$, $2\alpha_5 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8$, $\underline{2\alpha_5 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8}$,
$\boxed{x_{44}} = [[x_4, [x_2, x_3]], [[x_5, x_8], [x_6, x_7]]]$, $\boxed{x_{45}} = [[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]$, $\boxed{x_{46}} = [[x_7, [x_5, x_6]], [[x_3, x_4], [x_5, x_8]]]$, $x_{47} = [[x_8, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]]$, $\boxed{x_{48}} = [[x_8, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]$, $\boxed{x_{49}} = [[x_8, [x_5, x_6]], [[x_2, x_3], [x_4, x_5]]]$, $x_{50} = [[x_8, [x_5, x_6]], [[x_4, x_5], [x_6, x_7]]]$,	$\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8}$, $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7}$, $\underline{2\alpha_5 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8}$, $2\alpha_4 + 2\alpha_5 + \alpha_3 + \alpha_6 + \alpha_8$, $\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_8}$, $\underline{2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8}$, $2\alpha_5 + 2\alpha_6 + \alpha_4 + \alpha_7 + \alpha_8$,
$x_{51} = [[x_8, [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]]$, $\boxed{x_{52}} = [[x_8, [x_6, [x_4, x_5]]], [[x_2, x_3], [x_4, x_5]]]$, $x_{53} = [[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, x_7]]]$, $x_{54} = [[[x_2, x_3], [x_4, x_5]], [[x_5, x_8], [x_6, x_7]]]$, $x_{55} = [[[x_3, x_4], [x_5, x_6]], [[x_5, x_8], [x_6, x_7]]]$, $\boxed{x_{56}} = [[[x_3, x_4], [x_5, x_8]], [[x_4, x_5], [x_6, x_7]]]$,	$2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8$, $\underline{2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_8}$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$, $2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8$, $2\alpha_5 + 2\alpha_6 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8$, $\underline{2\alpha_4 + 2\alpha_5 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8}$,
$x_{57} =$ $[[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, [x_4, x_5]]]]$, $\boxed{x_{58}} =$ $[[[x_3, x_4], [x_5, x_8]], [[x_5, x_6], [x_4, [x_2, x_3]]]]$, $x_{59} =$ $[[[x_4, x_5], [x_6, x_7]], [[x_3, x_4], [x_8, [x_5, x_6]]]]$, $x_{60} =$ $[[[x_4, x_5], [x_6, x_7]], [[x_5, x_8], [x_4, [x_2, x_3]]]]$, $\boxed{x_{61}} =$ $[[[x_5, x_8], [x_6, x_7]], [[x_4, x_5], [x_3, [x_1, x_2]]]]$, $\boxed{x_{62}} =$ $[[[x_5, x_8], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]]$,	$2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_8$, $\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6 + \alpha_8}$, $\underline{2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_3 + \alpha_7 + \alpha_8}$, $2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8$, $\underline{2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8}$, $\underline{2\alpha_5 + 2\alpha_6 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8}$,

$x_{63} = [[[x_5, x_8], [x_6, x_7]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$	$2\alpha_5 + 2\alpha_6 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8,$
$x_{64} = [[[x_3, x_4], [x_8, [x_5, x_6]]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6 + \alpha_8,$
$\boxed{x_{65}} = [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_6, x_7], [x_8, [x_4, x_5]]]],$	$\underline{2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8},$
$\boxed{x_{66}} = [[[x_5, x_6], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_8, [x_4, x_5]]]],$	$\underline{2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_2 + \alpha_3 + \alpha_7 + \alpha_8},$
$x_{67} = [[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_5, [x_3, x_4]]]],$	$2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8,$
$\boxed{x_{68}} = [[[x_5, x_8], [x_6, [x_4, x_5]]], [[x_6, x_7], [x_5, [x_3, x_4]]]],$	$\underline{2\alpha_4 + 2\alpha_6 + 3\alpha_5 + \alpha_3 + \alpha_7 + \alpha_8},$
$\boxed{x_{69}} = [[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$	$\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_8},$
$x_{70} = [[[x_5, x_8], [x_6, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]],$	$2\alpha_4 + 2\alpha_6 + 3\alpha_5 + \alpha_2 + \alpha_3 + \alpha_7 + \alpha_8,$
$\boxed{x_{71}} = [[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]]],$	$\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8},$
$\boxed{x_{72}} = [[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]]],$	$\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_2 + \alpha_7 + \alpha_8},$
$x_{73} = [[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$	$2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_7 + \alpha_8,$
$x_{74} = [[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]],$	$2\alpha_4 + 2\alpha_6 + 2\alpha_8 + 3\alpha_5 + \alpha_3 + \alpha_7,$
$x_{75} = [[[x_6, x_7], [[x_3, x_4], [x_5, x_8]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$	$2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_1 + \alpha_2 + \alpha_7 + \alpha_8,$
$\boxed{x_{76}} = [[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]]],$	$\underline{2\alpha_4 + 2\alpha_6 + 3\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_7 + \alpha_8},$
$x_{77} = [[[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]],$	$2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_6 + \alpha_7 + \alpha_8,$
$x_{78} = [[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]],$	$2\alpha_3 + 2\alpha_4 + 2\alpha_6 + 3\alpha_5 + \alpha_2 + \alpha_7 + \alpha_8,$
$\boxed{x_{79}} = [[[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]]],$	$\underline{2\alpha_4 + 2\alpha_6 + 2\alpha_8 + 3\alpha_5 + \alpha_2 + \alpha_3 + \alpha_7},$
$x_{80} = [[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_8, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]]],$	$2\alpha_3 + 2\alpha_6 + 3\alpha_4 + 3\alpha_5 + \alpha_2 + \alpha_7 + \alpha_8,$
$\boxed{x_{81}} = [[[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]],$	$\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_1 + \alpha_7 + \alpha_8},$
$\boxed{x_{82}} = [[[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]], [[x_4, [x_2, x_3]], [[x_5, x_8], [x_6, x_7]]]],$	$\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_6 + 2\alpha_8 + 3\alpha_5 + \alpha_2 + \alpha_7},$
$\boxed{x_{83}} = [[[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]],$	$\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_6 + 3\alpha_5 + \alpha_1 + \alpha_2 + \alpha_7 + \alpha_8},$
$x_{84} = [[[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]], [[x_8, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]],$	$2\alpha_4 + 2\alpha_6 + 2\alpha_8 + 3\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_7,$

