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RESEARCH CONTRIBUTION

Shifted Quantum Affine Algebras: Integral Forms in Type *A*

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To Rafail Kalmanovich Gordin on his 70th birthday.

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Abstract

We define an integral form of shifted quantum affine algebras of type A and construct Poincaré–Birkhoff–Witt–Drinfeld bases for them. When the shift is trivial, our integral form coincides with the RTT integral form. We prove that these integral forms are closed with respect to the coproduct and shift homomorphisms. We prove that the homomorphism from our integral form to the corresponding quantized K-theoretic Coulomb branch of a quiver gauge theory is always surjective. In one particular case we identify this Coulomb branch with the extended quantum universal enveloping algebra of type A. Finally, we obtain the rational (homological) analogues of the above results [proved earlier in Kamnitzer et al. (Proc Am Math Soc **146**(2):861–874, 2018a; On category $\mathcal{O}$ for affine Grassmannian slices and categorified tensor products. arXiv:1806.07519, 2018b) via different techniques].

Keywords Shifted Yangians · Shifted quantum affine algebras · Coulomb branch · Evaluation homomorphism · Drinfeld-Gavarini duality · PBWD bases

Michael Finkelberg is grateful to his high school teacher Rafail Kalmanovich Gordin for showing him the beauty of geometry. Since the high school years, he was suffering from an unrequited love for geometry, the present contribution being but a pathetic illustration of it.

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1 Introduction

1.1 Summary

This paper is a sequel to Finkelberg and Tsymbaliuk (2017), where we initiated the study of shifted quantum affine algebras. Recall that the shifted quantum affine algebra

$U_{\boldsymbol{v}}^{\mu}$ depends on a coweight μ of a semisimple Lie algebra $\mathfrak{g}$, and in case $\mu = 0$ it is just a central extension of the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{g})$ over the field $\mathbb{C}(\boldsymbol{v})$. Let us represent μ in the form $\mu = \lambda - \alpha$, where λ is a dominant coweight of $\mathfrak{g}$, and α is a sum of positive coroots. Also, let us assume from now on that $\mathfrak{g}$ is simply-laced. Then λ encodes a framing of a Dynkin quiver of $\mathfrak{g}$, and α encodes the dimension vector of a representation of this quiver. Let $\mathcal{A}^{\boldsymbol{v}}$ stand for the quantized K-theoretic Coulomb branch of the corresponding $3d$ $\mathcal{N} = 4$ SUSY quiver gauge theory. It is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra, and we denote $\mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}} := \mathcal{A}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$. One of the main motivations for our study of shifted quantum affine algebras was the existence of a homomorphism $\overline{\Phi}^{\lambda}_{\mu} : U_{\boldsymbol{v}}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}] \to \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$, where N is the total dimension of the framing. We conjectured that this homomorphism is surjective and also conjectured an explicit description of its kernel. In other words, we gave a conjectural presentation of $\mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$ by generators and relations as a truncated shifted quantum affine algebra U^{λ}_{μ}.

It is very much desirable to have a similar presentation for the genuine quantized K-theoretic Coulomb branch $\mathcal{A}^{\boldsymbol{v}}$ (e.g. in order to study the non-quantized K-theoretic Coulomb branch at $\boldsymbol{v} = 1$). To this end, it is necessary to construct an integral form (a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra) $\mathfrak{U}_{\boldsymbol{v}}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}] \subset U_{\boldsymbol{v}}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}]$ such that $\overline{\Phi}^{\lambda}_{\mu}(\mathfrak{U}_{\boldsymbol{v}}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}]) = \mathcal{A}^{\boldsymbol{v}}$ and the specialization $\mathfrak{U}_{\boldsymbol{v}=1}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}]$ is a commutative $\mathbb{C}$-algebra. Then $\mathcal{A}^{\boldsymbol{v}}$ would be represented as an explicit quotient algebra $\mathfrak{U}^{\lambda}_{\mu}$.

In the present paper, we restrict ourselves to the case $\mathfrak{g} = \mathfrak{sl}_n$, and propose a definition of the desired integral form $\mathfrak{U}_{\boldsymbol{v}}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}]$. It possesses a PBWD (Poincaré–Birkhoff–Witt–Drinfeld) $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-base, cf. Tsymbaliuk (2018). We prove the surjectivity of $\overline{\Phi}^{\lambda}_{\mu} : \mathfrak{U}_{\boldsymbol{v}}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}] \to \mathcal{A}^{\boldsymbol{v}}$ in Theorem 4.15. Unfortunately, we are still unable to say much about the kernel ideal of $\overline{\Phi}^{\lambda}_{\mu} : \mathfrak{U}_{\boldsymbol{v}}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}] \to \mathcal{A}^{\boldsymbol{v}}$ in the general case. The only case when we were able to determine the kernel ideal explicitly is $\mathfrak{g} = \mathfrak{sl}_n$, $\mu = 0$, $\lambda = n\omega_{n-1}$ (a multiple of the last fundamental coweight). Then the corresponding truncated shifted quantum affine $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra $\mathfrak{U}^{\lambda}_{\mu}$ is isomorphic to an integral form $\widetilde{\mathfrak{U}}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ of an extended version $\widetilde{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ of the quantized universal enveloping algebra of $\mathfrak{sl}_n$. More precisely, the Harish-Chandra center Z of $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ is isomorphic to the ring of symmetric polynomials $\left(\mathbb{C}(\boldsymbol{v})[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_n^{\pm 1}]\right)^{\Sigma_n}/(\mathtt{z}_1 \cdots \mathtt{z}_n - 1)$, and $\widetilde{U}_{\boldsymbol{v}}(\mathfrak{sl}_n) := U_{\boldsymbol{v}}(\mathfrak{sl}_n) \otimes_Z \mathbb{C}(\boldsymbol{v})[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_n^{\pm 1}]/(\mathtt{z}_1 \cdots \mathtt{z}_n - 1)$, cf. Beilinson and Ginzburg (1999). The corresponding integral form $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{sl}_n) = \widetilde{\mathfrak{U}}_{\boldsymbol{v}}(\mathfrak{sl}_n) \cap U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ of the non-extended quantized universal enveloping algebra $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ is nothing but the RTT integral form $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{sl}_n)$. It is free over $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ and admits a PBW basis. The truncation homomorphism $U_{\boldsymbol{v}}^{0}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_n^{\pm 1}] \to \widetilde{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ factors through Jimbo's evaluation homomorphism $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_n^{\pm 1}] \to \widetilde{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ of Jimbo (1986), and $\mathfrak{U}^{0}_{\boldsymbol{v}}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_n^{\pm 1}]$ is nothing but the pull-back of the RTT integral form of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_n^{\pm 1}]$ along the projection $U_{\boldsymbol{v}}^{0}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_n^{\pm 1}] \twoheadrightarrow U_{\boldsymbol{v}}(L\mathfrak{sl}_n)[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_n^{\pm 1}]$. In fact, our definition of the integral form $\mathfrak{U}_{\boldsymbol{v}}^{\mu}[\mathtt{z}_1^{\pm 1}, \dots, \mathtt{z}_N^{\pm 1}]$ for general μ was found as a straightforward generalization of the RTT integral form expressed in terms of a PBWD basis.

Note that $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ possesses three different integral forms:

(a) Lusztig's $U \otimes_{\mathbb{Z}} \mathbb{C}$ of (Lusztig 1990a, 0.4);
(b) Lusztig's ${}_{\mathcal{A}}\mathbf{O} \otimes_{\mathbb{Z}} \mathbb{C}$ of (Lusztig 1993, 29.5.1) (its specialization at $\boldsymbol{v} = 1$ is the commutative ring of functions $\mathbb{C}[\mathrm{SL}(n)]$);
(c) $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ (its specialization at $\boldsymbol{v} = 1$ is the commutative ring of functions on the big Bruhat cell of $\mathrm{SL}(n)$). It is dual to (a) with respect to a natural $\mathbb{C}(\boldsymbol{v})$-valued pairing on $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$.

We expect that $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is dual to the integral form of Chari and Pressley (1997) and (Grojnowski 1994, §7.8) of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ with respect to the new Drinfeld pairing, cf. (Grojnowski 1994, Lemma 9.1).

Finally, recall that in Finkelberg and Tsymbaliuk (2017) we have constructed the comultiplication $\mathbb{C}(\boldsymbol{v})$-algebra homomorphisms (in case $\mathfrak{g} = \mathfrak{sl}_n$) $\Delta_{\mu_1,\mu_2} : U_{\boldsymbol{v}}^{\mu_1+\mu_2} \to U_{\boldsymbol{v}}^{\mu_1} \otimes U_{\boldsymbol{v}}^{\mu_2}$ for any coweights μ_1, μ_2. We prove in Theorem 4.23 that this coproduct preserves our integral forms, and induces the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra homomorphisms $\Delta_{\mu_1,\mu_2} : \mathfrak{U}_{\boldsymbol{v}}^{\mu_1+\mu_2} \to \mathfrak{U}_{\boldsymbol{v}}^{\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mu_2}$.

To simplify the exposition of the paper, we start by establishing the rational/homological counterparts of the aforementioned results, proved earlier in Kamnitzer et al. (2018a, b) using different techniques.

In Appendix A, we collect the relevant results on shifted Yangians and Drinfeld–Gavarini duals, which are used in Sect. 2. Our objectives are twofold. First, we establish the PBW property for the Drinfeld–Gavarini dual (Proposition A.2) and apply it to the Yangians (Theorems A.7, A.10). Second, we identify two different approaches (of Kamnitzer et al. 2014; Braverman et al. 2016; Finkelberg et al. 2018) towards dominantly shifted Yangians of semisimple Lie algebras (Theorem A.12).

In Appendix B, we provide a short proof of the well-known PBW property for the Yangian $Y_\hbar(\mathfrak{g})$, since the original proof of Levendorskii (1993) contains a gap.

1.2 Outline of the Paper

• In Sect. 2.1, we recall the RTT Yangians $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$, $Y^{\mathrm{rtt}}_\hbar(\mathfrak{sl}_n)$ and their $\mathbb{C}[\hbar]$-subalgebras $\mathbf{Y}^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$, $\mathbf{Y}^{\mathrm{rtt}}_\hbar(\mathfrak{sl}_n)$. Since the terminology varies in the literature, we shall stress right away that the former two are quantizations of the universal enveloping $U(\mathfrak{gl}_n[t])$, $U(\mathfrak{sl}_n[t])$ (see Remark 2.2), while the latter two quantize the algebras of functions on the congruence subgroups $\mathrm{GL}(n)[[t^{-1}]]_1$, $\mathrm{SL}(n)[[t^{-1}]]_1$ (see Remark 2.4) and can be viewed as the Drinfeld–Gavarini dual Gavarini (2002) of the former, see Appendices A.1, A.6.

In Sect. 2.2, we recall the standard definition of the quantum minors and the quantum determinant of $T(z)$, as well as the description of the center $ZY^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$. All of this is crucially used in Sect. 2.10.

In Sect. 2.3, we recall the RTT evaluation homomorphism $\mathrm{ev}^{\mathrm{rtt}} : Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n) \twoheadrightarrow U(\mathfrak{gl}_n)$ as well as the induced homomorphism between their $\mathbb{C}[\hbar]$-subalgebras $\mathrm{ev}^{\mathrm{rtt}} : \mathbf{Y}^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n) \twoheadrightarrow \mathbb{U}(\mathfrak{gl}_n)$. The main result of this subsection provides a "minimalistic description" of the kernels of these homomorphisms, see Theorems 2.15 and 2.17 (the former is essentially due to Brundan and Kleshchev 2006).

In Sect. 2.4, we recall the Drinfeld Yangians $Y_\hbar(\mathfrak{gl}_n)$ and $Y_\hbar(\mathfrak{sl}_n)$. The isomorphism $\Upsilon : Y_\hbar(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$ (see Theorem 2.18) is due to Iohara (1996) and is essentially a Yangian counterpart of Ding and Frenkel (1993). Following Kamnitzer

et al. (2014), we define their $\mathbb{C}[\hbar]$-subalgebras $\mathbf{Y}_\hbar(\mathfrak{gl}_n)$, $\mathbf{Y}_\hbar(\mathfrak{sl}_n)$, and the main result identifies the former with $\mathbf{Y}^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$ via the isomorphism Υ, see Proposition 2.21 (a straightforward proof is sketched right after it, while a more conceptual one is provided in Appendix A.6).

In Sect. 2.5, we recall the evaluation homomorphism $\mathrm{ev} : Y_\hbar(\mathfrak{sl}_n) \to U(\mathfrak{sl}_n)$ of Drinfeld (1985) and verify its compatibility with $\mathrm{ev}^{\mathrm{rtt}}$ via Υ, see Theorem 2.25.

In Sects. 2.6 and 2.7, we recall two alternative definitions of the shifted Yangian $\mathbf{Y}_\mu$ for a general shift μ and for a dominant shift μ, respectively (μ is an element of the coweight lattice). The fact that those two approaches are indeed equivalent for dominant shifts is the subject of Theorem 2.31, the proof of which is presented in Appendix A, see Theorem A.12.

In Sects. 2.8 and 2.9, we recall two key constructions of (Braverman et al. 2016, Appendix B): the homomorphism $\Phi^\lambda_\mu : \mathbf{Y}_\mu[z_1, \dots, z_N] \to \widetilde{\mathcal{A}}_\hbar$ of Theorem 2.34, which factors through the quantized Coulomb branch $\mathcal{A}_\hbar$ giving rise to the homomorphism $\overline{\Phi}^\lambda_\mu : \mathbf{Y}_\mu[z_1, \dots, z_N] \to \mathcal{A}_\hbar$. The main result of this subsection, Proposition 2.36 due to Kamnitzer et al. (2018b), establishes the surjectivity of $\overline{\Phi}^\lambda_\mu$ in type A. An alternative proof of this result is outlined in Remark 4.16 and crucially utilizes the shuffle realizations of $Y_\hbar(\mathfrak{sl}_n)$, $\mathbf{Y}_\hbar(\mathfrak{sl}_n)$ of (Tsymbaliuk 2018, §6).

In Sect. 2.10, we prove a reduced version of the conjectured description (Braverman et al. 2016, Remark B.21) of $\mathrm{Ker}(\Phi^\lambda_\mu)$ as an explicit truncation ideal $\mathcal{I}^\lambda_\mu$ in the particular case $\mu = 0$, $\lambda = n\omega_{n-1}$ (which corresponds to the dimension vector $(1, 2, \dots, n-1)$ and the framing $(0, \dots, 0, n)$), see Theorem 2.39. An alternative proof of this result was given earlier in Kamnitzer et al. (2018a). The key ingredient in our proof, Theorem 2.41, identifies the reduced truncation ideal $\underline{\mathcal{I}}^{n\omega_{n-1}}_0$ with the kernel of a certain version of the evaluation homomorphism ev. This culminates in Corollary 2.44, where we identify the corresponding reduced Coulomb branch $\underline{\mathcal{A}}_\hbar$ with the integral form of the extended (in the sense of Beilinson and Ginzburg (1999)) universal enveloping algebra of $\mathfrak{sl}_n$.

• In Sect. 3.1, we recall the RTT integral form $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ following Faddeev et al. (1989); Ding and Frenkel (1993). The latter is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra, which can be thought of as a quantization of the algebra of functions on the big Bruhat cell in $\mathrm{GL}(n)$ (see (3.5) and Remark 3.15) as $\boldsymbol{v} \to 1$.

In Sect. 3.2, we recall the RTT integral form $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ following Faddeev et al. (1989), Ding and Frenkel (1993). The latter is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra, which can be thought of as a quantization of the algebra of functions on the thick slice ${}^\dagger\mathcal{W}_0$ of (Finkelberg and Tsymbaliuk 2017, 4(viii)) (see (3.10) and Remark 3.26) as $\boldsymbol{v} \to 1$.

In Sect. 3.3, we recall the RTT evaluation homomorphism $\mathrm{ev}^{\mathrm{rtt}} : \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \twoheadrightarrow \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$. The main result of this subsection provides a "minimalistic description" of the kernel of this homomorphism, see Theorem 3.7.

In Sect. 3.4, we recall the Drinfeld-Jimbo quantum $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$, $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ defined over $\mathbb{C}(\boldsymbol{v})$, and an isomorphism $\Upsilon : U_{\boldsymbol{v}}(\mathfrak{gl}_n) \xrightarrow{\sim} \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$ of Ding and Frenkel (1993) (see Theorem 3.9). We introduce $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebras $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)$, $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ in Definition 3.10, and identify the former with $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ via Υ, see

Proposition 3.11. Finally, linear $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-bases of $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)$, $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ are constructed in Theorem 3.14.

In Sect. 3.5, we recall the Drinfeld-Jimbo quantum loop algebras $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ defined over $\mathbb{C}(\boldsymbol{v})$, and an isomorphism $\Upsilon : U_{\boldsymbol{v}}(L\mathfrak{gl}_n) \xrightarrow{\sim} \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$ of Ding and Frenkel (1993) (see Theorem 3.17). Following Tsymbaliuk (2018), we introduce $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebras $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ in Definition 3.19, and identify the former with $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ via Υ, see Proposition 3.20. Finally, based on Theorem 3.25 (proved in Tsymbaliuk 2018), we construct linear $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-bases of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ in Theorem 3.24.

In Sect. 3.6, we recall the shuffle realizations of $U^{>}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ and its integral form $\mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ as recently established in Tsymbaliuk (2018), see Theorems 3.28, 3.30 and Proposition 3.29. This is crucially used in Sect. 4.

In Sect. 3.7, we recall the evaluation homomorphism $\mathrm{ev} : U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \to U_{\boldsymbol{v}}(\mathfrak{gl}_n)$ of Jimbo (1986) (see Theorem 3.32) and verify its compatibility with (a $\mathbb{C}(\boldsymbol{v})$-extension of) $\mathrm{ev}^{\mathrm{rtt}}$ via Υ, see Theorem 3.33.

In Sect. 3.8, we recall the standard definition of the quantum minors and the quantum determinant of $T^{\pm}(z)$, as well as the description of the center of $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$. All of this is crucially used in Sect. 4.3.

In Sect. 3.9, we slightly generalize the algebras of the previous subsections, which is needed for Sect. 4.3.

• In Sect. 4.1, we recall the notion of shifted quantum affine algebras of Finkelberg and Tsymbaliuk (2017): $U^{\mathrm{sc},\mu}_{\boldsymbol{v}}$ and $U^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ (depending on a coweight μ). We introduce their $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebras $\mathfrak{U}^{\mathrm{sc},\mu}_{\boldsymbol{v}}$, $\mathfrak{U}^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ and construct linear $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-bases for those in Theorem 4.4. We also recall the homomorphism $\widetilde{\Phi}^{\lambda}_{\mu} : U^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ of Finkelberg and Tsymbaliuk (2017) (see Theorem 4.1).

In Sect. 4.2, we recall the notion of the (extended) quantized K-theoretic Coulomb branch $\mathcal{A}^{\boldsymbol{v}}$ (which is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra) and the fact that $\widetilde{\Phi}^{\lambda}_{\mu}$ gives rise to a homomorphism $\overline{\Phi}^{\lambda}_{\mu} : U^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \to \mathcal{A}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$. In Proposition 4.9 we prove that the integral form $\mathfrak{U}^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ is mapped to $\mathcal{A}^{\boldsymbol{v}}$ under $\overline{\Phi}^{\lambda}_{\mu}$, which is based on explicit formulas (4.6, 4.7). In Theorem 4.11, we provide a shuffle interpretation of the homomorphism $\widetilde{\Phi}^{\lambda}_{\mu}$ when restricted to either positive or negative halves of $U^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$. In Proposition 4.12, we combine this result with the shuffle description of the integral forms $\mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, $\mathfrak{U}^{<}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ to compute $\widetilde{\Phi}^{\lambda}_{\mu}$-images of certain elements in $\mathfrak{U}^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$. Combining this computation with the ideas of Cautis and Williams (2018), we finally prove that $\overline{\Phi}^{\lambda}_{\mu} : \mathfrak{U}^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \to \mathcal{A}^{\boldsymbol{v}}$ is surjective, see Theorem 4.15.

In Sect. 4.3, we prove a reduced version of the integral counterpart of (Finkelberg and Tsymbaliuk 2017, Conjecture 8.14), see Conjecture 4.17, which identifies $\mathrm{Ker}(\Phi^{\lambda}_{\mu})$ with an explicit truncation ideal $\mathfrak{I}^{\lambda}_{\mu}$ in the particular case $\mu = 0$, $\lambda = n\omega_{n-1}$ (which corresponds to the dimension vector $(1, 2, \ldots, n-1)$ and the framing $(0, \ldots, 0, n)$), see Theorem 4.18. The key ingredient in our proof, Theorem 4.19, identifies the reduced truncation ideal $\mathfrak{I}^{n\omega_{n-1}}_{0}$ with the kernel of a certain version

of the evaluation homomorphism ev. This culminates in Corollary 4.22, where we identify the corresponding reduced quantized Coulomb branch $\underline{\mathcal{A}}^{\boldsymbol{v}}$ with the extended version (in the sense of Beilinson and Ginzburg (1999)) of $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$.

In Sect. 4.4, we prove that the $\mathbb{C}(\boldsymbol{v})$-algebra homomorphisms $\Delta_{\mu_1,\mu_2} : U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2} \to U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes U_{\boldsymbol{v}}^{\mathrm{sc},\mu_2}$ of (Finkelberg and Tsymbaliuk 2017, Theorem 10.26) generalizing the Drinfeld-Jimbo coproduct on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ give rise to $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra homomorphisms $\Delta_{\mu_1,\mu_2} : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2} \to \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2}$, see Theorem 4.23. We also prove that the integral forms $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\bullet}$ are intertwined by the shift homomorphisms of (Finkelberg and Tsymbaliuk 2017, Lemma 10.24), see Lemma 4.31.

• In Appendix A.1, we recall the notion of the Drinfeld–Gavarini dual A' of a Hopf algebra A defined over $\mathbb{C}[\hbar]$, see (A.1, A.2).

In Appendix A.2, following the ideas of Gavarini (2002), we establish a PBW theorem for the Drinfeld–Gavarini dual A' of a Hopf algebra A satisfying Assumptions (As1)–(As3), see Proposition A.2. This yields an explicit description of A'.

In Appendix A.3, assuming that the Hopf algebra A is in addition graded (see assumption (As4)), we identify its Drinfeld–Gavarini dual A' with the Rees algebra of the specialization $A_{\hbar=1}$ with respect to the filtration (A.10), see Proposition A.4.

In Appendix A.4, we briefly recall the Yangian $Y_\hbar = Y_\hbar(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ (generalizing the case $\mathfrak{g} = \mathfrak{sl}_n$ featuring in Sect. 2) and its key relevant properties.

In Appendix A.5, we verify that the aforementioned Assumptions (As1)–(As3) hold for $Y_\hbar$, hence, Proposition A.2 applies. This culminates in the explicit description of the Drinfeld–Gavarini dual $Y'_\hbar$ (thus filling in the gap of the description of $Y'_\hbar$ given just before (Kamnitzer et al. 2014, Theorem 3.5)) and establishes a PBW theorem for it, see Theorem A.7. The validity of the assumption (As4) for $Y_\hbar$ and Proposition A.4 yield a Rees algebra description of $Y'_\hbar$, see Corollary A.8.

In Appendix A.6, we verify that Assumptions (As1)–(As3) hold for the RTT Yangian $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$. This gives rise to the identification of its Drinfeld–Gavarini dual $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)'$ with the subalgebra $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of Definition 2.3, as well as establishes the PBW theorem (that we referred to in Sect. 2) for the latter, see Theorem A.10. As an immediate corollary, we also deduce a new conceptual proof of Proposition 2.21.

In Appendix A.7, we compare two definitions of dominantly shifted Yangians for any semisimple Lie algebra $\mathfrak{g}$: the Rees algebra construction of Section 2.6 (following the approach undertaken in Braverman et al. (2016); Finkelberg et al. (2018)) and the subalgebra construction of Sect. 2.7 (following the original approach of Kamnitzer et al. (2014)). Our main result, Theorem A.12 (generalizing Theorem 2.31 stated for $\mathfrak{g} = \mathfrak{sl}_n$) provides an identification of these two definitions.

In Appendix A.8, we introduce one more definition of the shifted Yangian and prove in Theorem A.17 that it is equivalent to the Rees algebra construction.

• In Appendix B.1, we state a simple but useful general result, Lemma B.1, relating the specializations of the graded $\mathbb{C}[\hbar]$-algebra at $\hbar = 0$ and $\hbar = 1$. This is needed for Theorem B.2.

In Appendix B.2, we recall the basic facts about $Y = Y_{\hbar=1}$.

In Appendix B.3, we establish the PBW theorem for Y (thus filling in the gap of Levendorskii (1993), though our proof is different), see Theorem B.2, which allows us to immediately deduce the PBW theorem for the Yangian $Y_\hbar$, see Theorem B.3.

2 Shifted Yangian

This section is a rational/cohomological prototype of Sects. 3, 4.

2.1 The RTT Yangian of $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$

Let $\hbar$ be a formal variable. Consider the *rational* R-matrix

$$R_{\mathrm{rat}}(z) = R^{\hbar}_{\mathrm{rat}}(z) = 1 - \frac{\hbar}{z} P \tag{2.1}$$

which is an element of $\mathbb{C}[\hbar]\otimes_{\mathbb{C}}(\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$, where $P = \sum_{i,j} E_{ij}\otimes E_{ji} \in (\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$ is the permutation operator. It satisfies the famous *Yang-Baxter equation with a spectral parameter*:

$$R_{\mathrm{rat};12}(u)R_{\mathrm{rat};13}(u+v)R_{\mathrm{rat};23}(v) = R_{\mathrm{rat};23}(v)R_{\mathrm{rat};13}(u+v)R_{\mathrm{rat};12}(u). \tag{2.2}$$

Following Faddeev et al. (1989), define the *RTT Yangian of* $\mathfrak{gl}_n$, denoted by $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$, to be the associative $\mathbb{C}[\hbar]$-algebra generated by $\{t^{(r)}_{ij}\}^{r\geq 1}_{1\leq i,j\leq n}$ subject to the following defining relations:

$$R_{\mathrm{rat}}(z-w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{\mathrm{rat}}(z-w). \tag{2.3}$$

Here $T(z)$ is the series in z^{-1} with coefficients in the algebra $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)\otimes \mathrm{End}\,\mathbb{C}^n$, defined by $T(z) = \sum_{i,j} t_{ij}(z)\otimes E_{ij}$ with $t_{ij}(z) := \delta_{ij} + \hbar\sum_{r>0} t^{(r)}_{ij} z^{-r}$. Multiplying both sides of (2.3) by $z-w$, we obtain an equality of series in z, w with coefficients in $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)\otimes(\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$.

Let $ZY^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$ denote the center of $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$. Explicitly, $ZY^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n) \simeq \mathbb{C}[\hbar][d_1, d_2, \ldots]$ with d_r defined via qdet $T(z) = 1 + \hbar\sum_{r\geq 1} d_r z^{-r}$, see Definition 2.9 and Proposition 2.10.

For any formal series $f(z) \in 1 + \frac{\hbar}{z}\mathbb{C}[\hbar][[z^{-1}]]$, the assignment

$$T(z) \mapsto f(z)T(z) \tag{2.4}$$

defines an algebra automorphism of $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$.

Definition 2.1 The $\mathbb{C}[\hbar]$-subalgebra $Y^{\mathrm{rtt}}_\hbar(\mathfrak{sl}_n)$ of $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$ formed by all the elements fixed under all automorphisms (2.4) is called the *RTT Yangian of* $\mathfrak{sl}_n$.

Analogously to (Molev 2007, Theorem 1.8.2),[1] we have a $\mathbb{C}[\hbar]$-algebra isomorphism

$$Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n) \simeq Y^{\mathrm{rtt}}_\hbar(\mathfrak{sl}_n)\otimes_{\mathbb{C}[\hbar]} ZY^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n). \tag{2.5}$$

[1] We note that the $\mathbb{C}$-algebras of *loc.cit.* are the quotients of their $\mathbb{C}[\hbar]$-counterparts above by $(\hbar - 1)$.

Hence, there is a natural projection $\pi : Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \twoheadrightarrow Y_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)$ with $\mathrm{Ker}(\pi) = (d_1, d_2, \ldots)$.

Remark 2.2 Note that the assignment $t_{ij}^{(r)} \mapsto E_{ij} \cdot t^{r-1}$ gives rise to a $\mathbb{C}$-algebra isomorphism $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)/(\hbar) \xrightarrow{\sim} U(\mathfrak{gl}_n[t])$. This explains why $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ is usually treated as a quantization of the universal enveloping algebra $U(\mathfrak{gl}_n[t])$.

Definition 2.3 Let $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ be the $\mathbb{C}[\hbar]$-subalgebra of $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ generated by $\{\hbar t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 1}$.

Let us note right away that (2.4) with $f(z) \in 1 + \frac{\hbar}{z}\mathbb{C}[\hbar][[z^{-1}]]$ defines an algebra automorphism of $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$. As in Definition 2.1, define $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)$ to be the $\mathbb{C}[\hbar]$-subalgebra of $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ formed by all the elements fixed under these automorphisms. We also note that the center $Z\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ is explicitly given by $Z\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \simeq \mathbb{C}[\hbar][\hbar d_1, \hbar d_2, \ldots]$ (clearly $\{\hbar d_r\}_{r\ge 1} \subset \mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$). Finally, we also have a $\mathbb{C}[\hbar]$-algebra isomorphism $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \simeq \mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n) \otimes_{\mathbb{C}[\hbar]} Z\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$, cf. (2.5). Hence, there is a natural projection $\pi : \mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \twoheadrightarrow \mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)$ with $\mathrm{Ker}(\pi) = (\hbar d_1, \hbar d_2, \ldots)$.

Remark 2.4 In contrast to Remark 2.2, we note that the assignment $\hbar t_{ij}^{(r)} \mapsto \mathfrak{t}_{ij}^{(r)}$ gives rise to a $\mathbb{C}$-algebra isomorphism $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)/(\hbar) \simeq \mathbb{C}[\mathfrak{t}_{ij}^{(r)}]_{1\le i,j\le n}^{r\ge 1}$. In other words, $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ can be treated as a quantization of the algebra of functions on the congruence subgroup $\mathrm{GL}(n)[[t^{-1}]]_1 :=$ the kernel of the evaluation homomorphism $\mathrm{GL}(n)[[t^{-1}]] \to \mathrm{GL}(n)$.

2.2 Quantum Minors of $T(z)$

We recall the notion of quantum minors following (Molev 2007, §1.6). This generalizes qdet $T(z)$ featuring in Sect. 2.1, and will be used in the proof of Theorem 2.41. For $1 < r \le n$, define $R(z_1, \ldots, z_r) \in (\mathrm{End}\ \mathbb{C}^n)^{\otimes r}$ via

$$R(z_1, \ldots, z_r) := (R_{r-1,r})(R_{r-2,r}R_{r-2,r-1}) \cdots (R_{1r} \cdots R_{12}) \text{ with}$$
$$R_{ij} := R_{\mathrm{rat};ij}(z_i - z_j).$$

The following is implied by (2.2) and (2.3), cf. (Molev 2007, Proposition 1.6.1):

Lemma 2.5 $R(z_1, \ldots, z_r)T_1(z_1) \cdots T_r(z_r) = T_r(z_r) \cdots T_1(z_1)R(z_1, \ldots, z_r)$.

Let $A_r \in (\mathrm{End}\ \mathbb{C}^n)^{\otimes r}$ denote the image of the antisymmetrizer $\sum_{\sigma\in\Sigma_r}(-1)^\sigma \cdot \sigma \in \mathbb{C}[\Sigma_r]$ under the natural action of the symmetric group Σ_r on $(\mathbb{C}^n)^{\otimes r}$. Recall the following classical observation, cf. (Molev 2007, Proposition 1.6.2):

Proposition 2.6 $R(z, z-\hbar, \ldots, z-(r-1)\hbar) = A_r$.

Combining Lemma 2.5 and Proposition 2.6, we obtain the following

Corollary 2.7 *We have*

$$A_r T_1(z)T_2(z-\hbar) \cdots T_r(z-(r-1)\hbar) = T_r(z-(r-1)\hbar) \cdots T_2(z-\hbar)T_1(z)A_r. \quad (2.6)$$

The operator of (2.6) can be written as $\sum t_{b_1\dots b_r}^{a_1\dots a_r}(z)\otimes E_{a_1,b_1}\otimes\dots\otimes E_{a_r,b_r}$ with $t_{b_1\dots b_r}^{a_1\dots a_r}(z)\in \mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[[z^{-1}]]$ and the sum taken over all $a_1,\dots,a_r,b_1,\dots,b_r\in\{1,\dots,n\}$.

Definition 2.8 The coefficients $t_{b_1\dots b_r}^{a_1\dots a_r}(z)$ are called the *quantum minors* of $T(z)$.

In the particular case $r=n$, the image of the operator A_n acting on $(\mathbb{C}^n)^{\otimes n}$ is 1-dimensional. Hence $A_nT_1(z)\cdots T_n(z-(n-1)\hbar)=A_n\cdot\mathrm{qdet}\,T(z)$ with $\mathrm{qdet}\,T(z)\in\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[[z^{-1}]]$. We note that $\mathrm{qdet}\,T(z)=t_{1\dots n}^{1\dots n}(z)$ in the above notations.

Definition 2.9 $\mathrm{qdet}\,T(z)$ is called the *quantum determinant* of $T(z)$.

Since $t_{ij}(z)\in\delta_{ij}+\hbar Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[[z^{-1}]]$, it is clear that $\mathrm{qdet}\,T(z)\in 1+\hbar Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[[z^{-1}]]$. Hence, it is of the form $\mathrm{qdet}\,T(z)=1+\hbar\sum_{r\geq 1}d_rz^{-r}$ with $d_r\in Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$. The following result is well-known, cf. (Molev 2007, Theorem 1.7.5):

Proposition 2.10 *The elements $\{d_r\}_{r\geq 1}$ are central, algebraically independent, and generate the center $ZY_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$. In other words, we have a $\mathbb{C}[\hbar]$-algebra isomorphism $ZY_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)\simeq\mathbb{C}[\hbar][d_1,d_2,\dots]$.*

2.3 The RTT Evaluation Homomorphism $\mathrm{ev}^{\mathrm{rtt}}$

Definition 2.11 Let $U(\mathfrak{gl}_n)$ be the universal enveloping algebra of $\mathfrak{gl}_n$ over $\mathbb{C}[\hbar]$.

Recall the following two standard relations between $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ and $U(\mathfrak{gl}_n)$:

Lemma 2.12 *(a) The assignment $E_{ij}\mapsto t_{ij}^{(1)}$ gives rise to a $\mathbb{C}[\hbar]$-algebra embedding*

$$\iota: U(\mathfrak{gl}_n)\hookrightarrow Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n).$$

(b) The assignment $t_{ij}^{(r)}\mapsto\delta_{r,1}E_{ij}$ gives rise to a $\mathbb{C}[\hbar]$-algebra epimorphism

$$\mathrm{ev}^{\mathrm{rtt}}: Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)\twoheadrightarrow U(\mathfrak{gl}_n).$$

The homomorphism $\mathrm{ev}^{\mathrm{rtt}}$ is called the *RTT evaluation homomorphism*.

Remark 2.13 (a) The composition $\mathrm{ev}^{\mathrm{rtt}}\circ\iota$ is the identity endomorphism of $U(\mathfrak{gl}_n)$.
(b) Define $T:=\sum_{i,j}E_{ij}\otimes E_{ij}\in U(\mathfrak{gl}_n)\otimes\mathrm{End}\,\mathbb{C}^n$. Then $\mathrm{ev}^{\mathrm{rtt}}: T(z)\mapsto 1+\frac{\hbar}{z}T$.

Let $\mathbb{U}(\mathfrak{gl}_n)$ be the $\mathbb{C}[\hbar]$-subalgebra of $U(\mathfrak{gl}_n)$ generated by $\{\hbar x\}_{x\in\mathfrak{gl}_n}$. It is isomorphic to the $\hbar$-deformed universal enveloping algebra:

$$\mathbb{U}(\mathfrak{gl}_n)\simeq T(\mathfrak{gl}_n)/(\langle\{xy-yx-\hbar[x,y]\}_{x,y\in\mathfrak{gl}_n}\rangle),$$

where $T(\mathfrak{gl}_n)$ denotes the tensor algebra of $\mathfrak{gl}_n$ over $\mathbb{C}[\hbar]$. We note that the homomorphisms ι and $\mathrm{ev}^{\mathrm{rtt}}$ of Lemma 2.12 give rise to $\mathbb{C}[\hbar]$-algebra homomorphisms

$$\iota:\mathbb{U}(\mathfrak{gl}_n)\hookrightarrow\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)\ \text{and}\ \mathrm{ev}^{\mathrm{rtt}}:\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)\twoheadrightarrow\mathbb{U}(\mathfrak{gl}_n). \tag{2.7}$$

The PBW theorems for $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ (see Proposition A.9, cf. (Molev 2007, Theorem 1.4.1)) and $U(\mathfrak{gl}_n)$ imply the following simple result:

Lemma 2.14 $\mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}} : Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n))$ *is the* 2*-sided ideal generated by* $\{t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 2}$.

However, we will need an alternative description of this kernel $\mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}})$, essentially due to (Brundan and Kleshchev 2006, Section 6) (by taking further Rees algebras).

Theorem 2.15 *Let* I *denote the* 2*-sided ideal of* $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ *generated by* $\{t_{11}^{(r)}\}_{r\ge 2}$. *Then* $\mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}} : Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)) = I$.

Proof Recall that (2.3) is equivalent to

$$(z-w)[t_{ij}(z), t_{kl}(w)] = \hbar(t_{kj}(z)t_{il}(w) - t_{kj}(w)t_{il}(z))$$

for any $1 \le i, j, k, l \le n$, which in turn is equivalent to (cf. (Molev 2007, Proposition 1.1.2))

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \hbar \sum_{a=1}^{\min(r,s)} \left(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)} \right), \tag{2.8}$$

where we set $t_{ij}^{(0)} := \hbar^{-1}\delta_{ij}$.

- Set $i = j = k = 1, l > 1, s = 1$ in (2.8) to get $[t_{11}^{(r)}, t_{1l}^{(1)}] = t_{1l}^{(r)}$. Hence $\{t_{1l}^{(r)}\}_{l>1}^{r\ge 2} \subset I$.
- Set $i = j = l = 1, k > 1, s = 1$ in (2.8) to get $[t_{11}^{(r)}, t_{k1}^{(1)}] = -t_{k1}^{(r)}$. Hence $\{t_{k1}^{(r)}\}_{k>1}^{r\ge 2} \subset I$.
- Set $i = l = 1, j = k = 2, s = 1$ in (2.8) to get $[t_{12}^{(r)}, t_{21}^{(1)}] = t_{11}^{(r)} - t_{22}^{(r)}$. Hence $\{t_{22}^{(r)}\}_{r\ge 2} \subset I$.

One can now apply the above three verifications with all lower indices increased by 1. Proceeding further step by step, we obtain $\{t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 2} \subset I$.

This completes our proof of Theorem 2.15. □

Likewise, the PBW theorems for $\mathbb{U}(\mathfrak{gl}_n)$ and $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ [see Theorem A.10, cf. (Molev 2007, Theorem 1.4.1)] imply the following result:

Lemma 2.16 $\mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}} : \mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \to \mathbb{U}(\mathfrak{gl}_n))$ *is the* 2*-sided ideal generated by* $\{\hbar t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 2}$.

The following alternative description follows immediately from Theorem 2.15:

Theorem 2.17 $\mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}} : \mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \to \mathbb{U}(\mathfrak{gl}_n)) = \mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \cap I$.

2.4 The Drinfeld Yangian of $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$

Following Drinfeld (1988) [cf. Iohara (1996), Molev (2007)], define the Yangian of $\mathfrak{gl}_n$, denoted by $Y_\hbar(\mathfrak{gl}_n)$, to be the associative $\mathbb{C}[\hbar]$-algebra generated by $\{e_i^{(r)}, f_i^{(r)}, \zeta_j^{(r)}\}_{1\le i<n, 1\le j\le n}^{r\ge 0}$ with the following defining relations:

$$
\begin{aligned}
&\left[\zeta_j^{(r)}, \zeta_{j'}^{(s)}\right] = 0,\\
&\left[e_i^{(r+1)}, e_{i'}^{(s)}\right] - \left[e_i^{(r)}, e_{i'}^{(s+1)}\right] = \frac{c_{ii'}\hbar}{2}\left(e_i^{(r)} e_{i'}^{(s)} + e_{i'}^{(s)} e_i^{(r)}\right),\\
&\left[f_i^{(r+1)}, f_{i'}^{(s)}\right] - \left[f_i^{(r)}, f_{i'}^{(s+1)}\right] = -\frac{c_{ii'}\hbar}{2}\left(f_i^{(r)} f_{i'}^{(s)} + f_{i'}^{(s)} f_i^{(r)}\right),\\
&\left[\zeta_j^{(0)}, e_i^{(r)}\right] = (-\delta_{ji} + \delta_{j,i+1}) e_i^{(r)},\ \left[\zeta_j^{(0)}, f_i^{(r)}\right] = (\delta_{ji} - \delta_{j,i+1}) f_i^{(r)},\\
&\left[\zeta_j^{(s+1)}, e_i^{(r)}\right] - \left[\zeta_j^{(s)}, e_i^{(r+1)}\right] = \hbar\cdot\left(-\delta_{ji}\zeta_j^{(s)} e_i^{(r)} + \delta_{j,i+1}/2\cdot(\zeta_j^{(s)} e_i^{(r)} + e_i^{(r)}\zeta_j^{(s)})\right),\\
&\left[\zeta_j^{(s+1)}, f_i^{(r)}\right] - \left[\zeta_j^{(s)}, f_i^{(r+1)}\right] = \hbar\cdot\left(\delta_{ji}\zeta_j^{(s)} f_i^{(r)} - \delta_{j,i+1}/2\cdot(\zeta_j^{(s)} f_i^{(r)} + f_i^{(r)}\zeta_j^{(s)})\right),\\
&\left[e_i^{(r)}, f_{i'}^{(s)}\right] = \delta_{ii'} h_i^{(r+s)},\\
&\left[e_i^{(r)}, e_{i'}^{(s)}\right] = 0 \text{ and } \left[f_i^{(r)}, f_{i'}^{(s)}\right] = 0 \text{ if } c_{ii'} = 0,\\
&\left[e_i^{(r_1)}, \left[e_i^{(r_2)}, e_{i'}^{(s)}\right]\right] + \left[e_i^{(r_2)}, \left[e_i^{(r_1)}, e_{i'}^{(s)}\right]\right] = 0 \text{ if } c_{ii'} = -1,\\
&\left[f_i^{(r_1)}, \left[f_i^{(r_2)}, f_{i'}^{(s)}\right]\right] + \left[f_i^{(r_2)}, \left[f_i^{(r_1)}, f_{i'}^{(s)}\right]\right] = 0 \text{ if } c_{ii'} = -1,
\end{aligned}
\tag{2.9}
$$

where $(c_{ii'})_{i,i'=1}^{n-1}$ denotes the Cartan matrix of $\mathfrak{sl}_n$ and $\{h_i^{(r)}\}_{1\le i<n}^{r\in\mathbb{N}}$ are the coefficients of the generating series $h_i(z) = 1 + \hbar\sum_{r\ge 0} h_i^{(r)} z^{-r-1}$ determined via $h_i(z) := (\zeta_i(z))^{-1}\zeta_{i+1}(z-\hbar/2)$. Here the generating series $e_i(z)$, $f_i(z)$ $(1 \le i < n)$ and $\zeta_j(z)$ $(1 \le j \le n)$ are defined via

$$
e_i(z) := \hbar\sum_{r\ge 0} e_i^{(r)} z^{-r-1}, \quad f_i(z) := \hbar\sum_{r\ge 0} f_i^{(r)} z^{-r-1}, \quad \zeta_j(z) := 1 + \hbar\sum_{r\ge 0}\zeta_j^{(r)} z^{-r-1}.
$$

The $\mathbb{C}[\hbar]$-subalgebra of $Y_\hbar(\mathfrak{gl}_n)$ generated by $\{e_i^{(r)}, f_i^{(r)}, h_i^{(r)}\}_{1\le i<n}^{r\ge 0}$ is isomorphic to the Yangian of $\mathfrak{sl}_n$, denoted by $Y_\hbar(\mathfrak{sl}_n)$. To be more precise, this recovers the new Drinfeld realization of $Y_\hbar(\mathfrak{sl}_n)$, see Drinfeld (1988). The latter also admits the original *J-presentation* with generators $\{x,\ J(x)\}_{x\in\mathfrak{sl}_n}$ and a certain list of the defining relations which we shall skip, see Drinfeld (1985).