$\boxed{x_{85}} = [[[x_4,[x_2,x_3]],[[x_5,x_8],[x_6,x_7]]],$ $[[x_8,[x_4,x_5]],[[x_3,x_4],[x_5,x_6]]]],$	$\underline{2\alpha_3+2\alpha_6+2\alpha_8+3\alpha_4+3\alpha_5}\ \underline{+\alpha_2+\alpha_7},$
$\boxed{x_{86}} = [[[x_7,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]],$ $[[x_8,[x_4,x_5]],[[x_3,x_4],[x_5,x_6]]]],$	$\underline{2\alpha_3+2\alpha_6+3\alpha_4+3\alpha_5+\alpha_1+\alpha_2}$ $\underline{+\alpha_7+\alpha_8},$
$x_{87} = [[[x_7,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]],$ $[[x_8,[x_5,x_6]],[[x_2,x_3],[x_4,x_5]]]],$	$2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_6+3\alpha_5+\alpha_1+\alpha_7+\alpha_8,$
$x_{88} = [[[x_7,[x_5,x_6]],[[x_3,x_4],[x_5,x_8]]],$ $[[x_8,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]]],$	$2\alpha_3+2\alpha_4+2\alpha_6+2\alpha_8+3\alpha_5+\alpha_1+\alpha_2+\alpha_7,$
$x_{89} = [[[x_7,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]],$ $[[x_8,[x_6,[x_4,x_5]]],[[x_2,x_3],[x_4,x_5]]]],$	$2\alpha_2+2\alpha_3+2\alpha_6+3\alpha_4+3\alpha_5+\alpha_1+\alpha_7+\alpha_8,$
$x_{90} = [[[x_8,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]],$ $[[[x_3,x_4],[x_5,x_8]],[[x_4,x_5],[x_6,x_7]]]],$	$2\alpha_3+2\alpha_6+2\alpha_8+3\alpha_4+3\alpha_5+\alpha_1+\alpha_2+\alpha_7,$
$\boxed{x_{91}} = [[[x_8,[x_5,x_6]],[[x_2,x_3],[x_4,x_5]]],$ $[[[x_1,x_2],[x_3,x_4]],[[x_5,x_8],[x_6,x_7]]]],$	$\underline{2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_6+2\alpha_8}$ $\underline{+3\alpha_5+\alpha_1+\alpha_7},$
$x_{92} = [[[x_8,[x_5,x_6]],[[x_2,x_3],[x_4,x_5]]],$ $[[[x_3,x_4],[x_5,x_8]],[[x_4,x_5],[x_6,x_7]]]],$	$2\alpha_3+2\alpha_6+2\alpha_8+3\alpha_4+4\alpha_5+\alpha_2+\alpha_7,$
$x_{93} = [[[x_7,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]],$ $[[[x_3,x_4],[x_5,x_8]],[[x_5,x_6],$ $[x_4,[x_2,x_3]]]]],$	$2\alpha_2+2\alpha_6+3\alpha_3+3\alpha_4+3\alpha_5+\alpha_1+\alpha_7+\alpha_8,$
$\boxed{x_{94}} = [[[x_8,[x_5,x_6]],[[x_4,x_5],[x_3,[x_1,x_2]]]],$ $[[[x_3,x_4],[x_5,x_8]],[[x_4,x_5],[x_6,x_7]]]],$	$\underline{2\alpha_3+2\alpha_6+2\alpha_8+3\alpha_4+4\alpha_5+\alpha_1}$ $\underline{+\alpha_2+\alpha_7},$
$\boxed{x_{95}} = [[[x_8,[x_6,[x_4,x_5]]],[[x_2,x_3],[x_4,x_5]]],$ $[[[x_1,x_2],[x_3,x_4]],[[x_5,x_8],[x_6,x_7]]]],$	$\underline{2\alpha_2+2\alpha_3+2\alpha_6+2\alpha_8+3\alpha_4+3\alpha_5}$ $\underline{+\alpha_1+\alpha_7},$
$\boxed{x_{96}} = [[[x_8,[x_6,[x_4,x_5]]],[[x_2,x_3],[x_4,x_5]]],$ $[[[x_3,x_4],[x_5,x_6]],[[x_5,x_8],[x_6,x_7]]]],$	$\underline{2\alpha_3+2\alpha_8+3\alpha_4+3\alpha_6}\ \underline{+4\alpha_5+\alpha_2+\alpha_7},$
$\boxed{x_{97}} = [[[[x_1,x_2],[x_3,x_4]],[[x_5,x_8],[x_6,x_7]]],$ $[[[x_3,x_4],[x_5,x_8]],[[x_5,x_6],$ $[x_4,[x_2,x_3]]]]],$	$\underline{2\alpha_2+2\alpha_6+2\alpha_8+3\alpha_3+3\alpha_4+3\alpha_5}$ $\underline{+\alpha_1+\alpha_7},$
$x_{98} = [[[[x_2,x_3],[x_4,x_5]],[[x_5,x_8],[x_6,x_7]]],$ $[[[x_1,x_2],[x_3,x_4]],[[x_5,x_8],$ $[x_6,[x_4,x_5]]]]],$	$2\alpha_2+2\alpha_3+2\alpha_6+2\alpha_8+3\alpha_4+4\alpha_5+\alpha_1+\alpha_7,$
$x_{99} = [[[[x_3,x_4],[x_5,x_6]],[[x_5,x_8],[x_6,x_7]]],$ $[[[x_1,x_2],[x_3,x_4]],[[x_5,x_8],$ $[x_6,[x_4,x_5]]]]],$	$2\alpha_3+2\alpha_8+3\alpha_4+3\alpha_6+4\alpha_5+\alpha_1+\alpha_2+\alpha_7,$
$x_{100} = [[[[x_3,x_4],[x_5,x_8]],[[x_4,x_5],[x_6,x_7]]],$ $[[[x_5,x_8],[x_6,x_7]],[[x_5,x_6],$ $[x_4,[x_2,x_3]]]]],$	$2\alpha_3+2\alpha_7+2\alpha_8+3\alpha_4+3\alpha_6+4\alpha_5+\alpha_2,$
$\boxed{x_{101}} = [[[[x_1,x_2],[x_3,x_4]],[[x_5,x_8],$ $[x_6,[x_4,x_5]]]],[[[x_5,x_8],[x_6,x_7]],$ $[[x_5,x_6],[x_4,[x_2,x_3]]]]],$	$\underline{2\alpha_2+2\alpha_3+2\alpha_8+3\alpha_4+3\alpha_6}$ $\underline{+4\alpha_5+\alpha_1+\alpha_7},$
$x_{102} = [[[[x_3,x_4],[x_5,x_8]],[[x_5,x_6],$ $[x_4,[x_2,x_3]]]],[[[x_5,x_8],[x_6,x_7]],$ $[[x_4,x_5],[x_3,[x_1,x_2]]]]],$	$2\alpha_2+2\alpha_6+2\alpha_8+3\alpha_3+3\alpha_4+4\alpha_5+\alpha_1+\alpha_7,$
$\boxed{x_{103}} = [[[[x_4,x_5],[x_6,x_7]],[[x_3,x_4],$ $[x_8,[x_5,x_6]]]],[[[x_5,x_8],[x_6,x_7]],$ $[[x_4,x_5],[x_3,[x_1,x_2]]]]],$	$\underline{2\alpha_3+2\alpha_7+2\alpha_8+3\alpha_4+3\alpha_6}$ $\underline{+4\alpha_5+\alpha_1+\alpha_2},$

$x_{104} = [[[[x_4, x_5], [x_6, x_7]], [[x_5, x_8], [x_4, [x_2, x_3]]]], [[[x_3, x_4], [x_8, [x_5, x_6]]], [[x_4, x_5], [x_3, [x_1, x_2]]]]],$	$2\alpha_2+2\alpha_6+2\alpha_8+3\alpha_3+4\alpha_4+4\alpha_5+\alpha_1+\alpha_7,$
$\boxed{x_{105}} = [[[[x_5, x_8], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]], [[[x_3, x_4], [x_8, [x_5, x_6]]], [[x_4, x_5], [x_3, [x_1, x_2]]]]],$	$\underline{2\alpha_2 + 2\alpha_8 + 3\alpha_3 + 3\alpha_4 + 3\alpha_6 +4\alpha_5 + \alpha_1 + \alpha_7},$
$x_{106} = [[[[x_5, x_8], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]], [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_6, x_7], [x_8, [x_4, x_5]]]]],$	$2\alpha_2+2\alpha_3+2\alpha_7+2\alpha_8+3\alpha_4+3\alpha_6+4\alpha_5+\alpha_1,$
$x_{107} = [[[[x_5, x_8], [x_6, x_7]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]], [[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_5, [x_3, x_4]]]]],$	$2\alpha_2+2\alpha_7+2\alpha_8+3\alpha_3+3\alpha_4+3\alpha_6+4\alpha_5+\alpha_1,$
$\boxed{x_{108}} = [[[[x_3, x_4], [x_8, [x_5, x_6]]], [[x_4, x_5], [x_3, [x_1, x_2]]]], [[[x_5, x_6], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_8, [x_4, x_5]]]]],$	$\underline{2\alpha_2 + 2\alpha_8 + 3\alpha_3 + 3\alpha_6 + 4\alpha_4 +4\alpha_5 + \alpha_1 + \alpha_7},$
$x_{109} = [[[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_5, [x_3, x_4]]]], [[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]]],$	$2\alpha_2+2\alpha_7+2\alpha_8+3\alpha_3+3\alpha_6+4\alpha_4+4\alpha_5+\alpha_1,$
$x_{110} = [[[[x_5, x_8], [x_6, [x_4, x_5]]], [[x_6, x_7], [x_5, [x_3, x_4]]]], [[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]]],$	$2\alpha_2+2\alpha_8+3\alpha_3+3\alpha_6+4\alpha_4+5\alpha_5+\alpha_1+\alpha_7,$
$\boxed{x_{111}} = [[[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]], [[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]]],$	$\underline{2\alpha_2 + 3\alpha_3 + 3\alpha_6 + 3\alpha_8 + 4\alpha_4 + 5\alpha_5 +\alpha_1 + \alpha_7},$
$\boxed{x_{112}} = [[[[x_5, x_8], [x_6, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]], [[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]]]],$	$\underline{2\alpha_2 + 2\alpha_7 + 2\alpha_8 + 3\alpha_3 + 3\alpha_6 +4\alpha_4 + 5\alpha_5 + \alpha_1},$
$x_{113} = [[[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]], [[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]]],$	$2\alpha_2+2\alpha_7+2\alpha_8+3\alpha_3+4\alpha_4+4\alpha_6+5\alpha_5+\alpha_1,$
$x_{114} = [[[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]], [[[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]]],$	$2\alpha_2+2\alpha_7+3\alpha_3+3\alpha_6+3\alpha_8+4\alpha_4+5\alpha_5+\alpha_1,$
$\boxed{x_{115}} = [[[[x_6, x_7], [[x_3, x_4], [x_5, x_8]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]], [[[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]]]],$	$\underline{2\alpha_2 + 2\alpha_7 + 3\alpha_3 + 3\alpha_8 + 4\alpha_4 +4\alpha_6 + 5\alpha_5 + \alpha_1},$
$x_{116} = [[[[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]]], [[[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]]],$	$2\alpha_2+2\alpha_7+3\alpha_3+3\alpha_8+4\alpha_4+4\alpha_6+6\alpha_5+\alpha_1,$
$x_{117} = [[[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_8, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]]], [[[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]], [[x_8, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]]],$	$2\alpha_2+2\alpha_7+3\alpha_3+3\alpha_8+4\alpha_6+5\alpha_4+6\alpha_5+\alpha_1,$

$x_{118} = [[[[x_5,[x_3,x_4]],[x_8,[x_5,x_6]]],$ $[[x_7,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]]],$ $[[[x_4,[x_2,x_3]],[[x_5,x_8],[x_6,x_7]]],$ $[[x_8,[x_4,x_5]],[[x_3,x_4],[x_5,x_6]]]]],$	$2\alpha_2+2\alpha_7+3\alpha_8+4\alpha_3+4\alpha_6+5\alpha_4+6\alpha_5+\alpha_1,$
$\boxed{x_{119}} = [[[[x_4,[x_2,x_3]],[[x_5,x_8],[x_6,x_7]]],$ $[[x_8,[x_4,x_5]],[[x_3,x_4],[x_5,x_6]]]],$ $[[[x_7,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]],$ $[[x_8,[x_5,x_6]],[[x_2,x_3],[x_4,x_5]]]]],$	$\underline{2\alpha_7+3\alpha_2+3\alpha_8+4\alpha_3+4\alpha_6}$ $\underline{+5\alpha_4+6\alpha_5+\alpha_1},$
$x_{120} = [[[[x_7,[x_5,x_6]],[[x_3,x_4],[x_5,x_8]]],$ $[[x_8,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]]],$ $[[[x_7,[x_5,x_6]],[[x_1,x_2],[x_3,x_4]]],$ $[[x_8,[x_6,[x_4,x_5]]],[[x_2,x_3],[x_4,x_5]]]]]$	$2\alpha_1+2\alpha_7+3\alpha_2+3\alpha_8+4\alpha_3+4\alpha_6+$ $5\alpha_4+6\alpha_5$

4.16.10 $\mathfrak{e}(8,8)$ of sdim $= 120|128$

In the $\mathbb{Z}$-grading with the 1st CM with $\deg e_8^{\pm} = \pm 1$ and $\deg e_i^{\pm} = 0$ for $i \neq 8$, we have $\mathfrak{g}_0 = \mathfrak{gl}(8) = \mathfrak{gl}(V)$. There are different isomorphisms between $\mathfrak{g}_0$ and $\mathfrak{gl}(8)$; using the one where $h_i = E_{i,i} + E_{i+1,i+1}$ for all $i = 1,\dots,7$, and $h_8 = E_{6,6}+E_{7,7}+E_{8,8}$, we see that, as modules over $\mathfrak{gl}(V)$,

$$\begin{array}{lll}\mathfrak{g}_1 = \bigwedge^5 V^*; & \mathfrak{g}_2 = \bigwedge^6 V; & \mathfrak{g}_3 = V;\\ \mathfrak{g}_{-1} = \bigwedge^5 V; & \mathfrak{g}_{-2} = \bigwedge^6 V^*; & \mathfrak{g}_{-3} = V^*.\end{array}$$

We can also set $h_8 = E_{1,1} + E_{2,2} + E_{3,3} + E_{4,4} + E_{5,5}$. Then we get

$$\begin{array}{lll}\mathfrak{g}_1 = \bigwedge^3 V; & \mathfrak{g}_2 = \bigwedge^6 V; & \mathfrak{g}_3 = \bigwedge^7 V^*;\\ \mathfrak{g}_{-1} = \bigwedge^3 V^*; & \mathfrak{g}_{-2} = \bigwedge^6 V^*; & \mathfrak{g}_{-3} = \bigwedge^7 V.\end{array}$$

The algebra $\mathfrak{g}_{\bar 0}$ is isomorphic to $\mathfrak{o}_{\Pi}^{(2)}(16) \oplus \mathbb{K}d$, where $d = E_{6,6} + \cdots + E_{13,13}$, and $\mathfrak{g}_{\bar 1}$ is an irreducible $\mathfrak{g}_{\bar 0}$-module with the highest weight element x_{120} of weight $(1,0,\dots,0)$ with respect to $h_1,\dots,h_8$; $\mathfrak{g}_{\bar 1}$ also possesses a lowest weight vector.

the root vectors	the roots
$\boxed{x_1}, \boxed{x_2}, \boxed{x_3}, \boxed{x_4}, \boxed{x_5}, \boxed{x_6}, \boxed{x_7}, \boxed{x_8},$	$\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3}, \underline{\alpha_4}, \underline{\alpha_5}, \underline{\alpha_6}, \underline{\alpha_7}, \underline{\alpha_8},$
$x_9 = [x_1,x_2], x_{10} = [x_2,x_3], x_{11} = [x_3,x_4], x_{12} = [x_4,x_5],$ $x_{13} = [x_5,x_6], x_{14} = [x_5,x_8], x_{15} = [x_6,x_7],$	$\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_4,$ $\alpha_4+\alpha_5, \alpha_5+\alpha_6, \alpha_5+\alpha_8,$ $\alpha_6+\alpha_7,$
$\boxed{x_{16}} = [x_3,[x_1,x_2]], \boxed{x_{17}} = [x_4,[x_2,x_3]],$ $\boxed{x_{18}} = [x_5,[x_3,x_4]], \boxed{x_{19}} = [x_6,[x_4,x_5]],$ $\boxed{x_{20}} = [x_7,[x_5,x_6]], \boxed{x_{21}} = [x_8,[x_4,x_5]],$ $\boxed{x_{22}} = [x_8,[x_5,x_6]],$	$\underline{\alpha_1+\alpha_2+\alpha_3}, \underline{\alpha_2+\alpha_3+\alpha_4},$ $\underline{\alpha_3+\alpha_4+\alpha_5}, \underline{\alpha_4+\alpha_5+\alpha_6},$ $\underline{\alpha_5+\alpha_6+\alpha_7}, \underline{\alpha_4+\alpha_5+\alpha_8},$ $\underline{\alpha_5+\alpha_6+\alpha_8},$
$x_{23} = [x_8,[x_6,[x_4,x_5]]], x_{24} = [[x_1,x_2],[x_3,x_4]],$ $x_{25} = [[x_2,x_3],[x_4,x_5]], x_{26} = [[x_3,x_4],[x_5,x_6]],$ $x_{27} = [[x_3,x_4],[x_5,x_8]], x_{28} = [[x_4,x_5],[x_6,x_7]],$ $x_{29} = [[x_5,x_8],[x_6,x_7]],$	$\alpha_4+\alpha_5+\alpha_6+\alpha_8,$ $\alpha_1+\alpha_2+\alpha_3+\alpha_4,$ $\alpha_2+\alpha_3+\alpha_4+\alpha_5,$ $\alpha_3+\alpha_4+\alpha_5+\alpha_6,$ $\alpha_3+\alpha_4+\alpha_5+\alpha_8,$ $\alpha_4+\alpha_5+\alpha_6+\alpha_7,$ $\alpha_5+\alpha_6+\alpha_7+\alpha_8,$