To relate $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ and $Y_\hbar(\mathfrak{gl}_n)$, consider the Gauss decomposition of $T(z)$ of Sect. 2.1:

$$
T(z) = F(z)\cdot G(z)\cdot E(z).
$$

Here $F(z)$, $G(z)$, $E(z)$ are the series in z^{-1} with coefficients in the algebra $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)\otimes \mathrm{End}\,\mathbb{C}^n$ which are of the form

$$
\begin{aligned}
F(z) &= \sum_i E_{ii} + \sum_{i>j} f_{ij}(z)\cdot E_{ij},\ G(z) = \sum_i g_i(z)\cdot E_{ii},\\
E(z) &= \sum_i E_{ii} + \sum_{i<j} e_{ij}(z)\cdot E_{ij}.
\end{aligned}
$$

Theorem 2.18 (Iohara 1996; cf. Ding and Frenkel 1993) *There is a unique* $\mathbb{C}[\hbar]$*-algebra isomorphism*

$$\Upsilon : Y_\hbar(\mathfrak{gl}_n) \xrightarrow{\sim} Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$$

defined by

$$e_i(z) \mapsto e_{i,i+1}(z+i\hbar/2),\ f_i(z) \mapsto f_{i+1,i}(z+i\hbar/2),\ \zeta_j(z) \mapsto g_j(z+j\hbar/2). \quad (2.10)$$

As an immediate corollary, $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ is realized as a $\mathbb{C}[\hbar]$-subalgebra of $Y_\hbar(\mathfrak{gl}_n)$. To describe this subalgebra explicitly, define the elements $\{E^{(r)}_{\alpha^\vee}, F^{(r)}_{\alpha^\vee}\}^{r\geq 0}_{\alpha^\vee\in\Delta^+}$ of $Y_\hbar(\mathfrak{gl}_n)$ via

$$\begin{aligned} E^{(r)}_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_i^\vee} &:= \Big[\cdots\Big[e^{(r)}_{j,}, e^{(0)}_{j+1}\Big], \cdots, e^{(0)}_i\Big], \\ F^{(r)}_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_i^\vee} &:= \Big[f^{(0)}_i, \cdots, \Big[f^{(0)}_{j+1}, f^{(r)}_j\Big]\cdots\Big]. \end{aligned} \quad (2.11)$$

Here $\{\alpha_i^\vee\}_{i=1}^{n-1}$ are the standard simple roots of $\mathfrak{sl}_n$, and Δ^+ denotes the set of positive roots, $\Delta^+ = \{\alpha_j^\vee + \alpha_{j+1}^\vee + \cdots + \alpha_i^\vee\}_{1\leq j\leq i\leq n-1}$.

Definition 2.19 (a) Let $\mathbf{Y}_\hbar(\mathfrak{gl}_n)$ be the $\mathbb{C}[\hbar]$-subalgebra of $Y_\hbar(\mathfrak{gl}_n)$ generated by

$$\left\{\hbar E^{(r)}_{\alpha^\vee}, \hbar F^{(r)}_{\alpha^\vee}\right\}^{r\geq 0}_{\alpha^\vee\in\Delta^+} \cup \left\{\hbar\zeta^{(r)}_j\right\}^{r\geq 0}_{1\leq j\leq n}. \quad (2.12)$$

(b) Let $\mathbf{Y}_\hbar(\mathfrak{sl}_n)$ be the $\mathbb{C}[\hbar]$-subalgebra of $Y_\hbar(\mathfrak{sl}_n)$ generated by

$$\left\{\hbar E^{(r)}_{\alpha^\vee}, \hbar F^{(r)}_{\alpha^\vee}\right\}^{r\geq 0}_{\alpha^\vee\in\Delta^+} \cup \left\{\hbar h^{(r)}_i\right\}^{r\geq 0}_{1\leq i<n}. \quad (2.13)$$

Remark 2.20 The subalgebra $\mathbf{Y}_\hbar(\mathfrak{gl}_n)$ is free over $\mathbb{C}[\hbar]$ and the ordered PBW monomials in the generators (2.12) form its basis. This can be derived similarly to Theorem 3.24, cf. (Tsymbaliuk 2018, Theorem 6.8). An alternative proof (valid for all Yangians) is provided in Appendix A, see Theorem A.7.

Proposition 2.21 $\mathbf{Y}_\hbar(\mathfrak{gl}_n) = \Upsilon^{-1}(\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n))$.

The proof of Proposition 2.21 follows immediately from Proposition 2.22 and Corollary 2.23 below. To state those, let us express the matrix coefficients of $F(z), G(z), E(z)$ as series in z^{-1} with coefficients in $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$:

$$e_{ij}(z) = \hbar\sum_{r\geq 1} e^{(r)}_{ij} z^{-r}, \quad f_{ij}(z) = \hbar\sum_{r\geq 1} f^{(r)}_{ij} z^{-r}, \quad g_i(z) = 1 + \hbar\sum_{r\geq 1} g^{(r)}_i z^{-r}. \quad (2.14)$$

The proof of the following result is analogous to that of Proposition 3.21 (actually it is much simpler), and we leave details to the interested reader:

Proposition 2.22 *For any $1 \leq j < i < n$, the following equalities hold in $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$:*

$$e_{j,i+1}(z) = \Big[e_{ji}(z), e^{(1)}_{i,i+1}\Big], \quad f_{i+1,j}(z) = \Big[f^{(1)}_{i+1,i}, f_{ij}(z)\Big]. \tag{2.15}$$

Corollary 2.23 *For any $1 \leq j \leq i < n$ and $r \geq 1$, the following equalities hold:*

$$\begin{aligned} e^{(r)}_{j,i+1} &= \Big[\cdots\Big[e^{(r)}_{j,j+1}, e^{(1)}_{j+1,j+2}\Big], \cdots, e^{(1)}_{i,i+1}\Big], \\ f^{(r)}_{i+1,j} &= \Big[f^{(1)}_{i+1,i}, \cdots, \Big[f^{(1)}_{j+2,j+1}, f^{(r)}_{j+1,j}\Big]\cdots\Big]. \end{aligned} \tag{2.16}$$

Remark 2.24 A more conceptual and computation-free proof of Proposition 2.21 is provided in the end of Appendix A.6.

2.5 The Drinfeld Evaluation Homomorphism ev

While the universal enveloping algebra (over $\mathbb{C}[\hbar]$) $U(\mathfrak{g})$ is always embedded into the Yangian $Y_\hbar(\mathfrak{g})$, in type A there also exists a $\mathbb{C}[\hbar]$-algebra epimorphism

$$\mathrm{ev} : Y_\hbar(\mathfrak{sl}_n) \twoheadrightarrow U(\mathfrak{sl}_n)$$

discovered in (Drinfeld 1985, Theorem 9). This homomorphism is given in the J-presentation of $Y_\hbar(\mathfrak{sl}_n)$. We shall skip explicit formulas, referring the reader to Drinfeld (1985) and (Chari and Pressley 1994, Proposition 12.1.15).

Define $s_i \in Y_\hbar(\mathfrak{sl}_n)$ via

$$s_i := h^{(1)}_i - \frac{\hbar}{2}(h^{(0)}_i)^2, \tag{2.17}$$

so that

$$[s_i, e^{(r)}_{i'}] = c_{ii'} e^{(r+1)}_{i'}, \quad [s_i, f^{(r)}_{i'}] = -c_{ii'} f^{(r+1)}_{i'}.$$

As a result, $Y_\hbar(\mathfrak{sl}_n)$ is generated by $\{e^{(0)}_i, f^{(0)}_i, s_1\}_{i=1}^{n-1}$. We will need the following explicit formulas:

$$\mathrm{ev}(e^{(0)}_i) = E_{i,i+1}, \ \mathrm{ev}(f^{(0)}_i) = E_{i+1,i}, \ \mathrm{ev}(s_1) = \frac{\hbar}{2}(\omega_2 h_1 - E_{12}E_{21} - E_{21}E_{12}), \tag{2.18}$$

where $h_1 = E_{11} - E_{22}$, $\omega_2 = E_{11} + E_{22} - \frac{2}{n} I_n$, $I_n = E_{11} + \cdots + E_{nn}$. The last equality of (2.18) is verified by a straightforward computation (sketched in Appel and Gautam 2017, §5.7).

Let $\widetilde{\gamma} : U(\mathfrak{gl}_n) \twoheadrightarrow U(\mathfrak{sl}_n)$ be the $\mathbb{C}[\hbar]$-algebra epimorphism defined by $\widetilde{\gamma}(X) = X - \frac{\mathrm{tr}(X)}{n} \cdot I_n$ for $X \in \mathfrak{gl}_n$. We also define a $\mathbb{C}[\hbar]$-algebra embedding $\widetilde{\Upsilon} : Y_\hbar(\mathfrak{sl}_n) \hookrightarrow Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$ as a composition of an automorphism of $Y_\hbar(\mathfrak{sl}_n)$ defined by $e_i(z) \mapsto e_i(z-\hbar)$, $f_i(z) \mapsto f_i(z-\hbar)$, $h_i(z) \mapsto h_i(z-\hbar)$, a natural embedding $Y_\hbar(\mathfrak{sl}_n) \hookrightarrow Y_\hbar(\mathfrak{gl}_n)$, and the isomorphism $\Upsilon : Y_\hbar(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)$ of Theorem 2.18.

The key result of this subsection establishes the relation between the evaluation homomorphism ev and the RTT evaluation homomorphism $\mathrm{ev}^{\mathrm{rtt}}$ of Lemma 2.12(b):

Theorem 2.25 *The following diagram is commutative:*

$$\begin{array}{ccc} Y_\hbar(\mathfrak{sl}_n) & \xrightarrow{\widetilde{\Upsilon}} & Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \\ \downarrow{\scriptstyle \mathrm{ev}} & & {\scriptstyle \mathrm{ev}^{\mathrm{rtt}}}\downarrow \\ U(\mathfrak{sl}_n) & \xleftarrow{\widetilde{\gamma}} & U(\mathfrak{gl}_n) \end{array} \tag{2.19}$$

Proof It suffices to verify $\widetilde{\gamma}(\mathrm{ev}^{\mathrm{rtt}}(\widetilde{\Upsilon}(X))) = \mathrm{ev}(X)$ for all $X \in \{e_i^{(0)}, f_i^{(0)}, s_1\}_{i=1}^{n-1}$. This equality is obvious for $e_i^{(0)}, f_i^{(0)}$, hence, it remains to verify it for $X = s_1$.

Note that $\widetilde{\Upsilon}(h_1(z)) = g_1(z-\hbar/2)^{-1}g_2(z-\hbar/2)$. Using the notations of (2.14), this implies

$$\widetilde{\Upsilon}(h_1^{(0)}) = g_2^{(1)} - g_1^{(1)},$$
$$\widetilde{\Upsilon}(h_1^{(1)}) = \hbar\left((g_1^{(1)})^2 - g_1^{(1)}g_2^{(1)} + \frac{g_2^{(1)} - g_1^{(1)}}{2}\right) + \left(g_2^{(2)} - g_1^{(2)}\right),$$

so that

$$\widetilde{\Upsilon}(s_1) = \frac{\hbar}{2}\left((g_1^{(1)})^2 - (g_2^{(1)})^2 + g_2^{(1)} - g_1^{(1)}\right) + \left(g_2^{(2)} - g_1^{(2)}\right).$$

On the other hand, considering the Gauss decomposition of the matrix $1+\hbar T z^{-1} = \mathrm{ev}^{\mathrm{rtt}}(T(z))$ of Remark 2.13(b), we find $\mathrm{ev}^{\mathrm{rtt}} : g_1^{(1)} \mapsto E_{11}$, $g_1^{(2)} \mapsto 0$, $g_2^{(1)} \mapsto E_{22}$, $g_2^{(2)} \mapsto -\hbar E_{21}E_{12}$. Therefore, we obtain

$$\begin{aligned} \mathrm{ev}^{\mathrm{rtt}}(\widetilde{\Upsilon}(s_1)) &= \frac{\hbar}{2}(E_{11}^2 - E_{22}^2 + E_{22} - E_{11} - 2E_{21}E_{12}) \\ &= \frac{\hbar}{2}(E_{11}^2 - E_{22}^2 - E_{12}E_{21} - E_{21}E_{12}). \end{aligned}$$

Applying $\widetilde{\gamma}$, we finally get

$$\widetilde{\gamma}(\mathrm{ev}^{\mathrm{rtt}}(\widetilde{\Upsilon}(s_1))) = \frac{\hbar}{2}(\omega_2 h_1 - E_{12}E_{21} - E_{21}E_{12}) = \mathrm{ev}(s_1),$$

due to the last formula of (2.18).

This completes our proof of Theorem 2.25. □

2.6 The Shifted Yangian, Construction I

In this subsection, we recall the notion of shifted Yangians following (Braverman et al. 2016, Appendix B).

First, recall that given a $\mathbb{C}$-algebra A with an algebra filtration $F^\bullet A = \cdots \subseteq F^{-1}A \subseteq F^0 A \subseteq F^1 A \subseteq \cdots$ which is separated and exhaustive (that is, $\cap_k F^k A = 0$ and $\cup_k F^k A = A$), we define the *Rees algebra* of A to be the graded $\mathbb{C}[\hbar]$-algebra $\mathrm{Rees}^{F^\bullet} A := \bigoplus_k \hbar^k F^k A$, viewed as a subalgebra of $A[\hbar, \hbar^{-1}]$.

Following (Braverman et al. 2016, Definition B.1), define the *Cartan doubled Yangian* $Y_\infty = Y_\infty(\mathfrak{sl}_n)$ to be the $\mathbb{C}$-algebra generated by $\{E_i^{(r)}, F_i^{(r)}, H_i^{(s)}\}_{1\le i\le n-1}^{r\ge 1, s\in\mathbb{Z}}$ with the following defining relations:

$$
\begin{aligned}
&\left[H_i^{(s)}, H_j^{(s')}\right] = 0,\\
&\left[E_i^{(r)}, F_j^{(r')}\right] = \delta_{ij} H_i^{(r+r'-1)},\\
&\left[H_i^{(s+1)}, E_j^{(r)}\right] - \left[H_i^{(s)}, E_j^{(r+1)}\right] = \frac{c_{ij}}{2}\left(H_i^{(s)} E_j^{(r)} + E_j^{(r)} H_i^{(s)}\right),\\
&\left[H_i^{(s+1)}, F_j^{(r)}\right] - \left[H_i^{(s)}, F_j^{(r+1)}\right] = -\frac{c_{ij}}{2}\left(H_i^{(s)} F_j^{(r)} + F_j^{(r)} H_i^{(s)}\right),\\
&\left[E_i^{(r+1)}, E_j^{(r')}\right] - \left[E_i^{(r)}, E_j^{(r'+1)}\right] = \frac{c_{ij}}{2}(E_i^{(r)} E_j^{(r')} + E_j^{(r')} E_i^{(r)}),\\
&\left[F_i^{(r+1)}, F_j^{(r')}\right] - \left[F_i^{(r)}, F_j^{(r'+1)}\right] = -\frac{c_{ij}}{2}\left(F_i^{(r)} F_j^{(r')} + F_j^{(r')} F_i^{(r)}\right),\\
&\left[E_i^{(r)}, E_j^{(r')}\right] = 0 \text{ and } \left[F_i^{(r)}, F_j^{(r')}\right] = 0 \text{ if } c_{ij} = 0,\\
&\left[E_i^{(r_1)}, \left[E_i^{(r_2)}, E_j^{(r')}\right]\right] + \left[E_i^{(r_2)}, \left[E_i^{(r_1)}, E_j^{(r')}\right]\right] = 0 \text{ if } c_{ij} = -1,\\
&\left[F_i^{(r_1)}, \left[F_i^{(r_2)}, F_j^{(r')}\right]\right] + \left[F_i^{(r_2)}, \left[F_i^{(r_1)}, F_j^{(r')}\right]\right] = 0 \text{ if } c_{ij} = -1.
\end{aligned}
\tag{2.20}
$$

Fix a coweight μ of $\mathfrak{sl}_n$ and set $b_i := \alpha_i^\vee(\mu)$. Following (Braverman et al. 2016, Definition B.2), define $Y_\mu = Y_\mu(\mathfrak{sl}_n)$ as the quotient of Y_∞ by the relations $H_i^{(r)} = 0$ for $r < -b_i$ and $H_i^{(-b_i)} = 1$.

Analogously to (2.11), define the elements $\{E_{\alpha^\vee}^{(r)}, F_{\alpha^\vee}^{(r)}\}_{\alpha^\vee\in\Delta^+}^{r\ge 1}$ of Y_μ via

$$
\begin{aligned}
E_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_i^\vee}^{(r)} &:= \left[\cdots\left[E_j^{(r)}, E_{j+1}^{(1)}\right], \cdots, E_i^{(1)}\right],\\
F_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_i^\vee}^{(r)} &:= \left[F_i^{(1)}, \cdots, \left[F_{j+1}^{(1)}, F_j^{(r)}\right]\cdots\right].
\end{aligned}
\tag{2.21}
$$

Choose any total ordering on the following set of *PBW generators*:

$$
\left\{E_{\alpha^\vee}^{(r)}\right\}_{\alpha^\vee\in\Delta^+}^{r\ge 1} \cup \left\{F_{\alpha^\vee}^{(r)}\right\}_{\alpha^\vee\in\Delta^+}^{r\ge 1} \cup \left\{H_i^{(r)}\right\}_{1\le i\le n-1}^{r>-b_i}. \tag{2.22}
$$

The following PBW property of Y_μ was established in (Finkelberg et al. 2018, Corollary 3.15):

Theorem 2.26 (Finkelberg et al. 2018) *For an arbitrary coweight μ, the ordered PBW monomials in the generators (2.22) form a $\mathbb{C}$-basis of Y_μ.*

Fix a pair of coweights μ_1, μ_2 such that $\mu_1 + \mu_2 = \mu$. Following (Finkelberg et al. 2018, §5.4), consider the filtration $F^{\bullet}_{\mu_1,\mu_2} Y_\mu$ of Y_μ by defining degrees of the PBW generators as follows:

$$\deg E^{(r)}_{\alpha^\vee} = \alpha^\vee(\mu_1) + r, \ \deg F^{(r)}_{\alpha^\vee} = \alpha^\vee(\mu_2) + r, \ \deg H^{(r)}_i = \alpha^\vee_i(\mu) + r. \quad (2.23)$$

More precisely, $F^k_{\mu_1,\mu_2} Y_\mu$ is defined as the span of all ordered PBW monomials whose total degree is at most k.

According to Finkelberg et al. (2018), this defines an algebra filtration and the Rees algebras $\mathrm{Rees}^{F^{\bullet}_{\mu_1,\mu_2}} Y_\mu$ are canonically isomorphic for any choice of μ_1, μ_2 as above.

Definition 2.27 Define the *shifted Yangian* $\mathbf{Y}_\mu = \mathbf{Y}_\mu(\mathfrak{sl}_n)$ via $\mathbf{Y}_\mu := \mathrm{Rees}^{F^{\bullet}_{\mu_1,\mu_2}} Y_\mu$.

2.7 The Shifted Yangian with a Dominant Shift, Construction II

Let us now recall an alternative (historically the first) definition of the dominantly shifted Yangians proposed in Kamnitzer et al. (2014). Fix a dominant coweight μ of $\mathfrak{sl}_n$ and set $b_i := \alpha^\vee_i(\mu)$ (the dominance condition on μ is equivalent to $b_i \geq 0$ for all i). Let $Y_{\mu,\hbar}$ be the associative $\mathbb{C}[\hbar]$-algebra generated by $\{e^{(r)}_i, f^{(r)}_i, h^{(s_i)}_i\}^{r\geq 0, s_i \geq -b_i}_{1\leq i\leq n-1}$ with the following defining relations:

$$\begin{aligned}
&\left[h^{(s)}_i, h^{(s')}_j\right] = 0,\\
&\left[e^{(r)}_i, f^{(r')}_j\right] = \begin{cases} h^{(r+r')}_i, & \text{if } i = j \text{ and } r + r' \geq -b_i \\ 0, & \text{otherwise} \end{cases},\\
&\left[h^{(-b_i)}_i, e^{(r)}_j\right] = c_{ij} e^{(r)}_j,\\
&\left[h^{(s+1)}_i, e^{(r)}_j\right] - \left[h^{(s)}_i, e^{(r+1)}_j\right] = \frac{c_{ij}\hbar}{2}\left(h^{(s)}_i e^{(r)}_j + e^{(r)}_j h^{(s)}_i\right),\\
&\left[h^{(-b_i)}_i, f^{(r)}_j\right] = -c_{ij} f^{(r)}_j,\\
&\left[h^{(s+1)}_i, f^{(r)}_j\right] - \left[h^{(s)}_i, f^{(r+1)}_j\right] = -\frac{c_{ij}\hbar}{2}\left(h^{(s)}_i f^{(r)}_j + f^{(r)}_j h^{(s)}_i\right),\\
&\left[e^{(r+1)}_i, e^{(r')}_j\right] - \left[e^{(r)}_i, e^{(r'+1)}_j\right] = \frac{c_{ij}\hbar}{2}\left(e^{(r)}_i e^{(r')}_j + e^{(r')}_j e^{(r)}_i\right),\\
&\left[f^{(r+1)}_i, f^{(r')}_j\right] - \left[f^{(r)}_i, f^{(r'+1)}_j\right] = -\frac{c_{ij}\hbar}{2}\left(f^{(r)}_i f^{(r')}_j + f^{(r')}_j f^{(r)}_i\right),\\
&\left[e^{(r)}_i, e^{(r')}_j\right] = 0 \text{ and } \left[e^{(r)}_i, e^{(r')}_j\right] = 0 \text{ if } c_{ij} = 0,\\
&\left[e^{(r_1)}_i, \left[e^{(r_2)}_i, e^{(r')}_j\right]\right] + \left[e^{(r_2)}_i, \left[e^{(r_1)}_i, e^{(r')}_j\right]\right] = 0 \text{ if } c_{ij} = -1,\\
&\left[f^{(r_1)}_i, \left[f^{(r_2)}_i, f^{(r')}_j\right]\right] + \left[f^{(r_2)}_i, \left[f^{(r_1)}_i, f^{(r')}_j\right]\right] = 0 \text{ if } c_{ij} = -1.
\end{aligned} \quad (2.24)$$

Remark 2.28 The main differences between (2.24) and (2.20) are: (1) all indices r, s are shifted by -1, (2) $\hbar$ appears in the right-hand sides to make the equations look homogeneous.

Analogously to (2.11, 2.21), define the elements $\{e^{(r)}_{\alpha^\vee}, f^{(r)}_{\alpha^\vee}\}^{r\geq 0}_{\alpha^\vee\in\Delta^+}$ of $Y_{\mu,\hbar}$ via

$$\begin{aligned} e^{(r)}_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_i^\vee} &:= \Big[\cdots\Big[e_j^{(r)}, e_{j+1}^{(0)}\Big], \cdots, e_i^{(0)}\Big], \\ f^{(r)}_{\alpha_j^\vee+\alpha_{j+1}^\vee+\cdots+\alpha_i^\vee} &:= \Big[f_i^{(0)}, \cdots, \Big[f_{j+1,}^{(0)}, f_j^{(r)}\Big]\cdots\Big]. \end{aligned} \tag{2.25}$$

Choose any total ordering on the following set of *PBW generators*:

$$\{e^{(r)}_{\alpha^\vee}\}^{r\geq 0}_{\alpha^\vee\in\Delta^+} \cup \{f^{(r)}_{\alpha^\vee}\}^{r\geq 0}_{\alpha^\vee\in\Delta^+} \cup \{h_i^{(s_i)}\}^{s_i\geq -b_i}_{1\leq i\leq n-1}. \tag{2.26}$$

The following is analogous to Theorem 2.26:

Theorem 2.29 *For an arbitrary dominant coweight μ, the ordered PBW monomials in the generators (2.26) form a basis of a free $\mathbb{C}[\hbar]$-module $Y_{\mu,\hbar}$.*

Proof Arguing as in Finkelberg et al. (2018, Proposition 3.13), it is easy to check that $Y_{\mu,\hbar}$ is spanned by the ordered PBW monomials. To prove the linear independence of the ordered PBW monomials, it suffices to verify that their images are linearly independent when we specialize $\hbar$ to any nonzero complex number (cf. our proof of Theorem A.9). The latter holds for $\hbar = 1$ (and thus for any $\hbar \neq 0$, since all such specializations are isomorphic), due to Theorem 2.26 and the isomorphism $Y_{\mu,\hbar}/(\hbar - 1) \simeq Y_\mu$. □

Following Kamnitzer et al. (2014, §3D,3F),[2] we introduce the following:

Definition 2.30 Let $\mathbf{Y}'_\mu$ be the $\mathbb{C}[\hbar]$-subalgebra of $Y_{\mu,\hbar}$ generated by

$$\Big\{\hbar e^{(r)}_{\alpha^\vee}\Big\}^{r\geq 0}_{\alpha^\vee\in\Delta^+} \cup \Big\{\hbar f^{(r)}_{\alpha^\vee}\Big\}^{r\geq 0}_{\alpha^\vee\in\Delta^+} \cup \Big\{\hbar h_i^{(s_i)}\Big\}^{s_i\geq -b_i}_{1\leq i\leq n-1}.$$

The following is the main result of this subsection:

Theorem 2.31 *For any dominant coweight μ, there is a canonical $\mathbb{C}[\hbar]$-algebra isomorphism*

$$\mathbf{Y}_\mu \simeq \mathbf{Y}'_\mu.$$

This provides an identification of two different approaches towards the dominantly shifted Yangians (which was missing in the literature, to our surprise). A proof of this result, generalized to any semisimple Lie algebra $\mathfrak{g}$, is presented in Appendix A.7, see Theorem A.12.

[2] Let us emphasize that (Kamnitzer et al. 2014, Theorem 3.5) is wrong, as pointed out in Braverman et al. (2016). That is, it does not include a complete set of relations, except when $\mathfrak{g} = \mathfrak{sl}_2$.

2.8 Homomorphism Φ_{μ}^{λ}

Let us recall the construction of (Braverman et al. 2016, Appendix B) for the type A_{n-1} Dynkin diagram with arrows pointing $i \to i+1$ for $1 \leq i \leq n-2$. We fix a dominant coweight λ and a coweight μ of $\mathfrak{sl}_n$, such that $\lambda - \mu = \sum_{i=1}^{n-1} a_i \alpha_i$ with $a_i \in \mathbb{N}$, where $\{\alpha_i\}_{i=1}^{n-1}$ are the simple coroots of $\mathfrak{sl}_n$. We set $a_0 := 0$, $a_n := 0$. We also fix a sequence $\underline{\lambda} = (\omega_{i_1}, \ldots, \omega_{i_N})$ of fundamental coweights, such that $\sum_{s=1}^{N} \omega_{i_s} = \lambda$.

Consider the $\mathbb{C}$-algebra

$$\widetilde{\mathcal{A}} = \mathbb{C}[z_1, \ldots, z_N]\langle w_{i,r}, \mathsf{u}_{i,r}^{\pm 1}, (w_{i,r} - w_{i,s} + m)^{-1} \rangle_{1 \leq i \leq n-1, m \in \mathbb{Z}}^{1 \leq r \neq s \leq a_i}$$

with the defining relations $[\mathsf{u}_{i,r}^{\pm 1}, w_{j,s}] = \pm \delta_{ij} \delta_{rs} \mathsf{u}_{i,r}^{\pm 1}$. Define $W_0(z) := 1$, $W_n(z) := 1$, and

$$Z_i(z) := \prod_{1 \leq s \leq N}^{i_s = i} (z - z_s - 1/2), \quad W_i(z) := \prod_{r=1}^{a_i} (z - w_{i,r}), \quad W_{i,r}(z) := \prod_{1 \leq s \leq a_i}^{s \neq r} (z - w_{i,s}). \tag{2.27}$$

We define a filtration on $\widetilde{\mathcal{A}}$ by setting $\deg(z_s) = 1, \deg(w_{i,r}) = 1, \deg((w_{i,r} - w_{i,s} + m)^{-1}) = -1, \deg(\mathsf{u}_{i,r}^{\pm 1}) = 0$, and set $\mathcal{A}_\hbar := \text{Rees } \widetilde{\mathcal{A}}$. Explicitly, we have

$$\widetilde{\mathcal{A}}_\hbar \simeq \mathbb{C}[\hbar][z_1, \ldots, z_N]\langle w_{i,r}, \mathsf{u}_{i,r}^{\pm 1}, \hbar^{-1}, (w_{i,r} - w_{i,s} + m\hbar)^{-1} \rangle_{1 \leq i \leq n-1, m \in \mathbb{Z}}^{1 \leq r \neq s \leq a_i}$$

with the defining relations $[\mathsf{u}_{i,r}^{\pm 1}, w_{j,s}] = \pm \hbar \delta_{ij} \delta_{rs} \mathsf{u}_{i,r}^{\pm 1}$.

Remark 2.32 By abuse of notation, for a generator x which lives in a filtered degree k (but not in a filtered degree $k-1$) we write x for the element $\hbar^k x$ in the corresponding Rees algebra.

We also need the larger algebra $Y_\mu[z_1, \ldots, z_N] := Y_\mu \otimes_{\mathbb{C}} \mathbb{C}[z_1, \ldots, z_N]$. Define new *Cartan* generators $\{A_i^{(r)}\}_{1 \leq i < n}^{r \geq 1}$ via

$$H_i(z) = Z_i(z) \cdot \frac{\prod_{j-i} (z - 1/2)^{a_j}}{z^{a_i} (z-1)^{a_i}} \cdot \frac{\prod_{j-i} A_j(z - 1/2)}{A_i(z) A_i(z-1)}, \tag{2.28}$$

where $H_i(z) := z^{b_i} + \sum_{r > -b_i} H_i^{(r)} z^{-r}$ and $A_i(z) := 1 + \sum_{r \geq 1} A_i^{(r)} z^{-r}$. The generating series $E_i(z), F_i(z)$ are defined via $E_i(z) := \sum_{r \geq 1} E_i^{(r)} z^{-r}$ and $F_i(z) := \sum_{r \geq 1} F_i^{(r)} z^{-r}$.

The following result is due to (Braverman et al. 2016, Theorem B.15) (for earlier results in this direction see Gerasimov et al. 2005; Kamnitzer et al. 2014):

Theorem 2.33 (Braverman et al. 2016) *There exists a unique homomorphism*

$$\Phi_{\mu}^{\underline{\lambda}} : Y_\mu[z_1, \ldots, z_N] \longrightarrow \widetilde{\mathcal{A}}$$

of filtered $\mathbb{C}$-algebras, such that

$$A_i(z) \mapsto z^{-a_i} W_i(z),$$
$$E_i(z) \mapsto -\sum_{r=1}^{a_i} \frac{Z_i(w_{i,r}) W_{i-1}(w_{i,r}-1/2)}{(z-w_{i,r}) W_{i,r}(w_{i,r})} u_{i,r}^{-1},$$
$$F_i(z) \mapsto \sum_{r=1}^{a_i} \frac{W_{i+1}(w_{i,r}+1/2)}{(z-w_{i,r}-1) W_{i,r}(w_{i,r})} u_{i,r}.$$

We extend the filtration $F^\bullet_{\mu_1,\mu_2}$ on Y_μ to $Y_\mu[z_1,\dots,z_N]$ by setting $\deg(z_s)=1$, and define $\mathbf{Y}_\mu[z_1,\dots,z_N] := \mathrm{Rees}^{F^\bullet_{\mu_1,\mu_2}} Y_\mu[z_1,\dots,z_N]$ (which is independent of the choice of μ_1,μ_2 up to a canonical isomorphism). Applying the Rees functor to Theorem 2.33, we obtain

Theorem 2.34 (Braverman et al. 2016) *There exists a unique graded $\mathbb{C}[\hbar][z_1,\dots,z_N]$-algebra homomorphism*

$$\Phi^{\lambda}_{\mu} : \mathbf{Y}_\mu[z_1,\dots,z_N] \longrightarrow \widetilde{\mathcal{A}}_\hbar,$$

such that

$$A_i(z) \mapsto z^{-a_i} W_i(z),$$
$$E_i(z) \mapsto -\sum_{r=1}^{a_i} \frac{Z_i(w_{i,r}) W_{i-1}(w_{i,r}-\hbar/2)}{(z-w_{i,r}) W_{i,r}(w_{i,r})} u_{i,r}^{-1},$$
$$F_i(z) \mapsto \sum_{r=1}^{a_i} \frac{W_{i+1}(w_{i,r}+\hbar/2)}{(z-w_{i,r}-\hbar) W_{i,r}(w_{i,r})} u_{i,r}.$$

Remark 2.35 Following Remark 2.32, we note that the defining formulas of $W_i(z)$, $W_{i,r}(z)$ in $\widetilde{\mathcal{A}}_\hbar$ are given again by (2.27). In contrast, $Z_i(z) = \prod_{1\le s\le N}^{i_s=i}(z-z_s-\hbar/2)$, cf. (2.27).

2.9 Coulomb Branch

Following Braverman et al. (2016, 2019), let $\mathcal{A}_\hbar$ denote the quantized Coulomb branch. We choose a basis $w_1,\dots,w_N$ in $W=\bigoplus_{i=1}^{n-1} W_i$ such that $w_s \in W_{i_s}$, where i_s are chosen as in Sect. 2.8. Then $\mathcal{A}_\hbar$ is defined as $\mathcal{A}_\hbar := H_\bullet^{(\mathrm{GL}(V)\times T_W)_{\mathcal{O}}\rtimes\mathbb{C}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$, where $\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}}$ is the variety of triples, T_W is the maximal torus of $\mathrm{GL}(W) = \prod_{i=1}^{n-1}\mathrm{GL}(W_i)$, and $\mathrm{GL}(V)=\prod_{i=1}^{n-1}\mathrm{GL}(V_i)$. We identify $H^\bullet_{T_W}(\mathrm{pt}) = \mathbb{C}[z_1,\dots,z_N]$ and $H^\bullet_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[\hbar]$. Recall a $\mathbb{C}[\hbar][z_1,\dots,z_N]$-algebra embedding $\mathbf{z}^*(\iota_*)^{-1} : \mathcal{A}_\hbar \hookrightarrow \widetilde{\mathcal{A}}_\hbar$, which takes the homological grading on $\mathcal{A}_\hbar$ to the above grading on $\widetilde{\mathcal{A}}_\hbar$.

According to Braverman et al. (2016, Theorem B.18), the homomorphism $\Phi_\mu^\lambda : \mathbf{Y}_\mu[z_1, \dots, z_N] \to \widetilde{\mathcal{A}}_\hbar$ factors through $\mathcal{A}_\hbar$. In other words, there is a unique graded $\mathbb{C}[\hbar][z_1, \dots, z_N]$-algebra homomorphism $\overline{\Phi}_\mu^\lambda : \mathbf{Y}_\mu[z_1, \dots, z_N] \to \mathcal{A}_\hbar$, such that the composition $\mathbf{Y}_\mu[z_1, \dots, z_N] \xrightarrow{\overline{\Phi}_\mu^\lambda} \mathcal{A}_\hbar \xrightarrow{\mathbf{z}^*(\iota_*)^{-1}} \widetilde{\mathcal{A}}_\hbar$ coincides with Φ_μ^λ.

The following result is due to Kamnitzer et al. (2018b, Corollary 4.10) (see Remark 4.16 for an alternative proof, based on the shuffle realizations of $Y_\hbar(\mathfrak{sl}_n)$, $\mathbf{Y}_\hbar(\mathfrak{sl}_n)$ of Tsymbaliuk (2018, §6)):

Proposition 2.36 (Kamnitzer et al. (2018b)) $\overline{\Phi}_\mu^\lambda : \mathbf{Y}_\mu[z_1, \dots, z_N] \to \mathcal{A}_\hbar$ *is surjective.*

Lemma 2.37 *For any $1 \leq j \leq i < n$ and $r \geq 1$, the following equalities hold:*

$$\Phi_\mu^\lambda \left(E^{(r)}_{\alpha_j^\vee + \alpha_{j+1}^\vee + \dots + \alpha_i^\vee} \right) = (-1)^{i-j+1} \times \sum_{\substack{1 \leq r_j \leq a_j \\ \dots \\ 1 \leq r_i \leq a_i}} \frac{W_{j-1}(w_{j,r_j} - \frac{\hbar}{2}) \prod_{k=j}^{i-1} W_{k,r_k}(w_{k+1,r_{k+1}} - \frac{\hbar}{2})}{\prod_{k=j}^{i} W_{k,r_k}(w_{k,r_k})} \cdot \prod_{k=j}^{i} Z_k(w_{k,r_k}) \cdot w_{j,r_j}^{r-1} \cdot \prod_{k=j}^{i} u_{k,r_k}^{-1}, \tag{2.29}$$

$$\Phi_\mu^\lambda \left(F^{(r)}_{\alpha_j^\vee + \alpha_{j+1}^\vee + \dots + \alpha_i^\vee} \right) = (-1)^{i-j} \times \sum_{\substack{1 \leq r_j \leq a_j \\ \dots \\ 1 \leq r_i \leq a_i}} \frac{\prod_{k=j+1}^{i} W_{k,r_k}(w_{k-1,r_{k-1}} + \frac{\hbar}{2}) W_{i+1}(w_{i,r_i} + \frac{\hbar}{2})}{\prod_{k=j}^{i} W_{k,r_k}(w_{k,r_k})} \cdot (w_{j,r_j} + \hbar)^{r-1} \cdot \prod_{k=j}^{i} u_{k,r_k}. \tag{2.30}$$

Proof Straightforward computation. □

Remark 2.38 For $1 \leq j \leq i < n$, we consider a coweight $\lambda_{ji} = (0, \dots, 0, \varpi_{j,1}, \dots, \varpi_{i,1}, 0, \dots, 0)$ (resp. $\lambda_{ji}^* = (0, \dots, 0, \varpi_{j,1}^*, \dots, \varpi_{i,1}^*, 0, \dots, 0)$) of $\mathrm{GL}(V) = \mathrm{GL}(V_1) \times \dots \times \mathrm{GL}(V_{n-1})$. The corresponding orbits $\mathrm{Gr}_{\mathrm{GL}(V)}^{\lambda_{ji}}, \mathrm{Gr}_{\mathrm{GL}(V)}^{\lambda_{ji}^*} \subset \mathrm{Gr}_{\mathrm{GL}(V)}$ are closed, and let $\mathcal{R}_{\lambda_{ji}}, \mathcal{R}_{\lambda_{ji}^*}$ denote their preimages in the variety of triples $\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}}$. Then, Lemma 2.37 implies

$$\overline{\Phi}_\mu^\lambda \left(E^{(r)}_{\alpha_j^\vee + \alpha_{j+1}^\vee + \dots + \alpha_i^\vee} \right) = (-1)^{\sum_{k=j}^{i} a_k} (c_1(\mathcal{S}_j) + \hbar)^{r-1} \cap [\mathcal{R}_{\lambda_{ji}^*}],$$

$$\overline{\Phi}_\mu^\lambda \left(F^{(r)}_{\alpha_j^\vee + \alpha_{j+1}^\vee + \dots + \alpha_i^\vee} \right) = (-1)^{\sum_{k=j+1}^{i+1} a_k} (c_1(\mathcal{Q}_j) + \hbar)^{r-1} \cap [\mathcal{R}_{\lambda_{ji}}].$$

2.10 Explicit Description for $\mu = 0$, $\lambda = n\omega_{n-1}$

Following Braverman et al. (2016), define the *truncation ideal* $\mathcal{I}_\mu^\lambda$ as the 2-sided ideal of $\mathbf{Y}_\mu[z_1, \ldots, z_N]$ generated over $\mathbb{C}[\hbar][z_1, \ldots, z_N]$ by $\{A_i^{(r)}\}_{1\le i\le n-1}^{r>a_i}$. This ideal is discussed extensively in Kamnitzer et al. (2014). The inclusion $\mathcal{I}_\mu^\lambda \subset \mathrm{Ker}(\Phi_\mu^\lambda)$ is clear, while the opposite inclusion was conjectured in Braverman et al. (2016, Remark B.21). This conjecture is proved for dominant μ in Kamnitzer et al. (2018a).

The goal of this subsection is to provide an alternative proof of a *reduced version* of that equality in the particular case $\mu = 0, \lambda = n\omega_{n-1}$ (so that $N = n$ and $a_i = i$ for $1 \le i < n$; recall that $a_0 = 0, a_n = 0$). Here, a reduced version means that we impose an extra relation $\sum_{i=1}^n z_i = 0$ in all algebras. We use $\underline{\mathcal{I}}_0^{n\omega_{n-1}}$ to denote the reduced version of the corresponding truncation ideal, while $\underline{\Phi}_0^{n\omega_{n-1}}$ denotes the resulting homomorphism between the reduced algebras.