$\boxed{x_{30}} = [[x_3, x_4], [x_8, [x_5, x_6]]],$	$\underline{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_8},$
$\boxed{x_{31}} = [[x_4, x_5], [x_3, [x_1, x_2]]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5},$
$\boxed{x_{32}} = [[x_5, x_6], [x_4, [x_2, x_3]]],$	$\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6},$
$\boxed{x_{33}} = [[x_5, x_8], [x_4, [x_2, x_3]]],$	$\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8},$
$\boxed{x_{34}} = [[x_5, x_8], [x_6, [x_4, x_5]]],$	$\underline{2\alpha_5 + \alpha_4 + \alpha_6 + \alpha_8},$
$\boxed{x_{35}} = [[x_6, x_7], [x_5, [x_3, x_4]]],$	$\underline{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$
$\boxed{x_{36}} = [[x_6, x_7], [x_8, [x_4, x_5]]],$	$\underline{\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8},$
$x_{37} = [[x_6, x_7], [[x_3, x_4], [x_5, x_8]]],$	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8,$
$x_{38} = [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]],$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$
$x_{39} = [[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]],$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8,$
$x_{40} = [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]],$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$
$x_{41} = [[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]],$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_8,$
$x_{42} = [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]],$	$2\alpha_5 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8,$
$x_{43} = [[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]],$	$2\alpha_5 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8,$
$\boxed{x_{44}} = [[x_4, [x_2, x_3]], [[x_5, x_8], [x_6, x_7]]],$	$\underline{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8},$
$\boxed{x_{45}} = [[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7},$
$\boxed{x_{46}} = [[x_7, [x_5, x_6]], [[x_3, x_4], [x_5, x_8]]],$	$\underline{2\alpha_5 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8},$
$\boxed{x_{47}} = [[x_8, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]],$	$\underline{2\alpha_4 + 2\alpha_5 + \alpha_3 + \alpha_6 + \alpha_8},$
$\boxed{x_{48}} = [[x_8, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]],$	$\underline{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_8},$
$\boxed{x_{49}} = [[x_8, [x_5, x_6]], [[x_2, x_3], [x_4, x_5]]],$	$\underline{2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8},$
$\boxed{x_{50}} = [[x_8, [x_5, x_6]], [[x_4, x_5], [x_6, x_7]]],$	$\underline{2\alpha_5 + 2\alpha_6 + \alpha_4 + \alpha_7 + \alpha_8},$
$x_{51} = [[x_8, [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8,$
$x_{52} = [[x_8, [x_6, [x_4, x_5]]], [[x_2, x_3], [x_4, x_5]]],$	$2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_8,$
$x_{53} = [[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, x_7]]],$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8,$
$x_{54} = [[[x_2, x_3], [x_4, x_5]], [[x_5, x_8], [x_6, x_7]]],$	$2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8,$
$x_{55} = [[[x_3, x_4], [x_5, x_6]], [[x_5, x_8], [x_6, x_7]]],$	$2\alpha_5 + 2\alpha_6 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8,$
$x_{56} = [[[x_3, x_4], [x_5, x_8]], [[x_4, x_5], [x_6, x_7]]],$	$2\alpha_4 + 2\alpha_5 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8,$
$\boxed{x_{57}} = [[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, [x_4, x_5]]]],$	$\underline{2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_8},$
$\boxed{x_{58}} = [[[x_3, x_4], [x_5, x_8]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$	$\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6 + \alpha_8},$
$\boxed{x_{59}} = [[[x_4, x_5], [x_6, x_7]], [[x_3, x_4], [x_8, [x_5, x_6]]]],$	$\underline{2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_3 + \alpha_7 + \alpha_8},$
$\boxed{x_{60}} = [[[x_4, x_5], [x_6, x_7]], [[x_5, x_8], [x_4, [x_2, x_3]]]],$	$\underline{2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8},$
$\boxed{x_{61}} = [[[x_5, x_8], [x_6, x_7]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$	$\underline{2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8},$
$\boxed{x_{62}} = [[[x_5, x_8], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]],$	$\underline{2\alpha_5 + 2\alpha_6 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8},$

$x_{63} = [[[x_5, x_8], [x_6, x_7]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $x_{64} = [[[x_3, x_4], [x_8, [x_5, x_6]]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$ $x_{65} = [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_6, x_7], [x_8, [x_4, x_5]]]],$ $x_{66} = [[[x_5, x_6], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_8, [x_4, x_5]]]],$ $x_{67} = [[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_5, [x_3, x_4]]]],$ $x_{68} = [[[x_5, x_8], [x_6, [x_4, x_5]]], [[x_6, x_7], [x_5, [x_3, x_4]]]],$	$2\alpha_5 + 2\alpha_6 + \alpha_1 + \alpha_2 + \alpha_3 +$ $\alpha_4 + \alpha_7 + \alpha_8, 2\alpha_3 + 2\alpha_4 +$ $2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_6 + \alpha_8,$ $2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 +$ $\alpha_6 + \alpha_7 + \alpha_8, 2\alpha_4 + 2\alpha_5 +$ $2\alpha_6 + \alpha_2 + \alpha_3 + \alpha_7 + \alpha_8, 2\alpha_3 +$ $2\alpha_4 + 2\alpha_5 + \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8,$ $2\alpha_4 + 2\alpha_6 + 3\alpha_5 + \alpha_3 + \alpha_7 + \alpha_8,$
$\boxed{x_{69}} =$ $[[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $\boxed{x_{70}} =$ $[[[x_5, x_8], [x_6, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]],$ $\boxed{x_{71}} =$ $[[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]]],$ $\boxed{x_{72}} =$ $[[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]]],$ $\boxed{x_{73}} =$ $[[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $\boxed{x_{74}} =$ $[[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]],$	$\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1}$ $\underline{+\alpha_6 + \alpha_8},$ $\underline{2\alpha_4 + 2\alpha_6 + 3\alpha_5 + \alpha_2 + \alpha_3}$ $\underline{+\alpha_7 + \alpha_8},$ $\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 + \alpha_2}$ $\underline{+\alpha_6 + \alpha_7 + \alpha_8},$ $\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6}$ $\underline{+\alpha_2 + \alpha_7 + \alpha_8},$ $\underline{2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_1 + \alpha_2}$ $\underline{+\alpha_3 + \alpha_7 + \alpha_8},$ $\underline{2\alpha_4 + 2\alpha_6 + 2\alpha_8 + 3\alpha_5}$ $\underline{+\alpha_3 + \alpha_7},$
$x_{75} =$ $[[[x_6, x_7], [[x_3, x_4], [x_5, x_8]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $x_{76} =$ $[[[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]], [[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]]],$ $x_{77} =$ $[[[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]],$ $x_{78} =$ $[[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]],$ $x_{79} =$ $[[[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]]],$	$2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_1 +$ $\alpha_2 + \alpha_7 + \alpha_8, 2\alpha_4 + 2\alpha_6 +$ $3\alpha_5 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_7 + \alpha_8,$ $2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_1 +$ $\alpha_6 + \alpha_7 + \alpha_8, 2\alpha_3 + 2\alpha_4 +$ $2\alpha_6 + 3\alpha_5 + \alpha_2 + \alpha_7 + \alpha_8,$ $2\alpha_4 + 2\alpha_6 + 2\alpha_8 + 3\alpha_5 + \alpha_2 +$ $\alpha_3 + \alpha_7,$
$\boxed{x_{80}} = [[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_8, [x_4, x_5]],$ $[[x_3, x_4], [x_5, x_6]]]],$ $\boxed{x_{81}} = [[[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]],$ $[[x_1, x_2], [x_3, x_4]]]],$ $\boxed{x_{82}} = [[[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]], [[x_4, [x_2, x_3]],$ $[[x_5, x_8], [x_6, x_7]]]],$ $\boxed{x_{83}} = [[[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]],$ $[[x_1, x_2], [x_3, x_4]]]],$ $\boxed{x_{84}} = [[[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]], [[x_8, [x_5, x_6]],$ $[[x_1, x_2], [x_3, x_4]]]],$	$\underline{2\alpha_3 + 2\alpha_6 + 3\alpha_4 + 3\alpha_5}$ $\underline{+\alpha_2 + \alpha_7 + \alpha_8},$ $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5}$ $\underline{+2\alpha_6 + \alpha_1 + \alpha_7 + \alpha_8},$ $\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_6 + 2\alpha_8}$ $\underline{+3\alpha_5 + \alpha_2 + \alpha_7},$ $\underline{2\alpha_3 + 2\alpha_4 + 2\alpha_6 + 3\alpha_5}$ $\underline{+\alpha_1 + \alpha_2 + \alpha_7 + \alpha_8},$ $\underline{2\alpha_4 + 2\alpha_6 + 2\alpha_8 + 3\alpha_5}$ $\underline{+\alpha_1 + \alpha_2 + \alpha_3 + \alpha_7},$
$x_{85} = [[[x_4, [x_2, x_3]], [[x_5, x_8], [x_6, x_7]]], [[x_8, [x_4, x_5]],$ $[[x_3, x_4], [x_5, x_6]]]],$ $x_{86} = [[[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]], [[x_8, [x_4, x_5]],$ $[[x_3, x_4], [x_5, x_6]]]],$ $x_{87} = [[[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]], [[x_8, [x_5, x_6]],$ $[[x_2, x_3], [x_4, x_5]]]],$ $x_{88} = [[[x_7, [x_5, x_6]], [[x_3, x_4], [x_5, x_8]]], [[x_8, [x_5, x_6]],$ $[[x_1, x_2], [x_3, x_4]]]],$	$2\alpha_3 + 2\alpha_6 + 2\alpha_8 + 3\alpha_4 + 3\alpha_5 +$ $\alpha_2 + \alpha_7, 2\alpha_3 + 2\alpha_6 + 3\alpha_4 +$ $3\alpha_5 + \alpha_1 + \alpha_2 + \alpha_7 + \alpha_8,$ $2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_6 +$ $3\alpha_5 + \alpha_1 + \alpha_7 + \alpha_8,$ $2\alpha_3 + 2\alpha_4 + 2\alpha_6 + 2\alpha_8 +$ $3\alpha_5 + \alpha_1 + \alpha_2 + \alpha_7,$

$\boxed{x_{89}}$ $=$ $[[[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]], [[x_8, [x_6, [x_4, x_5]]], [[x_2, x_3], [x_4, x_5]]]],$ $\boxed{x_{90}}$ $=$ $[[[x_8, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]], [[[x_3, x_4], [x_5, x_8]], [[x_4, x_5], [x_6, x_7]]]],$ $\boxed{x_{91}}$ $=$ $[[[x_8, [x_5, x_6]], [[x_2, x_3], [x_4, x_5]]], [[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, x_7]]]],$ $\boxed{x_{92}}$ $=$ $[[[x_8, [x_5, x_6]], [[x_2, x_3], [x_4, x_5]]], [[[x_3, x_4], [x_5, x_8]], [[x_4, x_5], [x_6, x_7]]]],$	$\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_6 + 3\alpha_4 + 3\alpha_5 + \alpha_1 + \alpha_7 + \alpha_8}$, $\underline{2\alpha_3 + 2\alpha_6 + 2\alpha_8 + 3\alpha_4 + 3\alpha_5 + \alpha_1 + \alpha_2 + \alpha_7}$, $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_6 + 2\alpha_8 + 3\alpha_5 + \alpha_1 + \alpha_7}$, $\underline{2\alpha_3 + 2\alpha_6 + 2\alpha_8 + 3\alpha_4 + 4\alpha_5 + \alpha_2 + \alpha_7}$,
$x_{93} =$ $[[[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]], [[[x_3, x_4], [x_5, x_8]], [[x_5, x_6], [x_4, [x_2, x_3]]]]],$ $x_{94} = [[[x_8, [x_5, x_6]], [[x_4, x_5], [x_3, [x_1, x_2]]]], [[[x_3, x_4], [x_5, x_8]], [[x_4, x_5], [x_6, x_7]]]],$ $x_{95} = [[[x_8, [x_6, [x_4, x_5]]], [[x_2, x_3], [x_4, x_5]]], [[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, x_7]]]],$ $x_{96} = [[[x_8, [x_6, [x_4, x_5]]], [[x_2, x_3], [x_4, x_5]]], [[[x_3, x_4], [x_5, x_6]], [[x_5, x_8], [x_6, x_7]]]],$	$2\alpha_2 + 2\alpha_6 + 3\alpha_3 + 3\alpha_4 + 3\alpha_5 + \alpha_1 + \alpha_7 + \alpha_8$, $2\alpha_3 + 2\alpha_6 + 2\alpha_8 + 3\alpha_4 + 4\alpha_5 + \alpha_1 + \alpha_2 + \alpha_7$, $2\alpha_2 + 2\alpha_3 + 2\alpha_6 + 2\alpha_8 + 3\alpha_4 + 3\alpha_5 + \alpha_1 + \alpha_7$, $2\alpha_3 + 2\alpha_8 + 3\alpha_4 + 3\alpha_6 + 4\alpha_5 + \alpha_2 + \alpha_7$,
$\boxed{x_{97}}$ $=$ $[[[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, x_7]]], [[[x_3, x_4], [x_5, x_8]], [[x_5, x_6], [x_4, [x_2, x_3]]]]],$ $\boxed{x_{98}}$ $=$ $[[[[x_2, x_3], [x_4, x_5]], [[x_5, x_8], [x_6, x_7]]], [[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, [x_4, x_5]]]]],$ $\boxed{x_{99}}$ $=$ $[[[[x_3, x_4], [x_5, x_6]], [[x_5, x_8], [x_6, x_7]]], [[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, [x_4, x_5]]]]],$ $\boxed{x_{100}}$ $=$ $[[[[x_3, x_4], [x_5, x_8]], [[x_4, x_5], [x_6, x_7]]], [[[x_5, x_8], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]]],$	$\underline{2\alpha_2 + 2\alpha_6 + 2\alpha_8 + 3\alpha_3 + 3\alpha_4 + 3\alpha_5 + \alpha_1 + \alpha_7}$, $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_6 + 2\alpha_8 + 3\alpha_4 + 4\alpha_5 + \alpha_1 + \alpha_7}$, $\underline{2\alpha_3 + 2\alpha_8 + 3\alpha_4 + 3\alpha_6 + 4\alpha_5 + \alpha_1 + \alpha_2 + \alpha_7}$, $\underline{2\alpha_3 + 2\alpha_7 + 2\alpha_8 + 3\alpha_4 + 3\alpha_6 + 4\alpha_5 + \alpha_2}$,
$x_{101} =$ $[[[[x_1, x_2], [x_3, x_4]], [[x_5, x_8], [x_6, [x_4, x_5]]]], [[[x_5, x_8], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]]],$ $x_{102} =$ $[[[[x_3, x_4], [x_5, x_8]], [[x_5, x_6], [x_4, [x_2, x_3]]]], [[[x_5, x_8], [x_6, x_7]], [[x_4, x_5], [x_3, [x_1, x_2]]]]],$ $x_{103} =$ $[[[[x_4, x_5], [x_6, x_7]], [[x_3, x_4], [x_8, [x_5, x_6]]]], [[[x_5, x_8], [x_6, x_7]], [[x_4, x_5], [x_3, [x_1, x_2]]]]],$	$2\alpha_2 + 2\alpha_3 + 2\alpha_8 + 3\alpha_4 + 3\alpha_6 + 4\alpha_5 + \alpha_1 + \alpha_7$, $2\alpha_2 + 2\alpha_6 + 2\alpha_8 + 3\alpha_3 + 3\alpha_4 + 4\alpha_5 + \alpha_1 + \alpha_7$, $2\alpha_3 + 2\alpha_7 + 2\alpha_8 + 3\alpha_4 + 3\alpha_6 + 4\alpha_5 + \alpha_1 + \alpha_2$,
$\boxed{x_{104}}$ $=$ $[[[[x_4, x_5], [x_6, x_7]], [[x_5, x_8], [x_4, [x_2, x_3]]]], [[[x_3, x_4], [x_8, [x_5, x_6]]], [[x_4, x_5], [x_3, [x_1, x_2]]]]],$ $\boxed{x_{105}}$ $=$ $[[[[x_5, x_8], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]], [[[x_3, x_4], [x_8, [x_5, x_6]]], [[x_4, x_5], [x_3, [x_1, x_2]]]]],$ $\boxed{x_{106}}$ $=$ $[[[[x_5, x_8], [x_6, x_7]], [[x_5, x_6], [x_4, [x_2, x_3]]]], [[[x_4, x_5], [x_3, [x_1, x_2]]], [[x_6, x_7], [x_8, [x_4, x_5]]]]],$	$\underline{2\alpha_2 + 2\alpha_6 + 2\alpha_8 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + \alpha_1 + \alpha_7}$, $\underline{2\alpha_2 + 2\alpha_8 + 3\alpha_3 + 3\alpha_4 + 3\alpha_6 + 4\alpha_5 + \alpha_1 + \alpha_7}$, $\underline{2\alpha_2 + 2\alpha_3 + 2\alpha_7 + 2\alpha_8 + 3\alpha_4 + 3\alpha_6 + 4\alpha_5 + \alpha_1}$,