The forthcoming discussion is very close to Brundan and Kleshchev (2006) and Webster et al. (2017), while we choose to present it in full details as it will be generalized along the same lines to the trigonometric counterpart in Sect. 4.3.

Theorem 2.39 $\underline{\mathcal{I}}_0^{n\omega_{n-1}} = \mathrm{Ker}(\underline{\Phi}_0^{n\omega_{n-1}})$.

Our proof of this result is based on the identification of the reduced truncation ideal $\underline{\mathcal{I}}_0^{n\omega_{n-1}}$ with the kernel of a certain version of the evaluation homomorphism ev, which is of independent interest.

Recall the commutative diagram (2.19) of Theorem 2.25. Adjoining extra variables $\{z_i\}_{i=1}^n$ subject to $\sum_{i=1}^n z_i = 0$, we obtain the following commutative diagram:

$$\begin{array}{ccc} Y_\hbar(\mathfrak{sl}_n)[z_1, \ldots, z_n]/(\sum z_i) & \xrightarrow{\mathrm{ev}} & U(\mathfrak{sl}_n)[z_1, \ldots, z_n]/(\sum z_i) \\ \downarrow \widetilde{\Upsilon} & & \widetilde{\gamma} \uparrow \\ Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[z_1, \ldots, z_n]/(\sum z_i) & \xrightarrow{\mathrm{ev}^{\mathrm{rtt}}} & U(\mathfrak{gl}_n)[z_1, \ldots, z_n]/(\sum z_i) \end{array} \tag{2.31}$$

where $U(\mathfrak{gl}_n)[z_1, \ldots, z_n]/(\sum z_i) := U(\mathfrak{gl}_n) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar][z_1, \ldots, z_n]/(\sum z_i)$ and the other three algebras are defined likewise.

Recall the isomorphism $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \simeq Y_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n) \otimes_{\mathbb{C}[\hbar]} ZY_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of (2.5), which after adjoining extra variables $\{z_i\}_{i=1}^n$ subject to $\sum_{i=1}^n z_i = 0$ gives rise to an algebra isomorphism $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[z_1, \ldots, z_n]/(\sum z_i) \simeq Y_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n) \otimes_{\mathbb{C}[\hbar]} ZY_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar][z_1, \ldots, z_n]/(\sum z_i)$. Let $\widetilde{\Delta}_n(z)$ denote the quantum determinant of the matrix $zT(z)$, which is explicitly given by $\widetilde{\Delta}_n(z) = z(z-\hbar)(z-2\hbar)\cdots(z-(n-1)\hbar) \cdot \mathrm{qdet}\, T(z)$. According to Proposition 2.10, the center $ZY_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ is a polynomial algebra in $\{\widetilde{d}_r\}_{r=1}^\infty$, where $\widetilde{d}_r$ are defined via $z^{-n}\widetilde{\Delta}_n(z + \frac{n-1}{2}\hbar) = 1 + \hbar\sum_{r\ge 1} \widetilde{d}_r z^{-r}$. Let $\mathcal{J}$ be the 2-sided ideal of $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[z_1, \ldots, z_n]/(\sum z_i)$ generated by $\{\widetilde{d}_r\}_{r>n} \cup \{\widetilde{d}_r - \hbar^{-1}e_r(-\hbar z_1, \ldots, -\hbar z_n)\}_{r=1}^n$, where $e_r(\bullet)$ denotes the r-th elementary symmetric polynomial. The ideal $\mathcal{J}$ is chosen so that $z^{-n}\widetilde{\Delta}_n(z + \frac{n-1}{2}\hbar) - \prod_{s=1}^n(1 - \frac{\hbar z_s}{z}) \in \mathcal{J}[[z^{-1}]]$. Let $\pi : Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[z_1, \ldots, z_n]/(\sum z_i) \twoheadrightarrow Y_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)[z_1, \ldots, z_n]/(\sum z_i)$ be the natural projection along $\mathcal{J}$. Set $X_r := \mathrm{ev}^{\mathrm{rtt}}(\widetilde{d}_r)$

(note that $X_r = 0$ for $r > n$). Then, the center of $U(\mathfrak{gl}_n)[z_1, \dots, z_n]/(\sum z_i)$ is isomorphic to $\mathbb{C}[\hbar][z_1 \dots, z_n, X_1, \dots, X_n]/(\sum z_i)$.

Recall the *extended enveloping algebra* $\widetilde{U}(\mathfrak{sl}_n)$ of Beilinson and Ginzburg (1999), defined as the central reduction of $U(\mathfrak{gl}_n)[z_1, \dots, z_n]/(\sum z_i)$ by the 2-sided ideal generated by $\{X_r - \hbar^{-1} e_r(-\hbar z_1, \dots, -\hbar z_n)\}_{r=1}^n$ (the appearance of $\mathfrak{sl}_n$ is due to the fact that $X_1 = 0$). By abuse of notation, we denote the corresponding projection $U(\mathfrak{gl}_n)[z_1, \dots, z_n]/(\sum z_i) \twoheadrightarrow \widetilde{U}(\mathfrak{sl}_n)$ by π again. We denote the composition $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[z_1, \dots, z_n]/(\sum z_i) \xrightarrow{\mathrm{ev}^{\mathrm{rtt}}} U(\mathfrak{gl}_n)[z_1, \dots, z_n]/(\sum z_i) \xrightarrow{\pi} \widetilde{U}(\mathfrak{sl}_n)$ by $\overline{\mathrm{ev}}^{\mathrm{rtt}}$. It factors through $\pi : Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[z_1, \dots, z_n]/(\sum z_i) \to Y_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i)$, and we denote the corresponding homomorphism $Y_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) \to \widetilde{U}(\mathfrak{sl}_n)$ by $\overline{\mathrm{ev}}^{\mathrm{rtt}}$ again. The algebra $\widetilde{U}(\mathfrak{sl}_n)$ can be also realized as the central reduction of $U(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i)$ by the 2-sided ideal generated by $\{\bar{X}_r - \hbar^{-1} e_r(-\hbar z_1, \dots, -\hbar z_n)\}_{r=2}^n$, where $\bar{X}_r = \widetilde{\gamma}(X_r)$, see Sect. 2.5. We denote the corresponding projection $U(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) \twoheadrightarrow \widetilde{U}(\mathfrak{sl}_n)$ by π again. Finally, we denote the composition $Y_\hbar(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) \xrightarrow{\mathrm{ev}} U(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) \xrightarrow{\pi} \widetilde{U}(\mathfrak{sl}_n)$ by $\overline{\mathrm{ev}}$.

Summarizing all the above, we obtain the following commutative diagram:

$$\begin{array}{ccc}
Y_\hbar(\mathfrak{sl}_n)[z_1, \dots, z_n]/\left(\sum z_i\right) & \xrightarrow{\overline{\mathrm{ev}}} & \widetilde{U}(\mathfrak{sl}_n) \\
\downarrow \widetilde{\Upsilon} & & \widetilde{\gamma} \uparrow \wr \\
Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[z_1, \dots, z_n]/(\sum z_i) & \xrightarrow{\overline{\mathrm{ev}}^{\mathrm{rtt}}} & \widetilde{U}(\mathfrak{sl}_n) \\
\downarrow \pi & & \| \\
Y_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) & \xrightarrow{\overline{\mathrm{ev}}^{\mathrm{rtt}}} & \widetilde{U}(\mathfrak{sl}_n)
\end{array} \tag{2.32}$$

We note that the vertical arrows on the right are isomorphisms, as well as the composition $\pi \circ \widetilde{\Upsilon} : Y_\hbar(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) \xrightarrow{\sim} Y_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i)$ on the left.

The commutative diagram (2.32) in turn gives rise to the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{Y}_\hbar(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) & \xrightarrow{\overline{\mathrm{ev}}} & \widetilde{\mathbb{U}}(\mathfrak{sl}_n) \\
\downarrow \widetilde{\Upsilon} & & \widetilde{\gamma} \uparrow \wr \\
\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)[z_1, \dots, z_n]/(\sum z_i) & \xrightarrow{\overline{\mathrm{ev}}^{\mathrm{rtt}}} & \widetilde{\mathbb{U}}(\mathfrak{sl}_n) \\
\downarrow \pi & & \| \\
\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) & \xrightarrow{\overline{\mathrm{ev}}^{\mathrm{rtt}}} & \widetilde{\mathbb{U}}(\mathfrak{sl}_n)
\end{array} \tag{2.33}$$

Here we use the following notations:

- $\widetilde{\mathbb{U}}(\mathfrak{sl}_n)$ denotes the reduced extended version of $\mathbb{U}(\mathfrak{sl}_n)$, or alternatively it can be viewed as a $\mathbb{C}[\hbar]$-subalgebra of $\widetilde{U}(\mathfrak{sl}_n)$ generated by $\{\hbar x\}_{x \in \mathfrak{sl}_n} \cup \{\hbar z_i\}_{i=1}^n$.

• $\mathbf{Y}_\hbar(\mathfrak{sl}_n)[z_1,\dots,z_n]/(\sum z_i) := \mathbf{Y}_\hbar(\mathfrak{sl}_n)\otimes_{\mathbb{C}[\hbar]}\mathbb{C}[\hbar][z_1,\dots,z_n]/(\sum z_i)$, or alternatively it can be viewed as a $\mathbb{C}[\hbar]$-subalgebra of $Y_\hbar(\mathfrak{sl}_n)[z_1,\dots,z_n]/(\sum z_i)$ generated by $\{\hbar E^{(r)}_{\alpha^\vee}, \hbar F^{(r)}_{\alpha^\vee}\}^{r\geq 0}_{\alpha^\vee\in\Delta^+} \cup \{\hbar h^{(r)}_i\}^{r\geq 0}_{1\leq i<n} \cup \{\hbar z_i\}^n_{i=1}$. Following our conventions of Remark 2.32, we shall denote $\hbar z_i$ simply by z_i.
• $\mathbf{Y}^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)[z_1,\dots,z_n]/(\sum z_i) := \mathbf{Y}^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)\otimes_{\mathbb{C}[\hbar]}\mathbb{C}[\hbar][z_1,\dots,z_n]/(\sum z_i)$, or alternatively it can be viewed as a $\mathbb{C}[\hbar]$-subalgebra of $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}_n)[z_1,\dots,z_n]/(\sum z_i)$ generated by $\{\hbar t^{(r)}_{ij}\}^{r\geq 1}_{1\leq i,j\leq n} \cup \{\hbar z_i\}^n_{i=1}$. Here we denote $\hbar z_i$ simply by z_i as above.

Remark 2.40 Note that $\widetilde{\Upsilon}$ in (2.33) is well-defined, due to Proposition 2.21 (see also our discussion in Appendix A.6).

Theorem 2.41 $\mathcal{I}_0^{n\omega_{n-1}} = \mathrm{Ker}\left(\overline{\mathrm{ev}} : \mathbf{Y}_\hbar(\mathfrak{sl}_n)[z_1,\dots,z_n]/(\sum z_i) \to \widetilde{\mathbb{U}}(\mathfrak{sl}_n)\right)$.

Proof In the particular case $\mu = 0, \lambda = n\omega_{n-1}$, we note that $Z_1(z) = \dots = Z_{n-2}(z) = 1$, $Z_{n-1}(z) = \prod_{s=1}^n (z-\hbar/2-z_s)$ and $a_k = k$ $(1\leq k\leq n-1)$. Let us introduce extra currents $A_0(z)$, $A_n(z)$ via $A_0(z) := 1$, $A_n(z) := \prod_{s=1}^n (1-z_s/z)$. Then, formula (2.28) relating the generating series $\{H_k(z)\}_{k=1}^{n-1}$ to $\{A_k(z)\}_{k=1}^{n-1}$ can be uniformly written as

$$H_k(z) = \frac{(z-\frac{\hbar}{2})^{2k}}{z^k(z-\hbar)^k}\cdot\frac{A_{k-1}(z-\frac{\hbar}{2})A_{k+1}(z-\frac{\hbar}{2})}{A_k(z)A_k(z-\hbar)} \quad \text{for any } 1\leq k\leq n-1. \tag{2.34}$$

Let $\Delta_k(z)$ denote the k-th principal quantum minor $t^{1\dots k}_{1\dots k}(z)$ of $T(z)$, see Definition 2.8. According to Molev (2007), the following equality holds:

$$\Upsilon(H_k(z)) = \frac{\Delta_{k-1}(z+\frac{k-1}{2}\hbar)\Delta_{k+1}(z+\frac{k+1}{2}\hbar)}{\Delta_k(z+\frac{k-1}{2}\hbar)\Delta_k(z+\frac{k+1}{2}\hbar)}.$$

This immediately implies

$$\widetilde{\Upsilon}(H_k(z)) = \frac{\Delta_{k-1}(z+\frac{k-3}{2}\hbar)\Delta_{k+1}(z+\frac{k-1}{2}\hbar)}{\Delta_k(z+\frac{k-3}{2}\hbar)\Delta_k(z+\frac{k-1}{2}\hbar)}.$$

Generalizing $\widetilde{\Delta}_n(z)$, define $\widetilde{\Delta}_k(z)$ as the k-th principal quantum minor of the matrix $zT(z)$. Explicitly, we have $\widetilde{\Delta}_k(z) = z(z-\hbar)\cdots(z-(k-1)\hbar)\cdot\Delta_k(z)$. Then, we get

$$\widetilde{\Upsilon}(H_k(z)) = \frac{\widetilde{\Delta}_{k-1}(z+\frac{k-3}{2}\hbar)\widetilde{\Delta}_{k+1}(z+\frac{k-1}{2}\hbar)}{\widetilde{\Delta}_k(z+\frac{k-3}{2}\hbar)\widetilde{\Delta}_k(z+\frac{k-1}{2}\hbar)}.$$

Finally, define $\hat{\Delta}_k(z) := z^{-k}\widetilde{\Delta}_k(z+\frac{k-1}{2}\hbar)$. Then, the above formula reads as

$$\widetilde{\Upsilon}(H_k(z)) = \frac{(z-\frac{\hbar}{2})^{2k}}{z^k(z-\hbar)^k}\cdot\frac{\hat{\Delta}_{k-1}(z-\frac{\hbar}{2})\hat{\Delta}_{k+1}(z-\frac{\hbar}{2})}{\hat{\Delta}_k(z)\hat{\Delta}_k(z-\hbar)}.$$

By abuse of notation, let us denote the image $\pi(\hat{\Delta}_k(z))$ by $\hat{\Delta}_k(z)$ again. Note that $\hat{\Delta}_n(z) = A_n(z)$, due to our definition of π. Combining this with (2.34), we obtain the following result:

Corollary 2.42 *Under the isomorphism*

$$\pi \circ \widetilde{\Upsilon} : \mathbf{Y}_\hbar(\mathfrak{sl}_n)[z_1, \dots, z_n]/(z_1 + \cdots + z_n) \xrightarrow{\sim} \mathbf{Y}^{\mathrm{rtt}}_\hbar(\mathfrak{sl}_n)[z_1, \dots, z_n]/(z_1 + \cdots + z_n),$$

the generating series $A_k(z)$ are mapped into $\hat{\Delta}_k(z)$, that is, $\pi \circ \widetilde{\Upsilon}(A_k(z)) = \hat{\Delta}_k(z)$.

Define $\mathsf{T} \in U(\mathfrak{sl}_n) \otimes \mathrm{End}(\mathbb{C}^n)$ via $\mathsf{T} := (\widetilde{\gamma} \otimes 1)(T)$ with $T = \sum_{i,j} E_{ij} \otimes E_{ij} \in U(\mathfrak{gl}_n) \otimes \mathrm{End}(\mathbb{C}^n)$ as in Remark 2.13(b). Set $\overline{\mathsf{T}}(z) := zI_n + \hbar\mathsf{T}$. Denote the k-th principal quantum minor of $\overline{\mathsf{T}}(z)$ by $\overline{\mathsf{T}}^{1\dots k}_{1\dots k}(z)$. The following is clear:

$$\overline{\mathrm{ev}}^{\mathrm{rtt}}(\hat{\Delta}_k(z)) = z^{-k}\overline{\mathsf{T}}^{1\dots k}_{1\dots k}\left(z + \frac{k-1}{2}\hbar\right). \tag{2.35}$$

Combining Corollary 2.42 with (2.35) and the commutativity of the diagram (2.33), we get

Corollary 2.43 $\overline{\mathrm{ev}}(A_i^{(r)}) = 0$ *for any* $1 \le i \le n-1, r > i$. *In particular,* $\underline{\mathcal{I}}_0^{n\omega_{n-1}} \subseteq \mathrm{Ker}(\overline{\mathrm{ev}})$.

The opposite inclusion $\underline{\mathcal{I}}_0^{n\omega_{n-1}} \supseteq \mathrm{Ker}(\overline{\mathrm{ev}})$ follows from Theorem 2.17 by noticing that $\hat{\Delta}_1(z) = t_{11}(z)$ and so $(\pi \circ \widetilde{\Upsilon})^{-1}(t_{11}^{(r)}) = A_1^{(r)} \in \underline{\mathcal{I}}_0^{n\omega_{n-1}}$ for $r > 1$.

This completes our proof of Theorem 2.41. □

Now we are ready to present the proof of Theorem 2.39.

Proof of Theorem 2.39 Consider a subtorus $T'_W = \{g \in T_W \,|\, \det(g) = 1\}$ of T_W, and define $\underline{\mathcal{A}}_\hbar := H_\bullet^{(\mathrm{GL}(V)\times T'_W)_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$, so that $\underline{\mathcal{A}}_\hbar \simeq \mathcal{A}_\hbar/(\sum z_i)$. After imposing $\sum z_i = 0$, the homomorphism $\underline{\Phi}_0^{n\omega_{n-1}} : \mathbf{Y}_\hbar(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) \to \widetilde{\mathcal{A}}_\hbar/(\sum z_i)$ is a composition of the surjective homomorphism $\overline{\underline{\Phi}}_0^{n\omega_{n-1}} : \mathbf{Y}_\hbar(\mathfrak{sl}_n)[z_1, \dots, z_n]/(\sum z_i) \to \underline{\mathcal{A}}_\hbar$ (see Proposition 2.36) and an embedding $\mathbf{z}^*(\iota_*)^{-1} : \underline{\mathcal{A}}_\hbar \hookrightarrow \widetilde{\mathcal{A}}_\hbar/(\sum z_i)$, so that $\mathrm{Ker}(\underline{\Phi}_0^{n\omega_{n-1}}) = \mathrm{Ker}(\overline{\underline{\Phi}}_0^{n\omega_{n-1}})$. The homomorphism $\overline{\underline{\Phi}}_0^{n\omega_{n-1}}$ factors through $\overline{\phi} : \widetilde{\mathbb{U}}(\mathfrak{sl}_n) \twoheadrightarrow \underline{\mathcal{A}}_\hbar$ (due to Theorem 2.41), and it remains to prove the injectivity of $\overline{\phi}$. Note that $\overline{\phi}$ is compatible with the gradings, and it is known to be an isomorphism modulo the ideal generated by $\hbar, z_1, \dots, z_n$, see e.g. (Braverman et al. 2017, Theorem 4.12): namely, both sides are isomorphic to the ring of functions on the nilpotent cone $\mathcal{N} \subset \mathfrak{sl}_n$. To prove the injectivity of $\overline{\phi}$ it suffices to identify the graded characters of the algebras in question. But both graded characters are equal to $\mathrm{char}\,\mathbb{C}[\mathcal{N}] \cdot \mathrm{char}\left(\mathbb{C}[\hbar, z_1, \dots, z_n]/(\sum z_i)\right)$.

This completes our proof of Theorem 2.39. □

Corollary 2.44 *The reduced quantized Coulomb branch $\underline{\mathcal{A}}_\hbar$ is explicitly given by $\underline{\mathcal{A}}_\hbar \simeq \widetilde{\mathbb{U}}(\mathfrak{sl}_n)$.*

3 Quantum Algebras

3.1 The RTT Integral Form of Quantum $\mathfrak{gl}_n$

Let $\boldsymbol{v}$ be a formal variable. Consider the R-matrix $R = R^{\boldsymbol{v}}$ given by

$$R = \boldsymbol{v}^{-1} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (\boldsymbol{v}^{-1} - \boldsymbol{v}) \sum_{i>j} E_{ij} \otimes E_{ji} \tag{3.1}$$

which is an element of $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}] \otimes_{\mathbb{C}} (\text{End } \mathbb{C}^n)^{\otimes 2}$. It satisfies the famous *Yang-Baxter equation*

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},$$

viewed as the equality in $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}] \otimes_{\mathbb{C}} (\text{End } \mathbb{C}^n)^{\otimes 3}$.

Following Faddeev et al. (1989), define the *RTT integral form of quantum* $\mathfrak{gl}_n$, denoted by $\mathfrak{U}_{\boldsymbol{v}}^{\text{rtt}}(\mathfrak{gl}_n)$, to be the associative $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra generated by $\{t_{ij}^{+}, t_{ij}^{-}\}_{i,j=1}^{n}$ with the following defining relations:

$$\begin{aligned} & t_{ii}^{\pm} t_{ii}^{\mp} = 1 \quad \text{for} \quad 1 \leq i \leq n, \\ & t_{ij}^{+} = t_{ji}^{-} = 0 \quad \text{for} \quad 1 \leq j < i \leq n, \\ & R T_1^{+} T_2^{+} = T_2^{+} T_1^{+} R, \quad R T_1^{-} T_2^{-} = T_2^{-} T_1^{-} R, \quad R T_1^{-} T_2^{+} = T_2^{+} T_1^{-} R. \end{aligned} \tag{3.2}$$

Here $T^{\pm}$ are the elements of the algebra $\mathfrak{U}_{\boldsymbol{v}}^{\text{rtt}}(\mathfrak{gl}_n) \otimes \text{End } \mathbb{C}^n$, defined by $T^{\pm} = \sum_{i,j} t_{ij}^{\pm} \otimes E_{ij}$. Thus, the last three defining relations of (3.2) should be viewed as equalities in $\mathfrak{U}_{\boldsymbol{v}}^{\text{rtt}}(\mathfrak{gl}_n) \otimes (\text{End } \mathbb{C}^n)^{\otimes 2}$.

For completeness of the picture, define $\widetilde{R} \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}] \otimes_{\mathbb{C}} (\text{End } \mathbb{C}^n)^{\otimes 2}$ via[3]

$$\widetilde{R} = \boldsymbol{v} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji}. \tag{3.3}$$

Lemma 3.1 *The following equalities hold:*

$$\widetilde{R} T_1^{+} T_2^{+} = T_2^{+} T_1^{+} \widetilde{R}, \quad \widetilde{R} T_1^{-} T_2^{-} = T_2^{-} T_1^{-} \widetilde{R}, \quad \widetilde{R} T_1^{+} T_2^{-} = T_2^{-} T_1^{+} \widetilde{R}. \tag{3.4}$$

Proof Multiplying the last equality of (3.2) by R^{-1} on the left and on the right, and conjugating further by the permutation operator $P = \sum_{i,j} E_{ij} \otimes E_{ji} \in (\text{End } \mathbb{C}^n)^{\otimes 2}$, we get

$$(P R^{-1} P^{-1}) T_1^{+} T_2^{-} = T_2^{-} T_1^{+} (P R^{-1} P^{-1}).$$

[3] Let us note right away that this $\widetilde{R}$ is denoted by R in Ding and Frenkel (1993, (2.2)) and Molev (2007, §1.15.1).

Since $\widetilde{R} = PR^{-1}P^{-1}$ (straightforward verification), we obtain the last equality of (3.4).

The other two equalities of (3.4) are proved analogously. □

Note that specializing $\boldsymbol{v}$ to 1, i.e. taking a quotient by $(\boldsymbol{v}-1)$, $R^{\boldsymbol{v}}$ specializes to the identity operator $\mathrm{I} = \sum_{i,j} E_{ii} \otimes E_{jj} \in (\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$, hence, the specializations of the generators $t_{ij}^{\pm}$ pairwise commute. In other words, we get the following isomorphism:

$$\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)/(\boldsymbol{v}-1) \simeq \mathbb{C}[t_{ij}^{+}, t_{ji}^{-}]_{1\le i\le j\le n} / \left(\left\langle t_{ii}^{\pm} t_{ii}^{\mp} - 1\right\rangle_{i=1}^{n}\right). \tag{3.5}$$

We also define the $\mathbb{C}(\boldsymbol{v})$-counterpart $U_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n) := \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$.

3.2 The RTT Integral Form of Quantum Affine $\mathfrak{gl}_n$

Consider the *trigonometric* R-matrix $R_{\mathrm{trig}}(z,w) = R_{\mathrm{trig}}^{\boldsymbol{v}}(z,w)$ given by

$$\begin{aligned} R_{\mathrm{trig}}(z,w) := (\boldsymbol{v}z - \boldsymbol{v}^{-1}w)\sum_{i=1}^{n} E_{ii}\otimes E_{ii} + (z-w)\sum_{i\neq j} E_{ii}\otimes E_{jj} \\ +(\boldsymbol{v}-\boldsymbol{v}^{-1})z\sum_{i<j} E_{ij}\otimes E_{ji} + (\boldsymbol{v}-\boldsymbol{v}^{-1})w\sum_{i>j} E_{ij}\otimes E_{ji} \end{aligned} \tag{3.6}$$

which is an element of $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]\otimes_{\mathbb{C}}(\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$, cf. (Ding and Frenkel 1993, (3.7)). It satisfies the famous *Yang-Baxter equation with a spectral parameter*:

$$R_{\mathrm{trig};12}(u,v)R_{\mathrm{trig};13}(u,w)R_{\mathrm{trig};23}(v,w) = R_{\mathrm{trig};23}(v,w)R_{\mathrm{trig};13}(u,w)R_{\mathrm{trig};12}(u,v). \tag{3.7}$$

Following Faddeev et al. (1989), Ding and Frenkel (1993), define the *RTT integral form of quantum loop* $\mathfrak{gl}_n$, denoted by $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$, to be the associative $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra generated by $\{t_{ij}^{\pm}[\pm r]\}_{1\le i,j\le n}^{r\in\mathbb{N}}$ with the following defining relations:

$$\begin{aligned} &t_{ii}^{\pm}[0]t_{ii}^{\mp}[0] = 1 \ \text{ for } 1\le i\le n, \\ &t_{ij}^{+}[0] = t_{ji}^{-}[0] = 0 \ \text{ for } 1\le j<i\le n, \\ &R_{\mathrm{trig}}(z,w)T_1^{+}(z)T_2^{+}(w) = T_2^{+}(w)T_1^{+}(z)R_{\mathrm{trig}}(z,w), \\ &R_{\mathrm{trig}}(z,w)T_1^{-}(z)T_2^{-}(w) = T_2^{-}(w)T_1^{-}(z)R_{\mathrm{trig}}(z,w), \\ &R_{\mathrm{trig}}(z,w)T_1^{-}(z)T_2^{+}(w) = T_2^{+}(w)T_1^{-}(z)R_{\mathrm{trig}}(z,w). \end{aligned} \tag{3.8}$$

Here $T^{\pm}(z)$ are the series in $z^{\mp 1}$ with coefficients in the algebra $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)\otimes\mathrm{End}\,\mathbb{C}^n$, defined by $T^{\pm}(z) = \sum_{i,j} t_{ij}^{\pm}(z)\otimes E_{ij}$ with $t_{ij}^{\pm}(z) := \sum_{r\ge 0} t_{ij}^{\pm}[\pm r]z^{\mp r}$. Thus, the last three relations should be viewed as equalities of series in z, w with coefficients in $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)\otimes(\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$.

In contrast to Lemma 3.1, we have the following result (cf. (Gow and Molev 2010, (2.45))):

Lemma 3.2 *For any $\epsilon, \epsilon' \in \{\pm\}$, the following holds:*

$$R_{\mathrm{trig}}(z,w)T_1^{\epsilon}(z)T_2^{\epsilon'}(w) = T_2^{\epsilon'}(w)T_1^{\epsilon}(z)R_{\mathrm{trig}}(z,w). \tag{3.9}$$

Proof Multiplying the last equality of (3.8) by $R_{\mathrm{trig}}^{-1}(z,w)$ on the left and on the right, and conjugating further by the permutation operator $P = \sum_{i,j} E_{ij} \otimes E_{ji} \in (\mathrm{End}\ \mathbb{C}^n)^{\otimes 2}$, we get

$$(PR_{\mathrm{trig}}^{-1}(z,w)P^{-1})T_1^{+}(w)T_2^{-}(z) = T_2^{-}(z)T_1^{+}(w)(PR_{\mathrm{trig}}^{-1}(z,w)P^{-1}).$$

Combining this with the equality

$$R_{\mathrm{trig}}(z,w) = (\boldsymbol{v}z - \boldsymbol{v}^{-1}w)(\boldsymbol{v}w - \boldsymbol{v}^{-1}z)\cdot PR_{\mathrm{trig}}^{-1}(w,z)P^{-1},$$

we derive the validity of (3.9) for the only remaining case $\epsilon = +, \epsilon' = -$. □

Note that specializing $\boldsymbol{v}$ to 1, i.e. taking a quotient by $(\boldsymbol{v}-1)$, $R_{\mathrm{trig}}^{\boldsymbol{v}}(z,w)$ specializes to $(z-w)\mathrm{I} = (z-w)\sum_{i,j} E_{ii} \otimes E_{jj} \in (\mathrm{End}\ \mathbb{C}^n)^{\otimes 2}$, hence, the specializations of the generators $t_{ij}^{\pm}[\pm r]$ pairwise commute. In other words, we get the following isomorphism:

$$\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)/(\boldsymbol{v}-1) \simeq \mathbb{C}\left[t_{ji}^{\pm}[\pm r]\right]_{1\le j,i\le n}^{r\ge 0} \Big/ \left(\langle t_{ij}^{+}[0], t_{ji}^{-}[0], t_{kk}^{\pm}[0]t_{kk}^{\mp}[0]-1\rangle_{1\le j<i\le n}^{1\le k\le n}\right). \tag{3.10}$$

We also define the $\mathbb{C}(\boldsymbol{v})$-counterpart $U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n) := \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$.

3.3 The RTT Evaluation Homomorphism $\mathrm{ev}^{\mathrm{rtt}}$

Recall the following two standard relations between $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$ and $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)$, cf. Lemma 2.12.

Lemma 3.3 *The assignment $t_{ij}^{\pm} \mapsto t_{ij}^{\pm}[0]$ gives rise to a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra embedding*

$$\iota : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n) \hookrightarrow \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n).$$

Proof The above assignment is compatible with defining relations (3.2), hence, it gives rise to a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra homomorphism $\iota : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n) \to \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$. The injectivity of ι follows from the PBW theorems for $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)$ and $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$ of (Gow and Molev 2010, Proposition 2.1, Theorem 2.11). □

Lemma 3.4 *For $a \in \mathbb{C}^{\times}$, the assignment $T^{+}(z) \mapsto T^{+} - aT^{-}z^{-1}$, $T^{-}(z) \mapsto T^{-} - a^{-1}T^{+}z$ gives rise to a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra epimorphism*

$$\mathrm{ev}_a^{\mathrm{rtt}} : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n) \twoheadrightarrow \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n).$$

Proof The above assignment is compatible with defining relations (3.8), due to (3.2), (3.4), and the equality $R_{\mathrm{trig}}(z,w) = (z-w)R + (\boldsymbol{v}-\boldsymbol{v}^{-1})zP$ relating the two R-matrices, cf. (Hopkins 2007, Lemma 1.11). The resulting homomorphism $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \to \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ is clearly surjective. □

We will denote the *RTT evaluation homomorphism* $\mathrm{ev}^{\mathrm{rtt}}_1$ simply by $\mathrm{ev}^{\mathrm{rtt}}$.

Remark 3.5 (a) For any $a \in \mathbb{C}^\times$, the homomorphism $\mathrm{ev}^{\mathrm{rtt}}_a$ equals the composition of $\mathrm{ev}^{\mathrm{rtt}}$ and the automorphism of $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ given by $T^{\pm}(z) \mapsto T^{\pm}(a^{-1}z)$.
(b) The composition $\mathrm{ev}^{\mathrm{rtt}}_a \circ \iota$ is the identity endomorphism of $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ for any $a \in \mathbb{C}^\times$.

The PBW theorems for $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ and $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ of (Gow and Molev 2010, Proposition 2.1, Theorem 2.11) imply the following simple result, cf. Lemma 2.14:

Lemma 3.6 *The kernel of* $\mathrm{ev}^{\mathrm{rtt}}$ *is the* 2*-sided ideal generated by the following elements:*

$$\left\{t^+_{ij}[r],\, t^+_{ii}[s],\, t^+_{ji}[s],\, t^-_{ji}[-r],\, t^-_{ii}[-s],\, t^-_{ij}[-s]\right\}^{\substack{r\geq 1\\ s\geq 2}}_{i<j} \bigcup \left\{t^+_{ji}[1] + t^-_{ji}[0],\, t^-_{ij}[-1] + t^+_{ij}[0]\right\}_{i\leq j}. \tag{3.11}$$

However, we will need an alternative description of this kernel $\mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}})$, cf. Theorem 2.15:

Theorem 3.7 $\mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}}) = \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \cap I$, *where* I *is the* 2*-sided ideal of* $U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ *generated by* $\{t^+_{11}[s], t^-_{11}[-s]\}_{s\geq 2} \cup \{t^+_{11}[1] + t^-_{11}[0],\, t^-_{11}[-1] + t^+_{11}[0]\}$.

Proof Note that the ideal I is in the kernel of $\mathbb{C}(\boldsymbol{v})$-extended evaluation homomorphism $\mathrm{ev}^{\mathrm{rtt}} : U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \to U^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$, hence, the inclusion $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \cap I \subset \mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}})$. To prove the opposite inclusion $\mathrm{Ker}(\mathrm{ev}^{\mathrm{rtt}}) \subset \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \cap I$, it suffices to verify that all elements of (3.11) belong to I. We write $x \underset{I}{\equiv} y$ if $x - y \in I$.

• *Verification of* $t^+_{1j}[r] \in I$ *for all* $j > 1, r \geq 1$.

Comparing the matrix coefficients $\langle v_1 \otimes v_1| \cdots |v_1 \otimes v_j\rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = +$, we get $(\boldsymbol{v}z - \boldsymbol{v}^{-1}w)t^+_{11}(z)t^+_{1j}(w) = (z-w)t^+_{1j}(w)t^+_{11}(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})wt^+_{11}(w)t^+_{1j}(z)$. Evaluating the coefficients of $z^{-r}w^1$ in both sides of this equality, we find

$$-\boldsymbol{v}^{-1}t^+_{11}[r]t^+_{1j}[0] = -t^+_{1j}[0]t^+_{11}[r] + (\boldsymbol{v}-\boldsymbol{v}^{-1})t^+_{11}[0]t^+_{1j}[r]$$
$$\Longrightarrow t^+_{1j}[r] = \frac{t^-_{11}[0]\cdot[t^+_{1j}[0], t^+_{11}[r]]_{\boldsymbol{v}^{-1}}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}.$$

We claim that $[t^+_{1j}[0], t^+_{11}[r]]_{\boldsymbol{v}^{-1}} \in I$. This is clear for $r > 1$ as $t^+_{11}[r] \in I$. For $r = 1$, we note that $[t^+_{1j}[0], t^+_{11}[1]]_{\boldsymbol{v}^{-1}} \underset{I}{\equiv} -[t^+_{1j}[0], t^-_{11}[0]]_{\boldsymbol{v}^{-1}} = -(t^+_{11}[0])^{-1}\cdot[t^+_{11}[0], t^+_{1j}[0]]_{\boldsymbol{v}^{-1}}\cdot(t^+_{11}[0])^{-1}$. Finally, comparing the coefficients of z^1w^0 (instead of $z^{-r}w^1$) in the above equality, we immediately find $[t^+_{11}[0], t^+_{1j}[0]]_{\boldsymbol{v}^{-1}} = 0$. This completes our proof of the remaining inclusion $t^+_{1j}[1] \in I$.

• *Verification of* $t^+_{j1}[s] \in I$ *for all* $j > 1, s \geq 2$.

Comparing the matrix coefficients $\langle v_1 \otimes v_j | \cdots | v_1 \otimes v_1 \rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = +$, we get $(z-w)t^+_{11}(z)t^+_{j1}(w) + (\boldsymbol{v} - \boldsymbol{v}^{-1})zt^+_{j1}(z)t^+_{11}(w) = (\boldsymbol{v}z - \boldsymbol{v}^{-1}w)t^+_{j1}(w)t^+_{11}(z)$. Evaluating the coefficients of $z^{-r}w^0$ in both sides of this equality, we find

$$-t^+_{11}[r]t^+_{j1}[1] + (\boldsymbol{v} - \boldsymbol{v}^{-1})t^+_{j1}[r+1]t^+_{11}[0] = -\boldsymbol{v}^{-1}t^+_{j1}[1]t^+_{11}[r]$$
$$\Rightarrow t^+_{j1}[r+1] = \frac{[t^+_{11}[r], t^+_{j1}[1]]_{\boldsymbol{v}^{-1}} \cdot t^-_{11}[0]}{\boldsymbol{v} - \boldsymbol{v}^{-1}}.$$

We claim that $[t^+_{11}[r], t^+_{j1}[1]]_{\boldsymbol{v}^{-1}} \in I$ for $r = s-1 \geq 1$. This is clear for $r > 1$ as $t^+_{11}[r] \in I$. For $r = 1$, we note that $[t^+_{11}[1], t^+_{j1}[1]]_{\boldsymbol{v}^{-1}} \underset{I}{\equiv} -[t^-_{11}[0], t^+_{j1}[1]]_{\boldsymbol{v}^{-1}} = -(t^+_{11}[0])^{-1} \cdot [t^+_{j1}[1], t^+_{11}[0]]_{\boldsymbol{v}^{-1}} \cdot (t^+_{11}[0])^{-1}$. Finally, comparing the coefficients of z^0w^0 (instead of $z^{-r}w^0$) in the above equality, we immediately find $[t^+_{j1}[1], t^+_{11}[0]]_{\boldsymbol{v}^{-1}} = 0$. This implies the remaining inclusion $t^+_{j1}[2] \in I$.

• *Verification of* $t^+_{22}[s] \in I$ *for all* $s \geq 2$.

Comparing the matrix coefficients $\langle v_2 \otimes v_1 | \cdots | v_1 \otimes v_2 \rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = +$, we get

$$(z-w)t^+_{21}(z)t^+_{12}(w) + (\boldsymbol{v} - \boldsymbol{v}^{-1})wt^+_{11}(z)t^+_{22}(w)$$
$$= (z-w)t^+_{12}(w)t^+_{21}(z) + (\boldsymbol{v} - \boldsymbol{v}^{-1})wt^+_{11}(w)t^+_{22}(z).$$

Evaluating the coefficients of $z^{-s}w^1$ in both sides of this equality, we find

$$-t^+_{21}[s]t^+_{12}[0] + (\boldsymbol{v} - \boldsymbol{v}^{-1})t^+_{11}[s]t^+_{22}[0]$$
$$= -t^+_{12}[0]t^+_{21}[s] + (\boldsymbol{v} - \boldsymbol{v}^{-1})t^+_{11}[0]t^+_{22}[s].$$

Since $t^+_{11}[s], t^+_{21}[s] \in I$ for $s \geq 2$ by above, we immediately get the inclusion $t^+_{22}[s] \in I$.

• *Verification of* $t^+_{22}[1] + t^-_{22}[0] \in I$.

Comparing the matrix coefficients $\langle v_2 \otimes v_1 | \cdots | v_1 \otimes v_2 \rangle$ of both sides of the equality (3.9) with $\epsilon = -, \epsilon' = +$, we get

$$(z-w)t^-_{21}(z)t^+_{12}(w) + (\boldsymbol{v} - \boldsymbol{v}^{-1})wt^-_{11}(z)t^+_{22}(w)$$
$$= (z-w)t^+_{12}(w)t^-_{21}(z) + (\boldsymbol{v} - \boldsymbol{v}^{-1})wt^+_{11}(w)t^-_{22}(z).$$

Evaluating the coefficients of z^0w^0 in both sides of this equality, we find

$$-t^-_{21}[0]t^+_{12}[1] + (\boldsymbol{v} - \boldsymbol{v}^{-1})t^-_{11}[0]t^+_{22}[1]$$
$$= -t^+_{12}[1]t^-_{21}[0] + (\boldsymbol{v} - \boldsymbol{v}^{-1})t^+_{11}[1]t^-_{22}[0].$$

Since $t^+_{12}[1], t^+_{11}[1] + t^-_{11}[0] \in I$, we immediately get the inclusion $t^+_{22}[1] + t^-_{22}[0] \in I$.

• *Verification of* $t_{j1}^+[1] + t_{j1}^-[0] \in I$ *for all* $j > 1$.

Comparing the matrix coefficients $\langle v_1 \otimes v_j| \cdots |v_1 \otimes v_1\rangle$ of both sides of the equality (3.9) with $\epsilon = +, \epsilon' = -$, we get

$$(z-w)t_{11}^+(z)t_{j1}^-(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zt_{j1}^+(z)t_{11}^-(w) = (\boldsymbol{v}z - \boldsymbol{v}^{-1}w)t_{j1}^-(w)t_{11}^+(z).$$

Evaluating the coefficients of z^0w^0 in both sides of this equality, we find

$$t_{11}^+[1]t_{j1}^-[0] + (\boldsymbol{v}-\boldsymbol{v}^{-1})t_{j1}^+[1]t_{11}^-[0] = \boldsymbol{v}t_{j1}^-[0]t_{11}^+[1].$$

Since $t_{11}^+[1] + t_{11}^-[0] \in I$, we get $t_{j1}^+[1] \underset{I}{\equiv} \frac{[t_{11}^-[0], t_{j1}^-[0]]_{\boldsymbol{v}} \cdot t_{11}^+[0]}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$. On the other hand, comparing the matrix coefficients $\langle v_1 \otimes v_j| \cdots |v_1 \otimes v_1\rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = -$, we get $(z-w)t_{11}^-(z)t_{j1}^-(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zt_{j1}^-(z)t_{11}^-(w) = (\boldsymbol{v}z - \boldsymbol{v}^{-1}w)t_{j1}^-(w)t_{11}^-(z)$. Evaluating the coefficients of z^1w^0 in both sides of this equality, we find

$$\begin{aligned} t_{11}^-[0]t_{j1}^-[0] + (\boldsymbol{v}-\boldsymbol{v}^{-1})t_{j1}^-[0]t_{11}^-[0] &= \boldsymbol{v}t_{j1}^-[0]t_{11}^-[0] \\ \Longrightarrow t_{j1}^-[0] &= -\frac{[t_{11}^-[0], t_{j1}^-[0]]_{\boldsymbol{v}} \cdot t_{11}^+[0]}{\boldsymbol{v}-\boldsymbol{v}^{-1}}. \end{aligned}$$

Hence, the inclusion $t_{j1}^+[1] + t_{j1}^-[0] \in I$.