$x_{107} = [[[[x_5, x_8], [x_6, x_7]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $[[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_5, [x_3, x_4]]]]],$ $x_{108} = [[[[x_3, x_4], [x_8, [x_5, x_6]]], [[x_4, x_5], [x_3, [x_1, x_2]]]],$ $[[[x_5, x_6], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_8, [x_4, x_5]]]]],$	$2\alpha_2 + 2\alpha_7 + 2\alpha_8 + 3\alpha_3 +$ $3\alpha_4 + 3\alpha_6 + 4\alpha_5 + \alpha_1,$ $2\alpha_2 + 2\alpha_8 + 3\alpha_3 + 3\alpha_6 +$ $4\alpha_4 + 4\alpha_5 + \alpha_1 + \alpha_7,$
$\boxed{x_{109}} = [[[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_6, x_7], [x_5, [x_3, x_4]]]],$ $[[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_3, [x_1, x_2]],$ $[x_6, [x_4, x_5]]]]],$ $\boxed{x_{110}} = [[[[x_5, x_8], [x_6, [x_4, x_5]]], [[x_6, x_7], [x_5, [x_3, x_4]]]],$ $[[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]],$ $[x_6, [x_4, x_5]]]]],$	$\underline{2\alpha_2 + 2\alpha_7 + 2\alpha_8 + 3\alpha_3}$ $\underline{+3\alpha_6 + 4\alpha_4 + 4\alpha_5 + \alpha_1},$ $\underline{2\alpha_2 + 2\alpha_8 + 3\alpha_3 + 3\alpha_6}$ $\underline{+4\alpha_4 + 5\alpha_5 + \alpha_1 + \alpha_7},$
$x_{111} = [[[[x_5, x_8], [x_4, [x_2, x_3]]], [[x_3, [x_1, x_2]], [x_6, [x_4, x_5]]]],$ $[[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]]],$ $x_{112} = [[[[x_5, x_8], [x_6, [x_4, x_5]]], [[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]],$ $[[[x_6, x_7], [x_5, [x_3, x_4]]], [[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]]]],$	$2\alpha_2 + 3\alpha_3 + 3\alpha_6 + 3\alpha_8 +$ $4\alpha_4 + 5\alpha_5 + \alpha_1 + \alpha_7,$ $2\alpha_2 + 2\alpha_7 + 2\alpha_8 + 3\alpha_3 +$ $3\alpha_6 + 4\alpha_4 + 5\alpha_5 + \alpha_1,$
$\boxed{x_{113}} = [[[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_3, [x_1, x_2]],$ $[x_6, [x_4, x_5]]]], [[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]],$ $[[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]]]],$ $\boxed{x_{114}} = [[[[x_6, x_7], [x_8, [x_4, x_5]]], [[x_5, [x_3, x_4]],$ $[x_8, [x_5, x_6]]]], [[[x_3, [x_1, x_2]], [x_8, [x_4, x_5]]],$ $[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]]]],$	$\underline{2\alpha_2 + 2\alpha_7 + 2\alpha_8 + 3\alpha_3}$ $\underline{+4\alpha_4 + 4\alpha_6 + 5\alpha_5 + \alpha_1},$ $\underline{2\alpha_2 + 2\alpha_7 + 3\alpha_3 + 3\alpha_6}$ $\underline{+3\alpha_8 + 4\alpha_4 + 5\alpha_5 + \alpha_1},$
$x_{115} = [[[[x_6, x_7], [[x_3, x_4], [x_5, x_8]]], [[x_3, [x_1, x_2]],$ $[x_6, [x_4, x_5]]]], [[[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]],$ $[[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]]]],$	$2\alpha_2 + 2\alpha_7 + 3\alpha_3 + 3\alpha_8 +$ $4\alpha_4 + 4\alpha_6 + 5\alpha_5 + \alpha_1,$
$\boxed{x_{116}} = [[[[x_4, [x_2, x_3]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]],$ $[x_8, [x_4, x_5]]]], [[[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]],$ $[[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]]],$	$\underline{2\alpha_2 + 2\alpha_7 + 3\alpha_3 + 3\alpha_8}$ $\underline{+4\alpha_4 + 4\alpha_6 + 6\alpha_5 + \alpha_1},$
$x_{117} = [[[[x_4, [x_2, x_3]], [x_7, [x_5, x_6]]], [[x_8, [x_4, x_5]],$ $[[x_3, x_4], [x_5, x_6]]]], [[[x_7, [x_5, x_6]], [x_8, [x_4, x_5]]],$ $[[x_8, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]]]],$	$2\alpha_2 + 2\alpha_7 + 3\alpha_3 + 3\alpha_8 +$ $4\alpha_6 + 5\alpha_4 + 6\alpha_5 + \alpha_1,$
$\boxed{x_{118}} = [[[[x_5, [x_3, x_4]], [x_8, [x_5, x_6]]], [[x_7, [x_5, x_6]],$ $[[x_1, x_2], [x_3, x_4]]]], [[[x_4, [x_2, x_3]], [[x_5, x_8],$ $[x_6, x_7]]], [[x_8, [x_4, x_5]], [[x_3, x_4], [x_5, x_6]]]]],$	$\underline{2\alpha_2 + 2\alpha_7 + 3\alpha_8 + 4\alpha_3}$ $\underline{+4\alpha_6 + 5\alpha_4 + 6\alpha_5 + \alpha_1},$
$x_{119} = [[[[x_4, [x_2, x_3]], [[x_5, x_8], [x_6, x_7]]], [[x_8, [x_4, x_5]],$ $[[x_3, x_4], [x_5, x_6]]]], [[[x_7, [x_5, x_6]], [[x_1, x_2], [x_3, x_4]]],$ $[[x_8, [x_5, x_6]], [[x_2, x_3], [x_4, x_5]]]]],$	$2\alpha_7 + 3\alpha_2 + 3\alpha_8 + 4\alpha_3 +$ $4\alpha_6 + 5\alpha_4 + 6\alpha_5 + \alpha_1,$
$\boxed{x_{120}} = [[[[x_7, [x_5, x_6]], [[x_3, x_4], [x_5, x_8]]], [[x_8, [x_5, x_6]],$ $[[x_1, x_2], [x_3, x_4]]]], [[[x_7, [x_5, x_6]], [[x_1, x_2],$ $[x_3, x_4]]], [[x_8, [x_6, [x_4, x_5]]], [[x_2, x_3], [x_4, x_5]]]]]$	$\underline{2\alpha_1 + 2\alpha_7 + 3\alpha_2 + 3\alpha_8}$ $\underline{+4\alpha_3 + 4\alpha_6 + 5\alpha_4 + 6\alpha_5}$