One can now apply the above five verifications with all lower indices increased by 1 to prove the inclusions $t_{2j}^+[r], t_{j2}^+[s], t_{33}^+[s], t_{33}^+[1] + t_{33}^-[0], t_{j2}^+[1] + t_{j2}^-[0] \in I$ for any $j > 2, r \geq 1, s \geq 2$. Proceeding further step by step, we obtain $\{t_{ij}^+[r], t_{ii}^+[s], t_{ji}^+[s]\}_{i<j}^{r\geq 1, s\geq 2} \cup \{t_{ji}^+[1] + t_{ji}^-[0]\}_{i\leq j} \subset I$. The proof of the remaining inclusion $\{t_{ji}^-[-r], t_{ii}^-[-s], t_{ij}^-[-s]\}_{i<j}^{r\geq 1, s\geq 2} \cup \{t_{ij}^-[-1] + t_{ij}^+[0]\}_{i\leq j} \subset I$ is analogous and we leave details to the interested reader.

This completes our proof of Theorem 3.7. □

3.4 The Drinfeld–Jimbo Quantum $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$

Following Jimbo (1986), define the quantum $\mathfrak{gl}_n$, denoted by $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$, to be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{E_i, F_i, t_j, t_j^{-1}\}_{1\leq i<n}^{1\leq j\leq n}$ with the following defining relations:

$$\begin{aligned} &t_j t_j^{-1} = t_j^{-1} t_j = 1,\ t_j t_{j'} = t_{j'} t_j, \\ &t_j E_i = \boldsymbol{v}^{-\delta_{ji}+\delta_{j,i+1}} E_i t_j,\ t_j F_i = \boldsymbol{v}^{\delta_{ji}-\delta_{j,i+1}} F_i t_j, \\ &E_i F_{i'} - F_{i'} E_i = \delta_{ii'} \frac{K_i - K_i^{-1}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}, \\ &E_i E_{i'} = E_{i'} E_i \text{ and } F_i F_{i'} = F_{i'} F_i \text{ if } c_{ii'} = 0, \\ &E_i^2 E_{i'} - (\boldsymbol{v}+\boldsymbol{v}^{-1}) E_i E_{i'} E_i + E_{i'} E_i^2 = 0 \text{ if } c_{ii'} = -1, \\ &F_i^2 F_{i'} - (\boldsymbol{v}+\boldsymbol{v}^{-1}) F_i F_{i'} F_i + F_{i'} F_i^2 = 0 \text{ if } c_{ii'} = -1, \end{aligned} \tag{3.12}$$

where $K_i := t_i^{-1} t_{i+1}$ and $(c_{ii'})_{i,i'=1}^{n-1}$ denotes the Cartan matrix of $\mathfrak{sl}_n$.

Remark 3.8 We note that our generators $E_i, F_i, t_j^{\pm 1}$ correspond to the generators $f_i, e_i, \boldsymbol{v}^{\pm H_j}$ of Ding and Frenkel (1993, Definition 2.3), respectively.

The $\mathbb{C}(\boldsymbol{v})$-subalgebra of $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$ generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i=1}^{n-1}$ is isomorphic to the Drinfeld-Jimbo quantum $\mathfrak{sl}_n$, denoted by $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$, see Drinfeld (1985), Jimbo (1986).

The following well-known result was conjectured in Faddeev et al. (1989) and proved in Ding and Frenkel (1993, Theorem 2.1):

Theorem 3.9 (Ding and Frenkel 1993) *There is a unique $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism*

$$\Upsilon : U_{\boldsymbol{v}}(\mathfrak{gl}_n) \xrightarrow{\sim} U_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)$$

defined by

$$t_j^{\pm 1} \mapsto t_{jj}^{\pm}, \quad E_i \mapsto \frac{t_{ii}^- t_{i,i+1}^+}{\boldsymbol{v} - \boldsymbol{v}^{-1}}, \quad F_i \mapsto \frac{t_{i+1,i}^- t_{ii}^+}{\boldsymbol{v}^{-1} - \boldsymbol{v}}. \tag{3.13}$$

As an immediate corollary, $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)$ is realized as a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$. To describe this subalgebra explicitly, define the elements $\{E_{j,i+1}, F_{i+1,j}\}_{1 \le j \le i < n}$ of $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$ via

$$\begin{aligned} E_{j,i+1} &:= (\boldsymbol{v} - \boldsymbol{v}^{-1})[E_i, \cdots, [E_{j+1}, E_j]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}, \\ F_{i+1,j} &:= (\boldsymbol{v}^{-1} - \boldsymbol{v})[\cdots [F_j, F_{j+1}]_{\boldsymbol{v}}, \cdots, F_i]_{\boldsymbol{v}}, \end{aligned} \tag{3.14}$$

where $[a, b]_x := ab - x \cdot ba$. In particular, $E_{i,i+1} = (\boldsymbol{v} - \boldsymbol{v}^{-1}) E_i$ and $F_{i+1,i} = (\boldsymbol{v}^{-1} - \boldsymbol{v}) F_i$.

Definition 3.10 (a) Let $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ be the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$ generated by

$$\{E_{j,i+1}, F_{i+1,j}\}_{1 \le j \le i < n} \cup \{t_j^{\pm 1}\}_{1 \le j \le n}. \tag{3.15}$$

(b) Let $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ be the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ generated by

$$\{E_{j,i+1}, F_{i+1,j}\}_{1 \le j \le i < n} \cup \{K_i^{\pm 1}\}_{1 \le i < n}. \tag{3.16}$$

Proposition 3.11 $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n) = \Upsilon^{-1}(\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n))$.

This result follows immediately from Proposition 3.12 and Corollary 3.13 below. To state those, define the elements $\{\tilde{e}_{j,i+1}, \tilde{f}_{i+1,j}\}_{1 \le j \le i < n}$ of $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)$ via

$$\tilde{e}_{j,i+1} := t_{jj}^- t_{j,i+1}^+, \quad \tilde{f}_{i+1,j} := t_{i+1,j}^- t_{jj}^+. \tag{3.17}$$

Proposition 3.12 *For any $1 \leq j < i < n$, the following equalities hold in $U_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)$:*

$$\tilde{e}_{j,i+1} = \frac{[\tilde{e}_{i,i+1}, \tilde{e}_{ji}]_{\boldsymbol{v}^{-1}}}{\boldsymbol{v} - \boldsymbol{v}^{-1}}, \quad \tilde{f}_{i+1,j} = \frac{[\tilde{f}_{ij}, \tilde{f}_{i+1,i}]_{\boldsymbol{v}}}{\boldsymbol{v}^{-1} - \boldsymbol{v}}.$$

The proof of this result is analogous to that of Proposition 3.21 below (and actually it can be deduced from the latter by using the embedding $\iota : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n) \hookrightarrow \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$ of Lemma 3.3).

Corollary 3.13 $E_{j,i+1} = \Upsilon^{-1}(\tilde{e}_{j,i+1})$, $F_{i+1,j} = \Upsilon^{-1}(\tilde{f}_{i+1,j})$ *for any* $1 \leq j \leq i < n$.

Proof For a fixed $1 \leq i < n$, this follows by a decreasing induction in j. The base of the induction $j = i$ is due to (3.13), while the induction step follows from Proposition 3.12. □

We order $\{E_{j,i+1}\}_{1 \leq j \leq i < n}$ in the following way: $E_{j,i+1} \leq E_{j',i'+1}$ if $j < j'$, or $j = j', i \leq i'$. Likewise, we order $\{F_{i+1,j}\}_{1 \leq j \leq i < n}$ so that $F_{i+1,j} \geq F_{i'+1,j'}$ if $j < j'$, or $j = j', i \leq i'$. Finally, we choose any total ordering of the Cartan generators $\{t_j\}_{1 \leq j \leq n}$ of $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ (or $\{K_i\}_{1 \leq i < n}$ of $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$). Having specified these three total orderings, elements $F \cdot H \cdot E$ with F, E, H being ordered monomials in $\{F_{i+1,j}\}_{1 \leq j \leq i < n}$, $\{E_{j,i+1}\}_{1 \leq j \leq i < n}$, and the Cartan generators $\{t_j^{\pm 1}\}_{1 \leq j \leq n}$ of $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ (or $\{K_i^{\pm 1}\}_{1 \leq i < n}$ of $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$), respectively, are called the *ordered PBW monomials* (in the corresponding generators). The proof of the following result is analogous to that of Theorem 3.24 below and is based on Proposition 3.11, we leave details to the interested reader.

Theorem 3.14 *(a) The ordered PBW monomials in $\{F_{i+1,j}, t_k^{\pm 1}, E_{j,i+1}\}_{1 \leq j \leq i < n}^{1 \leq k \leq n}$ form a basis of a free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)$.*
(b) The ordered PBW monomials in $\{F_{i+1,j}, K_k^{\pm 1}, E_{j,i+1}\}_{1 \leq j \leq i < n}^{1 \leq k < n}$ form a basis of a free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{sl}_n)$.

Remark 3.15 We note that $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n) \simeq \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)$ quantizes the algebra of functions on the big Bruhat cell in $\mathrm{GL}(n)$, that is $\mathfrak{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)/(\boldsymbol{v} - 1) \simeq \mathbb{C}[N_- T N_+]$, due to (3.5) and the PBW theorem of (Gow and Molev 2010, Proposition 2.1). Here N_- (resp. N_+) denotes the subgroup of strictly lower (resp. strictly upper) triangular matrices, and T denotes the diagonal torus of $\mathrm{GL}(n)$.

Remark 3.16 For a complete picture, let us recall in which sense $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$ is usually treated as a quantization of the universal enveloping algebra $U(\mathfrak{gl}_n)$. Let $\mathsf{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ be the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$ generated by $\{t_j^{\pm 1}\}_{j=1}^n$ and the divided powers $\{E_i^{(m)}, F_i^{(m)}\}_{1 \leq i < n}^{m \geq 1}$. According to Lusztig (1990b, Proposition 2.3(a)) (cf. Jimbo (1986)), the subalgebra $\mathsf{U}_{\boldsymbol{v}}^{<}(\mathfrak{gl}_n)$ (resp. $\mathsf{U}_{\boldsymbol{v}}^{>}(\mathfrak{gl}_n)$) of $\mathsf{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ generated by $\{F_i^{(m)}\}_{1 \leq i < n}^{m \geq 1}$ (resp. $\{E_i^{(m)}\}_{1 \leq i < n}^{m \geq 1}$) is a free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-module with a basis consisting of the ordered products of the divided powers of the root generators $F'_{i+1,j} := [\cdots [F_j, F_{j+1}]_{\boldsymbol{v}}, \ldots, F_i]_{\boldsymbol{v}}$ (resp. $E'_{j,i+1} := [E_i, \ldots, [E_{j+1}, E_j]_{\boldsymbol{v}^{-1}} \ldots]_{\boldsymbol{v}^{-1}}$). Specializing $\boldsymbol{v}$ to 1, we have $t_j^2 = 1$ in a $\mathbb{C}$-algebra $\mathsf{U}_1(\mathfrak{gl}_n) := \mathsf{U}_{\boldsymbol{v}}(\mathfrak{gl}_n)/(\boldsymbol{v} - 1)$.

Specializing further t_j to 1, we get a $\mathbb{C}$-algebra isomorphism $\mathsf{U}_1(\mathfrak{gl}_n)/(\langle t_j-1\rangle_{j=1}^n) \simeq U(\mathfrak{gl}_n)$, under which $E'_{j,i+1} \mapsto (-1)^{i-j}E_{j,i+1}$, $F'_{i+1,j} \mapsto (-1)^{i-j}E_{i+1,j}$.

3.5 The Drinfeld Quantum Affine $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$

Following Drinfeld (1988), define the quantum loop $\mathfrak{gl}_n$, denoted by $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, to be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{e_{i,r},\, f_{i,r},\, \varphi^+_{j,s},\, \varphi^-_{j,-s}\}^{r\in\mathbb{Z},s\in\mathbb{N}}_{1\le i<n,1\le j\le n}$ with the following defining relations (cf. (Ding and Frenkel 1993, Definition 3.1)):

$$
\begin{aligned}
&[\varphi^{\epsilon}_j(z), \varphi^{\epsilon'}_{j'}(w)] = 0,\ \ \varphi^{\pm}_{j,0}\cdot\varphi^{\mp}_{j,0} = 1,\\
&(z-\boldsymbol{v}^{c_{ii'}}w)e_i(z)e_{i'}(w) = (\boldsymbol{v}^{c_{ii'}}z-w)e_{i'}(w)e_i(z),\\
&(\boldsymbol{v}^{c_{ii'}}z-w)f_i(z)f_{i'}(w) = (z-\boldsymbol{v}^{c_{ii'}}w)f_{i'}(w)f_i(z),\\
&(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)^{\delta_{ji}}(z-\boldsymbol{v}w)^{\delta_{j,i+1}}\varphi^{\epsilon}_j(z)e_i(w) = (z-w)^{\delta_{ji}}(\boldsymbol{v}z-w)^{\delta_{j,i+1}}e_i(w)\varphi^{\epsilon}_j(z),\\
&(z-w)^{\delta_{ji}}(\boldsymbol{v}z-w)^{\delta_{j,i+1}}\varphi^{\epsilon}_j(z)f_i(w) = (\boldsymbol{v}z-\boldsymbol{v}^{-1}w)^{\delta_{ji}}(z-\boldsymbol{v}w)^{\delta_{j,i+1}}f_i(w)\varphi^{\epsilon}_j(z),\\
&[e_i(z),\, f_{i'}(w)] = \frac{\delta_{ii'}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\delta\left(\frac{z}{w}\right)\left(\psi^+_i(z)-\psi^-_i(z)\right),\\
&e_i(z)e_{i'}(w) = e_{i'}(w)e_i(z) \text{ and } f_i(z)f_{i'}(w) = f_{i'}(w)f_i(z) \text{ if } c_{ii'}=0,\\
&[e_i(z_1),[e_i(z_2),e_{i'}(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} + [e_i(z_2),[e_i(z_1),e_{i'}(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0 \text{ if } c_{ii'}=-1,\\
&[f_i(z_1),[f_i(z_2),f_{i'}(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} + [f_i(z_2),[f_i(z_1),f_{i'}(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0 \text{ if } c_{ii'}=-1,
\end{aligned}
\tag{3.18}
$$

where the generating series are defined as follows:

$$
e_i(z) := \sum_{r\in\mathbb{Z}} e_{i,r}z^{-r},\quad f_i(z) := \sum_{r\in\mathbb{Z}} f_{i,r}z^{-r},\quad \varphi^{\pm}_i(z) := \sum_{s\ge 0}\varphi^{\pm}_{i,\pm s}z^{\mp s},\ \delta(z) := \sum_{r\in\mathbb{Z}} z^r,
$$

and $\psi^{\pm}_i(z) = \sum_{s\ge 0}\psi^{\pm}_{i,\pm s}z^{\mp s}$ is determined via $\psi^{\pm}_i(z) := (\varphi^{\pm}_i(z))^{-1}\varphi^{\pm}_{i+1}(\boldsymbol{v}^{-1}z)$. We will also need *Drinfeld half-currents* $e^{\pm}_i(z)$, $f^{\pm}_i(z)$ defined via

$$
e^+_i(z) := \sum_{r\ge 0} e_{i,r}z^{-r},\quad e^-_i(z) := -\sum_{r<0} e_{i,r}z^{-r},\quad f^+_i(z) := \sum_{r>0} f_{i,r}z^{-r},\ f^-_i(z) := -\sum_{r\le 0} f_{i,r}z^{-r},
$$

so that $e_i(z) = e^+_i(z) - e^-_i(z)$, $f_i(z) = f^+_i(z) - f^-_i(z)$.

The $\mathbb{C}(\boldsymbol{v})$-subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ generated by $\{e_{i,r},\, f_{i,r},\, \psi^{\pm}_{i,\pm s}\}^{r\in\mathbb{Z},s\in\mathbb{N}}_{1\le i<n}$ is isomorphic to the quantum loop $\mathfrak{sl}_n$, denoted by $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. To be more precise, this recovers the new Drinfeld realization of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, see Drinfeld (1988). The latter also admits the original Drinfeld-Jimbo realization with the generators $\{E_i, F_i, K_i^{\pm 1}\}_{i\in[n]}$ (here $[n] := \{0,1,\dots,n-1\}$ viewed as mod n residues) and with the defining relations exactly as in (3.12), but with $(c_{ii'})_{i,i'\in[n]}$ denoting the Cartan matrix of $\widehat{\mathfrak{sl}}_n$. We prefer to keep the same notation $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ for these two realizations. However, we will need an explicit identification which expresses the Drinfeld-Jimbo generators in terms of

the "loop" generators (featuring in the new Drinfeld realization), see Drinfeld (1988), Jing (1998):

$$\begin{aligned}
&E_i \mapsto e_{i,0},\ F_i \mapsto f_{i,0},\ K_i^{\pm 1} \mapsto \psi_{i,0}^{\pm} \text{ for } i \in [n]\backslash\{0\},\\
&K_0^{\pm 1} \mapsto \psi_{1,0}^{\mp} \cdots \psi_{n-1,0}^{\mp},\\
&E_0 \mapsto (-\boldsymbol{v})^{-n+2} \cdot [\cdots [f_{1,1}, f_{2,0}]_{\boldsymbol{v}}, \cdots, f_{n-1,0}]_{\boldsymbol{v}} \cdot \psi_{1,0}^{-} \cdots \psi_{n-1,0}^{-},\\
&F_0 \mapsto (-\boldsymbol{v})^{n} \cdot [e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} \cdot \psi_{1,0}^{+} \cdots \psi_{n-1,0}^{+}.
\end{aligned} \tag{3.19}$$

The relation between the algebras $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ and $U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$ was conjectured in Faddeev et al. (1989) and proved in Ding and Frenkel (1993, Main Theorem). To state the result, consider the Gauss decomposition of the matrices $T^{\pm}(z)$ of Sect. 3.2:

$$T^{\pm}(z) = \widetilde{F}^{\pm}(z) \cdot \widetilde{G}^{\pm}(z) \cdot \widetilde{E}^{\pm}(z).$$

Here $\widetilde{F}^{\pm}(z), \widetilde{G}^{\pm}(z), \widetilde{E}^{\pm}(z)$ are the series in $z^{\mp 1}$ with coefficients in the algebra $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n) \otimes \mathrm{End}\, \mathbb{C}^n$ which are of the form

$$\begin{aligned}
\widetilde{F}^{\pm}(z) &= \sum_i E_{ii} + \sum_{i>j} \tilde{f}_{ij}^{\pm}(z) \cdot E_{ij},\ \widetilde{G}^{\pm}(z) = \sum_i \tilde{g}_i^{\pm}(z) \cdot E_{ii},\\
\widetilde{E}^{\pm}(z) &= \sum_i E_{ii} + \sum_{i<j} \tilde{e}_{ij}^{\pm}(z) \cdot E_{ij}.
\end{aligned}$$

Theorem 3.17 (Ding and Frenkel 1993) *There is a unique* $\mathbb{C}(\boldsymbol{v})$*-algebra isomorphism*

$$\Upsilon : U_{\boldsymbol{v}}(L\mathfrak{gl}_n) \xrightarrow{\sim} U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$$

defined by

$$e_i^{\pm}(z) \mapsto \frac{\tilde{e}_{i,i+1}^{\pm}(\boldsymbol{v}^i z)}{\boldsymbol{v} - \boldsymbol{v}^{-1}},\quad f_i^{\pm}(z) \mapsto \frac{\tilde{f}_{i+1,i}^{\pm}(\boldsymbol{v}^i z)}{\boldsymbol{v} - \boldsymbol{v}^{-1}},\quad \varphi_j^{\pm}(z) \mapsto \tilde{g}_j^{\pm}(\boldsymbol{v}^j z). \tag{3.20}$$

Remark 3.18 To compare with the notations of Ding and Frenkel (1993), we note that our generating series $e_i(z), f_i(z), \varphi_j^{\pm}(z)$ of $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ correspond to $\frac{X_i^-(\boldsymbol{v}^i z)}{\boldsymbol{v}^{-1} - \boldsymbol{v}}, \frac{X_i^+(\boldsymbol{v}^i z)}{\boldsymbol{v}^{-1} - \boldsymbol{v}}, k_j^{\mp}(\boldsymbol{v}^j z)$ of Ding and Frenkel (1993, Definition 3.1), respectively. Likewise, our matrices $T^+(z)$ and $T^-(z)$ of Sect. 3.2 correspond to $L^-(z)$ and $L^+(z)$ of Ding and Frenkel (1993, Definition 3.2), respectively. After these identifications, we see that Theorem 3.17 is just Ding and Frenkel (1993, Main Theorem) (for the trivial central charge).

As an immediate corollary, $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$ is realized as a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$. To describe this subalgebra explicitly, define the elements $\{E_{j,i+1}^{(r)},$

$F^{(r)}_{i+1,j}\}^{r\in\mathbb{Z}}_{1\le j\le i<n}$ of $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ via

$$\begin{aligned} E^{(r)}_{j,i+1} &:= (\boldsymbol{v}-\boldsymbol{v}^{-1})[e_{i,0},\cdots,[e_{j+1,0},e_{j,r}]_{\boldsymbol{v}^{-1}}\cdots]_{\boldsymbol{v}^{-1}},\\ F^{(r)}_{i+1,j} &:= (\boldsymbol{v}^{-1}-\boldsymbol{v})[\cdots[f_{j,r},f_{j+1,0}]_{\boldsymbol{v}},\cdots,f_{i,0}]_{\boldsymbol{v}}. \end{aligned} \tag{3.21}$$

These elements with $r=0,\pm 1$ played an important role in Finkelberg and Tsymbaliuk (2017, Section 10, Appendix G). We also note that $E^{(r)}_{i,i+1}=(\boldsymbol{v}-\boldsymbol{v}^{-1})e_{i,r}$ and $F^{(r)}_{i+1,i}=(\boldsymbol{v}^{-1}-\boldsymbol{v})f_{i,r}$.

Definition 3.19 (a) Let $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ be the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ generated by

$$\{E^{(r)}_{j,i+1},F^{(r)}_{i+1,j}\}^{r\in\mathbb{Z}}_{1\le j\le i<n}\cup\{\varphi^{\pm}_{j,\pm s}\}^{s\in\mathbb{N}}_{1\le j\le n}. \tag{3.22}$$

(b) Let $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ be the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by

$$\{E^{(r)}_{j,i+1},F^{(r)}_{i+1,j}\}^{r\in\mathbb{Z}}_{1\le j\le i<n}\cup\{\psi^{\pm}_{i,\pm s}\}^{s\in\mathbb{N}}_{1\le i<n}. \tag{3.23}$$

The following result can be viewed as a trigonometric counterpart of Proposition 2.21:

Proposition 3.20 $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)=\Upsilon^{-1}(\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n))$.

The proof of Proposition 3.20 follows immediately from Proposition 3.21 and Corollary 3.23 below. To state those, let us express the matrix coefficients of $\widetilde{F}^{\pm}(z)$, $\widetilde{G}^{\pm}(z)$, $\widetilde{E}^{\pm}(z)$ as series in $z^{\mp 1}$ with coefficients in $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$:

$$\begin{aligned} &\tilde{e}^{+}_{ij}(z)=\sum_{r\ge 0}\tilde{e}^{(r)}_{ij}z^{-r},\quad \tilde{e}^{-}_{ij}(z)=\sum_{r<0}\tilde{e}^{(r)}_{ij}z^{-r},\\ &\tilde{f}^{+}_{ij}(z)=\sum_{r>0}\tilde{f}^{(r)}_{ij}z^{-r},\quad \tilde{f}^{-}_{ij}(z)=\sum_{r\le 0}\tilde{f}^{(r)}_{ij}z^{-r},\\ &\tilde{g}^{\pm}_{i}(z)=\tilde{g}^{\pm}_{i}+\sum_{r>0}\tilde{g}^{(\pm r)}_{i}z^{\mp r}. \end{aligned}$$

The following result generalizes (Finkelberg and Tsymbaliuk 2017, Proposition G.9):

Proposition 3.21 *For any $1\le j<i<n$, the following equalities hold in $U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$:*

$$\tilde{e}^{+}_{j,i+1}(z)=\frac{[\tilde{e}^{(0)}_{i,i+1},\tilde{e}^{+}_{ji}(z)]_{\boldsymbol{v}^{-1}}}{\boldsymbol{v}-\boldsymbol{v}^{-1}},\quad \tilde{e}^{-}_{j,i+1}(z)=\frac{[\tilde{e}^{(0)}_{i,i+1},\tilde{e}^{-}_{ji}(z)]_{\boldsymbol{v}^{-1}}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}, \tag{3.24}$$

$$\tilde{f}^{+}_{i+1,j}(z)=\frac{[\tilde{f}^{+}_{ij}(z),\tilde{f}^{(0)}_{i+1,i}]_{\boldsymbol{v}}}{\boldsymbol{v}^{-1}-\boldsymbol{v}},\quad \tilde{f}^{-}_{i+1,j}(z)=\frac{[\tilde{f}^{-}_{ij}(z),\tilde{f}^{(0)}_{i+1,i}]_{\boldsymbol{v}}}{\boldsymbol{v}^{-1}-\boldsymbol{v}}. \tag{3.25}$$

Proof For any $1 \leq i < n$, we proceed by an increasing induction in j.

• *Verification of the first formula in* (3.24).

Comparing the matrix coefficients $\langle v_j \otimes v_i | \cdots | v_i \otimes v_{i+1} \rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = +$, we get

$$\begin{aligned}(z-w)t^+_{ji}(z)t^+_{i,i+1}(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zt^+_{ii}(z)t^+_{j,i+1}(w)\\ = (z-w)t^+_{i,i+1}(w)t^+_{ji}(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})wt^+_{ii}(w)t^+_{j,i+1}(z).\end{aligned}$$

Evaluating the terms with w^1 in both sides of this equality, we find

$$-t^+_{ji}(z)t^+_{i,i+1}[0] = -t^+_{i,i+1}[0]t^+_{ji}(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})t^+_{ii}[0]t^+_{j,i+1}(z).$$

Note that $t^+_{ji}(z)t^+_{ii}[0] = \boldsymbol{v}^{-1}t^+_{ii}[0]t^+_{ji}(z)$. To see the latter, we compare the matrix coefficients $\langle v_j \otimes v_i | \cdots | v_i \otimes v_i \rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = +$, and then evaluate the terms with w^1 as above. Combining this with $t^+_{i,i+1}[0] = \tilde{g}^+_i \tilde{e}^{(0)}_{i,i+1} = t^+_{ii}[0]\tilde{e}^{(0)}_{i,i+1}$, we deduce

$$t^+_{j,i+1}(z) = \frac{[\tilde{e}^{(0)}_{i,i+1}, t^+_{ji}(z)]_{\boldsymbol{v}^{-1}}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}. \tag{3.26}$$

Recall that

$$\begin{aligned}t^+_{ji}(z) &= \tilde{g}^+_j(z)\tilde{e}^+_{ji}(z) + \sum_{1\leq k\leq j-1} \tilde{f}^+_{jk}(z)\tilde{g}^+_k(z)\tilde{e}^+_{ki}(z),\\ t^+_{j,i+1}(z) &= \tilde{g}^+_j(z)\tilde{e}^+_{j,i+1}(z) + \sum_{1\leq k\leq j-1} \tilde{f}^+_{jk}(z)\tilde{g}^+_k(z)\tilde{e}^+_{k,i+1}(z).\end{aligned}$$

Let us further note that $\tilde{e}^{(0)}_{i,i+1}$ commutes with $\tilde{f}^+_{jk}(z)$ (since by the induction assumption the latter can be expressed via $\tilde{f}^{(\bullet)}_{s,s-1}$ which clearly commute with $\tilde{e}^{(0)}_{i,i+1}$ for $s \leq j$) and with $\tilde{g}^+_k(z)$ for $k \leq j$. By the induction assumption $\frac{[\tilde{e}^{(0)}_{i,i+1}, \tilde{e}^+_{ki}(z)]_{\boldsymbol{v}^{-1}}}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = \tilde{e}^+_{k,i+1}(z)$ for $k < j$. Hence, we get

$$\begin{aligned}&\tilde{g}^+_j(z)\tilde{e}^+_{j,i+1}(z) + \sum_{k=1}^{j-1} \tilde{f}^+_{jk}(z)\tilde{g}^+_k(z)\tilde{e}^+_{k,i+1}(z)\\ &= \frac{\tilde{g}^+_j(z)[\tilde{e}^{(0)}_{i,i+1}, \tilde{e}^+_{ji}(z)]_{\boldsymbol{v}^{-1}}}{\boldsymbol{v}-\boldsymbol{v}^{-1}} + \sum_{k=1}^{j-1} \tilde{f}^+_{jk}(z)\tilde{g}^+_k(z)\tilde{e}^+_{k,i+1}(z),\end{aligned}$$

which implies the first equality in (3.24).

• *Verification of the second formula in* (3.24).

Comparing the matrix coefficients $\langle v_j \otimes v_i | \cdots | v_i \otimes v_{i+1} \rangle$ of both sides of the equality (3.9) with $\epsilon = -, \epsilon' = +$, we get

$$\begin{aligned}(z-w)t^-_{ji}(z)t^+_{i,i+1}(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zt^-_{ii}(z)t^+_{j,i+1}(w)\\ = (z-w)t^+_{i,i+1}(w)t^-_{ji}(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})wt^+_{ii}(w)t^-_{j,i+1}(z).\end{aligned}$$

Evaluating the terms with w^1 in both sides of this equality, we find

$$-t^-_{ji}(z)t^+_{i,i+1}[0] = -t^+_{i,i+1}[0]t^-_{ji}(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})t^+_{ii}[0]t^-_{j,i+1}(z).$$

Note that $t^-_{ji}(z)t^+_{ii}[0] = \boldsymbol{v}^{-1}t^+_{ii}[0]t^-_{ji}(z)$ (which follows by comparing the matrix coefficients $\langle v_j \otimes v_i | \cdots | v_i \otimes v_i \rangle$ of both sides of the equality (3.9) with $\epsilon = -, \epsilon' = +$, and then evaluating the terms with w^1 as above). Combining this with $t^+_{i,i+1}[0] = \tilde{g}^+_i \tilde{e}^{(0)}_{i,i+1} = t^+_{ii}[0]\tilde{e}^{(0)}_{i,i+1}$, we obtain

$$t^-_{j,i+1}(z) = \frac{[\tilde{e}^{(0)}_{i,i+1}, t^-_{ji}(z)]_{\boldsymbol{v}^{-1}}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}. \tag{3.27}$$

This implies the second equality in (3.24) via the same inductive arguments as above.

• *Verification of the first formula in* (3.25).

Comparing the matrix coefficients $\langle v_{i+1} \otimes v_i | \cdots | v_i \otimes v_j \rangle$ of both sides of the equality (3.9) with $\epsilon = -, \epsilon' = +$, we get

$$\begin{aligned}(z-w)t^-_{i+1,i}(z)t^+_{ij}(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})wt^-_{ii}(z)t^+_{i+1,j}(w)\\ = (z-w)t^+_{ij}(w)t^-_{i+1,i}(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zt^+_{ii}(w)t^-_{i+1,j}(z).\end{aligned}$$

Evaluating the terms with z^0 in both sides of this equality, we find

$$-t^-_{i+1,i}[0]t^+_{ij}(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})t^-_{ii}[0]t^+_{i+1,j}(w) = -t^+_{ij}(w)t^-_{i+1,i}[0].$$

Note that $t^-_{ii}[0]t^+_{i+1,j}(z) = t^+_{i+1,j}(z)t^-_{ii}[0]$ and $t^-_{ii}[0]t^+_{ij}(z) = \boldsymbol{v}t^+_{ij}(z)t^-_{ii}[0]$. To see these equalities, we compare the matrix coefficients $\langle v_{i+1} \otimes v_i | \cdots | v_j \otimes v_i \rangle$ and $\langle v_i \otimes v_i | \cdots | v_j \otimes v_i \rangle$ of both sides of the equality (3.9) with $\epsilon = +, \epsilon' = -$, and then evaluate the terms with w^0 as above. Combining this with $t^-_{i+1,i}[0] = \tilde{f}^{(0)}_{i+1,i}\tilde{g}^-_i = \tilde{f}^{(0)}_{i+1,i}t^-_{ii}[0]$, we deduce

$$t^+_{i+1,j}(w) = \frac{[t^+_{ij}(w), \tilde{f}^{(0)}_{i+1,i}]_{\boldsymbol{v}}}{\boldsymbol{v}^{-1}-\boldsymbol{v}}. \tag{3.28}$$

Recall that

$$t_{ij}^{+}(w) = \tilde{f}_{ij}^{+}(w)\tilde{g}_{j}^{+}(w) + \sum_{1\le k\le j-1} \tilde{f}_{ik}^{+}(w)\tilde{g}_{k}^{+}(w)\tilde{e}_{kj}^{+}(w),$$
$$t_{i+1,j}^{+}(w) = \tilde{f}_{i+1,j}^{+}(w)\tilde{g}_{j}^{+}(w) + \sum_{1\le k\le j-1} \tilde{f}_{i+1,k}^{+}(w)\tilde{g}_{k}^{+}(w)\tilde{e}_{kj}^{+}(w).$$

We further note that $\tilde{f}_{i+1,i}^{(0)}$ commutes with $\tilde{e}_{kj}^{+}(z)$ (since by the induction assumption the latter can be expressed via $\tilde{e}_{s-1,s}^{(\bullet)}$ which clearly commute with $\tilde{f}_{i+1,i}^{(0)}$ for $s \le j$) and with $\tilde{g}_{k}^{+}(w)$ for $k \le j$. By the induction assumption $\frac{[\tilde{f}_{ik}^{+}(w), \tilde{f}_{i+1,i}^{(0)}]_{\boldsymbol{v}}}{\boldsymbol{v}^{-1}-\boldsymbol{v}} = \tilde{f}_{i+1,k}^{+}(w)$ for $k < j$. Hence, we finally get

$$\tilde{f}_{i+1,j}^{+}(w)\tilde{g}_{j}^{+}(w) + \sum_{1\le k\le j-1} \tilde{f}_{i+1,k}^{+}(w)\tilde{g}_{k}^{+}(w)\tilde{e}_{kj}^{+}(w)$$
$$= \frac{[\tilde{f}_{ij}^{+}(w), \tilde{f}_{i+1,i}^{(0)}]_{\boldsymbol{v}} \cdot \tilde{g}_{j}^{+}(w)}{\boldsymbol{v}^{-1}-\boldsymbol{v}} + \sum_{1\le k\le j-1} \tilde{f}_{i+1,k}^{+}(w)\tilde{g}_{k}^{+}(w)\tilde{e}_{kj}^{+}(w),$$

which implies the first equality in (3.25).

• *Verification of the second formula in* (3.25).

Comparing the matrix coefficients $\langle v_{i+1} \otimes v_i| \cdots |v_i \otimes v_j\rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = -$, we get

$$(z-w)t_{i+1,i}^{-}(z)t_{ij}^{-}(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})wt_{ii}^{-}(z)t_{i+1,j}^{-}(w)$$
$$= (z-w)t_{ij}^{-}(w)t_{i+1,i}^{-}(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zt_{ii}^{-}(w)t_{i+1,j}^{-}(z).$$

Evaluating the terms with z^0 in both sides of this equality, we find

$$-t_{i+1,i}^{-}[0]t_{ij}^{-}(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})t_{ii}^{-}[0]t_{i+1,j}^{-}(w) = -t_{ij}^{-}(w)t_{i+1,i}^{-}[0].$$

Note that $t_{ii}^{-}[0]t_{i+1,j}^{-}(z) = t_{i+1,j}^{-}(z)t_{ii}^{-}[0]$ and $t_{ii}^{-}[0]t_{ij}^{-}(z) = \boldsymbol{v}t_{ij}^{-}(z)t_{ii}^{-}[0]$. To see these equalities, we compare the matrix coefficients $\langle v_{i+1} \otimes v_i| \cdots |v_j \otimes v_i\rangle$ and $\langle v_i \otimes v_i| \cdots |v_j \otimes v_i\rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = -$, and then evaluate the terms with w^0 as above. Combining this with $t_{i+1,i}^{-}[0] = \tilde{f}_{i+1,i}^{(0)}\tilde{g}_{i}^{-} = \tilde{f}_{i+1,i}^{(0)}t_{ii}^{-}[0]$, we obtain

$$t_{i+1,j}^{-}(w) = \frac{[t_{ij}^{-}(w), \tilde{f}_{i+1,i}^{(0)}]_{\boldsymbol{v}}}{\boldsymbol{v}^{-1}-\boldsymbol{v}}. \tag{3.29}$$

This implies the second equality in (3.25) via the same inductive arguments as above.

This completes our proof of Proposition 3.21. □

Corollary 3.22 *For any $1 \leq j \leq i < n$ and $r \in \mathbb{Z}$, the following equalities hold:*

$$\begin{aligned} \tilde{e}^{(r)}_{j,i+1} &:= (\boldsymbol{v}-\boldsymbol{v}^{-1})^{j-i}[\tilde{e}^{(0)}_{i,i+1}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(r)}_{j,j+1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}, \\ \tilde{f}^{(r)}_{i+1,j} &:= (\boldsymbol{v}^{-1}-\boldsymbol{v})^{j-i}[\cdots[\tilde{f}^{(r)}_{j+1,j}, \tilde{f}^{(0)}_{j+2,j+1}]_{\boldsymbol{v}}, \cdots, \tilde{f}^{(0)}_{i+1,i}]_{\boldsymbol{v}}. \end{aligned} \tag{3.30}$$

Combining these explicit formulas with (3.20), we obtain

Corollary 3.23 *For any $1 \leq j \leq i < n$ and $r \in \mathbb{Z}$, we have the following equalities:*

$$E^{(r)}_{j,i+1} = (-1)^{\delta_{r<0}} \boldsymbol{v}^{-jr} \cdot \Upsilon^{-1}(\tilde{e}^{(r)}_{j,i+1}), \ F^{(r)}_{i+1,j} = (-1)^{\delta_{r>0}} \boldsymbol{v}^{-jr} \cdot \Upsilon^{-1}(\tilde{f}^{(r)}_{i+1,j}). \tag{3.31}$$

We now apply Proposition 3.20 to construct bases of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ and $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. It will be convenient to relabel the Cartan generators via $\varphi_{i,r} := \begin{cases} \varphi^+_{i,r}, & \text{if } r \geq 0 \\ \varphi^-_{i,r}, & \text{if } r < 0 \end{cases}$, $\psi_{i,r} := \begin{cases} \psi^+_{i,r}, & \text{if } r \geq 0 \\ \psi^-_{i,r}, & \text{if } r < 0 \end{cases}$, so that $(\varphi_{i,0})^{-1} = \varphi^-_{i,0}, (\psi_{i,0})^{-1} = \psi^-_{i,0}$. We order the elements $\{E^{(r)}_{j,i+1}\}^{r\in\mathbb{Z}}_{1\leq j\leq i<n}$ in the following way: $E^{(r)}_{j,i+1} \leq E^{(r')}_{j',i'+1}$ if $j < j'$, or $j = j', i < i'$, or $j = j', i = i', r \leq r'$. Likewise, we order $\{F^{(r)}_{i+1,j}\}^{r\in\mathbb{Z}}_{1\leq j\leq i<n}$ so that $F^{(r)}_{i+1,j} \geq F^{(r')}_{i'+1,j'}$ if $j < j'$, or $j = j', i < i'$, or $j = j', i = i', r \leq r'$. Finally, we choose any total ordering of the Cartan generators $\{\varphi_{j,s}\}^{s\in\mathbb{Z}}_{1\leq j\leq n}$ of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ (or $\{\psi_{i,s}\}^{s\in\mathbb{Z}}_{1\leq i<n}$ of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$). Having specified these three total orderings, elements $F\cdot H\cdot E$ with F, E, H being ordered monomials in $\{F^{(r)}_{i+1,j}\}^{r\in\mathbb{Z}}_{1\leq j\leq i<n}, \{E^{(r)}_{j,i+1}\}^{r\in\mathbb{Z}}_{1\leq j\leq i<n}$, and the Cartan generators $\{\varphi_{j,s}\}^{s\in\mathbb{Z}}_{1\leq j\leq n}$ of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ (or $\{\psi_{i,s}\}^{s\in\mathbb{Z}}_{1\leq i<n}$ of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$), respectively, are called the *ordered PBWD monomials* (in the corresponding generators).

Theorem 3.24 *(a) The ordered PBWD monomials in $\{F^{(r)}_{i+1,j}, \varphi_{k,s}, E^{(r)}_{j,i+1}\}^{r,s\in\mathbb{Z}}_{1\leq j\leq i<n, 1\leq k\leq n}$ form a basis of a free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$.*
(b) The ordered PBWD monomials in $\{F^{(r)}_{i+1,j}, \psi_{k,s}, E^{(r)}_{j,i+1}\}^{r,s\in\mathbb{Z}}_{1\leq j\leq i<n, 1\leq k<n}$ form a basis of a free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

This result generalizes (and its proof is actually based on) (Tsymbaliuk 2018, Theorems 2.15, 2.17, 2.19). To recall these theorems in the full generality (which is needed for the further use), let us generalize the elements $\{E^{(r)}_{j,i+1}, F^{(r)}_{i+1,j}\}^{r\in\mathbb{Z}}_{1\leq j\leq i<n}$ first. For every pair $1 \leq j \leq i < n$ and any $r \in \mathbb{Z}$, we choose a *decomposition* $\underline{r} = (r_j, \ldots, r_i) \in \mathbb{Z}^{i-j+1}$ such that $r = r_j + r_{j+1} + \cdots + r_i$. We define

$$\begin{aligned} E_{j,i+1}(\underline{r}) &:= (\boldsymbol{v}-\boldsymbol{v}^{-1})[e_{i,r_i}, \cdots, [e_{j+1,r_{j+1}}, e_{j,r_j}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}, \\ F_{i+1,j}(\underline{r}) &:= (\boldsymbol{v}^{-1}-\boldsymbol{v})[\cdots[f_{j,r_j}, f_{j+1,r_{j+1}}]_{\boldsymbol{v}}, \cdots, f_{i,r_i}]_{\boldsymbol{v}}. \end{aligned} \tag{3.32}$$

In the particular case $r_j = r, r_{j+1} = \cdots = r_i = 0$, we recover $E^{(r)}_{j,i+1}, F^{(r)}_{i+1,j}$ of (3.21).