5 Root Systems of Lie Algebras of the Form $\mathfrak{g}(A)$ with Indecomposable *A*

5.1 $\mathfrak{wk}(3; a)$ and $\mathfrak{wk}(4; a)$, Where $\alpha \neq 0, 1$ and $p = 2$

These Lie algebras are desuperizations (which means we forget squaring and only consider the brackets, see Bouarroudj et al. 2015) of $\mathfrak{bgl}(3; a)$ and $\mathfrak{bgl}(4; a)$, respectively, so they have the same root systems, see Sects. 4.16.2 and 4.16.3.

5.2 $\mathbf{F}(\mathfrak{oo}(1|2n))$, where F is the Desuperization Functor, $p = 2$

(In Weisfeiler and Kac (1971), this simple Lie algebra is denoted Δ_n.) Its root system is the same as that of $\mathfrak{o}(2n + 1)$, see Bouarroudj et al. (2015).

5.3 $\mathfrak{br}(2; \varepsilon)$, Where $\varepsilon \neq 0$ and $p = 3$

These Lie algebras (described in Bouarroudj et al. (2011)) have the same root system as $\mathfrak{o}(5)$.

5.4 $\mathfrak{br}(3)$, $p = 3$

We consider the following (one of the two) Cartan matrix (Skryabin was the first to describe the Cartan matrices of $\mathfrak{br}(3)$, see Skryabin (1993)). The corresponding Chevalley basis is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & \bar{0} \end{pmatrix}$$

the root vectors	the roots
$x_1,\ x_2,\ x_3$	$\alpha_1,\ \alpha_2,\ \alpha_3$
$x_4 = [x_1, x_2], \quad x_5 = [x_2, x_3]$	$\alpha_1 + \alpha_2,\ \alpha_2 + \alpha_3$
$x_6 = [x_3, [x_1, x_2]], \quad x_7 = [x_3, [x_2, x_3]]$	$\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + 2\alpha_3$
$x_8 = [x_3, [x_3, [x_1, x_2]]]$	$\alpha_1 + \alpha_2 + 2\alpha_3$
$x_9 = [[x_2, x_3], [x_3, [x_1, x_2]]]$	$\alpha_1 + 2\alpha_2 + 2\alpha_3$
$x_{10} = [[x_3, [x_1, x_2]], [x_3, [x_2, x_3]]]$	$\alpha_1 + 2\alpha_2 + 3\alpha_3$
$x_{11} = [[x_3, [x_2, x_3]], [x_3, [x_3, [x_1, x_2]]]]$	$\alpha_1 + 2\alpha_2 + 4\alpha_3$
$x_{12} = [[x_3, [x_2, x_3]], [[x_2, x_3], [x_3, [x_1, x_2]]]]$	$\alpha_1 + 3\alpha_2 + 4\alpha_3$
$x_{13} = [[x_3, [x_3, [x_1, x_2]]], [[x_2, x_3], [x_3, [x_1, x_2]]]]$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3$

6 Appendix. On Restrictedness (from Bouarroudj et al. 2015)

In 2005, P. Deligne wrote several comments to a draft of Lebedev and Leites (2006), see his Appendix in Lebedev and Leites (2006). In particular, a part of his advice (in our words) was: “Over $\mathbb{K}$, to classify ALL simple Lie (super)algebras and their representations are, perhaps, not very reasonable problems, and definitely very tough; investigate first the **restricted** case: it is related to geometry, meaningful and of interest”.

Having cited Deligne's words in our papers devoted to classification of simple finite-dimensional modular Lie (super)algebras we were rebuffed by referees: non-restricted Lie (super)algebras are often needed as well, at least, to describe the restricted ones! See also studies of other topics, e.g., of p-groups, see Kostrikin (1996).

But what is restrictedness if $p = 2$? We consider here only the versions of restrictedness relevant for the exceptional cases; certain serial Lie (super)algebras can have still **other** types of restrictedness, see Bouarroudj et al. (2015).

6.1 Restrictedness on Lie Algebras

Let the ground field $\mathbb{K}$ be of characteristic $p > 0$, and $\mathfrak{g}$ a Lie algebra. For every $x \in \mathfrak{g}$, the operator $(\mathrm{ad}\,_x)^p$ is a derivation of $\mathfrak{g}$. If this derivation is an inner one, i.e., there is a map (called *p-structure*) $[p] : \mathfrak{g} \longrightarrow \mathfrak{g},\ x \mapsto x^{[p]}$ such that

$$[x^{[p]}, y] = (\mathrm{ad}\,_x)^p(y) \quad \text{for any } x, y \in \mathfrak{g}, \tag{23}$$
$$(ax)^{[p]} = a^p x^{[p]} \quad \text{for any } a \in \mathbb{K},\ x \in \mathfrak{g}, \tag{24}$$
$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{1\leq i\leq p-1} s_i(x, y) \quad \text{for any } x, y \in \mathfrak{g}, \tag{25}$$

where $is_i(x, y)$ is the coefficient of λ^{i-1} in $(\mathrm{ad}\,_{\lambda x+y})^{p-1}(x)$, then the Lie algebra $\mathfrak{g}$ is said to be *restricted* or *having a p-structure*.

6.1.1 Remarks

(1) If the Lie algebra $\mathfrak{g}$ is centerless, then the condition (23) implies conditions (24) and (25).

A p-structure on a given Lie algebra $\mathfrak{g}$ does not have to be unique; all p-structures on $\mathfrak{g}$ agree modulo center. Hence, on any simple Lie algebra, there is not more than one p-structure.

(2) According to Strade and Farnsteiner (1988, Th. 2.3, p. 71), the following condition, due to Jacobson, is sufficient for a Lie algebra $\mathfrak{g}$ to have a p-structure: for a basis $\{g_i\}_{i\in I}$ of $\mathfrak{g}$, there exist elements $g_i^{[p]}$ such that

$$[g_i^{[p]}, y] = (\mathrm{ad}\,_{g_i})^p(y) \quad \text{for any } y \in \mathfrak{g}.$$

6.1.2 Restricted Modules

A $\mathfrak{g}$-module M over a restricted Lie algebra $\mathfrak{g}$, and the representation ρ defining M, are said to be *restricted* or having a *p-structure* if

$$\rho(x^{[p]}) = (\rho(x))^p \quad \text{for any } x \in \mathfrak{g}.$$

6.2 Lie Superalgebras

Naively, the definition of *Lie superalgebra* is the same for any $p \neq 2$. Let us point at the subtleties for $p = 2$. For any p, a *Lie superalgebra* is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that the even part $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, the odd part $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module (made into the two-sided one by *anti*-symmetry, i.e., $[y, x] = -[x, y]$ for any $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}_{\bar{1}}$), and a *squaring* defined on $\mathfrak{g}_{\bar{1}}$ as a map $S^2(\mathfrak{g}_{\bar{1}}) \longrightarrow \mathfrak{g}_{\bar{0}}$:

$$\begin{gathered} x \mapsto x^2 \in \mathfrak{g}_{\bar{0}} \quad \text{such that } (ax)^2 = a^2x^2 \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } a \in \mathbb{K}, \text{ and} \\ [x, y] := (x + y)^2 - x^2 - y^2 \text{ is a bilinear form on } \mathfrak{g}_{\bar{1}} \text{ with values in } \mathfrak{g}_{\bar{0}}. \end{gathered}$$

(This extra requirement on squaring is needed, say, over $\mathbb{Z}/2$ where not any quadratic form that vanishes at the origin yields a bilinear form $[-, -]$.)

The Jacobi identity involving odd elements takes the following form:

$$[x^2, y] = [x, [x, y]] \text{ for any } x \in \mathfrak{g}_{\bar{1}}, y \in \mathfrak{g}.$$

For any Lie **super**algebra $\mathfrak{g}$, its *derived algebras* are defined to be (for $i \geq 0$)

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = \begin{cases} [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] & \text{for } p \neq 2, \\ [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathrm{Span}\{g^2 \mid g \in \mathfrak{g}^{(i)}_{\bar{1}}\} & \text{for } p = 2. \end{cases} \tag{26}$$

6.3 The $p|2p$-Structure or Restricted Lie Superalgebra

For a Lie superalgebra $\mathfrak{g}$ of characteristic $p > 0$, let the Lie algebra $\mathfrak{g}_{\bar{0}}$ be restricted and

$$[x^{[p]}, y] = (\mathrm{ad}_x)^p(y) \quad \text{for any } x \in \mathfrak{g}_{\bar{0}}, \; y \in \mathfrak{g}. \tag{27}$$

This gives rise to the map (recall that the bracket of odd elements is the polarization of the squaring $x \mapsto x^2$)

$$[2p] : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}, \quad x \mapsto (x^2)^{[p]},$$

satisfying the condition

$$[x^{[2p]}, y] = (\mathrm{ad}_x)^{2p}(y) \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}, \; y \in \mathfrak{g}. \tag{28}$$

The pair of maps $[p]$ and $[2p]$ is called a *p-structure* (or, sometimes, a *$p|2p$-structure*) on $\mathfrak{g}$, and $\mathfrak{g}$ is said to be *restricted*. It suffices to determine the $p|2p$-structure on any basis of $\mathfrak{g}$; on simple Lie superalgebras there are not more than one $p|2p$-structure.

6.3.1 Remark

If (28) is not satisfied, the p-structure on $\mathfrak{g}_{\bar{0}}$ does not have to generate a $p|2p$-structure on $\mathfrak{g}$: even if the actions of $(\mathrm{ad}_x)^p$ and $\mathrm{ad}_{x^{[p]}}$ coincide on $\mathfrak{g}_{\bar{0}}$, they do not have to coincide on the whole of $\mathfrak{g}$. The *restricted universal enveloping algebra* $U^{[p]}(\mathfrak{g})$ defined for Lie algebras $\mathfrak{g}$ as the quotient of the universal enveloping $U(\mathfrak{g})$ modulo the two-sided ideal generated by $g^{\otimes p} - g^{[p]}$ for any $g \in \mathfrak{g}$ should be defined for the Lie superalgebra $\mathfrak{g}$ as the quotient of $U(\mathfrak{g})$ modulo the two-sided ideal $\mathfrak{i}$ generated by $g^{\otimes p} - g^{[p]}$ for any $g \in \mathfrak{g}_{\bar{0}}$. The seemingly needed further factorization modulo the two-sided ideal generated by the elements $g^{\otimes 2p} - g^{[2p]}$ for any $g \in \mathfrak{g}_{\bar{1}}$ is not needed: these elements are in $\mathfrak{i}$ automatically, as is not difficult to show.

6.3.2 Restricted Modules

A $\mathfrak{g}$-module M corresponding to a representation ρ of the restricted Lie superalgebra $\mathfrak{g}$ is said to be *restricted* or having a *$p|2p$-structure* if

$$\begin{aligned} \rho(x^{[p]}) &= (\rho(x))^p \quad \text{for any } x \in \mathfrak{g}_{\bar{0}}, \\ \rho(x^{[2p]}) &= (\rho(x))^{2p} \quad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

6.4 On 2|2-Structures on Lie Superalgebras (for $p = 2$)

Let $p = 2$, a Lie superalgebra $\mathfrak{g}$ have a 2|4-structure, and $\mathbf{F}(\mathfrak{g})$ be the Lie algebra one gets from $\mathfrak{g}$ by forgetting the squaring and considering only brackets by setting $[x, x] := 2x^2 = 0$ for x odd. Then, $\mathbf{F}(\mathfrak{g})$ has a 2-structure given by

$$\begin{aligned} &\text{the “2” part of 2|4-structure on } \mathfrak{g}_{\bar{0}};\\ &\text{the squaring on } \mathfrak{g}_{\bar{1}}, \text{ i.e., } x^{[2]} := x^2;\\ &\text{the rule } (x+y)^{[2]} := \begin{cases} x^{[2]} + y^{[2]} + [x, y] & \text{if } x, y \in \mathfrak{g}_{\bar{0}}, \\ x^2 + y^{[2]} + [x, y] & \text{if } x \in \mathfrak{g}_{\bar{1}}, y \in \mathfrak{g}_{\bar{0}}, \\ x^2 + y^2 + [x, y] & \text{if } x, y \in \mathfrak{g}_{\bar{1}}. \end{cases} \end{aligned} \tag{29}$$

(Actually, the first and the third cases in (29) are redundant. If x and y are both in $\mathfrak{g}_{\bar{0}}$ or both in $\mathfrak{g}_{\bar{1}}$, then $x + y$ is homogenous, and $(x + y)^{[2]}$ in $\mathbf{F}(\mathfrak{g})$ is already given by $(x + y)^{[2]}$ or $(x + y)^2$, accordingly.) So one can say that if $p = 2$, then the restricted Lie superalgebra (i.e., the one with a 2|4-structure) also has a 2|2-*structure* which is defined even on inhomogeneous elements (unlike $p|2p$-structures). In future, for Lie superalgebras with 2|2-structure, we write $x^{[2]}$ instead of x^2 for any odd or inhomogeneous $x \in \mathfrak{g}$. The analog of sufficient condition 2) of Remarks 6.1.1 holds.

6.4.1 Restricted Modules

A $\mathfrak{g}$-module M corresponding to a representation ρ of the Lie superalgebra $\mathfrak{g}$ with 2|2-structure is said to be *restricted* or having a 2|2-*structure* if

$$\rho(x^{[2]}) = (\rho(x))^2 \quad \text{for any } x \in \mathfrak{g}.$$

6.5 Restrictedness of Lie (super)algebras with Cartan Matrix, and of Their Relatives

Let R be the set of all roots of $\mathfrak{g}$ and $\mathfrak{h}$ the maximal torus.

6.5.1 Proposition

Bouarroudj et al. (2015) (1) If $p > 2$ (or $p = 2$ and $A_{ii} \neq \bar{1}$ or 1 for any i) and $\mathfrak{g}(A)$ is a Lie (super)algebra, then $\mathfrak{g}(A)$ has a p-structure (resp. $p|2p$-structure) such that

$$\begin{aligned}
&(x_\alpha)^{[p]} = 0 \text{ for any even } \alpha \in R \text{ and } x_\alpha \in \mathfrak{g}_\alpha,\\
&(x_\alpha)^{[2p]} = 0 \text{ for any odd } \alpha \in R \text{ and } x_\alpha \in \mathfrak{g}_\alpha,\\
&\mathfrak{h}^{[p]} \subset \mathfrak{h}.
\end{aligned} \tag{30}$$

(2) If $A_{ij} \in \mathbb{Z}/p$ for all i, j, then the derived Lie (super)algebra $\mathfrak{g}^{(1)}(A)$ inherits the p-structure (resp. $p|2p$-structure) of $\mathfrak{g}(A)$ (assuming $\mathfrak{g}(A)$ has one), and we can make the 3rd line of Eq. (30) precise:

$$h_i^{[p]} = h_i \text{ for any basis element } h_i \in \mathfrak{h}. \tag{31}$$

(3) The quotient modulo center of any Lie (super)algebra $\mathfrak{g}$ with a p-structure (resp. $p|2p$-structure) always inherits the p-structure (resp. $p|2p$-structure) of $\mathfrak{g}$.

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