Let $U_{\boldsymbol{v}}^<(L\mathfrak{gl}_n)$ and $U_{\boldsymbol{v}}^>(L\mathfrak{gl}_n)$ be the $\mathbb{C}(\boldsymbol{v})$-subalgebras of $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ generated by $\{f_{i,r}\}_{1\le i<n}^{r\in\mathbb{Z}}$ and $\{e_{i,r}\}_{1\le i<n}^{r\in\mathbb{Z}}$, respectively. Let $\mathfrak{U}_{\boldsymbol{v}}^<(L\mathfrak{gl}_n)$ and $\mathfrak{U}_{\boldsymbol{v}}^>(L\mathfrak{gl}_n)$ be the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebras of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ generated by $\{F^{(r)}_{i+1,j}\}_{1\le j\le i<n}^{r\in\mathbb{Z}}$ and $\{E^{(r)}_{j,i+1}\}_{1\le j\le i<n}^{r\in\mathbb{Z}}$, respectively.

Theorem 3.25 (Tsymbaliuk 2018) *For any $1\le j\le i<n$ and $r\in\mathbb{Z}$, choose a decomposition $\underline{r}$ as above.*
(a) The ordered PBWD monomials in $\{E_{j,i+1}(\underline{r})\}_{1\le j\le i<n}^{r\in\mathbb{Z}}$ form a basis of a free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}^>(L\mathfrak{gl}_n)$.
(b) The ordered PBWD monomials in $\{E_{j,i+1}(\underline{r})\}_{1\le j\le i<n}^{r\in\mathbb{Z}}$ form a basis of a $\mathbb{C}(\boldsymbol{v})$-vector space $U_{\boldsymbol{v}}^>(L\mathfrak{gl}_n)$.
(c) The ordered PBWD monomials in $\{F_{i+1,j}(\underline{r})\}_{1\le j\le i<n}^{r\in\mathbb{Z}}$ form a basis of a free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}^<(L\mathfrak{gl}_n)$.
(d) The ordered PBWD monomials in $\{F_{i+1,j}(\underline{r})\}_{1\le j\le i<n}^{r\in\mathbb{Z}}$ form a basis of a $\mathbb{C}(\boldsymbol{v})$-vector space $U_{\boldsymbol{v}}^<(L\mathfrak{gl}_n)$.
(e) The ordered PBWD monomials in $\{F^{(r)}_{i+1,j},\ \varphi_{k,s},\ E^{(r)}_{j,i+1}\}_{1\le j\le i<n,1\le k\le n}^{r,s\in\mathbb{Z}}$ form a basis of the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$.

Proof of Theorem 3.24 Due to Theorem 3.25, it suffices to verify that all unordered products $E^{(r)}_{j,i+1}\varphi^{\pm}_{j',\pm s}$, $\varphi^{\pm}_{j',\pm s}F^{(r)}_{i+1,j}$, $E^{(r)}_{j,i+1}F^{(s)}_{i'+1,j'}$ are equal to $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-linear combinations of the ordered PBWD monomials. The verification for the first two cases is simple. Indeed, we can always move $\varphi^{\pm}_{j',0}$ to the left or to the right acquiring an appropriate power of $\boldsymbol{v}$. As for the other Cartan generators, it is more convenient to work with another choice of Cartan generators $h_{j',\pm s}$ defined via $\varphi^{\pm}_{j'}(z)=\varphi^{\pm}_{j',0}\exp(\sum_{s>0}h_{j',s}z^{\mp s})$. These generators satisfy simple commutation relations: $[h_{j',s},e_{i,r}]=c(i,j',r,s)e_{i,r+s}$, $[h_{j',s},f_{i,r}]=-c(i,j',r,s)f_{i,r+s}$ for certain $c(i,j',r,s)\in\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$. Therefore, $E^{(r)}_{j,i+1}h_{j',s}-h_{j',s}E^{(r)}_{j,i+1}$ is a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-linear combination of the terms of the form $E_{j,i+1}(\underline{r+s})$ for various decompositions of $r+s$ into the sum of $i-j+1$ integers, hence, the claim for $E^{(r)}_{j,i+1}h_{j',s}$. The case of $h_{j',s}F^{(r)}_{i+1,j}$ is analogous.

Thus, it remains to verify that $E^{(r)}_{j,i}F^{(s)}_{i',j'}$ is a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-linear combination of the ordered PBWD monomials. First, let us note that if $j\ge i'$ or $j'\ge i$, then $E^{(r)}_{j,i}F^{(s)}_{i',j'}=F^{(s)}_{i',j'}E^{(r)}_{j,i}$ and the latter is already an ordered PBWD monomial. Hence, from now on we shall assume $i'>j, i>j'$. There are four cases to consider: (1) $r\ge 0, s>0$, (2) $r<0, s>0$, (3) $r\ge 0, s\le 0$, (4) $r<0, s\le 0$. For simplicity of the current exposition, we shall treat only the first case, while the proof is similar in the remaining three cases. Thus, we assume $r\ge 0, s>0$ from now on. The proof will proceed by an increasing induction in $r+s$, then by an increasing induction in j', and finally by an increasing induction in r.

Our proof is based on Proposition 3.20. In particular, applying Corollary 3.23 to $E^{(r)}_{j,i}F^{(s)}_{i',j'}$, the question is reduced to the proof of the fact that $\tilde{e}^{(r)}_{ji}\tilde{f}^{(s)}_{i'j'}$ is a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-linear combination of monomials in the generators $\{\tilde{e}^{(\bullet)}_{\bullet,\bullet},\tilde{f}^{(\bullet)}_{\bullet,\bullet},\tilde{g}^{\pm}_{\bullet},\tilde{g}^{(\bullet)}_{\bullet}\}$ (ordered accordingly).

Recall that $t_{ji}^+(z) = \tilde{g}_j^+(z)\tilde{e}_{ji}^+(z) + \sum_{k=1}^{j-1} \tilde{f}_{jk}^+(z)\tilde{g}_k^+(z)\tilde{e}_{ki}^+(z)$, which immediately implies

$$t_{ji}^+[r] = \tilde{g}_j^+\tilde{e}_{ji}^{(r)} + \sum_{0\le r'<r} \tilde{g}_j^{(r-r')}\tilde{e}_{ji}^{(r')} + \sum_{k=1}^{j-1} \sum_{r_1>0,r_2\ge 0,r_3\ge 0}^{r_1+r_2+r_3=r} \tilde{f}_{jk}^{(r_1)}\tilde{g}_k^{(r_2)}\tilde{e}_{ki}^{(r_3)}, \tag{3.33}$$

where $\tilde{g}_k^{(0)}$ denotes $\tilde{g}_k^+$. Likewise,

$$t_{i'j'}^+(w) = \tilde{f}_{i'j'}^+(w)\tilde{g}_{j'}^+(w) + \sum_{k'=1}^{j'-1} \tilde{f}_{i'k'}^+(w)\tilde{g}_{k'}^+(w)\tilde{e}_{k'j'}^+(w)$$

implies

$$t_{i'j'}^+[s] = \tilde{f}_{i'j'}^{(s)}\tilde{g}_{j'}^+ + \sum_{0<s'<s} \tilde{f}_{i'j'}^{(s')}\tilde{g}_{j'}^{(s-s')} + \sum_{k'=1}^{j'-1} \sum_{s_1>0,s_2\ge 0,s_3\ge 0}^{s_1+s_2+s_3=s} \tilde{f}_{i'k'}^{(s_1)}\tilde{g}_{k'}^{(s_2)}\tilde{e}_{k'j'}^{(s_3)}. \tag{3.34}$$

Applying formulas (3.33, 3.34), let us now evaluate the product $t_{ji}^+[r]t_{i'j'}^+[s]$ and consider the corresponding unordered terms (we shall be ignoring the Cartan generators $\tilde{g}_\bullet^\pm$, $\tilde{g}_\bullet^{(\bullet)}$ since they can be moved to any side harmlessly as explained above). Besides for $\tilde{e}_{ji}^{(r)}\tilde{f}_{i'j'}^{(s)}$, all other terms will be either of the form $\tilde{e}_{ji}^{(r)}\tilde{f}_{i'k'}^{(s)}$ with $k' < j'$ or of the form $\tilde{e}_{\bullet,\bullet}^{(r')}\tilde{f}_{\bullet,\bullet}^{(s')}$ with $r'+s' < r+s$. By the induction assumption, the latter terms are $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combinations of the ordered monomials. Therefore, it suffices to prove that so is $t_{ji}^+[r]t_{i'j'}^+[s]$.

To verify the latter, we start by comparing the matrix coefficients $\langle v_j \otimes v_{i'}| \cdots |v_i \otimes v_{j'}\rangle$ of both sides of the equality (3.9) with $\epsilon = \epsilon' = +$:

$$\begin{aligned}(z-w)t_{ji}^+(z)t_{i'j'}^+(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zt_{i'i}^+(z)t_{jj'}^+(w)\\ = (z-w)t_{i'j'}^+(w)t_{ji}^+(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zt_{i'i}^+(w)t_{jj'}^+(z).\end{aligned}$$

Evaluating the coefficients of $z^{1-r}w^{-s}$ in both sides of this equality, we obtain

$$\begin{aligned}t_{ji}^+[r]t_{i'j'}^+[s] = (\boldsymbol{v}-\boldsymbol{v}^{-1})t_{i'i}^+[s]t_{jj'}^+[r] - (\boldsymbol{v}-\boldsymbol{v}^{-1})t_{i'i}^+[r]t_{jj'}^+[s]\\ +t_{ji}^+[r-1]t_{i'j'}^+[s+1] + t_{i'j'}^+[s]t_{ji}^+[r] - t_{i'j'}^+[s+1]t_{ji}^+[r-1].\end{aligned} \tag{3.35}$$

Let us now consider the unordered monomials appearing in each summand of the right-hand side of (3.35). First, we note that all the unordered monomials appearing in the last three summands are of the form $\tilde{e}_{\bullet,\bullet}^{(r')}\tilde{f}_{\bullet,\bullet}^{(s')}$ with either $r' = r-1$, $s' = s+1$ or with $r'+s' < r+s$, hence, they are $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combinations of the ordered monomials by the induction assumption. Let us now consider the unordered terms

appearing in $t^+_{i'i}[r]t^+_{jj'}[s]$. If $i' \geq i$, then clearly all the unordered terms are of the form $\tilde{e}^{(r')}_{\bullet,\bullet}\tilde{f}^{(s')}_{\bullet,\bullet}$ with $r'+s' < r+s$, to which the induction assumption applies. If $i' < i$, then all the unordered terms in $t^+_{i'i}[r]t^+_{jj'}[s]$ are either as above (to which the induction assumption applies) or of the form $\tilde{e}^{(r)}_{i'i}\tilde{f}^{(s)}_{jk}$ with $k < j$. As $i \geq i' > j > k$, we have $\tilde{e}^{(r)}_{i'i}\tilde{f}^{(s)}_{jk} = \tilde{f}^{(s)}_{jk}\tilde{e}^{(r)}_{i'i}$ (for any $k < j$) which is an ordered monomial. Therefore, we have eventually proved that $t^+_{i'i}[r]t^+_{jj'}[s]$ is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combination of the ordered monomials. Swapping r and s, we obtain the same result for $t^+_{i'i}[s]t^+_{jj'}[r]$.

Combining all the above, we see that $\tilde{e}^{(r)}_{ji}\tilde{f}^{(s)}_{i'j'}$ is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combination of the ordered monomials, hence, $E^{(r)}_{j,i}F^{(s)}_{i',j'}$ is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combination of the ordered PBWD monomials.

This completes our proof of Theorem 3.24. □

Remark 3.26 We note that $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ quantizes the algebra of functions on the thick slice ${}^{\dagger}\mathcal{W}_0$ of (Finkelberg and Tsymbaliuk 2017, 4(viii)), that is, $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)/(\boldsymbol{v}-1) \simeq \mathbb{C}[{}^{\dagger}\mathcal{W}_0]$.

Remark 3.27 For a complete picture, recall that $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is usually treated as a quantization of the universal enveloping algebra $U(L\mathfrak{sl}_n)$, cf. Remark 3.16. Let $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ be the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{K_j^{\pm 1}\}_{j\in[n]}$ and the divided powers $\{E_i^{(m)}, F_i^{(m)}\}^{m\geq 1}_{i\in[n]}$. Specializing $\boldsymbol{v}$ to 1, we have $K_j^2 = 1$ in a $\mathbb{C}$-algebra $\mathsf{U}_1(L\mathfrak{sl}_n) := \mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)/(\boldsymbol{v}-1)$. Specializing further K_j to 1, we get an algebra isomorphism $\mathsf{U}_1(L\mathfrak{sl}_n)/(\langle K_j - 1\rangle_{j\in[n]}) \simeq U(L\mathfrak{sl}_n)$. However, we are not aware of the description of $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ in the new Drinfeld realization. In particular, it would be interesting to find an explicit basis of $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ similar to that of Theorem 3.24.

3.6 Shuffle Algebra and its Integral Form

In this section, we recall the shuffle realizations of $U^{>}_{\boldsymbol{v}}(L\mathfrak{gl}_n), \mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ established in Tsymbaliuk (2018). Set $\Sigma_{(k_1,\ldots,k_{n-1})} := \Sigma_{k_1} \times \cdots \times \Sigma_{k_{n-1}}$ for $k_1, \ldots, k_{n-1} \in \mathbb{N}$. Consider an $\mathbb{N}^{n-1}$-graded $\mathbb{C}(\boldsymbol{v})$-vector space $\mathbb{S}^{(n)} = \bigoplus_{\underline{k}=(k_1,\ldots,k_{n-1})\in\mathbb{N}^{n-1}} \mathbb{S}^{(n)}_{\underline{k}}$, where $\mathbb{S}^{(n)}_{\underline{k}}$ consists of $\Sigma_{\underline{k}}$-symmetric rational functions in the variables $\{x_{i,r}\}^{1\leq r\leq k_i}_{1\leq i<n}$. We also fix a matrix of rational functions $(\zeta_{i,j}(z))^{n-1}_{i,j=1}$ by setting $\zeta_{i,j}(z) = \frac{z-\boldsymbol{v}^{-c_{ij}}}{z-1}$. Let us now introduce the bilinear *shuffle product* $\star$ on $\mathbb{S}^{(n)}$: given $F \in \mathbb{S}^{(n)}_{\underline{k}}$ and $G \in \mathbb{S}^{(n)}_{\underline{\ell}}$, define $F\star G \in \mathbb{S}^{(n)}_{\underline{k}+\underline{\ell}}$ via

$$(F\star G)(x_{1,1},\ldots,x_{1,k_1+\ell_1};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+\ell_{n-1}}) := \underline{k}!\cdot\underline{\ell}!\times$$
$$\mathrm{Sym}_{\Sigma_{\underline{k}+\underline{\ell}}}\left(F\left(\{x_{i,r}\}^{1\leq r\leq k_i}_{1\leq i<n}\right)G\left(\{x_{i',r'}\}^{k_{i'}<r'\leq k_{i'}+\ell_{i'}}_{1\leq i'<n}\right)\cdot\prod_{1\leq i<n}\prod_{1\leq i'<n}^{r'>k_{i'}}\prod_{r\leq k_i}\zeta_{i,i'}(x_{i,r}/x_{i',r'})\right). \tag{3.36}$$

Here $\underline{k}! := \prod_{i=1}^{n-1} k_i!$, while for $f \in \mathbb{C}(\{x_{i,1}, \ldots, x_{i,m_i}\}_{1\le i<n})$ we define its *symmetrization* via

$$\mathrm{Sym}_{\Sigma_{\underline{m}}}(f) := \frac{1}{\underline{m}!} \cdot \sum_{(\sigma_1,\ldots,\sigma_{n-1})\in\Sigma_{\underline{m}}} f\left(\{x_{i,\sigma_i(1)}, \ldots, x_{i,\sigma_i(m_i)}\}_{1\le i<n}\right).$$

This endows $\mathbb{S}^{(n)}$ with a structure of an associative $\mathbb{C}(\boldsymbol{v})$-algebra with the unit $\mathbf{1} \in \mathbb{S}^{(n)}_{(0,\ldots,0)}$.

We will be interested only in a certain $\mathbb{C}(\boldsymbol{v})$-subspace of $\mathbb{S}^{(n)}$, defined by the *pole* and *wheel conditions*:

• We say that $F \in \mathbb{S}^{(n)}_{\underline{k}}$ satisfies the *pole conditions* if

$$F = \frac{f(x_{1,1}, \ldots, x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2}\prod_{r\le k_i}^{r'\le k_{i+1}}(x_{i,r} - x_{i+1,r'})}, \text{ where } f \in \left(\mathbb{C}(\boldsymbol{v})\left[\{x_{i,r}^{\pm 1}\}_{1\le i<n}^{1\le r\le k_i}\right]\right)^{\Sigma_{\underline{k}}}. \quad (3.37)$$

• We say that $F \in \mathbb{S}^{(n)}_{\underline{k}}$ satisfies the *wheel conditions* if

$$F(\{x_{i,r}\}) = 0 \text{ once } x_{i,r_1} = \boldsymbol{v} x_{i+\epsilon,s} = \boldsymbol{v}^2 x_{i,r_2} \text{ for some } \epsilon, i, r_1, r_2, s, \quad (3.38)$$

where $\epsilon \in \{\pm 1\},\ 1 \le i, i+\epsilon < n,\ 1 \le r_1, r_2 \le k_i,\ 1 \le s \le k_{i+\epsilon}$.

Let $S^{(n)}_{\underline{k}} \subset \mathbb{S}^{(n)}_{\underline{k}}$ denote the $\mathbb{C}(\boldsymbol{v})$-subspace of all elements F satisfying these two conditions and set $S^{(n)} := \bigoplus_{\underline{k}\in\mathbb{N}^{n-1}} S^{(n)}_{\underline{k}}$. It is straightforward to check that the $\mathbb{C}(\boldsymbol{v})$-subspace $S^{(n)} \subset \mathbb{S}^{(n)}$ is $\star$-closed. The resulting associative $\mathbb{C}(\boldsymbol{v})$-algebra $\left(S^{(n)}, \star\right)$ is called the *shuffle algebra*. It is related to $U^>_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ via (Tsymbaliuk 2018, Theorem 3.5), (cf. (Negut 2013, Theorem 1.1)):

Theorem 3.28 (Tsymbaliuk 2018) *The assignment $e_{i,r} \mapsto x^r_{i,1}$ $(1 \le i < n, r \in \mathbb{Z})$ gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\Psi : U^>_{\boldsymbol{v}}(L\mathfrak{gl}_n) \xrightarrow{\sim} S^{(n)}$.*

For any $\underline{k} \in \mathbb{N}^{n-1}$, consider a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-submodule $\mathfrak{S}^{(n)}_{\underline{k}} \subset S^{(n)}_{\underline{k}}$ consisting of all *integral* elements, see Tsymbaliuk (2018, Definition 3.31). Set $\mathfrak{S}^{(n)} := \bigoplus_{\underline{k}\in\mathbb{N}^{n-1}} \mathfrak{S}^{(n)}_{\underline{k}}$ (it is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $S^{(n)}$ as follows from Theorem 3.30 below). While we skip an explicit definition of $\mathfrak{S}^{(n)}$ as it is quite involved, let us recall its relevant properties that were established in Tsymbaliuk (2018, Proposition 3.36):

Proposition 3.29 *(a) For any $1 \le \ell < n$, consider the linear map $\iota'_\ell : S^{(n)} \to S^{(n)}$ given by*

$$\iota'_\ell(F)\left(\{x_{i,r}\}_{1\le i<n}^{1\le r\le k_i}\right) := \prod_{r=1}^{k_\ell}\left(1 - x^{-1}_{\ell,r}\right) \cdot F\left(\{x_{i,r}\}_{1\le i<n}^{1\le r\le k_i}\right) \text{ for } F \in S^{(n)}_{\underline{k}}, \underline{k} \in \mathbb{N}^{n-1}. \quad (3.39)$$

Then

$$F \in \mathfrak{S}^{(n)} \Longleftrightarrow \iota'_\ell(F) \in \mathfrak{S}^{(n)}. \quad (3.40)$$

(b) For any $\underline{k} \in \mathbb{N}^{n-1}$ *and a collection* $g_i(\{x_{i,r}\}_{r=1}^{k_i}) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{x_{i,r}^{\pm 1}\}_{r=1}^{k_i}]^{\Sigma_{k_i}}$ $(1 \leq i < n)$, *set*

$$F := (\boldsymbol{v}-\boldsymbol{v}^{-1})^{k_1+\cdots+k_{n-1}} \cdot \frac{\prod_{i=1}^{n-1} \prod_{1\leq r\neq r'\leq k_i} (x_{i,r} - \boldsymbol{v}^{-2} x_{i,r'}) \cdot \prod_{i=1}^{n-1} g_i\left(\{x_{i,r}\}_{r=1}^{k_i}\right)}{\prod_{i=1}^{n-2} \prod_{1\leq r\leq k_i}^{1\leq r'\leq k_{i+1}} (x_{i,r} - x_{i+1,r'})}. \tag{3.41}$$

Then $F \in \mathfrak{S}_{\underline{k}}^{(n)}$.

According to Tsymbaliuk (2018, Theorem 3.34), the isomorphism Ψ of Theorem 3.28 identifies the integral forms $\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n) \subset U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n)$ and $\mathfrak{S}^{(n)} \subset S^{(n)}$:

Theorem 3.30 (Tsymbaliuk 2018) *The* $\mathbb{C}(\boldsymbol{v})$*-algebra isomorphism* $\Psi : U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n) \xrightarrow{\sim} S^{(n)}$ *gives rise to a* $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$*-algebra isomorphism* $\Psi : \mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n) \xrightarrow{\sim} \mathfrak{S}^{(n)}$.

We will crucially use this result in our proofs of Theorems 4.4, 4.15, 4.23.

Remark 3.31 For an algebra A, let A^{op} denote the opposite algebra. The assignment $f_{i,r} \mapsto e_{i,r}$ $(1 \leq i < n, r \in \mathbb{Z})$ gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $U_{\boldsymbol{v}}^{<}(L\mathfrak{gl}_n) \xrightarrow{\sim} U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n)^{\mathrm{op}}$ and a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra isomorphism $\mathfrak{U}_{\boldsymbol{v}}^{<}(L\mathfrak{gl}_n) \xrightarrow{\sim} \mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n)^{\mathrm{op}}$. Hence, Theorems 3.28 and 3.30 give rise to a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\Psi : U_{\boldsymbol{v}}^{<}(L\mathfrak{gl}_n) \xrightarrow{\sim} S^{(n),\mathrm{op}}$ and a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra isomorphism $\Psi : \mathfrak{U}_{\boldsymbol{v}}^{<}(L\mathfrak{gl}_n) \xrightarrow{\sim} \mathfrak{S}^{(n),\mathrm{op}}$ (by abuse of notation, we still denote them by Ψ).

3.7 The Jimbo Evaluation Homomorphism ev

While the quantum group $U_{\boldsymbol{v}}(\mathfrak{g})$ is always embedded into the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{g})$, in type A there also exist homomorphisms $U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \to U_{\boldsymbol{v}}(\mathfrak{gl}_n)$, discovered in Jimbo (1986). These homomorphisms are given in the Drinfeld-Jimbo realization of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

Theorem 3.32 (Jimbo 1986) *For any* $a \in \mathbb{C}^{\times}$*, there is a unique* $\mathbb{C}(\boldsymbol{v})$*-algebra homomorphism*

$$\mathrm{ev}_a : U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \to U_{\boldsymbol{v}}(\mathfrak{gl}_n)$$

defined by

$$\begin{aligned}
&E_i \mapsto E_i,\ F_i \mapsto F_i,\ K_i^{\pm 1} \mapsto K_i^{\pm 1} \text{ for } i \in [n]\backslash\{0\},\\
&K_0^{\pm 1} \mapsto K_1^{\mp 1} \cdots K_{n-1}^{\mp 1},\\
&E_0 \mapsto (-1)^n \boldsymbol{v}^{-n+1} a \cdot [\cdots [F_1, F_2]_{\boldsymbol{v}}, \cdots, F_{n-1}]_{\boldsymbol{v}} \cdot t_1^{-1} t_n^{-1},\\
&F_0 \mapsto (-1)^n \boldsymbol{v}^{n-1} a^{-1} \cdot [E_{n-1}, \cdots, [E_2, E_1]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} \cdot t_1 t_n.
\end{aligned} \tag{3.42}$$

The key result of this subsection identifies the *evaluation homomorphism* ev_a with the restriction of $\mathbb{C}(\boldsymbol{v})$-extended evaluation homomorphism $\mathrm{ev}_a^{\mathrm{rtt}}$ of Lemma 3.4.

Theorem 3.33 *The following diagram is commutative:*

$$\begin{array}{ccc} U_{\boldsymbol{v}}(L\mathfrak{sl}_n) & \xrightarrow{\Upsilon} & U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \\ \downarrow{\scriptstyle \mathrm{ev}_a} & & {\scriptstyle \mathrm{ev}^{\mathrm{rtt}}_a}\downarrow \\ U_{\boldsymbol{v}}(\mathfrak{gl}_n) & \xrightarrow[\Upsilon]{\sim} & U^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n) \end{array} \tag{3.43}$$

Proof It suffices to verify $\Upsilon^{-1}(\mathrm{ev}^{\mathrm{rtt}}_a(\Upsilon(X))) = \mathrm{ev}_a(X)$ for all $X \in \{E_i, F_i, K_i\}_{i\in[n]}$. The only nontrivial cases are $X = E_0$ or F_0, the verification for which is presented below.

• *Verification of* $\Upsilon^{-1}(\mathrm{ev}^{\mathrm{rtt}}_a(\Upsilon(E_0))) = \mathrm{ev}_a(E_0)$.
According to (3.31), we have

$$\Upsilon([\cdots[f_{1,1}, f_{2,0}]_{\boldsymbol{v}}, \cdots, f_{n-1,0}]_{\boldsymbol{v}}) = \frac{\Upsilon(F^{(1)}_{n1})}{\boldsymbol{v}^{-1} - \boldsymbol{v}} = \frac{\tilde{f}^{(1)}_{n1}}{\boldsymbol{v}(\boldsymbol{v} - \boldsymbol{v}^{-1})}.$$

On the other hand, we have $t^+_{n1}[1] = \tilde{f}^{(1)}_{n1}\tilde{g}^+_1 = \tilde{f}^{(1)}_{n1} \cdot t^+_{11}[0]$, so that $\mathrm{ev}^{\mathrm{rtt}}_a(\tilde{f}^{(1)}_{n1}) = -a \cdot t^-_{n1}(t^+_{11})^{-1}$. Note that $\Upsilon^{-1}((t^+_{kk})^{-1}) = t_k^{-1}$, while $\Upsilon^{-1}(t^-_{n1}) = (\boldsymbol{v}^{-1} - \boldsymbol{v}) \cdot [\cdots[F_1, F_2]_{\boldsymbol{v}}, \cdots, F_{n-1}]_{\boldsymbol{v}} \cdot t_1^{-1}$, due to Corollary 3.13. Combining all the above with (3.19), we finally obtain

$$\Upsilon^{-1}(\mathrm{ev}^{\mathrm{rtt}}_a(\Upsilon(E_0))) = (-1)^n \boldsymbol{v}^{-n+1} a \cdot [\cdots[F_1, F_2]_{\boldsymbol{v}}, \cdots, F_{n-1}]_{\boldsymbol{v}} \cdot t_1^{-1} t_n^{-1} = \mathrm{ev}_a(E_0).$$

• *Verification of* $\Upsilon^{-1}(\mathrm{ev}^{\mathrm{rtt}}_a(\Upsilon(F_0))) = \mathrm{ev}_a(F_0)$.
According to (3.31), we have

$$\Upsilon([e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}) = \frac{\Upsilon(E^{(-1)}_{1n})}{\boldsymbol{v} - \boldsymbol{v}^{-1}} = -\frac{\boldsymbol{v}\tilde{e}^{(-1)}_{1n}}{\boldsymbol{v} - \boldsymbol{v}^{-1}}.$$

On the other hand, $t^-_{1n}[-1] = \tilde{g}^-_1\tilde{e}^{(-1)}_{1n} = t^-_{11}[0]\tilde{e}^{(-1)}_{1n}$, so that $\mathrm{ev}^{\mathrm{rtt}}_a(\tilde{e}^{(-1)}_{1n}) = -a^{-1} \cdot (t^-_{11})^{-1}t^+_{1n}$. Note that $\Upsilon^{-1}((t^-_{kk})^{-1}) = t_k$, while $\Upsilon^{-1}(t^+_{1n}) = (\boldsymbol{v} - \boldsymbol{v}^{-1})t_1 \cdot [E_{n-1}, \cdots, [E_2, E_1]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}$, due to Corollary 3.13. Combining all the above with (3.19), we finally obtain

$$\begin{aligned} &\Upsilon^{-1}(\mathrm{ev}^{\mathrm{rtt}}_a(\Upsilon(F_0))) \\ &\quad = (-1)^n \boldsymbol{v}^{n+1} a^{-1} \cdot t_1^2 \cdot [E_{n-1}, \cdots, [E_2, E_1]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} \cdot t_1^{-1} t_n = \mathrm{ev}_a(F_0). \end{aligned}$$

This completes our proof of Theorem 3.33. □

We will denote the evaluation homomorphism ev_1 simply by ev.

3.8 Quantum Minors of $T^{\pm}(z)$

We recall the notion of quantum minors of $T^{\pm}(z)$ following Molev (2007, §1.15.6) and Hopkins (2007, Chapter 5) (though a slight change in our formulas is due to a different choice of the R-matrix). For $1 < r \leq n$, define $R(z_1, \dots, z_r) \in (\mathrm{End}\ \mathbb{C}^n)^{\otimes r}$ via

$$R(z_1, \dots, z_r) := (R_{r-1,r})(R_{r-2,r}R_{r-2,r-1}) \cdots (R_{1r} \cdots R_{12}) \text{ with}$$
$$R_{ij} := R_{\mathrm{trig};ij}(z_i, z_j).$$

The following is implied by the Yang–Baxter Eqs. (3.7) and (3.9):

Lemma 3.34 $R(z_1, \dots, z_r)T_1^{\pm}(z_1) \cdots T_r^{\pm}(z_r) = T_r^{\pm}(z_r) \cdots T_1^{\pm}(z_1)R(z_1, \dots, z_r)$.

Consider the $\boldsymbol{v}$-permutation operator $P^{\boldsymbol{v}} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ given by

$$P^{\boldsymbol{v}} = \sum_i E_{ii} \otimes E_{ii} + \boldsymbol{v} \sum_{i>j} E_{ij} \otimes E_{ji} + \boldsymbol{v}^{-1} \sum_{i<j} E_{ij} \otimes E_{ji}.$$

It gives rise to the action of the symmetric group Σ_r on $(\mathbb{C}^n)^{\otimes r}$ with transpositions $(i, i+1)$ acting via $P^{\boldsymbol{v}}_{i,i+1}$ (the operator $P^{\boldsymbol{v}}$ acting on the i-th and $(i+1)$-st factors of $\mathbb{C}^n$). Define the $\boldsymbol{v}$-antisymmetrizer $A_r^{\boldsymbol{v}} \in (\mathrm{End}\ \mathbb{C}^n)^{\otimes r}$ as the image of the antisymmetrizer $\sum_{\sigma \in \Sigma_r} (-1)^{\sigma} \cdot \sigma \in \mathbb{C}[\Sigma_r]$ under this action of Σ_r on $(\mathbb{C}^n)^{\otimes r}$. Recall the following classical observation [(cf. (Molev 2007, §1.15.6) and (Hopkins 2007, Lemma 5.5)]:

Theorem 3.35 $R(z, \boldsymbol{v}^2 z, \dots, \boldsymbol{v}^{2(r-1)} z) = \prod_{0 \leq i < j \leq r-1} (\boldsymbol{v}^{2i} - \boldsymbol{v}^{2j}) z^{\frac{r(r-1)}{2}} A_r^{\boldsymbol{v}}$.

Combining Lemma 3.34 and Theorem 3.35, we obtain the following

Corollary 3.36 *We have*

$$A_r^{\boldsymbol{v}} T_1^{\pm}(z) T_2^{\pm}(\boldsymbol{v}^2 z) \cdots T_r^{\pm}(\boldsymbol{v}^{2(r-1)} z) = T_r^{\pm}(\boldsymbol{v}^{2(r-1)} z) \cdots T_2^{\pm}(\boldsymbol{v}^2 z) T_1^{\pm}(z) A_r^{\boldsymbol{v}}. \tag{3.44}$$

The operator of (3.44) can be written as $\sum t^{a_1 \dots a_r;\pm}_{b_1 \dots b_r}(z) \otimes E_{a_1,b_1} \otimes \cdots \otimes E_{a_r,b_r}$ with $t^{a_1 \dots a_r;\pm}_{b_1 \dots b_r}(z) \in \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)[[z^{\mp 1}]]$ and the sum taken over all $a_1, \dots, a_r, b_1, \dots, b_r \in \{1, \dots, n\}$.

Definition 3.37 The coefficients $t^{a_1 \dots a_r;\pm}_{b_1 \dots b_r}(z)$ are called the *quantum minors* of $T^{\pm}(z)$.

In the particular case $r = n$, the image of the operator $A_n^{\boldsymbol{v}}$ acting on $(\mathbb{C}^n)^{\otimes n}$ is 1-dimensional. Hence $A_n^{\boldsymbol{v}} T_1^{\pm}(z) \cdots T_n^{\pm}(\boldsymbol{v}^{2(n-1)} z) = A_n^{\boldsymbol{v}} \cdot \mathrm{qdet}\ T^{\pm}(z)$ with $\mathrm{qdet}\ T^{\pm}(z) \in \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)[[z^{\mp 1}]]$. We note that $\mathrm{qdet}\ T^{\pm}(z) = t^{1 \dots n;\pm}_{1 \dots n}(z)$ in the above notations.

Definition 3.38 qdet $T^{\pm}(z)$ is called the *quantum determinant* of $T^{\pm}(z)$.

Define $d^{\pm}_{\pm r} \in \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ via $\mathrm{qdet}\, T^{\pm}(z) = \sum_{r\geq 0} d^{\pm}_{\pm r} z^{\mp r}$. The following result is a trigonometric counterpart of Proposition 2.10:

Proposition 3.39 *The elements $\{d^{\pm}_{\pm r}\}_{r\geq 0}$ are central, subject to the only defining relation $d^{+}_0 d^{-}_0 = 1$, and generate the center $Z\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ of $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$. In other words, we have a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra isomorphism*

$$Z\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{d^{\pm}_{\pm r}\}_{r\geq 0}]/(d^{+}_0 d^{-}_0 - 1).$$

3.9 Enhanced Algebras

In this section, we slightly generalize the algebras of the previous subsections as well as various relations between them. This is needed mostly for our discussions in Sect. 4.3.

- Let $\mathfrak{U}^{\mathrm{rtt},'}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ be a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra obtained from $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)$ by formally adjoining n-th roots of its central element $t := t^{+}_{11} \dots t^{+}_{nn} = (t^{-}_{11} \dots t^{-}_{nn})^{-1}$, that is, $\mathfrak{U}^{\mathrm{rtt},'}_{\boldsymbol{v}}(\mathfrak{gl}_n) = \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(\mathfrak{gl}_n)[t^{\pm 1/n}]$. Its $\mathbb{C}(\boldsymbol{v})$-counterpart is denoted by $U^{\mathrm{rtt},'}_{\boldsymbol{v}}(\mathfrak{gl}_n)$. Likewise, let $U'_{\boldsymbol{v}}(\mathfrak{gl}_n)$ be a $\mathbb{C}(\boldsymbol{v})$-algebra obtained from $U_{\boldsymbol{v}}(\mathfrak{gl}_n)$ by formally adjoining n-th roots of its central element $t := t_1 \dots t_n$, that is, $U'_{\boldsymbol{v}}(\mathfrak{gl}_n) = U_{\boldsymbol{v}}(\mathfrak{gl}_n)[t^{\pm 1/n}]$. Then the isomorphism of Theorem 3.9 gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\Upsilon : U'_{\boldsymbol{v}}(\mathfrak{gl}_n) \xrightarrow{\sim} U^{\mathrm{rtt},'}_{\boldsymbol{v}}(\mathfrak{gl}_n)$.
- Let $\mathfrak{U}^{\mathrm{rtt},'}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ be a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra obtained from $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ by formally adjoining n-th roots of its central element $t[0] := t^{+}_{11}[0] \dots t^{+}_{nn}[0] = (t^{-}_{11}[0] \dots t^{-}_{nn}[0])^{-1}$, that is, $\mathfrak{U}^{\mathrm{rtt},'}_{\boldsymbol{v}}(L\mathfrak{gl}_n) = \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)[(t[0])^{\pm 1/n}]$. Its $\mathbb{C}(\boldsymbol{v})$-counterpart is denoted by $U^{\mathrm{rtt},'}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$. Likewise, let $U'_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ be a $\mathbb{C}(\boldsymbol{v})$-algebra obtained from $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ by formally adjoining n-th roots of its central element $\varphi := \varphi^{+}_{1,0} \dots \varphi^{+}_{n,0} = (\varphi^{-}_{1,0} \dots \varphi^{-}_{n,0})^{-1}$, that is, $U'_{\boldsymbol{v}}(L\mathfrak{gl}_n) = U_{\boldsymbol{v}}(L\mathfrak{gl}_n)[\varphi^{\pm 1/n}]$. Then the isomorphism of Theorem 3.17 gives rise to an algebra isomorphism $\Upsilon : U'_{\boldsymbol{v}}(L\mathfrak{gl}_n) \xrightarrow{\sim} U^{\mathrm{rtt},'}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$.
- Let $U^{\mathrm{ad}}_{\boldsymbol{v}}(\mathfrak{sl}_n)$ be a $\mathbb{C}(\boldsymbol{v})$-algebra obtained from $U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ by adding extra generators $\{\phi_i^{\pm 1}\}_{i=1}^{n-1}$ subject to $K_i = \prod_{j=1}^{n-1} \phi_j^{c_{ji}}$, $\phi_i E_j = \boldsymbol{v}^{\delta_{ij}} E_j \phi_i$, $\phi_i F_j = \boldsymbol{v}^{-\delta_{ij}} F_j \phi_i$, $\phi_i \phi_j = \phi_j \phi_i$. Then, the natural embedding $U_{\boldsymbol{v}}(\mathfrak{sl}_n) \hookrightarrow U_{\boldsymbol{v}}(\mathfrak{gl}_n)$ gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra embedding $U^{\mathrm{ad}}_{\boldsymbol{v}}(\mathfrak{sl}_n) \hookrightarrow U'_{\boldsymbol{v}}(\mathfrak{gl}_n)$ via $\phi_i \mapsto t_1^{-1} \dots t_i^{-1} \cdot t^{i/n}$.
- Likewise, let $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ be a $\mathbb{C}(\boldsymbol{v})$-algebra obtained from $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ by adding extra generators $\{\phi_i^{\pm 1}\}_{i=1}^{n-1}$ subject to $\psi^{+}_{i,0} = \prod_{j=1}^{n-1} \phi_j^{c_{ji}}$, $\phi_i \psi^{\pm}_j(z) = \psi^{\pm}_j(z) \phi_i$, $\phi_i e_j(z) = \boldsymbol{v}^{\delta_{ij}} e_j(z) \phi_i$, $\phi_i f_j(z) = \boldsymbol{v}^{-\delta_{ij}} f_j(z) \phi_i$, $\phi_i \phi_j = \phi_j \phi_i$. Then, the natural embedding $U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra embedding $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow U'_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ via $\phi_i \mapsto \varphi^{-}_{1,0} \dots \varphi^{-}_{i,0} \cdot \varphi^{i/n}$.
- The homomorphisms $\mathrm{ev}^{\mathrm{rtt}}$, ev of Sects. 3.3, 3.7 extend to the homomorphisms of the corresponding enhanced algebras, so that (3.43) gives rise to the commutative

diagram

$$\begin{array}{ccc} U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n) & \xrightarrow{\mathrm{ev}} & U_{\boldsymbol{v}}^{'}(\mathfrak{gl}_n) \\ \downarrow \Upsilon & & \Upsilon \downarrow \wr \\ U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n) & \xrightarrow{\mathrm{ev}^{\mathrm{rtt}}} & U_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{gl}_n) \end{array} \tag{3.45}$$

- Let $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{sl}_n)$ (resp. $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)$) be the quotient of $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$ (resp. $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)$) by the relations qdet $T^{\pm}(z) = 1$ (resp. qdet $T^{\pm}(z) = 1$, $(t[0])^{1/n} = 1$). We denote its $\mathbb{C}(\boldsymbol{v})$-counterpart by $U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{sl}_n)$ (resp. $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)$). Clearly $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{sl}_n) \simeq \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{sl}_n) \simeq U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)$.
- The composition

$$U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n) \hookrightarrow U_{\boldsymbol{v}}^{'}(L\mathfrak{gl}_n) \xrightarrow{\sim} U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n) \twoheadrightarrow U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n) \tag{3.46}$$

  is a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism.
- Analogously to Definition 3.19, let $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)$ be the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)$ generated by $\{E_{j,i+1}^{(r)}, F_{i+1,j}^{(r)}\}_{1\le j\le i<n}^{r\in\mathbb{Z}} \cup \{\psi_{i,\pm s}^{\pm}\}_{1\le i<n}^{s>0} \cup \{\phi_i^{\pm 1}\}_{i=1}^{n-1}$. Then the $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism (3.46) gives rise to a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra isomorphism

$$\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n) \xrightarrow{\sim} \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n). \tag{3.47}$$

- Define the generating series $\varphi^{\pm}(z) = \varphi^{\pm} + \sum_{r\ge 1} \varphi_{\pm r} z^{\mp r}$ with coefficients in the algebra $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ (or $U_{\boldsymbol{v}}^{'}(L\mathfrak{gl}_n)$) via $\varphi^{\pm}(z) := \prod_{i=1}^{n} \varphi_i^{\pm}(\boldsymbol{v}^i z)$ (so that $\varphi^{\pm} = \varphi^{\pm 1}$). It is straightforward to check that all $\varphi_{\pm r}$ are central elements of $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ (or $U_{\boldsymbol{v}}^{'}(L\mathfrak{gl}_n)$). Moreover, it is known that the center $ZU_{\boldsymbol{v}}^{'}(L\mathfrak{gl}_n)$ of $U_{\boldsymbol{v}}^{'}(L\mathfrak{gl}_n)$ is a polynomial algebra in $\{\varphi_{\pm r}, \varphi^{\pm 1/n}\}_{r\ge 1}$ and

$$U_{\boldsymbol{v}}^{'}(L\mathfrak{gl}_n) \simeq U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}(\boldsymbol{v})} ZU_{\boldsymbol{v}}^{'}(L\mathfrak{gl}_n).$$

  The latter in turn gives rise to a trigonometric counterpart of (2.5):

$$U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n) \simeq U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}(\boldsymbol{v})} ZU_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n), \tag{3.48}$$

  where $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)$ is viewed as a subalgebra of $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)$ (rather than a quotient) via (3.46).

4 K-Theoretic Coulomb Branch of Type A Quiver Gauge Theory

4.1 Homomorphism $\widetilde{\Phi}_{\mu}^{\lambda}$

Let us recall the construction of Finkelberg and Tsymbaliuk (2017, §7) for the type A_{n-1} Dynkin diagram with arrows pointing $i \to i+1$ for $1 \le i \le n-2$. We use the same notations λ, μ, $\underline{\lambda}$, N, a_i as in Sect. 2.8 (in particular, we set $a_0 := 0$, $a_n := 0$).

Consider the associative $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra $\hat{\mathcal{A}}^{\boldsymbol{v}}$ generated by $\{D_{i,r}^{\pm 1}, \mathsf{w}_{i,r}^{\pm 1/2}\}_{1\le i\le n-1}^{1\le r\le a_i}$ such that $D_{i,r}\mathsf{w}_{i,r}^{1/2} = \boldsymbol{v}\mathsf{w}_{i,r}^{1/2}D_{i,r}$, while all other generators pairwise commute. Let $\widetilde{\mathcal{A}}^{\boldsymbol{v}}$ be the localization of $\hat{\mathcal{A}}^{\boldsymbol{v}}$ by the multiplicative set generated by $\{\mathsf{w}_{i,r} - \boldsymbol{v}^m\mathsf{w}_{i,s}\}_{1\le i<n, m\in\mathbb{Z}}^{1\le r\ne s\le a_i} \cup \{1-\boldsymbol{v}^m\}_{m\in\mathbb{Z}\setminus\{0\}}$. We define their $\mathbb{C}(\boldsymbol{v})$-counterparts $\hat{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}} := \hat{\mathcal{A}}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$ and $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}} := \widetilde{\mathcal{A}}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$. We also need the larger algebras $\widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] := \widetilde{\mathcal{A}}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ and $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] := \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}} \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$. Define $\mathsf{W}_0(z) := 1, \mathsf{W}_n(z) := 1$, and

$$\mathsf{Z}_i(z) := \prod_{1\le s\le N}^{i_s=i} \left(1 - \frac{\boldsymbol{v}\mathsf{z}_s}{z}\right), \quad \mathsf{W}_i(z) := \prod_{r=1}^{a_i} \left(1 - \frac{\mathsf{w}_{i,r}}{z}\right),$$

$$\mathsf{W}_{i,r}(z) := \prod_{1\le s\le a_i}^{s\ne r} \left(1 - \frac{\mathsf{w}_{i,s}}{z}\right).$$

To state (Finkelberg and Tsymbaliuk 2017, Theorem 7.1), we need the following modifications of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. First, recall the *simply-connected version of shifted quantum affine algebra* $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ introduced in Finkelberg and Tsymbaliuk (2017, §5(i)), which is a $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi^+_{i,s_i^+}, \psi^-_{i,-s_i^-}, (\psi^+_{i,0})^{-1}, (\psi^-_{i,b_i})^{-1}\}_{1\le i\le n-1}^{r\in\mathbb{Z}, s_i^+\ge 0, s_i^-\ge -b_i}$, where $b_i = \alpha_i^\vee(\mu)$ as in Sect. 2.6 with $\{\alpha_i^\vee\}_{i=1}^{n-1}$ denoting the simple positive roots of $\mathfrak{sl}_n$. Finally, we define $U_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ as a $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$-algebra obtained from $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] := U_{\boldsymbol{v}}^{\mathrm{sc},\mu} \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ by adding generators $\{(\phi_i^+)^{\pm 1}, (\phi_i^-)^{\pm 1}\}_{i=1}^{n-1}$ subject to the following extra relations:

$$\psi^+_{i,0} = (\phi_i^+)^2 \cdot \prod_{j-i} (\phi_j^+)^{-1}, \ (-\boldsymbol{v})^{-b_i} \prod_{1\le s\le N}^{i_s=i} \mathsf{z}_s^{-1} \cdot \psi^-_{i,b_i} = (\phi_i^-)^2 \cdot \prod_{j-i} (\phi_j^-)^{-1},$$
$$[\phi_i^\epsilon, \phi_{i'}^{\epsilon'}] = 0, \ \phi_i^\epsilon \psi_{i'}^{\epsilon'}(z) = \psi_{i'}^{\epsilon'}(z)\phi_i^\epsilon, \ \phi_i^\epsilon e_{i'}(z) = \boldsymbol{v}^{\epsilon\delta_{ii'}} e_{i'}(z)\phi_i^\epsilon,$$
$$\phi_i^\epsilon f_{i'}(z) = \boldsymbol{v}^{-\epsilon\delta_{ii'}} f_{i'}(z)\phi_i^\epsilon \tag{4.1}$$

for any $1 \le i, i' \le n-1$ and $\epsilon, \epsilon' \in \{\pm\}$.

Theorem 4.1 (Finkelberg and Tsymbaliuk 2017) *There exists a unique* $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$*-algebra homomorphism*

$$\widetilde{\Phi}_\mu^\lambda : U_{\boldsymbol{v}}^{\mathrm{ad},\mu}\left[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}\right] \longrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}\left[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}\right],$$

such that

$$e_i(z) \mapsto \frac{1}{\boldsymbol{v} - \boldsymbol{v}^{-1}} \prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{-1/2} \cdot \sum_{r=1}^{a_i} \delta\left(\frac{\mathsf{w}_{i,r}}{z}\right) \frac{\mathsf{Z}_i(\mathsf{w}_{i,r})}{\mathsf{W}_{i,r}(\mathsf{w}_{i,r})} \mathsf{W}_{i-1}(\boldsymbol{v}^{-1}\mathsf{w}_{i,r}) D_{i,r}^{-1},$$

$$f_i(z) \mapsto \frac{1}{1-\boldsymbol{v}^2} \prod_{t=1}^{a_{i+1}} w_{i+1,t}^{-1/2} \cdot \sum_{r=1}^{a_i} \delta\left(\frac{\boldsymbol{v}^2 w_{i,r}}{z}\right) \frac{1}{W_{i,r}(w_{i,r})} W_{i+1}(\boldsymbol{v} w_{i,r}) D_{i,r},$$

$$\psi_i^{\pm}(z) \mapsto \prod_{t=1}^{a_i} w_{i,t} \prod_{t=1}^{a_{i-1}} w_{i-1,t}^{-1/2} \prod_{t=1}^{a_{i+1}} w_{i+1,t}^{-1/2} \cdot \left(Z_i(z) \frac{W_{i-1}(\boldsymbol{v}^{-1}z) W_{i+1}(\boldsymbol{v}^{-1}z)}{W_i(z) W_i(\boldsymbol{v}^{-2}z)}\right)^{\pm},$$

$$(\phi_i^+)^{\pm 1} \mapsto \prod_{t=1}^{a_i} w_{i,t}^{\pm 1/2}, \ (\phi_i^-)^{\pm 1} \mapsto \prod_{t=1}^{a_i} w_{i,t}^{\mp 1/2}.$$

We write $\gamma(z)^{\pm}$ for the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

Remark 4.2 We note that the algebras $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ and $U_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ were denoted by $\mathfrak{U}_{0,\mu}^{\mathrm{sc}}$ and $\mathfrak{U}_{0,\mu}^{\mathrm{ad}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ in Finkelberg and Tsymbaliuk (2017). Moreover, we used a slightly different renormalization of ϕ_i^- in *loc.cit.*

In analogy with Definition 3.19, let us introduce integral forms of the shifted quantum affine algebras $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ and $U_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$.

Definition 4.3 (a) Let $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ be the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ generated by

$$\left\{E_{j,i+1}^{(r)}, F_{i+1,j}^{(r)}\right\}_{1\le j\le i<n}^{r\in\mathbb{Z}} \cup \left\{\psi_{i,s_i^+}^+, \psi_{i,-s_i^-}^-, (\psi_{i,0}^+)^{-1}, (\psi_{i,b_i}^-)^{-1}\right\}_{1\le i\le n-1}^{r\in\mathbb{Z}, s_i^+\ge 0, s_i^-\ge -b_i}. \tag{4.2}$$

(b) Let $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ be the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$-subalgebra of $U_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ generated by

$$\left\{E_{j,i+1}^{(r)}, F_{i+1,j}^{(r)}\right\}_{1\le j\le i<n}^{r\in\mathbb{Z}} \cup \left\{\psi_{i,s_i^+}^+, \psi_{i,-s_i^-}^-\right\}_{1\le i\le n-1}^{s_i^+>0, s_i^->-b_i} \cup \left\{(\phi_i^+)^{\pm 1}, (\phi_i^-)^{\pm 1}\right\}_{i=1}^{n-1}. \tag{4.3}$$

Here the elements $E_{j,i+1}^{(r)}, F_{i+1,j}^{(r)}$ are defined via (3.21). Recall the total orderings on the collections $\{E_{j,i+1}^{(r)}\}_{1\le j\le i<n}^{r\in\mathbb{Z}}$ and $\{F_{i+1,j}^{(r)}\}_{1\le j\le i<n}^{r\in\mathbb{Z}}$ which were introduced right before Theorem 3.24, and choose any total ordering on the corresponding Cartan generators. We introduce the *ordered PBWD monomials* (in the corresponding generators) accordingly. The following result generalizes Theorem 3.24 to the shifted setting.

Theorem 4.4 *(a) The ordered PBWD monomials in the elements (4.2) form a basis of a free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$.*
(b) The ordered PBWD monomials in the elements (4.3) form a basis of a free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$-module $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$.

Proof We will provide the proof only of part (a), since part (b) is proved analogously.

Following Finkelberg and Tsymbaliuk (2017, §5(i)), consider the $\mathbb{C}(\boldsymbol{v})$-subalgebras $U_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}$ and $U_{\boldsymbol{v}}^{\mathrm{sc},\mu;<}$ of $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ generated by $\{e_{i,r}\}_{1\le i\le n-1}^{r\in\mathbb{Z}}$ and $\{f_{i,r}\}_{1\le i\le n-1}^{r\in\mathbb{Z}}$, respectively, and let $U_{\boldsymbol{v}}^{\mathrm{sc},\mu;0}$ be the $\mathbb{C}(\boldsymbol{v})$-subalgebra of $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ generated by the Cartan

generators. According to Finkelberg and Tsymbaliuk (2017, Proposition 5.1), the multiplication map $m : U_{\boldsymbol{v}}^{\mathrm{sc},\mu;<} \otimes U_{\boldsymbol{v}}^{\mathrm{sc},\mu;0} \otimes U_{\boldsymbol{v}}^{\mathrm{sc},\mu;>} \to U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ is an isomorphism of $\mathbb{C}(\boldsymbol{v})$-vector spaces, and the subalgebras $U_{\boldsymbol{v}}^{\mathrm{sc},\mu;<}$, $U_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}$ are isomorphic to $U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n) \simeq U_{\boldsymbol{v}}^{<}(L\mathfrak{gl}_n)$, $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \simeq U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n)$, respectively. Combining this with Theorem 3.25(b,d), we immediately see that the ordered PBWD monomials in the elements (4.2) form a basis of a $\mathbb{C}(\boldsymbol{v})$-vector space $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$.

Therefore, as noted in the very beginning of our proof of Theorem 3.24, it suffices to verify that all unordered products $E_{j,i+1}^{(r)}\psi_{j',\pm s}^{\pm}$, $\psi_{j',\pm s}^{\pm}F_{i+1,j}^{(r)}$, $E_{j,i+1}^{(r)}F_{i'+1,j'}^{(s)}$ are equal to $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combinations of the ordered PBWD monomials. The first two cases are treated exactly as in our proof of Theorem 3.24. Hence, it remains to prove the following result:

Proposition 4.5 *All unordered products $E_{j,i+1}^{(r)}F_{i'+1,j'}^{(s)}$ are equal to $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combinations of the ordered PBWD monomials in the algebra $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$.*

The proof of Proposition 4.5 proceeds in four steps and is reminiscent of Finkelberg and Tsymbaliuk (2017, Appendix E).

Step 1: Case $\mu = 0$.

The fact that $E_{j,i+1}^{(r)}F_{i'+1,j'}^{(s)}$ equals a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combination of the ordered PBWD monomials in $U_{\boldsymbol{v}}^{\mathrm{sc},0}$ follows essentially from Theorem 3.24. To be more precise, recall the "extended" algebra $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},\mathrm{ext}}(L\mathfrak{gl}_n)$ of (Gow and Molev 2010, (2.15)): it is defined similarly to $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$, but we add extra generators $\{(t_{ii}^{\pm}[0])^{-1}\}_{i=1}^{n}$ and replace the first defining relation of (3.8) by

$$t_{ii}^{+}[0]t_{ii}^{-}[0] = t_{ii}^{-}[0]t_{ii}^{+}[0],\ t_{ii}^{\pm}[0](t_{ii}^{\pm}[0])^{-1} = (t_{ii}^{\pm}[0])^{-1}t_{ii}^{\pm}[0] = 1.$$

Set $U_{\boldsymbol{v}}^{\mathrm{rtt},\mathrm{ext}}(L\mathfrak{gl}_n) := \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},\mathrm{ext}}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$. Likewise, let $U_{\boldsymbol{v}}^{\mathrm{sc},0}(L\mathfrak{gl}_n)$ be a $\mathbb{C}(\boldsymbol{v})$-algebra obtained from $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ by formally adding generators $(\varphi_{j,0}^{\pm})^{-1}$ and ignoring $\varphi_{j,0}^{\pm}\varphi_{j,0}^{\mp} = 1$. Then, the isomorphism Υ of Theorem 3.17 gives rise to the $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism

$$\Upsilon^{\mathrm{ext}} : U_{\boldsymbol{v}}^{\mathrm{sc},0}(L\mathfrak{gl}_n) \xrightarrow{\sim} U_{\boldsymbol{v}}^{\mathrm{rtt},\mathrm{ext}}(L\mathfrak{gl}_n).$$

Hence, the arguments from our proof of Theorem 3.24 can be applied without any changes to prove Proposition 4.5 for $\mu = 0$.

Step 2: Reduction to $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$.

Consider the associative $\mathbb{C}(\boldsymbol{v})$-algebra $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ (resp. its $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra $\breve{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$), defined in the same way as $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ (resp. as $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$) but without the generators $\{(\psi_{i,0}^{+})^{-1}, (\psi_{i,b_i}^{-})^{-1}\}_{i=1}^{n-1}$, so that $U_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ is the localization of $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ by the multiplicative set S generated by $\{\psi_{i,0}^{+}, \psi_{i,b_i}^{-}\}_{i=1}^{n-1}$. Hence, Proposition 4.5 follows from its counterpart for $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$:

Proposition 4.6 *All unordered products $E_{j,i+1}^{(r)}F_{i'+1,j'}^{(s)}$ are equal to $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combinations of the ordered PBWD monomials in the algebra $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$*

We define the $\mathbb{C}(\boldsymbol{v})$-subalgebras $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}, \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;<}, \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;0}$ of $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ accordingly. Analogously to Finkelberg and Tsymbaliuk (2017, Proposition 5.1), the multiplication map $m : \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;<} \otimes \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;0} \otimes \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;>} \to \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ is an isomorphism of $\mathbb{C}(\boldsymbol{v})$-vector spaces, and the subalgebras $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;<}, \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}$ are isomorphic to $U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n) \simeq U_{\boldsymbol{v}}^{<}(L\mathfrak{gl}_n), U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \simeq U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n)$, respectively. Combining this with Theorem 3.25(b,d), we see that the ordered PBWD monomials form a basis of a $\mathbb{C}(\boldsymbol{v})$-vector space $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$. The following result generalizes the key verification in our proof of Theorem 3.24:

Lemma 4.7 *Proposition 4.6 holds for $\mu = 0$.*

Proof According to Step 1, $E_{j,i+1}^{(r)} F_{i'+1,j'}^{(s)} \in U_{\boldsymbol{v}}^{\mathrm{sc},0}$ equals a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combination of the ordered PBWD monomials in $U_{\boldsymbol{v}}^{\mathrm{sc},0}$. Hence, it suffices to show that none of these ordered monomials contains negative powers of either $\psi_{i,0}^{+}$ or $\psi_{i,0}^{-}$. Assume the contrary. For $1 \leq i < n$ and $\epsilon \in \{\pm\}$, choose $N_i^{\epsilon} \in \mathbb{N}$ so that $-N_i^{\epsilon}$ is the minimal of the negative powers of $\psi_{i,0}^{\epsilon}$ among the corresponding summands. Without loss of generality, we may assume that $N_1^{-} > 0$. Set $\psi := \prod_{i=1}^{n-1} \left((\psi_{i,0}^{+})^{N_i^{+}} (\psi_{i,0}^{-})^{N_i^{-}} \right) \in S$. Multiplying the equality in $U_{\boldsymbol{v}}^{\mathrm{sc},0}$ expressing $E_{j,i+1}^{(r)} F_{i'+1,j'}^{(s)}$ as a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combination of the ordered PBWD monomials by ψ, we obtain an equality in $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},0}$. Specializing further $\psi_{1,0}^{-}$ to 0, gives rise to an equality in $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},-\omega_1}$ (as before, ω_1 denotes the first fundamental coweight). As $N_1^{-} > 0$, the left-hand side specializes to zero. Meanwhile, every summand of the right-hand side specializes either to zero or to an ordered PBWD monomial in $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},-\omega_1}$. Note that there is at least one summand which does not specialize to zero, and the images of all those are pairwise distinct ordered PBWD monomials. This contradicts the fact (pointed out right before Lemma 4.7) that the ordered PBWD monomials form a basis of a $\mathbb{C}(\boldsymbol{v})$-vector space $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},-\omega_1}$. Hence, the contradiction.

This completes our proof of Lemma 4.7. □

Step 3: Case of antidominant μ.

For an antidominant μ, consider a $\mathbb{C}(\boldsymbol{v})$-algebra epimorphism $\pi_\mu : \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},0} \twoheadrightarrow \breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ defined by

$$e_{i,r} \mapsto e_{i,r}, \quad f_{i,r} \mapsto f_{i,r}, \quad \psi_{i,s}^{+} \mapsto \psi_{i,s}^{+},$$
$$\psi_{i,-s}^{-} \mapsto \begin{cases} \psi_{i,-s}^{-}, & \text{if } s \geq -b_i \\ 0, & \text{if otherwise} \end{cases} \quad \text{for } 1 \leq i < n, r \in \mathbb{Z}, s \in \mathbb{N}.$$

Using Lemma 4.7, let us express $E_{j,i+1}^{(r)} F_{i'+1,j'}^{(s)}$ as a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-linear combination of the ordered PBWD monomials in $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},0}$, and apply π_μ to the resulting equality in $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},0}$. Since $\pi_\mu(E_{j,i+1}^{(r)} F_{i'+1,j'}^{(s)}) = E_{j,i+1}^{(r)} F_{i'+1,j'}^{(s)}$ and π_μ maps ordered PBWD monomial in $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},0}$ either to the ordered PBWD monomial in $\breve{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ or to zero, we see that Proposition 4.6 holds for antidominant μ.

Step 4: General case.

Since Proposition 4.6 holds for antidominant μ (Step 3) and any coweight can be written as a sum of an antidominant coweight and several fundamental coweights ω_ℓ, it suffices to prove the following result:

Lemma 4.8 *If Proposition 4.6 holds for a coweight μ, then it also holds for the coweights $\mu + \omega_\ell$ $(1 \le \ell \le n-1)$.*

Proof Recall the *shift homomorphism* $\check{\iota}_{\mu+\omega_\ell,-\omega_\ell,0} : \check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell} \to \check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$ [cf. (Finkelberg and Tsymbaliuk 2017, Lemma 10.24, Appendix E)] defined explicitly via

$$e_{i,r} \mapsto e_{i,r} - \delta_{i,\ell} e_{i,r-1}, \ f_{i,r} \mapsto f_{i,r}, \ \psi^+_{i,s} \mapsto \psi^+_{i,s} - \delta_{i,\ell}\psi^+_{i,s-1},$$
$$\psi^-_{i,s} \mapsto \psi^-_{i,s} - \delta_{i,\ell}\psi^-_{i,s-1},$$

where we set $\psi^+_{\ell,-1} := 0$ and $\psi^-_{\ell,b_\ell+1} := 0$ in the right-hand sides.

First, we note that $\check{\iota}_{\mu+\omega_\ell,-\omega_\ell,0}(\check{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell}) \subset \check{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$. Indeed, $F^{(r)}_{i+1,j}$ is clearly fixed by $\check{\iota}_{\mu+\omega_\ell,-\omega_\ell,0}$, while $E^{(r)}_{j,i+1}$ is either fixed (if $\ell < j$ or $\ell > i$) or is mapped to $E^{(r)}_{j,i+1} - E_{j,i+1}(\underline{r-1})$ for a certain decomposition of $r-1$ (cf. formula (3.32) and the discussion preceding it), and is therefore still an element of $\check{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$, due to Theorem 3.25(a). Hence, applying our assumption to $\check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$, we see that $\check{\iota}_{\mu+\omega_\ell,-\omega_\ell,0}(E^{(r)}_{j,i+1}F^{(s)}_{i'+1,j'})$ equals a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-linear combination of the ordered PBWD monomials in $\check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$. On the other hand, let us write $E^{(r)}_{j,i+1}F^{(s)}_{i'+1,j'}$ as a $\mathbb{C}(\boldsymbol{v})$-linear combination of the ordered PBWD monomials in $\check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell}$ (such a presentation exists and is unique as the ordered PBWD monomials form a basis of a $\mathbb{C}(\boldsymbol{v})$-vector space $\check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell}$):

$$E^{(r)}_{j,i+1}F^{(s)}_{i'+1,j'} = \sum_{\underline{\alpha},\underline{\beta}^+,\underline{\beta}^-} F_{\underline{\alpha}}\psi^+_{\underline{\beta}^+}\psi^-_{\underline{\beta}^-}E(\underline{\alpha},\underline{\beta}^+,\underline{\beta}^-), \tag{4.4}$$

where $F_{\underline{\alpha}}, \psi^+_{\underline{\beta}^+}, \psi^-_{\underline{\beta}^-}$ range over all ordered monomials in $\{F^{(\bullet)}_{\bullet,\bullet}\}$, $\{\psi^+_{\bullet,\bullet}\}$, $\{\psi^-_{\bullet,\bullet}\}$, respectively, while $E(\underline{\alpha},\underline{\beta}^+,\underline{\beta}^-)$ are elements of $\check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell;>}$ and only finitely many of them are nonzero. From now on, we identify $\check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell;>} \simeq U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \simeq \check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}$, $\check{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell;>} \simeq \mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \simeq \check{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}$. Thus, it remains to verify the inclusions

$$E(\underline{\alpha},\underline{\beta}^+,\underline{\beta}^-) \in \mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \text{ for all } \underline{\alpha},\underline{\beta}^+,\underline{\beta}^-. \tag{4.5}$$

The proof of (4.5) utilizes the shuffle interpretations of both the subalgebras $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and the restriction of the shift homomorphism $\check{\iota}_{\mu+\omega_\ell,-\omega_\ell,0} : \check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell;>} \to \check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}$. Recall the $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\Psi : U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \xrightarrow{\sim} S^{(n)}$ of Theorem 3.28, which gives rise to a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra isomorphism $\Psi : \mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{gl}_n) \xrightarrow{\sim} \mathfrak{S}^{(n)}$, see Theorem 3.30. By the above discussion, applying $\check{\iota}_{\mu+\omega_\ell,-\omega_\ell,0}$ to the right-hand side of (4.4), we get a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-linear combination of the ordered PBWD monomials. Recall that $\check{\iota}_{\mu+\omega_\ell,-\omega_\ell,0}$ fixes all $F_{\underline{\alpha}}$, maps $\psi^+_{\underline{\beta}^+}$

to itself plus some smaller terms (wrt the ordering) and maps $\psi^-_{\underline{\beta}^-}$ to itself (with the indices of $\psi^-_{\ell,\bullet}$ shifted by -1) plus some smaller terms (wrt the ordering). Furthermore, according to Finkelberg and Tsymbaliuk (2017, Proposition I.4), the homomorphism $\check{\iota}_{\mu+\omega_\ell,-\omega_\ell,0} : \check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell;>} \to \check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}$ is intertwined (under the above identifications of $\check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\omega_\ell;>}$, $\check{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu;>}$ with $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \simeq S^{(n)}$) with the graded $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $\iota'_\ell : S^{(n)} \to S^{(n)}$ of (3.39). According to Proposition 3.29(a), $f \in \mathfrak{S}^{(n)}$ if and only if $\iota'_\ell(f) \in \mathfrak{S}^{(n)}$. Hence, a simple inductive argument (for every $\underline{\alpha}$, we use a descending induction in $\underline{\beta}^+$, and then a descending induction in $\underline{\beta}^-$) implies (4.5).

This implies the validity of Proposition 4.6 for the coweight $\mu+\omega_\ell$ $(1 \le \ell \le n-1)$. □

This completes our proof of Theorem 4.4. □

4.2 *K*-theoretic Coulomb Branch

Following Braverman et al. (2016, 2019) and using our notations of Sect. 2.9, consider the (extended) quantized K-theoretic Coulomb branch $\mathcal{A}^{\boldsymbol{v}} = K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$.

Here $\widetilde{\mathrm{GL}}(V)$ is a certain 2^{n-1}-cover of $\mathrm{GL}(V)$ and $\widetilde{\mathbb{C}}^\times$ is a two-fold cover of $\mathbb{C}^\times$, as defined in Finkelberg and Tsymbaliuk (2017, §8(i)) . We identify $K_{T_W}(\mathrm{pt}) = \mathbb{C}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ and $K_{\widetilde{\mathbb{C}}^\times}(\mathrm{pt}) = \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$. Recall a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$-algebra embedding $\mathbf{z}^*(\iota_*)^{-1} : \mathcal{A}^{\boldsymbol{v}} \hookrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ of Finkelberg and Tsymbaliuk (2017, §8(i)).

Set $\mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}} := \mathcal{A}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$. According to Finkelberg and Tsymbaliuk (2017, Theorem 8.5), the homomorphism $\widetilde{\Phi}^\lambda_\mu : U^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ factors through $\mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$ (embedded via $\mathbf{z}^*(\iota_*)^{-1}$). In other words, there is a unique homomorphism $\overline{\Phi}^\lambda_\mu : U^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \to \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$, such that the composition $U^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \xrightarrow{\overline{\Phi}^\lambda_\mu} \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}} \xrightarrow{\mathbf{z}^*(\iota_*)^{-1}} \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ coincides with $\widetilde{\Phi}^\lambda_\mu$.

Our next result establishes a certain integrality property of the homomorphism $\overline{\Phi}^\lambda_\mu$:

Proposition 4.9 $\overline{\Phi}^\lambda_\mu(\mathfrak{U}^{\mathrm{ad},\mu}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}])) \subset \mathcal{A}^{\boldsymbol{v}}$.

As the first ingredient of the proof, let us find explicit formulas for $\widetilde{\Phi}^\lambda_\mu(E^{(r)}_{j,i+1})$, $\widetilde{\Phi}^\lambda_\mu(F^{(r)}_{i+1,j})$.

Lemma 4.10 *For any $1 \le j \le i < n$ and $r \in \mathbb{Z}$, the following equalities hold:*

$$\widetilde{\Phi}^\lambda_\mu(E^{(r)}_{j,i+1}) = (-1)^{i-j} \cdot \prod_{t=1}^{a_i} w_{i,t} \prod_{k=j}^{i-1} \prod_{t=1}^{a_k} w_{k,t}^{1/2} \prod_{t=1}^{a_{j-1}} w_{j-1,t}^{-1/2}$$
$$\times \sum_{\substack{1\le r_j\le a_j \\ \dots \\ 1\le r_i\le a_i}} \frac{W_{j-1}(\boldsymbol{v}^{-1}w_{j,r_j}) \prod_{k=j}^{i-1} W_{k,r_k}(\boldsymbol{v}^{-1}w_{k+1,r_{k+1}})}{\prod_{k=j}^{i} W_{k,r_k}(w_{k,r_k})} \cdot \prod_{k=j}^{i} Z_k(w_{k,r_k}) \cdot \frac{w_{j,r_j}^{1+r}}{w_{i,r_i}} \cdot \prod_{k=j}^{i} D^{-1}_{k,r_k}, \tag{4.6}$$

$$\widetilde{\Phi}^{\lambda}_{\mu}(F^{(r)}_{i+1,j}) = (-1)^{i-j}\boldsymbol{v}^{j-1-i+2r}\cdot\prod_{k=j+1}^{i+1}\prod_{t=1}^{a_k} w_{k,t}^{-1/2}$$
$$\times\sum_{\substack{1\le r_j\le a_j\\ \dots\\ 1\le r_i\le a_i}}\frac{\prod_{k=j+1}^{i} W_{k,r_k}(\boldsymbol{v}w_{k-1,r_{k-1}})W_{i+1}(\boldsymbol{v}w_{i,r_i})}{\prod_{k=j}^{i} W_{k,r_k}(w_{k,r_k})}\cdot\frac{w_{i,r_i}}{w_{j,r_j}^{1-r}}\cdot\prod_{k=j}^{i} D_{k,r_k}. \tag{4.7}$$

Proof Straightforward computation. □

This lemma may be viewed as a trigonometric counterpart of Lemma 2.37.

Proof of Proposition 4.9 By explicit formulas of Theorem 4.1, we clearly have $\overline{\Phi}^{\lambda}_{\mu}((\phi^{\epsilon}_i)^{\pm 1}) \in \mathcal{A}^{\boldsymbol{v}}$ for $\epsilon = \pm$. Since $\widetilde{\Phi}^{\lambda}_{\mu}(\psi^{\pm}_{j,\pm s})$ are Laurent polynomials in $\{\mathsf{w}^{1/2}_{i,t}\}^{1\le t\le a_i}_{1\le i\le n-1}$ with coefficients in $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]$ and are symmetric in each family $\{\mathsf{w}^{1/2}_{i,t}\}_{t=1}^{a_i}$ $(1\le i<n)$, we immediately get $\overline{\Phi}^{\lambda}_{\mu}(\psi^{\pm}_{j,\pm s})\in\mathcal{A}^{\boldsymbol{v}}$. Hence, it remains to verify the inclusions $\overline{\Phi}^{\lambda}_{\mu}(E^{(r)}_{j,i+1}), \overline{\Phi}^{\lambda}_{\mu}(F^{(r)}_{i+1,j})\in\mathcal{A}^{\boldsymbol{v}}$ for all $1\le j\le i<n$ and $r\in\mathbb{Z}$.

Recall the setup of (Finkelberg and Tsymbaliuk 2017, §8(i)). For $1\le j\le i<n$, we consider a coweight $\lambda_{ji}=(0,\dots,0,\varpi_{j,1},\dots,\varpi_{i,1},0,\dots,0)$ (resp. $\lambda^*_{ji}=(0,\dots,0,\varpi^*_{j,1},\dots,\varpi^*_{i,1},0,\dots,0)$) of $\mathrm{GL}(V)=\mathrm{GL}(V_1)\times\cdots\times\mathrm{GL}(V_{n-1})$. The corresponding orbits $\mathrm{Gr}^{\lambda_{ji}}_{\mathrm{GL}(V)}, \mathrm{Gr}^{\lambda^*_{ji}}_{\mathrm{GL}(V)}\subset\mathrm{Gr}_{\mathrm{GL}(V)}$ are closed (they are products of the minuscule orbits, isomorphic to $\mathbb{P}^{a_j-1}\times\cdots\times\mathbb{P}^{a_i-1}$). Their preimages in the variety of triples $\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}}$ are denoted by $\mathcal{R}_{\lambda_{ji}}, \mathcal{R}_{\lambda^*_{ji}}$, respectively.

Then the right-hand side of (4.6) equals

$$\mathbf{z}^*(\iota_*)^{-1}\Big((-1)^{i-j}\mathrm{det}^{-1/2}_{j-1}\cdot\mathrm{det}^{1/2}_{j}\cdot\ldots\cdot\mathrm{det}^{1/2}_{i-1}\cdot\mathrm{det}_i\cdot\mathcal{O}_{\varpi^*_{j,1}}(-r-1)\boxtimes\mathcal{O}_{\varpi^*_{i,1}}(1)\Big), \tag{4.8}$$

while the right-hand side of (4.7) equals

$$\mathbf{z}^*(\iota_*)^{-1}\Big((-1)^{i-j}\boldsymbol{v}^{j-1-i+2r}\mathrm{det}^{-1/2}_{j+1}\cdot\ldots\cdot\mathrm{det}^{-1/2}_{i+1}\cdot\mathcal{O}_{\varpi_{j,1}}(r-1)\boxtimes\mathcal{O}_{\varpi_{i,1}}(1)\Big). \tag{4.9}$$

Here det_k stands for the determinant character of $\mathrm{GL}(V_k)$, while $\mathcal{O}_{\varpi_{k,1}}(s)$ stands for the class of the line bundle $\mathcal{O}(s)$ on $\mathrm{Gr}_{\varpi_{k,1}}\simeq\mathbb{P}^{a_k-1}$, and everything is pulled back to $\mathcal{R}_{\lambda_{ji}}$ (similarly for $\mathcal{O}_{\varpi^*_{k,1}}(s)$). □

To prove the main result of this subsection, let us obtain shuffle descriptions of the restrictions $\widetilde{\Phi}^{\lambda}_{\mu}: U^{\mathrm{ad},\mu;>}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]\to\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]$ and $\widetilde{\Phi}^{\lambda}_{\mu}: U^{\mathrm{ad},\mu;<}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]\to\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]$. In other words, evoking the isomorphism $\Psi: U^{>}_{\boldsymbol{v}}(L\mathfrak{gl}_n)\xrightarrow{\sim}S^{(n)}$ of Theorem 3.28 and the isomorphism $\Psi: U^{<}_{\boldsymbol{v}}(L\mathfrak{gl}_n)\xrightarrow{\sim}S^{(n),\mathrm{op}}$ of Remark 3.31, we compute the resulting homomorphisms

$$\widetilde{\Phi}^{\lambda}_{\mu} : S^{(n)}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \simeq U^{\mathrm{ad},\mu;>}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \longrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \tag{4.10}$$

and

$$\widetilde{\Phi}^{\lambda}_{\mu} : S^{(n),\mathrm{op}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \simeq U^{\mathrm{ad},\mu;<}_{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \longrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]. \tag{4.11}$$

For any $1 \le i < n$ and $1 \le r \le a_i$, we define $Y_{i,r}(z) := \frac{\mathsf{Z}_i(z)\mathsf{W}_{i-1}(\boldsymbol{v}^{-1}z)}{\mathsf{W}_{i,r}(z)}$, $Y'_{i,r}(z) := \frac{\mathsf{W}_{i+1}(\boldsymbol{v}z)}{\mathsf{W}_{i,r}(z)}$. We also recall the functions $\zeta_{i,j}(z) = \frac{z-\boldsymbol{v}^{-c_{ij}}}{z-1}$ of Sect. 3.6.

Theorem 4.11 *(a) For any $E \in S^{(n)}_{\underline{k}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$, its image under the homomorphism $\widetilde{\Phi}^{\lambda}_{\mu}$ of (4.10) equals*

$$\begin{aligned}
\widetilde{\Phi}^{\lambda}_{\mu}(E) = {} & \boldsymbol{v}^{-\sum_{i=1}^{n-1} k_i(k_i-1)}(\boldsymbol{v}-\boldsymbol{v}^{-1})^{-\sum_{i=1}^{n-1} k_i} \prod_{i=1}^{n-1}\prod_{r=1}^{a_i} \mathsf{w}_{i,r}^{k_i - \frac{1}{2}k_{i+1}} \\
& \times \sum_{\substack{m_r^{(i)} \in \mathbb{N} \\ m_1^{(1)}+\dots+m_{a_1}^{(1)}=k_1 \\ \dots \\ m_1^{(n-1)}+\dots+m_{a_{n-1}}^{(n-1)}=k_{n-1}}} \left(\prod_{i=1}^{n-1}\prod_{r=1}^{a_i}\prod_{p=1}^{m_r^{(i)}} Y_{i,r}(\boldsymbol{v}^{-2(p-1)}\mathsf{w}_{i,r}) \cdot E\left(\{\boldsymbol{v}^{-2(p-1)}\mathsf{w}_{i,r}\}_{\substack{1\le i<n \\ 1\le r\le a_i \\ 1\le p\le m_r^{(i)}}} \right) \right. \\
& \times \prod_{i=1}^{n-1}\prod_{r=1}^{a_i} \prod_{1\le p_1<p_2\le m_r^{(i)}} \zeta_{i,i}^{-1}(\boldsymbol{v}^{-2(p_1-1)}\mathsf{w}_{i,r}, \boldsymbol{v}^{-2(p_2-1)}\mathsf{w}_{i,r}) \\
& \times \prod_{i=1}^{n-1} \prod_{1\le r_1\ne r_2\le a_i} \prod_{1\le p_1\le m_{r_1}^{(i)}}^{1\le p_2\le m_{r_2}^{(i)}} \zeta_{i,i}^{-1}(\boldsymbol{v}^{-2(p_1-1)}\mathsf{w}_{i,r_1}, \boldsymbol{v}^{-2(p_2-1)}\mathsf{w}_{i,r_2}) \\
& \left. \times \prod_{i=1}^{n-2} \prod_{1\le r_1\le a_{i+1}}^{1\le r_2\le a_i} \prod_{1\le p_1\le m_{r_1}^{(i+1)}}^{1\le p_2\le m_{r_2}^{(i)}} \zeta_{i+1,i}^{-1}(\boldsymbol{v}^{-2(p_1-1)}\mathsf{w}_{i+1,r_1}, \boldsymbol{v}^{-2(p_2-1)}\mathsf{w}_{i,r_2}) \cdot \prod_{i=1}^{n-1}\prod_{r=1}^{a_i} D_{i,r}^{-m_r^{(i)}} \right).
\end{aligned} \tag{4.12}$$

(b) For any $F \in S^{(n),\mathrm{op}}_{\underline{k}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$, its image under the homomorphism $\widetilde{\Phi}^{\lambda}_{\mu}$ of (4.11) equals

$$\begin{aligned}
\widetilde{\Phi}^{\lambda}_{\mu}(F) = {} & (1-\boldsymbol{v}^2)^{-\sum_{i=1}^{n-1} k_i} \prod_{i=1}^{n-1}\prod_{r=1}^{a_i} \mathsf{w}_{i,r}^{-\frac{1}{2}k_{i-1}} \\
& \times \sum_{\substack{m_r^{(i)} \in \mathbb{N} \\ m_1^{(1)}+\dots+m_{a_1}^{(1)}=k_1 \\ \dots \\ m_1^{(n-1)}+\dots+m_{a_{n-1}}^{(n-1)}=k_{n-1}}} \left(\prod_{i=1}^{n-1}\prod_{r=1}^{a_i}\prod_{p=1}^{m_r^{(i)}} Y'_{i,r}(\boldsymbol{v}^{2(p-1)}\mathsf{w}_{i,r}) \cdot F\left(\{\boldsymbol{v}^{2p}\mathsf{w}_{i,r}\}_{\substack{1\le i<n \\ 1\le r\le a_i \\ 1\le p\le m_r^{(i)}}} \right) \right.
\end{aligned}$$

$$\times \prod_{i=1}^{n-1} \prod_{r=1}^{a_i} \prod_{1\le p_1<p_2\le m_r^{(i)}} \zeta_{i,i}^{-1}(\boldsymbol{v}^{2(p_2-1)}w_{i,r}, \boldsymbol{v}^{2(p_1-1)}w_{i,r})$$

$$\times \prod_{i=1}^{n-1} \prod_{1\le r_1\ne r_2\le a_i} \prod_{1\le p_1\le m_{r_1}^{(i)}} \prod_{1\le p_2\le m_{r_2}^{(i)}} \Big(\boldsymbol{v}^{-1}\cdot \zeta_{i,i}^{-1}(\boldsymbol{v}^{2(p_2-1)}w_{i,r_2}, \boldsymbol{v}^{2(p_1-1)}w_{i,r_1})\Big)$$

$$\times \prod_{i=1}^{n-2} \prod_{1\le r_1\le a_{i+1}} \prod_{1\le p_1\le m_{r_1}^{(i+1)}} \prod_{1\le p_2\le m_{r_2}^{(i)}} \zeta_{i+1,i}^{-1}(\boldsymbol{v}^{2(p_1-1)}w_{i+1,r_1}, \boldsymbol{v}^{2(p_2-1)}w_{i,r_2}) \cdot \prod_{i=1}^{n-1}\prod_{r=1}^{a_i} D_{i,r}^{m_r^{(i)}}\Bigg). \tag{4.13}$$

Proof (a) Let us denote the right-hand side of (4.12) by $\Theta(E)$. A tedious straightforward verification proves $\Theta(E\star E') = \Theta(E)\Theta(E')$, that is, Θ is a $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$-algebra homomorphism. On the other hand, $S^{(n)}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ is generated over $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ by its components $\{S^{(n)}_{1_i}\}_{i=1}^{n-1}$ with $1_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^{n-1}$ (here 1 stays at the i-th coordinate), due to the isomorphism $\Psi : U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n) \xrightarrow{\sim} S^{(n)}$. Comparing (4.12) with the formulas of Theorem 4.1, we immediately get $\Theta(E) = \widetilde{\Phi}^{\lambda}_{\mu}(E)$ for $E \in S^{(n)}_{1_i}$ $(1 \le i < n)$. Hence, we have $\Theta(E) = \widetilde{\Phi}^{\lambda}_{\mu}(E)$ for any $E \in S^{(n)}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$. This completes our proof of Theorem 4.11(a).

(b) The proof of Theorem 4.11(b) is completely analogous. □

For any $1 \le j \le i < n$, a vector $\underline{k} = (0, \dots, 0, k_j, \dots, k_i, 0, \dots, 0) \in \mathbb{N}^{n-1}$ with $1 \le k_\ell \le a_\ell$ $(j \le \ell \le i)$, a collection of integers $\gamma_\ell \in \mathbb{Z}$ $(j \le \ell \le i)$, and a collection of symmetric Laurent polynomials $g_\ell(\{x_{\ell,r}\}_{r=1}^{k_\ell}) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{x_{\ell,r}^{\pm 1}\}_{r=1}^{k_\ell}]^{\Sigma_{k_\ell}}$ $(j \le \ell \le i)$, consider shuffle elements $\widetilde{E} \in S^{(n)}_{\underline{k}}$ and $\widetilde{F} \in S^{(n),\mathrm{op}}_{\underline{k}}$ given by:

$$\widetilde{E} := (-1)^{\sum_{\ell=j}^{i-1} k_\ell k_{\ell+1}} \boldsymbol{v}^{\sum_{\ell=j}^{i} k_\ell(k_\ell-1)} (\boldsymbol{v}-\boldsymbol{v}^{-1})^{\sum_{\ell=j}^{i} k_\ell}$$
$$\times \frac{\prod_{\ell=j}^{i}\prod_{1\le r_1\ne r_2\le k_\ell}\left(x_{\ell,r_1}-\boldsymbol{v}^{-2}x_{\ell,r_2}\right)\cdot \prod_{\ell=j}^{i}\prod_{r=1}^{k_\ell} x_{\ell,r}^{\gamma_\ell+1+k_{\ell-1}-k_\ell}\cdot \prod_{\ell=j}^{j} g_\ell\left(\{x_{\ell,r}\}_{r=1}^{k_\ell}\right)}{\prod_{\ell=j}^{i-1}\prod_{1\le r_1\le k_\ell}^{1\le r_2\le k_{\ell+1}}\left(x_{\ell,r_1}-x_{\ell+1,r_2}\right)} \tag{4.14}$$

and

$$\widetilde{F} := (-1)^{\sum_{\ell=j}^{i} k_\ell} \boldsymbol{v}^{\sum_{\ell=j}^{i} k_\ell(k_\ell+k_{\ell+1}-2\gamma_\ell)} (\boldsymbol{v}-\boldsymbol{v}^{-1})^{\sum_{\ell=j}^{i} k_\ell}$$
$$\times \frac{\prod_{\ell=j}^{i}\prod_{1\le r_1\ne r_2\le k_\ell}(x_{\ell,r_1}-\boldsymbol{v}^{-2}x_{\ell,r_2})\cdot \prod_{\ell=j}^{i}\prod_{r=1}^{k_\ell} x_{\ell,r}^{\gamma_\ell+1+k_{\ell+1}-k_\ell}\cdot \prod_{\ell=j}^{j} g_\ell(\{\boldsymbol{v}^{-2}x_{\ell,r}\}_{r=1}^{k_\ell})}{\prod_{\ell=j}^{i-1}\prod_{1\le r_1\le k_\ell}^{1\le r_2\le k_{\ell+1}}(x_{\ell,r_1}-x_{\ell+1,r_2})}. \tag{4.15}$$

These elements obviously satisfy the pole conditions (3.37) as well as the wheel conditions (3.38), due to the presence of the factor $\prod_{\ell=j}^{i}\prod_{1\le r_1\ne r_2\le k_\ell}(x_{\ell,r_1}-\boldsymbol{v}^{-2}x_{\ell,r_2})$ in the right-hand sides of (4.14, 4.15). Moreover, $\widetilde{E} \in \mathfrak{S}^{(n)}_{\underline{k}}$ and $\widetilde{F} \in \mathfrak{S}^{(n),\mathrm{op}}_{\underline{k}}$, due to

Proposition 3.29(b). These elements are of crucial importance due to Proposition 4.12 and Remark 4.14, which play the key role in our proof of Theorem 4.15 below.

Proposition 4.12 *(a) For $\widetilde{E} \in \mathfrak{S}^{(n)}_{\underline{k}}$ given by (4.14), we have*

$$\widetilde{\Phi}^{\lambda}_{\mu}(\widetilde{E}) = \prod_{\ell=j-1}^{i} \prod_{r=1}^{a_\ell} w_{\ell,r}^{k_\ell - \frac{1}{2}k_{\ell+1}} \times \sum_{\substack{J_j \subset \{1,\dots,a_j\}: |J_j| = k_j \\ \dots \\ J_i \subset \{1,\dots,a_i\}: |J_i| = k_i}} \left(\frac{\prod_{r\in J_j}^{1\le s\le a_{j-1}} \left(1 - \frac{w_{j-1,s}}{\boldsymbol{v}^{-1} w_{j,r}}\right) \cdot \prod_{\ell=j}^{i-1} \prod_{r\in J_{\ell+1}}^{s\notin J_\ell} \left(1 - \frac{w_{\ell,s}}{\boldsymbol{v}^{-1} w_{\ell+1,r}}\right)}{\prod_{\ell=j}^{i} \prod_{r\in J_\ell}^{s\notin J_\ell} (1 - \frac{w_{\ell,s}}{w_{\ell,r}})} \right.$$
$$\left. \times \prod_{\ell=j}^{i} \prod_{r\in J_\ell} Z_\ell(w_{\ell,r}) \cdot \prod_{\ell=j}^{i} \prod_{r\in J_\ell} w_{\ell,r}^{\gamma_\ell} \cdot \prod_{\ell=j}^{i} g_\ell\left(\{w_{\ell,r}\}_{r\in J_\ell}\right) \cdot \prod_{\ell=j}^{i} \prod_{r\in J_\ell} D_{\ell,r}^{-1} \right). \tag{4.16}$$

(b) For $\widetilde{F} \in \mathfrak{S}^{(n),\mathrm{op}}_{\underline{k}}$ given by (4.15), we have

$$\widetilde{\Phi}^{\lambda}_{\mu}(\widetilde{F}) = \prod_{\ell=j+1}^{i+1} \prod_{r=1}^{a_\ell} w_{\ell,r}^{-\frac{1}{2}k_{\ell-1}} \times \sum_{\substack{J_j \subset \{1,\dots,a_j\}: |J_j| = k_j \\ \dots \\ J_i \subset \{1,\dots,a_i\}: |J_i| = k_i}} \left(\frac{\prod_{r\in J_i}^{1\le s\le a_{i+1}} \left(1 - \frac{w_{i+1,s}}{\boldsymbol{v} w_{i,r}}\right) \cdot \prod_{\ell=j+1}^{i} \prod_{r\in J_{\ell-1}}^{s\notin J_\ell} \left(1 - \frac{w_{\ell,s}}{\boldsymbol{v} w_{\ell-1,r}}\right)}{\prod_{\ell=j}^{i} \prod_{r\in J_\ell}^{s\notin J_\ell} \left(1 - \frac{w_{\ell,s}}{w_{\ell,r}}\right)} \right.$$
$$\left. \times \prod_{\ell=j}^{i} \prod_{r\in J_\ell} w_{\ell,r}^{\gamma_\ell} \cdot \prod_{\ell=j}^{i} g_\ell\left(\{w_{\ell,r}\}_{r\in J_\ell}\right) \cdot \prod_{\ell=j}^{i} \prod_{r\in J_\ell} D_{\ell,r} \right). \tag{4.17}$$

Proof The proof is straightforward and is based on (4.12, 4.13). Due to the presence of the factor $\prod_{\ell=j}^{i} \prod_{1\le r_1\ne r_2\le k_\ell} (x_{\ell,r_1} - \boldsymbol{v}^{-2} x_{\ell,r_2})$ in (4.14, 4.15), all the summands of (4.12, 4.13) with at least one index $m_r^{(\ell)} > 1$ actually vanish. This explains why the summations over all partitions of k_ℓ into the sum of a_ℓ nonnegative integers in (4.12, 4.13) are replaced by the summations over all subsets of $\{1, \dots, a_\ell\}$ of cardinality k_ℓ in (4.16, 4.17). □

Remark 4.13 In the particular case $k_\ell = 1, \gamma_\ell = (r+1)\delta_{\ell,j} - \delta_{\ell,i}, g_\ell = 1$ for $j \le \ell \le i$, the element $\widetilde{E}$ of (4.14) coincides with $\Psi((-1)^{i-j} E^{(r)}_{j,i+1})$. Likewise, in the particular case $k_\ell = 1, \gamma_\ell = (r-1)\delta_{\ell,j} + \delta_{\ell,i}, g_\ell = 1$ for $j \le \ell \le i$, the element $\widetilde{F}$ of (4.15) coincides with $\Psi((-1)^{i-j} \boldsymbol{v}^{i+1-j-2r} F^{(r)}_{i+1,j})$. Hence, Proposition 4.12 generalizes Lemma 4.10.

Remark 4.14 For any $1 \le j \le i < n$ and $\underline{k} \in \mathbb{N}^{n-1}$ as above, we consider a coweight $\kappa_{ji} = (0, \dots, 0, \varpi_{j,k_j}, \dots, \varpi_{i,k_i}, 0, \dots, 0)$ (resp. $\kappa^*_{ji} = (0, \dots, 0, \varpi^*_{j,k_j}, \dots, \varpi^*_{i,k_i},$

$0, \dots, 0)$) of $\mathrm{GL}(V)$, generalizing a coweight λ_{ji} (resp. λ^*_{ji}) from our proof of Proposition 4.9. The preimages of the corresponding orbits $\mathrm{Gr}^{\kappa_{ji}}_{\mathrm{GL}(V)}$, $\mathrm{Gr}^{\kappa^*_{ji}}_{\mathrm{GL}(V)}$ in the variety of triples $\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}}$ are denoted by $\mathcal{R}_{\kappa_{ji}}$, $\mathcal{R}_{\kappa^*_{ji}}$, respectively. Similarly to (4.8, 4.9), the right-hand sides of (4.16, 4.17) equal $\mathbf{z}^*(\iota_*)^{-1}$ of the appropriate classes in $K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\kappa^*_{ji}})$, $K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\kappa_{ji}})$. Moreover, any classes in these equivariant K-groups can be obtained this way for an appropriate choice of symmetric Laurent polynomials g_ℓ.

Our next result may be viewed as a trigonometric/K-theoretic counterpart of Proposition 2.36 as well as a generalization of Finkelberg and Tsymbaliuk (2017, Theorem 9.2) and Cautis and Williams (2018, Corollary 2.21):

Theorem 4.15 $\overline{\Phi}^{\lambda}_{\mu} : \mathfrak{U}^{\mathrm{ad},\mu}_{\boldsymbol{v}}[z_1^{\pm 1}, \dots, z_N^{\pm 1}] \to \mathcal{A}^{\boldsymbol{v}}$ *is surjective.*

Proof We need to prove that $K_{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{pt})$ together with RHS of (4.8, 4.9) generate $K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$. Recall the filtration by support on $K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$ defined in Braverman et al. (Braverman et al. 2019, §6(i)) (strictly speaking, it is defined on the equivariant Borel–Moore homology $H^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$, but the definition works word-for-word in the case of K-theory). It suffices to prove that the associated graded $\mathrm{gr}\, K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}}) = \bigoplus_\lambda K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{R}_\lambda)$ is generated by the right-hand sides of (4.16, 4.17) together with $K_{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{pt})$. Now the cone of dominant coweights of $\mathrm{GL}(V)$ is subdivided into chambers by the generalized root hyperplanes (Braverman et al. 2019, §5(i)). Recall that the generalized roots are either the roots $w_{i,r} - w_{i,s}$ $(1 \le i < n,\ 1 \le r \ne s \le a_i)$ of $\mathfrak{gl}(V)$ or the nonzero weights $w_{i,r}$, $w_{i,r} - w_{i+1,s}$ $(1 \le i < n,\ 1 \le r \le a_i,\ 1 \le s \le a_{i+1})$ of its module $\mathbf{N}$. Hence a chamber is cut out by the following conditions:

(a) For any pair of adjacent vertices i, j, we fix a *shuffle*, i.e. a permutation σ of $\{1, \dots, a_i, a_i + 1, \dots, a_i + a_j\}$ such that $\sigma(b) < \sigma(c)$ if $1 \le b < c \le a_i$ or $a_i < b < c \le a_i + a_j$. Then we require $\lambda^{(i)}_b \le \lambda^{(j)}_{b'}$ if $\sigma(b) > \sigma(a_i + b')$, and $\lambda^{(i)}_b \ge \lambda^{(j)}_{b'}$ if $\sigma(b) < \sigma(a_i + b')$.

(b) For any vertex i we fix a number $0 \le d_i \le a_i$ and require $\lambda^{(i)}_b \ge 0$ for $1 \le b \le d_i$, and $\lambda^{(i)}_b \le 0$ for $d_i < b \le a_i$.

So the chambers are numbered by the choices of shuffles for all the adjacent pairs $(i, j = i \pm 1)$ of vertices and the choices of numbers d_i for all the vertices. The intersection of a chamber C with the lattice of integral coweights is generated by the collections of fundamental coweights $(\varpi^{(i)}_{b_i})$ and the collections of dual coweights $(\varpi^{(i)*}_{c_i})$ (we allow $0 \le b_i, c_i \le a_i$) such that

(a) for any pair of adjacent vertices (i, j) and the corresponding shuffle σ, we have $\sigma(b) > \sigma(c)$ for any $1 \le b \le b_i$, $a_i + b_j < c \le a_i + a_j$ as well as for any $1 \le b \le a_i - c_i$, $a_i + a_j - c_j < c \le a_i + a_j$.

(b) For any vertex i and the corresponding number d_i, we have $b_i \le d_i < a_i - c_i$.

For any interval $[j, i] = \{j, j + 1, \dots, i\} \subset \{1, \dots, n - 1\}$, we consider collections of coweights $\kappa_{ji} = (0, \dots, 0, \varpi_{j,k_j}, \dots, \varpi_{i,k_i}, 0, \dots, 0)$ and

$\kappa_{ji}^* = (0, \dots, 0, \varpi_{j,k_j}^*, \dots, \varpi_{i,k_i}^*, 0, \dots, 0)$. According to Remark 4.14, any class in $K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\kappa_{ji}^*})$, $K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\kappa_{ji}})$ lies in the image of $\mathfrak{U}_{\mathfrak{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ under $\overline{\Phi}_\mu^\lambda$. According to the previous paragraph, for any chamber C, the equivariant K-groups $K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\kappa_{ji}^*})$ and $K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\kappa_{ji}})$ (we take all the collections $\kappa_{ji}, \kappa_{ji}^*$ generating C) generate the subring $\bigoplus_{\lambda\in C} K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_\lambda)$ of $\mathrm{gr}\, K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$. Indeed, if λ, μ lie in the same chamber C, then $(\pi^* c_\lambda) * (\pi^* c_\mu) = \pi^*(c_\lambda * c_\mu)$ (as in (Braverman et al. 2019, §6(i)), π stands for the projection $\mathcal{R}_\lambda \to \mathrm{Gr}^\lambda_{\mathrm{GL}(V)}$) for any classes $c_\lambda \in K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathrm{Gr}^\lambda_{\mathrm{GL}(V)})$, $c_\mu \in K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathrm{Gr}^\mu_{\mathrm{GL}(V)})$. And the equivariant K-theory of $\mathrm{Gr}_{\mathrm{GL}(V)}$ is generated by the equivariant K-groups of products of fundamental orbits by the argument in the beginning of the proof of Cautis and Williams (2018, Corollary 2.21). Hence, the appropriate classes in (4.16, 4.17) generate the entire associated graded ring $\mathrm{gr}\, K^{(\widetilde{\mathrm{GL}}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$ (cf. (Bullimore et al. 2017, §6.3), especially the last paragraph).

This completes our proof of Theorem 4.15. □

Remark 4.16 The above proof of Theorem 4.15 follows the one of Cautis and Williams (2018, Corollary 2.21), but crucially relies on the construction of certain elements of the integral form $\mathfrak{U}_{\mathfrak{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ whose shuffle realization is given by explicit formulas (4.14, 4.15) (let us emphasize that the explicit formulas for $\Psi^{-1}(\widetilde{E})$, $\Psi^{-1}(\widetilde{F})$ are not known). The same argument can be used to obtain a new proof of Proposition 2.36. To this end, let $W^{(n)} \supset \mathfrak{W}^{(n)}$ be the rational shuffle algebra and its integral form of Tsymbaliuk (2018, §6). Similar to Theorems 3.28, 3.30, there is a $\mathbb{C}[\hbar]$-algebra isomorphism $\Psi : Y_\hbar^>(\mathfrak{sl}_n) \xrightarrow{\sim} W^{(n)}$, which gives rise to a $\mathbb{C}[\hbar]$-algebra isomorphism $\Psi : \mathbf{Y}_\hbar^>(\mathfrak{sl}_n) \xrightarrow{\sim} \mathfrak{W}^{(n)}$, see Tsymbaliuk (2018, Theorems 6.20, 6.27). Then, for any $1 \le j \le i < n$, a vector $\underline{k} = (0, \dots, 0, k_j, \dots, k_i, 0, \dots, 0) \in \mathbb{N}^{n-1}$ with $1 \le k_\ell \le a_\ell$ $(j \le \ell \le i)$, and a collection of symmetric polynomials $g_\ell(\{x_{\ell,r}\}_{r=1}^{k_\ell}) \in \mathbb{C}[\hbar][\{x_{\ell,r}\}_{r=1}^{k_\ell}]^{\Sigma_{k_\ell}}$ $(j \le \ell \le i)$, consider shuffle elements $\widetilde{E} \in \mathfrak{W}^{(n)}_{\underline{k}}$ and $\widetilde{F} \in \mathfrak{W}^{(n),\mathrm{op}}_{\underline{k}}$ given by:

$$\widetilde{E} := \hbar^{\sum_{\ell=j}^i k_\ell} \cdot \frac{\prod_{\ell=j}^i \prod_{1\le r_1\ne r_2\le k_\ell} (x_{\ell,r_1} - x_{\ell,r_2} + \hbar) \cdot \prod_{\ell=j}^j g_\ell(\{x_{\ell,r}\}_{r=1}^{k_\ell})}{\prod_{\ell=j}^{i-1} \prod_{1\le r_1\le k_\ell}^{1\le r_2\le k_{\ell+1}} (x_{\ell,r_1} - x_{\ell+1,r_2})} \tag{4.18}$$

and

$$\widetilde{F} := \hbar^{\sum_{\ell=j}^i k_\ell} \cdot \frac{\prod_{\ell=j}^i \prod_{1\le r_1\ne r_2\le k_\ell} (x_{\ell,r_1} - x_{\ell,r_2} + \hbar) \cdot \prod_{\ell=j}^j g_\ell(\{\mathfrak{v}^{-2} x_{\ell,r}\}_{r=1}^{k_\ell})}{\prod_{\ell=j}^{i-1} \prod_{1\le r_1\le k_\ell}^{1\le r_2\le k_{\ell+1}} (x_{\ell,r_1} - x_{\ell+1,r_2})}. \tag{4.19}$$

These are the rational counterparts of the elements in (4.14, 4.15). Similar to Proposition 4.12, we have the following explicit formulas (generalizing Lemma 2.37, cf. Remark 4.13):

$$\Phi_{\mu}^{\lambda}(\Psi^{-1}(\widetilde{E})) = (-1)^{\sum_{\ell=j}^{i} k_\ell + \sum_{\ell=j}^{i-1} k_\ell k_{\ell+1}} \times \sum_{\substack{J_j \subset \{1,\dots,a_j\}: |J_j| = k_j \\ \dots \\ J_i \subset \{1,\dots,a_i\}: |J_i| = k_i}} \left(\frac{\prod_{r\in J_j}^{1\le s\le a_{j-1}} (w_{j,r} - w_{j-1,s} - \frac{\hbar}{2}) \cdot \prod_{\ell=j}^{i-1} \prod_{r\in J_{\ell+1}}^{s\notin J_\ell} (w_{\ell+1,r} - w_{\ell,s} - \frac{\hbar}{2})}{\prod_{\ell=j}^{i} \prod_{r\in J_\ell}^{s\notin J_\ell} (w_{\ell,r} - w_{\ell,s})} \right.$$
$$\left. \times \prod_{\ell=j}^{i} \prod_{r\in J_\ell} Z_\ell(w_{\ell,r}) \cdot \prod_{\ell=j}^{i} g_\ell(\{w_{\ell,r}\}_{r\in J_\ell}) \cdot \prod_{\ell=j}^{i} \prod_{r\in J_\ell} \mathsf{u}_{\ell,r}^{-1} \right) \tag{4.20}$$

and

$$\Phi_{\mu}^{\lambda}(\Psi^{-1}(\widetilde{F})) = \sum_{\substack{J_j \subset \{1,\dots,a_j\}: |J_j| = k_j \\ \dots \\ J_i \subset \{1,\dots,a_i\}: |J_i| = k_i}} \left(\frac{\prod_{r\in J_i}^{1\le s\le a_{i+1}} (w_{i,r} - w_{i+1,s} + \frac{\hbar}{2}) \cdot \prod_{\ell=j+1}^{i} \prod_{r\in J_{\ell-1}}^{s\notin J_\ell} (w_{\ell-1,r} - w_{\ell,s} + \frac{\hbar}{2})}{\prod_{\ell=j}^{i} \prod_{r\in J_\ell}^{s\notin J_\ell} (w_{\ell,r} - w_{\ell,s})} \right.$$
$$\left. \times \prod_{\ell=j}^{i} g_\ell(\{w_{\ell,r}\}_{r\in J_\ell}) \cdot \prod_{\ell=j}^{i} \prod_{r\in J_\ell} \mathsf{u}_{\ell,r} \right). \tag{4.21}$$

4.3 Explicit Description for $\mu = 0$, $\lambda = n\varpi_{n-1}$

Following Finkelberg and Tsymbaliuk (2017, §7(ii)), consider new Cartan generators $\{A_{i,\pm r}^{\pm}\}_{1\le i<n}^{r\ge 0}$ of $U_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ which are uniquely characterized by $A_{i,0}^{\pm} := (\phi_i^{\pm})^{-1}$ and

$$\psi_i^{+}(z) = \left(Z_i(z) \frac{\prod_{j-i} A_j^{+}(\boldsymbol{v}^{-1}z)}{A_i^{+}(z) A_i^{+}(\boldsymbol{v}^{-2}z)} \right)^{+},$$
$$\psi_i^{-}(z) = \left(\frac{\hat{\mathsf{Z}}_i(z) \prod_{1\le s\le N}^{i_s=i} \mathsf{z}_s}{(-z/\boldsymbol{v})^{\alpha_i^{\vee}(\mu)}} \cdot \frac{\prod_{j-i} A_j^{-}(\boldsymbol{v}^{-1}z)}{A_i^{-}(z) A_i^{-}(\boldsymbol{v}^{-2}z)} \right)^{-}, \tag{4.22}$$

where we set $A_i^{\pm}(z) := \sum_{r\ge 0} A_{i,\pm r}^{\pm} z^{\mp r}$ and $\hat{\mathsf{Z}}_i(z) := \prod_{1\le s\le N}^{i_s=i} (1 - \frac{z}{\boldsymbol{v}\mathsf{z}_s})$.

Following Finkelberg and Tsymbaliuk (2017, §8(iii)), define the *truncation ideal* $\mathcal{I}_{\mu}^{\lambda}$ as the 2-sided ideal of $U_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ generated over $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ by the following elements:

$$A_{i,0}^{\pm} A_{i,\pm a_i}^{\pm} - (-1)^{a_i},\ A_{i,\pm s}^{\pm},\ A_{i,a_i - r}^{+} - (-1)^{a_i} A_{i,-r}^{-}\ (0 \le r \le a_i < s). \tag{4.23}$$

For any $\underline{\lambda}$, μ, we have $\widetilde{\Phi}_{\mu}^{\lambda} : A_i^{+}(z) \mapsto \prod_{r=1}^{a_i} \mathsf{w}_{i,r}^{-1/2} \cdot \mathsf{W}_i(z)$, $A_i^{-}(z) \mapsto \prod_{r=1}^{a_i} \mathsf{w}_{i,t}^{1/2} \cdot \prod_{r=1}^{a_i} (1 - \frac{z}{\mathsf{w}_{i,r}})$. Hence $\mathcal{I}_{\mu}^{\lambda} \subset \mathrm{Ker}(\widetilde{\Phi}_{\mu}^{\lambda})$. The opposite inclusion is the subject of Finkelberg and Tsymbaliuk (2017, Conjecture 8.14).

Let us now formulate an integral version of this conjecture. Define the 2-sided ideal $\mathfrak{I}_\mu^\lambda$ of $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]$ as the intersection $\mathfrak{I}_\mu^\lambda := \mathfrak{I}_\mu^\lambda \cap \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]$. We also note that $\widetilde{\Phi}_\mu^\lambda(\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]) \subset \widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]$, due to Proposition 4.9 and the inclusion $\mathbf{z}^*(\iota_*)^{-1}(\mathcal{A}^{\boldsymbol{v}}) \subset \widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]$.

Conjecture 4.17 $\mathfrak{I}_\mu^\lambda = \mathrm{Ker}\left(\widetilde{\Phi}_\mu^\lambda : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},\mu}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}] \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_N^{\pm 1}]\right)$ *for all* $\underline{\lambda}, \mu$.

The goal of this subsection is to prove a *reduced version* of this equality in the particular case $\mu = 0, \lambda = n\omega_{n-1}$ (so that $N = n$ and $a_i = i$ for $1 \le i < n$; recall that $a_0 = 0, a_n = 0$). Here, a reduced version means that we impose an extra relation $\prod_{i=1}^n \mathsf{z}_i = 1$ in all our algebras. We use $\underline{\mathfrak{I}}_0^{n\omega_{n-1}}$ to denote the reduced version of the corresponding truncation ideal, while $\underline{\widetilde{\Phi}}_0^{n\omega_{n-1}}$ denotes the resulting homomorphism between the reduced algebras.

Theorem 4.18 $\underline{\mathfrak{I}}_0^{n\omega_{n-1}} = \mathrm{Ker}(\underline{\widetilde{\Phi}}_0^{n\omega_{n-1}})$.

Our proof of this result is based on the identification of the reduced truncation ideal $\underline{\mathfrak{I}}_0^{n\omega_{n-1}}$ with the kernel of a certain version of the evaluation homomorphism ev, which is of independent interest.

Recall the commutative diagram (3.45). Adjoining extra variables $\{\mathsf{z}_i^{\pm 1}\}_{i=1}^n$ subject to $\prod_{i=1}^n \mathsf{z}_i = 1$, we obtain the following commutative diagram:

$$\begin{array}{ccc} U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) & \xrightarrow{\mathrm{ev}} & U_{\boldsymbol{v}}'(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \\ \downarrow \Upsilon & & \Upsilon \downarrow \wr \\ U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) & \xrightarrow{\mathrm{ev}^{\mathrm{rtt}}} & U_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \end{array} \tag{4.24}$$

where

$$U_{\boldsymbol{v}}'(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) := U_{\boldsymbol{v}}'(\mathfrak{gl}_n) \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod_{i=1}^n \mathsf{z}_i - 1)$$

and the other three algebras are defined likewise.

Recall the isomorphism $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n) \simeq U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}(\boldsymbol{v})} ZU_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)$ of (3.48), which after adjoining extra variables $\{\mathsf{z}_i^{\pm 1}\}_{i=1}^n$ subject to $\prod_{i=1}^n \mathsf{z}_i = 1$ gives rise to an algebra isomorphism

$$\begin{aligned} &U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1\dots\mathsf{z}_n - 1) \simeq \\ &U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}(\boldsymbol{v})} ZU_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1\dots\mathsf{z}_n - 1), \end{aligned}$$

where $ZU_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)$ denotes the center of $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)$.

Let $\Delta_n^{\pm}(z)$ denote the quantum determinant qdet $T^{\pm}(z)$ of Definition 3.38, and set $\hat{\Delta}_n^{\pm}(z) := \Delta(\boldsymbol{v}^{1-n}z)$. According to Proposition 3.39, the center $ZU_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)$ is isomorphic to the quotient of the polynomial algebra in $\{(\hat{d}_0^{\pm})^{1/n}, \hat{d}_{\pm r}^{\pm}\}_{r\geq 1}$ by the relation $(\hat{d}_0^{+})^{1/n}(\hat{d}_0^{-})^{1/n} = 1$, that is,

$$ZU_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n) \simeq \mathbb{C}(\boldsymbol{v})[\{(\hat{d}_0^{\pm})^{1/n}, \hat{d}_{\pm r}^{\pm}\}_{r\geq 1}]/((\hat{d}_0^{+})^{1/n}(\hat{d}_0^{-})^{1/n} - 1),$$

where $\hat{d}_{\pm r}^{\pm}$ are defined via $\hat{\Delta}_n^{\pm}(z) = \sum_{r\geq 0} \hat{d}_{\pm r}^{\pm} z^{\mp r}$ and $(\hat{d}_0^{\pm})^{1/n} = (t[0])^{\pm 1/n}$. Let $\mathcal{J}$ be the 2-sided ideal of $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$ generated by the following elements:

$$\hat{d}_{\pm s}^{\pm},\ (\hat{d}_0^{\pm})^{1/n} - 1\ \ (s > n),$$
$$\hat{d}_r^{+} - (-1)^r e_r(\mathsf{z}_1, \ldots, \mathsf{z}_n),\ \hat{d}_{-r}^{-} - (-1)^r \mathsf{z}_1 \ldots \mathsf{z}_n e_r(\mathsf{z}_1^{-1}, \ldots, \mathsf{z}_n^{-1})\ \ (1 \leq r \leq n),$$

where $e_r(\bullet)$ denotes the r-th elementary symmetric polynomial. The ideal $\mathcal{J}$ is chosen so that $\hat{\Delta}_n^{+}(z) - \prod_{s=1}^{n}(1 - \mathsf{z}_s/z) \in \mathcal{J}[[z^{-1}]]$ and $\hat{\Delta}_n^{-}(z) - \prod_{s=1}^{n}(\mathsf{z}_s - z) \in \mathcal{J}[[z]]$. Let

$$\pi : U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1 \ldots \mathsf{z}_n - 1) \twoheadrightarrow$$
$$U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1 \ldots \mathsf{z}_n - 1)$$

be the natural projection along $\mathcal{J}$. Set $X_0^{1/n} := \mathrm{ev}^{\mathrm{rtt}}((\hat{d}_0^{+})^{1/n})$, $X_n^{1/n} := -\mathrm{ev}^{\mathrm{rtt}}((\hat{d}_0^{-})^{1/n})$, and $X_r := \mathrm{ev}^{\mathrm{rtt}}(\hat{d}_r^{+}) = (-1)^n \mathrm{ev}^{\mathrm{rtt}}(\hat{d}_{-n+r}^{-})$ for $0 \leq r \leq n$, where the last equality follows from the explicit formulas for $\mathrm{ev}^{\mathrm{rtt}}$ (which also imply $\mathrm{ev}^{\mathrm{rtt}}(\hat{d}_{\pm s}^{\pm}) = 0$ for $s > n$). Then, the center $ZU_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1 \ldots \mathsf{z}_n - 1)$ of $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1 \ldots \mathsf{z}_n - 1)$ is isomorphic to $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}, X_0^{1/n}, X_1, \ldots, X_{n-1}, X_n^{1/n}]/(X_0^{1/n} X_n^{1/n} + 1, \mathsf{z}_1 \ldots \mathsf{z}_n - 1)$.

Define the *extended quantized universal enveloping* $\widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$ as the central reduction of $U_{\boldsymbol{v}}'(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$ by the 2-sided ideal generated by

$$\Upsilon^{-1}(X_0^{1/n}) - 1,\ \Upsilon^{-1}(X_n^{1/n}) + 1,\ \Upsilon^{-1}(X_r) - (-1)^r e_r(\mathsf{z}_1, \ldots, \mathsf{z}_n)\ \ (0 < r < n),$$

cf. Beilinson and Ginzburg (1999) (the appearance of $\mathfrak{sl}_n$ is due to the fact that $\Upsilon^{-1}(X_0) = 1$). By abuse of notation, we denote the corresponding projection $U_{\boldsymbol{v}}'(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \twoheadrightarrow \widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$ by π again. Likewise, define $\widetilde{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n)$ as the central reduction of $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$ by the 2-sided ideal generated by $\{X_0^{1/n} - 1, X_n^{1/n} + 1, X_r - (-1)^r e_r(\mathsf{z}_1, \ldots, \mathsf{z}_n)\}_{r=1}^{n-1}$. By abuse of notation, we denote the corresponding projection $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \twoheadrightarrow \widetilde{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n)$ by π again. We denote the composition $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \xrightarrow{\mathrm{ev}^{\mathrm{rtt}}} U_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \xrightarrow{\pi} \widetilde{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n)$ by $\overline{\mathrm{ev}}^{\mathrm{rtt}}$. Note

that by construction it factors through $\pi : U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \to U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$, and we denote the corresponding homomorphism $U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \to \widetilde{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n)$ by $\overline{\mathrm{ev}}^{\mathrm{rtt}}$ again. Likewise, we denote the composition $U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \xrightarrow{\mathrm{ev}} U_{\boldsymbol{v}}'(\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \xrightarrow{\pi} \widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$ by $\overline{\mathrm{ev}}$.

Summarizing all the above, we obtain the following commutative diagram:

$$\begin{array}{ccc}
U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) & \xrightarrow{\overline{\mathrm{ev}}} & \widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n) \\
\downarrow \Upsilon & & \Upsilon \downarrow \wr \\
U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) & \xrightarrow{\overline{\mathrm{ev}}^{\mathrm{rtt}}} & \widetilde{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n) \\
\downarrow \pi & & \| \\
U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) & \xrightarrow{\overline{\mathrm{ev}}^{\mathrm{rtt}}} & \widetilde{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n)
\end{array} \tag{4.25}$$

Due to the isomorphism $U_v^{\mathrm{ad}}(L\mathfrak{sl}_n) \xrightarrow{\sim} U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)$ of (3.46), the composition of the left vertical arrows of (4.25) is an isomorphism:

$$\begin{aligned}
\pi \circ \Upsilon : U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1\ldots\mathsf{z}_n - 1) \\
\xrightarrow{\sim} U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1\ldots\mathsf{z}_n - 1).
\end{aligned}$$

The commutative diagram (4.25) in turn gives rise to the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) & \xrightarrow{\overline{\mathrm{ev}}} & \widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n) \\
\downarrow \Upsilon & & \Upsilon \downarrow \wr \\
\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) & \xrightarrow{\overline{\mathrm{ev}}^{\mathrm{rtt}}} & \widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n) \\
\downarrow \pi & & \| \\
\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) & \xrightarrow{\overline{\mathrm{ev}}^{\mathrm{rtt}}} & \widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n)
\end{array} \tag{4.26}$$

and the composition $\pi \circ \Upsilon$ on the left is again an algebra isomorphism.

Here we use the following notations:

- $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) := \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$, or alternatively it can be defined as a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$ generated by $\{E_{j,i+1}^{(r)}, F_{i+1,j}^{(r)}\}_{1\le j\le i<n}^{r\in\mathbb{Z}} \cup \{\psi_{i,\pm s}^{\pm}\}_{1\le i<n}^{s\ge 1} \cup \{\phi_i^{\pm 1}\}_{i=1}^{n-1} \cup \{\mathsf{z}_i^{\pm 1}\}_{i=1}^{n}$.
- $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) := \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$ or alternatively it can be viewed as a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of

$U_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{gl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$ generated by $\{t_{ij}^{\pm}[\pm r]\}_{1\le i,j\le n}^{r\in\mathbb{N}} \cup \{(t[0])^{\pm 1/n}\} \cup \{\mathsf{z}_i^{\pm 1}\}_{i=1}^n$.

• $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$ is defined similarly.

• $\widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$ denotes the reduced extended version of $\mathfrak{U}_{\boldsymbol{v}}'(\mathfrak{gl}_n)$, or alternatively it can be viewed as a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $\widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$ generated by $\{E_{j,i+1},\, F_{i+1,j}\}_{1\le j\le i<n} \cup \{\phi_i^{\pm 1}\}_{i=1}^{n-1} \cup \{\mathsf{z}_i^{\pm 1}\}_{i=1}^n$.

• $\widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n)$ denotes the reduced extended version of $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{gl}_n)$, or alternatively it can be viewed as a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $\widetilde{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(\mathfrak{sl}_n)$ generated by $\{t_{ij}^{\pm}\}_{i,j=1}^n \cup \{t^{\pm 1/n}\} \cup \{\mathsf{z}_i^{\pm 1}\}_{i=1}^n$.

Consider a natural projection

$$\varkappa : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},0}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1\dots\mathsf{z}_n - 1) \twoheadrightarrow \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\mathsf{z}_1\dots\mathsf{z}_n - 1) \tag{4.27}$$

whose kernel is a 2-sided ideal generated by $\{\phi_i^+\phi_i^- - 1\}_{i=1}^{n-1}$. Let $\widetilde{\mathrm{ev}}$ denote the composition $\overline{\mathrm{ev}} \circ \varkappa$. The following result can be viewed as a trigonometric counterpart of Theorem 2.41:

Theorem 4.19 $\mathfrak{I}_0^{n\omega_{n-1}} = \mathrm{Ker}\left(\widetilde{\mathrm{ev}} : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},0}[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \to \widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)\right)$.

Proof In the particular case $\mu = 0, \lambda = n\omega_{n-1}$, we note that $Z_1(z) = \dots = Z_{n-2}(z) = 1$, $Z_{n-1}(z) = \prod_{s=1}^n (1 - \frac{\mathsf{z}_s}{\boldsymbol{v}^{-1}z})$, $\hat{Z}_1(z) = \dots = \hat{Z}_{n-2}(z) = 1$, $\hat{Z}_{n-1}(z) = \prod_{s=1}^n (1 - \frac{\boldsymbol{v}^{-1}z}{\mathsf{z}_s})$. Let us introduce extra currents $A_0^{\pm}(z), A_n^{\pm}(z)$ via $A_0^{\pm}(z) := 1$, $A_n^+(z) := \prod_{s=1}^n (1 - \mathsf{z}_s/z)$, $A_n^-(z) = \prod_{s=1}^n (\mathsf{z}_s - z)$. Then, formula (4.22) relating the generating series $\{\psi_k^{\pm}(z)\}_{k=1}^{n-1}$ to $\{A_k^{\pm}(z)\}_{k=1}^{n-1}$ can be uniformly written as follows:

$$\psi_k^{\pm}(z) = \frac{A_{k-1}^{\pm}(\boldsymbol{v}^{-1}z)A_{k+1}^{\pm}(\boldsymbol{v}^{-1}z)}{A_k^{\pm}(z)A_k^{\pm}(\boldsymbol{v}^{-2}z)} \quad \text{for any} \quad 1 \le k \le n-1. \tag{4.28}$$

Denoting the $\varkappa$-images of $\psi_k^{\pm}(z)$, $A_k^{\pm}(z)$ again by $\psi_k^{\pm}(z)$, $A_k^{\pm}(z)$, we will view (4.28) from now on as an equality of the series with coefficients in the algebra $U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[\mathsf{z}_1^{\pm 1},\dots,\mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$.

Let $\Delta_k^{\pm}(z)$ denote the k-th principal quantum minor $t_{1\dots k}^{1\dots k;\pm}(z)$ of $T^{\pm}(z)$, see Definition 3.37. According to Molev (2007), the following equality holds:

$$\Upsilon(\psi_k^{\pm}(z)) = \frac{\Delta_{k-1}^{\pm}(\boldsymbol{v}^{1-k}z)\Delta_{k+1}^{\pm}(\boldsymbol{v}^{-1-k}z)}{\Delta_k^{\pm}(\boldsymbol{v}^{1-k}z)\Delta_k^{\pm}(\boldsymbol{v}^{-1-k}z)}.$$

Generalizing $\hat{\Delta}_n^{\pm}(z)$, define $\hat{\Delta}_k^{\pm}(z) := \Delta_k^{\pm}(\boldsymbol{v}^{1-k}z)$. Then, the above formula reads as

$$\Upsilon(\psi_k^{\pm}(z)) = \frac{\hat{\Delta}_{k-1}^{\pm}(\boldsymbol{v}^{-1}z)\hat{\Delta}_{k+1}^{\pm}(\boldsymbol{v}^{-1}z)}{\hat{\Delta}_k^{\pm}(z)\hat{\Delta}_k^{\pm}(\boldsymbol{v}^{-2}z)}.$$

By abuse of notation, let us denote the image $\pi(\hat{\Delta}_k^{\pm}(z))$ by $\hat{\Delta}_k^{\pm}(z)$ again. Note that $\hat{\Delta}_n^{\pm}(z) = A_n^{\pm}(z)$, due to our definition of π. Combining this with (4.28), we obtain the following result:

Corollary 4.20 *Under the isomorphism*

$$\pi \circ \Upsilon : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/(z_1 \dots z_n - 1)$$
$$\longrightarrow \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},'}(L\mathfrak{sl}_n)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/(z_1 \dots z_n - 1),$$

the generating series $A_k^{\pm}(z)$ are mapped into $\hat{\Delta}_k^{\pm}(z)$, that is, $\pi \circ \Upsilon(A_k^{\pm}(z)) = \hat{\Delta}_k^{\pm}(z)$.

Combining this result with the commutativity of the diagram (4.26) and the explicit formulas $\mathrm{ev}^{\mathrm{rtt}}(T^+(z)) = T^+ - T^- z^{-1}$, $\mathrm{ev}^{\mathrm{rtt}}(T^-(z)) = T^- - T^+ z$, we get

Corollary 4.21 $\underline{\mathfrak{I}}_0^{n\omega_{n-1}} \subseteq \mathrm{Ker}(\widetilde{\mathrm{ev}})$.

The opposite inclusion $\underline{\mathfrak{I}}_0^{n\omega_{n-1}} \supseteq \mathrm{Ker}(\widetilde{\mathrm{ev}})$ follows from the equality $\widetilde{\mathrm{ev}} = \overline{\mathrm{ev}} \circ \varkappa$, the obvious inclusion $\mathrm{Ker}(\varkappa) \subset \underline{\mathfrak{I}}_0^{n\omega_{n-1}}$, the commutativity of the diagrams (4.25, 4.26), and Theorem 3.7 by noticing that $\hat{\Delta}_1^{\pm}(z) = t_{11}^{\pm}(z)$ and so

$$(\pi \circ \Upsilon)^{-1}(t_{11}^{\pm}[\pm r]) = A_{1,\pm r}^{\pm} \in \varkappa(\underline{\mathfrak{I}}_0^{n\omega_{n-1}}) \text{ for } r > 1,$$
$$(\pi \circ \Upsilon)^{-1}(t_{11}^{\pm}[\pm 1] + t_{11}^{\mp}[0]) = A_{1,\pm 1}^{\pm} + A_{1,0}^{\mp} \in \varkappa(\underline{\mathfrak{I}}_0^{n\omega_{n-1}}).$$

This completes our proof of Theorem 4.19. □

Now we are ready to present the proof of Theorem 4.18.

Proof of Theorem 4.18 Recall the subtorus $T_W' = \{g \in T_W | \det(g) = 1\}$ of T_W, and define $\underline{\mathcal{A}}^{\boldsymbol{v}} := K^{(\widetilde{\mathrm{GL}}(V) \times T_W')_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^{\times}}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$, so that $\underline{\mathcal{A}}^{\boldsymbol{v}} \simeq \mathcal{A}^{\boldsymbol{v}}/(\prod \mathsf{z}_i - 1)$. After imposing $\prod \mathsf{z}_i = 1$, the homomorphism $\underline{\widetilde{\Phi}}_0^{n\omega_{n-1}} : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},0}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$ is a composition of the surjective homomorphism $\underline{\overline{\Phi}}_0^{n\omega_{n-1}} : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{ad},0}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1) \twoheadrightarrow \underline{\mathcal{A}}^{\boldsymbol{v}}$ (see Theorem 4.15) and an embedding $\mathbf{z}^*(\iota_*)^{-1} : \underline{\mathcal{A}}^{\boldsymbol{v}} \hookrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_n^{\pm 1}]/(\prod \mathsf{z}_i - 1)$, so that $\mathrm{Ker}(\underline{\widetilde{\Phi}}_0^{n\omega_{n-1}}) = \mathrm{Ker}(\underline{\overline{\Phi}}_0^{n\omega_{n-1}})$. The homomorphism $\underline{\overline{\Phi}}_0^{n\omega_{n-1}}$ factors through $\overline{\phi} : \widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n) \twoheadrightarrow \underline{\mathcal{A}}^{\boldsymbol{v}}$ (due to Theorem 4.19), and it remains to prove the injectivity of $\overline{\phi}$. Since both $\widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$ and $\underline{\mathcal{A}}^{\boldsymbol{v}}$ are free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-modules, $\mathrm{Ker}(\overline{\phi})$ is a flat $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-module. Hence, to prove the vanishing of $\mathrm{Ker}(\overline{\phi})$, it suffices to prove the vanishing of $\mathrm{Ker}(\overline{\phi}_{\mathrm{frac}} : \widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n) \twoheadrightarrow \underline{\mathcal{A}}_{\mathrm{frac}}^{\boldsymbol{v}})$.

To this end we will need the action of $U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_n)$ on the localized T_W-equivariant K-theory of the Laumon based complete quasiflags' moduli spaces $\mathfrak{Q}$, see e.g. (Finkelberg and Tsymbaliuk 2017, §12(v)). This action factors through the evaluation homomorphism and the action of $U_{\boldsymbol{v}}'(\mathfrak{gl}_n)$ on the T_W-equivariant K-theory in question, see (Finkelberg and Tsymbaliuk 2017, Remark 12.8(c)). According to Braverman and Finkelberg (2005, §2.26), the resulting $U_{\boldsymbol{v}}'(\mathfrak{gl}_n)$-module is nothing but the universal Verma module. It is known that the action of $U_{\boldsymbol{v}}'(\mathfrak{gl}_n)$ on the universal Verma module extends uniquely to the action of the *extended quantized universal enveloping* $\widetilde{U}_{\boldsymbol{v}}'(\mathfrak{gl}_n)$,

and the latter action is effective. This implies that the resulting action of $\widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$ on the localized T'_W-equivariant K-theory in question is also effective. According to Bullimore et al. (2018), the K-theoretic Coulomb branch $\underline{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$ acts naturally on the T'_W-equivariant K-theory in question, and the action of $\widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$ factors through the homomorphism $\overline{\phi}_{\mathrm{frac}} : \widetilde{U}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n) \twoheadrightarrow \underline{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$ (see Finkelberg and Tsymbaliuk (2017, Remark 12.8(c))). Hence, $\overline{\phi}_{\mathrm{frac}}$ is injective.

This completes our proof of Theorem 4.18. □

Corollary 4.22 *The reduced quantized K-theoretic Coulomb branch $\underline{\mathcal{A}}^{\boldsymbol{v}}$ is explicitly given by $\underline{\mathcal{A}}^{\boldsymbol{v}} \simeq \widetilde{\mathfrak{U}}_{\boldsymbol{v}}^{\mathrm{ad}}(\mathfrak{sl}_n)$.*

4.4 Coproduct on $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\boldsymbol{\mu}}$

In this subsection, we verify that the $\mathbb{C}(\boldsymbol{v})$-algebra homomorphisms Δ_{μ_1,μ_2}: $U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2} \to U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes U_{\boldsymbol{v}}^{\mathrm{sc},\mu_2}$ constructed in Finkelberg and Tsymbaliuk (2017, Theorem 10.26) give rise to the same named $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra homomorphisms $\Delta_{\mu_1,\mu_2} : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2} \to \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2}$. In other words, we have

Theorem 4.23 *For any coweights μ_1, μ_2, the image of the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2} \subset U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2}$ under the homomorphism Δ_{μ_1,μ_2} belongs to the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2} \subset U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes U_{\boldsymbol{v}}^{\mathrm{sc},\mu_2}$. This gives rise to the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra homomorphism*

$$\Delta_{\mu_1,\mu_2} : \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2} \to \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2}.$$

Before proving this result, let us recall the key properties of Δ_{μ_1,μ_2}. Define integers $b_{1,i} := \alpha_i^\vee(\mu_1)$, $b_{2,i} := \alpha_i^\vee(\mu_2)$ for $1 \le i < n$. The homomorphism $\Delta_{0,0}$ essentially coincides with the Drinfeld-Jimbo coproduct Δ on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

If μ_1 and μ_2 are antidominant (that is, $b_{1,i}, b_{2,i} \le 0$ for all i), then our construction of Δ_{μ_1,μ_2} in Finkelberg and Tsymbaliuk (2017, Theorem 10.22) is explicit and is based on the Levendorskii type presentation of antidominantly shifted quantum affine algebras, see Finkelberg and Tsymbaliuk (2017, Theorem 5.5). To state the key property of Δ_{μ_1,μ_2} (for antidominant μ_1 and μ_2) of Finkelberg and Tsymbaliuk (2017, Propositions H.1, H.22), we introduce the following notations:

• Let $U_{\boldsymbol{v}}^+$ and $U_{\boldsymbol{v}}^-$ be the *positive* and the *negative* Borel subalgebras in the Drinfeld-Jimbo realization of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, respectively. Explicitly, they are generated over $\mathbb{C}(\boldsymbol{v})$ by $\{e_{i,0}, (\psi_{i,0}^+)^{\pm 1}, F_{n1}^{(1)}\}_{i=1}^{n-1}$ and $\{f_{i,0}, (\psi_{i,0}^-)^{\pm 1}, E_{1n}^{(-1)}\}_{i=1}^{n-1}$, respectively.

• Likewise, let $U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1,\mu_2;+}$ and $U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1,\mu_2;-}$ be the $\mathbb{C}(\boldsymbol{v})$-subalgebras of $U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2}$ generated by $\{e_{i,0}, (\psi_{i,0}^+)^{\pm 1}, F_{n1}^{(1)}\}_{i=1}^{n-1}$ and $\{f_{i,b_{1,i}}, (\psi_{i,b_{1,i}+b_{2,i}}^-)^{\pm 1}, \hat{E}_{1n}^{(-1)}\}_{i=1}^{n-1}$, respectively. Here the element $\hat{E}_{1n}^{(-1)}$ is defined via $\hat{E}_{1n}^{(-1)} := (\boldsymbol{v}-\boldsymbol{v}^{-1})[e_{n-1,b_{2,n-1}}, \cdots, [e_{2,b_{2,2}}, e_{1,b_{2,1}-1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}$.

Proposition 4.24 (Finkelberg and Tsymbaliuk 2017) *(a) There are unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphisms*

$$J_{\mu_1,\mu_2}^+ : U_{\boldsymbol{v}}^+ \longrightarrow U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1,\mu_2;+}, \quad J_{\mu_1,\mu_2}^- : U_{\boldsymbol{v}}^- \longrightarrow U_{\boldsymbol{v}}^{\mathrm{sc},\mu_1,\mu_2;-}, \tag{4.29}$$

such that

$$J^+_{\mu_1,\mu_2} : e_{i,r} \mapsto e_{i,r},\ \psi^+_{i,0} \mapsto \psi^+_{i,0},\ F^{(1)}_{n1} \mapsto F^{(1)}_{n1} \ \text{ for } \ 1 \le i \le n-1, r \ge 0,$$
$$J^-_{\mu_1,\mu_2} : f_{i,s} \mapsto f_{i,s+b_{1,i}},\ \psi^-_{i,0} \mapsto \psi^-_{i,b_{1,i}+b_{2,i}},\ E^{(-1)}_{1n} \mapsto \hat{E}^{(-1)}_{1n}$$
$$\text{for } \ 1 \le i \le n-1, s \le 0.$$

(b) The following diagram is commutative:

$$\begin{array}{ccc} U^{\pm}_{\boldsymbol{v}} & \xrightarrow{\Delta} & U^{\pm}_{\boldsymbol{v}} \otimes U^{\pm}_{\boldsymbol{v}} \\ \downarrow{\scriptstyle J^{\pm}_{\mu_1,\mu_2}} & & \downarrow{\scriptstyle J^{\pm}_{\mu_1,0} \otimes J^{\pm}_{0,\mu_2}} \\ U^{\mathrm{sc},\mu_1,\mu_2;\pm}_{\boldsymbol{v}} & \xrightarrow{\Delta_{\mu_1,\mu_2}} & U^{\mathrm{sc},\mu_1,0;\pm}_{\boldsymbol{v}} \otimes U^{\mathrm{sc},0,\mu_2;\pm}_{\boldsymbol{v}} \end{array} \tag{4.30}$$

We shall crucially need the so-called *shift homomorphisms* ι_{μ,ν_1,ν_2} of Finkelberg and Tsymbaliuk (2017, Lemma 10.24) (which are injective due to Finkelberg and Tsymbaliuk (2017, Theorem 10.25, Appendix I)):

Proposition 4.25 (Finkelberg and Tsymbaliuk 2017) *For any coweight μ and antidominant coweights ν_1, ν_2, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra embedding*

$$\iota_{\mu,\nu_1,\nu_2} : U^{\mathrm{sc},\mu}_{\boldsymbol{v}} \hookrightarrow U^{\mathrm{sc},\mu+\nu_1+\nu_2}_{\boldsymbol{v}} \tag{4.31}$$

defined by

$$e_i(z) \mapsto (1-z^{-1})^{-\alpha_i^\vee(\nu_1)} e_i(z),\ f_i(z) \mapsto (1-z^{-1})^{-\alpha_i^\vee(\nu_2)} f_i(z),$$
$$\psi^{\pm}_i(z) \mapsto (1-z^{-1})^{-\alpha_i^\vee(\nu_1+\nu_2)} \psi^{\pm}_i(z).$$

In Finkelberg and Tsymbaliuk (2017), we used these shift homomorphisms to reduce the construction of Δ_{μ_1,μ_2} for general μ_1, μ_2 to the aforementioned case of antidominant μ_1, μ_2 by proving the following result:

Proposition 4.26 (Finkelberg and Tsymbaliuk 2017) *The homomorphisms $\{\Delta_{\mu_1,\mu_2}\}_{\mu_1,\mu_2}$ exist and are uniquely determined by the condition that they coincide with those constructed before for antidominant μ_1, μ_2 and that the following diagram is commutative for any antidominant ν_1, ν_2:*

$$\begin{array}{ccc} U^{\mathrm{sc},\mu_1+\mu_2}_{\boldsymbol{v}} & \xrightarrow{\Delta_{\mu_1,\mu_2}} & U^{\mathrm{sc},\mu_1}_{\boldsymbol{v}} \otimes U^{\mathrm{sc},\mu_2}_{\boldsymbol{v}} \\ \downarrow{\scriptstyle \iota_{\mu,\nu_2,\nu_1}} & & \downarrow{\scriptstyle \iota_{\mu_1,0,\nu_1} \otimes \iota_{\mu_2,\nu_2,0}} \\ U^{\mathrm{sc},\mu_1+\mu_2+\nu_1+\nu_2}_{\boldsymbol{v}} & \xrightarrow{\Delta_{\mu_1+\nu_1,\mu_2+\nu_2}} & U^{\mathrm{sc},\mu_1+\nu_1}_{\boldsymbol{v}} \otimes U^{\mathrm{sc},\mu_2+\nu_2}_{\boldsymbol{v}} \end{array} \tag{4.32}$$

Having summarized the key properties of the coproduct homomorphisms Δ_{μ_1,μ_2} of Finkelberg and Tsymbaliuk (2017), let us now proceed to the proof of Theorem 4.23.

Proof of Theorem 4.23 The proof proceeds in three steps (cf. our proof of Theorem 4.4).

Step 1: Case $\mu_1 = \mu_2 = 0$.

Under the embedding $\Upsilon : U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, the Drinfeld-Jimbo coproduct Δ on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is intertwined with the $\mathbb{C}(\boldsymbol{v})$-extension of the RTT-coproduct $\Delta^{\mathrm{rtt}} : \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \to \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \otimes \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ defined via $\Delta^{\mathrm{rtt}}(T^{\pm}(z)) = T^{\pm}(z) \otimes T^{\pm}(z)$, see Ding and Frenkel (1993). As the Υ-preimage of $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ coincides with $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ (due to Proposition 3.20 and the equality[4] $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n) = U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \cap \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$), we obtain $\Delta(\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)) \subset \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. This immediately implies the result of the theorem for $\mu_1 = \mu_2 = 0$, since $\Delta_{0,0}$ essentially coincides with Δ.[5]

Step 2: Case of antidominant μ_1, μ_2.

For any $1 \le j \le i < n$ and $\underline{r} = (r_j, \dots, r_i) \in \mathbb{Z}^{i-j+1}$, recall the elements $E_{j,i+1}(\underline{r}) \in \mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq \mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \simeq \mathfrak{U}^{\mathrm{sc},\mu_1+\mu_2;>}_{\boldsymbol{v}}$ and $F_{i+1,j}(\underline{r}) \in \mathfrak{U}^{<}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq \mathfrak{U}^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \simeq \mathfrak{U}^{\mathrm{sc},\mu_1+\mu_2;<}_{\boldsymbol{v}}$ defined in (3.32). We start with the following result:

Lemma 4.27 *(a) If $r_j, r_{j+1}, \dots, r_i \ge 0$, then $\Delta_{\mu_1,\mu_2}(E_{j,i+1}(\underline{r})) \in \mathfrak{U}^{\mathrm{sc},\mu_1}_{\boldsymbol{v}} \otimes \mathfrak{U}^{\mathrm{sc},\mu_2}_{\boldsymbol{v}}$.*
(b) If $r_j \le b_{1,j}, r_{j+1} \le b_{1,j+1}, \dots, r_i \le b_{1,i}$, then $\Delta_{\mu_1,\mu_2}(F_{i+1,j}(\underline{r})) \in \mathfrak{U}^{\mathrm{sc},\mu_1}_{\boldsymbol{v}} \otimes \mathfrak{U}^{\mathrm{sc},\mu_2}_{\boldsymbol{v}}$.

Proof (a) If $\underline{r} \in \mathbb{N}^{i-j+1}$, then clearly $E_{j,i+1}(\underline{r}) \in U^+_{\boldsymbol{v}} \cap \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. As $\Delta(U^+_{\boldsymbol{v}}) \subset U^+_{\boldsymbol{v}} \otimes U^+_{\boldsymbol{v}}$ and $\Delta(\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)) \subset \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ (see Step 1), we get $\Delta(E_{j,i+1}(\underline{r})) \in (U^+_{\boldsymbol{v}} \cap \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)) \otimes (U^+_{\boldsymbol{v}} \cap \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n))$. Combining the commutativity of the diagram (4.30) with the equality $J^+_{\mu_1,\mu_2}(E_{j,i+1}(\underline{r})) = E_{j,i+1}(\underline{r})$, it remains to prove the inclusion $J^+_{\nu_1,\nu_2}(U^+_{\boldsymbol{v}} \cap \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)) \subset \mathfrak{U}^{\mathrm{sc},\nu_1+\nu_2}_{\boldsymbol{v}}$ for antidominant ν_1, ν_2. The latter follows from Finkelberg and Tsymbaliuk (2017, Lemma H.9)[6] and the following result:

Lemma 4.28 *The $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra $U^+_{\boldsymbol{v}} \cap \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is generated by*

$$\{E^{(r)}_{j,i+1}\}^{r\in\mathbb{N}}_{1\le j\le i<n} \cup \{F^{(r)}_{i+1,j}\}^{r>0}_{1\le j\le i<n} \cup \{\psi^+_{i,s}, (\psi^+_{i,0})^{\pm 1}\}^{s>0}_{1\le i<n}.$$

Proof Recall the embedding $\Upsilon : U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$. Note that the Borel subalgebra $U^+_{\boldsymbol{v}}$ of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ coincides with the Υ-preimage of the $\mathbb{C}(\boldsymbol{v})$-subalgebra

[4] The equality $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n) = U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \cap \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ immediately follows from Theorem 3.24.

[5] To be more precise, one needs to replace $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \subset U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ by $\mathfrak{U}^{\mathrm{rtt,ext}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \subset U^{\mathrm{rtt,ext}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ introduced in Step 1 of our proof of Theorem 4.4, while $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \subset U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ should be replaced by $\mathfrak{U}^{\mathrm{sc},0}_{\boldsymbol{v}} \subset U^{\mathrm{sc},0}_{\boldsymbol{v}}$.

[6] Here we refer to the equalities $J^+_{\nu_1,\nu_2}(E^{(r)}_{j,i+1}) = E^{(r)}_{j,i+1}$, $J^+_{\nu_1,\nu_2}(F^{(s)}_{i+1,j}) = F^{(s)}_{i+1,j}$, $J^+_{\nu_1,\nu_2}(\psi^+_{i,s}) = \psi^+_{i,s}$ for any $1 \le j \le i < n$ and $r \ge 0, s \ge 1$. Actually, in Finkelberg and Tsymbaliuk (2017, Lemma H.9) we proved those only for $r = 0, s = 1$. However, since the matrix $([c_{ij}]_{\boldsymbol{v}})^{n-1}_{i,j=1}$ is non-degenerate, for every $1 \le i < n$ there is a unique $\mathbb{C}(\boldsymbol{v})$-linear combination of $\{(\psi^+_{j,0})^{-1}\psi^+_{j,1}\}^{n-1}_{j=1}$, denoted by $h^{\perp}_{i,1}$, such that $[h^{\perp}_{i,1}, e_{j,r}] = \delta_{ij} e_{j,r+1}$, $[h^{\perp}_{i,1}, f_{j,r}] = -\delta_{ij} f_{j,r+1}$. As $J^+_{\nu_1,\nu_2}(h^{\perp}_{i,1}) = h^{\perp}_{i,1}$ and the elements $E^{(r)}_{j,i+1}, F^{(s)}_{i+1,j}$ can be obtained by iteratively commuting $E^{(0)}_{j,i+1}, F^{(1)}_{i+1,j}$ with $h^{\perp}_{j,1}$, we immediately obtain the claimed equalities $J^+_{\nu_1,\nu_2}(E^{(r)}_{j,i+1}) = E^{(r)}_{j,i+1}$, $J^+_{\nu_1,\nu_2}(F^{(s)}_{i+1,j}) = F^{(s)}_{i+1,j}$ for any $r \ge 0, s \ge 1$. The remaining equality $J^+_{\nu_1,\nu_2}(\psi^+_{i,s}) = \psi^+_{i,s}$ follows from $\psi^+_{i,s} = (\boldsymbol{v} - \boldsymbol{v}^{-1})[e_{i,0}, f_{i,s}]$ for $s \ge 1$.

$U_{\boldsymbol{v}}^{\mathrm{rtt},+}(L\mathfrak{gl}_n)$ of $U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$ generated by $\{t_{ij}^+[r]\}_{1\le i,j\le n}^{r\in\mathbb{N}} \cup \{(t_{ii}^+[0])^{-1}\}_{i=1}^n$. Evoking the Gauss decomposition of $T^+(z)$, we see that the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt},+}(L\mathfrak{gl}_n) = U_{\boldsymbol{v}}^{\mathrm{rtt},+}(L\mathfrak{gl}_n) \cap \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$ is generated by $\{\tilde{e}_{j,i+1}^{(r)}\}_{1\le j\le i<n}^{r\in\mathbb{N}} \cup \{\tilde{f}_{i+1,j}^{(r)}\}_{1\le j\le i<n}^{r>0} \cup \{\tilde{g}_i^{(r)}, (\tilde{g}_i^+)^{\pm 1}\}_{1\le i<n}^{r>0}$. Combining this with Corollary 3.23 and the above equality $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n) = U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \cap \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ yields the claim. □

This completes our proof of part (a).

(b) The proof of part (b) is completely analogous and utilizes homomorphisms $\jmath_{\bullet,\bullet}^-$ instead. □

Let us now prove

$$\Delta_{\mu_1,\mu_2}(\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2}) \subset \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2} \tag{4.33}$$

for antidominant μ_1, μ_2 by induction in $-\mu_1-\mu_2$. The base of induction, $\mu_1 = \mu_2 = 0$, is established in Step 1. The following result establishes the induction step:

Proposition 4.29 *If (4.33) holds for a pair of antidominant coweights (μ_1, μ_2), then it also holds both for $(\mu_1, \mu_2 - \omega_\ell)$ and $(\mu_1 - \omega_\ell, \mu_2)$ for any $1 \le \ell \le n-1$.*

Proof We will prove this only for $(\mu_1, \mu_2 - \omega_\ell)$, since the verification for the second pair is completely analogous. For any $1 \le j \le i < n$ and $r \in \mathbb{Z}$, we pick a particular decomposition $\underline{r} = (r_j, \dots, r_i) \in \mathbb{Z}^{i-j+1}$ with $r_j + \dots + r_i = r$ as follows: we set $r_j = r, r_{j+1} = \dots = r_i = 0$ if $\ell < j$ or $\ell > i$, and we set $r_\ell = r, r_j = \dots = r_{\ell-1} = r_{\ell+1} = \dots = r_i = 0$ if $j \le \ell \le i$.

Identifying $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2;>} \simeq \mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \simeq \mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n)$, Theorem 3.25(a) guarantees that the ordered PBWD monomials in $E_{j,i+1}(\underline{r})$ form a basis of a free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-module $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2;>}$. Let us now apply the morphisms of the commutative diagram (4.32) with $\nu_1 = 0, \nu_2 = -\omega_\ell$ to the element $E_{j,i+1}(\underline{r})$. As $E_{j,i+1}(\underline{r}) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2}$, our assumption guarantees that $\Delta_{\mu_1,\mu_2}(E_{j,i+1}(\underline{r})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2}$. Meanwhile, for any coweight μ and antidominant coweights ν_1', ν_2', we have

$$\iota_{\mu,\nu_1',\nu_2'}(\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}) \subset \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\nu_1'+\nu_2'}, \tag{4.34}$$

since every generator $E_{j,i+1}(\underline{r})$ (resp. $F_{i+1,j}(\underline{r})$ or $\psi_{i,s}^{\pm}$) is mapped to a $\mathbb{C}$-linear combination of elements of the form $E_{j,i+1}(\underline{r}')$ (resp. $F_{i+1,j}(\underline{r}')$ or $\psi_{i,s'}^{\pm}$) for various $\underline{r}', s'$. Thus, we obtain

$$\begin{aligned} &\Delta_{\mu_1,\mu_2-\omega_\ell}(\iota_{\mu_1+\mu_2,-\omega_\ell,0}(E_{j,i+1}(\underline{r}))) \\ &\quad = (\mathrm{Id} \otimes \iota_{\mu_2,-\omega_\ell,0})(\Delta_{\mu_1,\mu_2}(E_{j,i+1}(\underline{r}))) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}. \end{aligned} \tag{4.35}$$

If $\ell < j$ or $\ell > i$, then $\iota_{\mu_1+\mu_2,-\omega_\ell,0}(E_{j,i+1}(\underline{r})) = E_{j,i+1}(\underline{r})$, and so $\Delta_{\mu_1,\mu_2-\omega_\ell}(E_{j,i+1}(\underline{r})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}$, due to (4.35). If $j \le \ell \le i$, then $\iota_{\mu_1+\mu_2,-\omega_\ell,0}(E_{j,i+1}(\underline{r})) = E_{j,i+1}(\underline{r}) - E_{j,i+1}(\underline{r-1})$, hence, $\Delta_{\mu_1,\mu_2-\omega_\ell}(E_{j,i+1}(\underline{r}) - E_{j,i+1}(\underline{r-1})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}$, due to (4.35). Combining this with

$\Delta_{\mu_1,\mu_2-\omega_\ell}(E_{j,i+1}(\underline{r})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}$ for $r \geq 0$, due to Lemma 4.27(a), we get $\Delta_{\mu_1,\mu_2-\omega_\ell}(E_{j,i+1}(\underline{r})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}$ for any $r \in \mathbb{Z}$. This completes the proof of the inclusion

$$\Delta_{\mu_1,\mu_2-\omega_\ell}(E_{j,i+1}(\underline{r})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell} \text{ for any } 1 \leq j \leq i < n, r \in \mathbb{Z}.$$

The proof of the inclusion

$$\Delta_{\mu_1,\mu_2-\omega_\ell}(F_{i+1,j}(\underline{r})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell} \text{ for any } 1 \leq j \leq i < n, r \in \mathbb{Z}$$

is analogous. However, to apply Lemma 4.27(b), we need another choice of decompositions $\underline{r}$. For any $1 \leq j \leq i < n$ and $r \in \mathbb{Z}$, we pick a decomposition $\underline{r} = (r_j, \dots, r_i) \in \mathbb{Z}^{i-j+1}$ with $r_j + \dots + r_i = r$ as follows: we set $r_j = r - b_{1,j+1} - \dots - b_{1,i}, r_{j+1} = b_{1,j+1}, \dots, r_i = b_{1,i}$ if $\ell < j$ or $\ell > i$, and we set $r_\ell = r - b_{1,j} - \dots - b_{1,\ell-1} - b_{1,\ell+1} - \dots - b_{1,i}, r_j = b_{1,j}, \dots, r_{\ell-1} = b_{1,\ell-1}, r_{\ell+1} = b_{1,\ell+1}, \dots, r_i = b_{1,i}$ if $j \leq \ell \leq i$.

Finally, we note that

$$\Delta_{\mu_1,\mu_2-\omega_\ell}(\iota_{\mu_1+\mu_2,-\omega_\ell,0}(\psi_{i,s}^{\pm})) = (\mathrm{Id}\otimes\iota_{\mu_2,-\omega_\ell,0})(\Delta_{\mu_1,\mu_2}(\psi_{i,s}^{\pm})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1}\otimes\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}. \tag{4.36}$$

Therefore, $\Delta_{\mu_1,\mu_2-\omega_\ell}(\psi_{i,s}^{\pm} - \delta_{i,\ell}\psi_{i,s-1}^{\pm}) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}$. This implies (after a simple induction in s for $i = \ell$) that $\Delta_{\mu_1,\mu_2-\omega_\ell}(\psi_{i,s}^{\pm}) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1}\otimes\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}$ for any i, s.

Thus, the images of all generators of $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2-\omega_\ell}$ under $\Delta_{\mu_1,\mu_2-\omega_\ell}$ belong to $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2-\omega_\ell}$. This implies the validity of (4.33) for the pair $(\mu_1, \mu_2 - \omega_\ell)$. □

This completes our proof of Theorem 4.23 for antidominant μ_1, μ_2.

Step 3: General case.

Having established (4.33) for all antidominant μ_1, μ_2 (Step 2), the validity of (4.33) for arbitrary μ_1, μ_2 is implied by the following result:

Lemma 4.30 *If (4.33) holds for a pair of coweights (μ_1, μ_2), then it also holds both for $(\mu_1, \mu_2 + \omega_\ell)$ and $(\mu_1 + \omega_\ell, \mu_2)$ for any $1 \leq \ell \leq n - 1$.*

Proof We will prove this only for $(\mu_1, \mu_2 + \omega_\ell)$, since the verification for the second pair is completely analogous. The commutativity of the diagram (4.32) implies the following equality: $\Delta_{\mu_1,\mu_2}(\iota_{\mu_1+\mu_2+\omega_\ell,-\omega_\ell,0}(E_{j,i+1}(\underline{r}))) = (\mathrm{Id} \otimes \iota_{\mu_2+\omega_\ell,-\omega_\ell,0})(\Delta_{\mu_1,\mu_2+\omega_\ell}(E_{j,i+1}(\underline{r})))$. Its left-hand side belongs to $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2}$, due to (4.34) and our assumption. However, the argument identical to the one used in Step 4 of our proof of Theorem 4.4 yields the following implication:

$$(\mathrm{Id} \otimes \iota_{\mu_2+\omega_\ell,-\omega_\ell,0})(X) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2} \Longrightarrow X \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2+\omega_\ell}.$$

This completes our proof of the inclusion $\Delta_{\mu_1,\mu_2+\omega_\ell}(E_{j,i+1}(\underline{r})) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1}\otimes\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2+\omega_\ell}$.

The verification of inclusions $\Delta_{\mu_1,\mu_2+\omega_\ell}(F_{i+1,j}(\underline{r}))$, $\Delta_{\mu_1,\mu_2+\omega_\ell}(\psi_{i,s}^{\pm}) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2+\omega_\ell}$ is analogous. Hence, the images of all generators of $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1+\mu_2+\omega_\ell}$ under

$\Delta_{\mu_1,\mu_2+\omega_\ell}$ belong to $\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_1} \otimes \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu_2+\omega_\ell}$. This implies the validity of (4.33) for the pair $(\mu_1, \mu_2 + \omega_\ell)$. □

This completes our proof of Theorem 4.23. □

We conclude this subsection with the following result:

Lemma 4.31 *For any μ, ν_1, ν_2, we have $\iota_{\mu,\nu_1,\nu_2}^{-1}(\mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\nu_1+\nu_2}) = \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$.*

Proof Since $\iota_{\mu+\nu_1+\nu_2,\nu_1',\nu_2'} \circ \iota_{\mu,\nu_1,\nu_2} = \iota_{\mu,\nu_1+\nu_1',\nu_2+\nu_2'}$ for any coweight μ and antidominant coweights ν_1, ν_2, ν_1', ν_2', it suffices to verify the claim for the simplest pairs $(\nu_1 = -\omega_\ell, \nu_2 = 0)$ and $(\nu_1 = 0, \nu_2 = -\omega_\ell)$, $1 \le \ell \le n-1$. In both cases, the inclusion

$$\{x \in U_{\boldsymbol{v}}^{\mathrm{sc},\mu} : \iota_{\mu,\nu_1,\nu_2}(x) \in \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu+\nu_1+\nu_2}\} \subset \mathfrak{U}_{\boldsymbol{v}}^{\mathrm{sc},\mu}$$

has been already used in Step 3 above and follows from the argument used in Step 4 of our proof of Theorem 4.4. The opposite inclusion is just (4.34).

This completes our proof of Lemma 4.31. □

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Appendices By Alexander Tsymbaliuk and AlexWeekes.

Appendix A: PBW Theorem and Rees Algebra Realization for the Drinfeld–Gavarini Dual, and the Shifted Yangian

In Kamnitzer et al. (2014), dominantly shifted Yangians were defined for any semisimple Lie algebra $\mathfrak{g}$, generalizing Brundan–Kleshchev's definition Brundan and Kleshchev (2006) for $\mathfrak{gl}_n$. Two issues with the definition given in Kamnitzer et al. (2014) are now clear:

(a) Kamnitzer et al. (2014, §3C) recalled Drinfeld–Gavarini duality, and an explicit description of the Drinfeld–Gavarini dual based on the discussion in Gavarini (2002, §3.5). However, additional assumptions seem necessary in order for this description to be correct.
(b) Applying the explicit description of (a), a presentation of the dominantly shifted Yangians was given in Kamnitzer et al. (2014, Theorem 3.5, Definition 3.10). But

it was incomplete, as it does not include a full set of relations. In fact, writing down a complete (explicit) set of relations seems very difficult (at least, in terms of new Drinfeld generators).

We rectify (a) in Proposition A.2, which is of independent interest. We then verify that this result applies to the Yangian, yielding Theorem A.7. This proves that the set of generators given just before Kamnitzer et al. (2014, Theorem 3.5) is indeed correct. We also verify that Proposition A.2 applies to the RTT Yangian $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$, which implies an identification of its Drinfeld–Gavarini dual with the subalgebra $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of Definition 2.3, establishes the PBW theorem for the latter (that we referred to in Sect. 2), and provides a conceptual proof of Proposition 2.21.

Another definition of the shifted Yangian (for an arbitrary, not necessarily dominant, shift) as a Rees algebra was given in Finkelberg et al. (2018, §5.4) and Braverman et al. (2016, Appendix B(i)), precisely to avoid the issues mentioned above. This raises

Question: Are these two definitions of the shifted Yangians equivalent for dominant shifts?

We answer this question in the affirmative in Theorem A.12, which generalizes Theorem 2.31 for any semisimple Lie algebra $\mathfrak{g}$.

We conclude this appendix with one more equivalent definition of the shifted Yangians, see Appendix A.8, in particular, Theorem A.17.

A.1 Drinfeld's Functor

Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{C}$. Assume that A is a *deformation quantization* of the Hopf algebra $U(\mathfrak{a})$,[7] over $\mathbb{C}[\hbar]$. In other words, A is a Hopf algebra over $\mathbb{C}[\hbar]$, and there is an isomorphism of Hopf algebras $A/\hbar A \simeq U(\mathfrak{a})$.

Denote the coproduct and the counit of A by Δ and ϵ, respectively. For any $n \geq 0$, let $\Delta^n : A \to A^{\otimes n}$ be the n-th iterated coproduct (tensor product over $\mathbb{C}[\hbar]$). It is defined inductively by $\Delta^0 = \epsilon$, $\Delta^1 = \mathrm{id}$, and $\Delta^n = (\Delta \otimes \mathrm{id}^{\otimes(n-2)}) \circ \Delta^{n-1}$. Define $\delta_n : A \to A^{\otimes n}$ via

$$\delta_n := (\mathrm{id} - \epsilon)^{\otimes n} \circ \Delta^n. \tag{A.1}$$

Drinfeld (1986) introduced functors on Hopf algebras, which have been studied extensively in work of Gavarini, see e.g. Gavarini (2002). In particular, the *Drinfeld–Gavarini dual* of A is the sub-Hopf algebra $A' \subset A$ defined by

$$A' = \left\{ a \in A : \delta_n(a) \in \hbar^n A^{\otimes n} \text{ for all } n \in \mathbb{N} \right\}. \tag{A.2}$$

As in the second part of the proof of Gavarini (2002, Proposition 2.6):

Lemma A.1 *For any $a, b \in A'$, we have $[a, b] \in \hbar A'$.*

The dual A' can be defined for any Hopf algebra A over $\mathbb{C}[\hbar]$. However, the case of the greatest interest is precisely when $A/\hbar A \simeq U(\mathfrak{a})$. In this case, one can prove that

[7] Here $U(\mathfrak{a})$ denotes the universal enveloping algebra of $\mathfrak{a}$ over $\mathbb{C}$, in contrast to Definition 2.11.

A' is a deformation quantization of the coordinate ring of a 'dual' algebraic group. This is a part of the *quantum duality principle*, see Gavarini (2002, Theorem 1.6) (called *Drinfeld–Gavarini duality* in Kamnitzer et al. (2014, §3C)).

A.2 PBW Basis for A'

We will now make some additional assumptions on A. Suppose that there exists a totally ordered set $(\mathcal{I}, \leq)$, and elements $\{x_i\}_{i\in\mathcal{I}} \subset A$. By an (ordered) *PBW monomial*, we mean any ordered monomial

$$x_{i_1} \dots x_{i_\ell} \in A \tag{A.3}$$

with $\ell \in \mathbb{N}$ and $i_1 \leq \dots \leq i_\ell$. Assume that:

$$\{x_i\}_{i\in\mathcal{I}} \text{ lifts a basis } \{\overline{x}_i\}_{i\in\mathcal{I}} \text{ for } \mathfrak{a}, \tag{As1}$$

$$A \text{ is free over } \mathbb{C}[\hbar], \text{ with a basis given by the PBW monomials in } \{x_i\}_{i\in\mathcal{I}}, \tag{As2}$$

$$\text{for all } i \in \mathcal{I}, \text{ we have } \hbar x_i \in A'. \tag{As3}$$

We will use the multi-index notation x^α to denote a PBW monomial $\prod_{i\in\mathcal{I}} x_i^{\alpha_i}$ in the PBW generators x_i. We write $|\alpha| = \sum_{i\in\mathcal{I}} \alpha_i$ for the total degree of x^α. Finally, for $\overline{a} \in U(\mathfrak{a})$ we denote by $\partial(\overline{a})$ its degree with respect to the usual filtration, i.e. the maximal value of $|\alpha|$ over all summands $\overline{x}^\alpha$ that appear in $\overline{a}$.

In Gavarini (2002, §3.5), an explicit description of A' is given in the *formal* case, i.e. when working with complete algebras over $\mathbb{C}[[\hbar]]$. The next result is inspired by this description, but with the aim of working instead over $\mathbb{C}[\hbar]$.

Proposition A.2 *Suppose that A satisfies Assumptions (As1)–(As3). Then A' is free over $\mathbb{C}[\hbar]$, with a basis given by the PBW monomials in the elements $\{\hbar x_i\}_{i\in\mathcal{I}}$. In particular, $A' \subset A$ is the $\mathbb{C}[\hbar]$-subalgebra generated by $\{\hbar x_i\}_{i\in\mathcal{I}}$.*

In the proof, we will make use of Etingof and Kazhdan (1996, Lemma 4.12) (cf. (Gavarini 2002, Lemma 3.3)):

Lemma A.3 *Let $a \in A'$ be non-zero, and write $a = \hbar^n b$ where $b \in A \backslash \hbar A$. Then $\partial(\overline{b}) \leq n$.*

Proof of Proposition A.2 Let $c \in A'$. By assumption (As2), we can write $c = \sum_{k,\alpha} c_{k,\alpha} \hbar^k x^\alpha$ for some $c_{k,\alpha} \in \mathbb{C}$ which are almost all zero. Since A' is an algebra, by assumption (As3) we know that $\hbar^k x^\alpha \in A'$ whenever $k \geq |\alpha|$. Subtracting all such elements from c, we conclude that the element

$$a = \sum_{k,\alpha: k<|\alpha|} c_{k,\alpha} \hbar^k x^\alpha \in A'. \tag{A.4}$$

Assume that $a \neq 0$. Choosing

$$n = \min\{k : \exists \alpha \text{ such that } k < |\alpha| \text{ and } c_{k,\alpha} \neq 0\}, \tag{A.5}$$

we can write $a = \hbar^n b$, where

$$b \in \sum_{\alpha: n < |\alpha|} c_{n,\alpha} x^\alpha + \hbar A. \tag{A.6}$$

From assumption (As1) it follows that

$$\overline{b} = \sum_{\alpha: n < |\alpha|} c_{n,\alpha} \overline{x}^\alpha \ \in \ U(\mathfrak{a}), \tag{A.7}$$

and so $\partial(\overline{b}) > n$. But by Lemma A.3 we should have $\partial(\overline{b}) \leq n$. So we conclude that $a = 0$.

This shows that the PBW monomials in $\{\hbar x_i\}_{i \in \mathcal{I}}$ span A' over $\mathbb{C}[\hbar]$. But they are also linearly independent, because of assumption (As2). Thus, they form a basis.

A.3 Rees Algebra Description of A'

In this subsection, we make a further assumption on A:

$$A \text{ is a graded Hopf algebra, with } \deg(\hbar) = 1 \text{ and } \{x_i\}_{i \in \mathcal{I}} \text{ being homogeneous.} \tag{As4}$$

Note that $A' \subset A$ is then a graded sub-Hopf algebra, and that the specializations of A, A' at $\hbar = 0$ inherit gradings. By Proposition A.2, we see that the inclusion $A' \subset A$ induces an isomorphism of their specializations at $\hbar = 1$:

$$A'/(\hbar - 1)A' \simeq A/(\hbar - 1)A. \tag{A.8}$$

Moreover, the images of their respective PBW generators and bases agree as $\hbar x_i + (\hbar - 1)A = x_i + (\hbar - 1)A$.

Denote the algebra in (A.8) by $A_{\hbar=1}$. If assumption (As4) holds, it follows that $A_{\hbar=1}$ inherits two filtrations $F'_\bullet A_{\hbar=1}$, $F_\bullet A_{\hbar=1}$, coming from A' and A, respectively. Denoting $\mathrm{d}_i = \deg x_i$, these filtrations may be defined explicitly in terms of the PBW monomials:

$$F_k A_{\hbar=1} = \mathrm{span}_{\mathbb{C}} \Big\{ x^\alpha + (\hbar - 1)A \ : \ \sum_i \mathrm{d}_i \alpha_i \leq k \Big\}, \tag{A.9}$$

$$F'_k A_{\hbar=1} = \mathrm{span}_{\mathbb{C}} \Big\{ x^\alpha + (\hbar - 1)A \ : \ \sum_i (\mathrm{d}_i + 1)\alpha_i \leq k \Big\}. \tag{A.10}$$

In particular, $F'_k A_{\hbar=1} \subset F_k A_{\hbar=1}$ for all $k \in \mathbb{Z}$.

By the above discussion, we obtain another description of A', as a Rees algebra:

Proposition A.4 *Suppose that A satisfies Assumptions (As1)–(As4). Then there is a canonical isomorphism of graded Hopf algebras*

$$A' \simeq \mathrm{Rees}^{F'_\bullet}(A_{\hbar=1}).$$

It is compatible with the canonical isomorphism $A \simeq \mathrm{Rees}^{F_\bullet}(A_{\hbar=1})$*, under the natural inclusions* $A' \subset A$ *and* $\mathrm{Rees}^{F'_\bullet}(A_{\hbar=1}) \subset \mathrm{Rees}^{F_\bullet}(A_{\hbar=1})$.

A.4 The Yangian of $\mathfrak{g}$

Consider the *Yangian* $Y_\hbar = Y_\hbar(\mathfrak{g})$ associated to a semisimple Lie algebra $\mathfrak{g}$. It is the associative $\mathbb{C}[\hbar]$-algebra with generators $\{e_i^{(r)}, h_i^{(r)}, f_i^{(r)}\}_{i\in I}^{r\in\mathbb{N}}$ (here I denotes the set of vertices of the Dynkin diagram of $\mathfrak{g}$), and relations as in Kamnitzer et al. (2014, §3A) and Guay et al. (2018, Definition 2.1), cf. (2.9). For each $i \in I$, define the element $s_i := h_i^{(1)} - \frac{\hbar}{2}(h_i^{(0)})^2 \in Y_\hbar$, cf. (2.17). Then $Y_\hbar$ is also generated by $\{e_i^{(0)}, h_i^{(0)}, f_i^{(0)}, s_i\}_{i\in I}$, cf. Sect. 2.5.

For each positive root $\alpha^\vee$ and $r \geq 0$, define the elements $e_{\alpha^\vee}^{(r)}, f_{\alpha^\vee}^{(r)}$ of $Y_\hbar$ via

$$\begin{aligned} e_{\alpha^\vee}^{(r)} &:= \left[\left[\cdots\left[e_{\alpha_{i_\ell}^\vee}^{(r)}, e_{\alpha_{i_{\ell-1}}^\vee}^{(0)}\right], \cdots, e_{\alpha_{i_2}^\vee}^{(0)}\right], e_{\alpha_{i_1}^\vee}^{(0)}\right], \\ f_{\alpha^\vee}^{(r)} &:= \left[f_{\alpha_{i_1}^\vee}^{(0)}, \left[f_{\alpha_{i_2}^\vee}^{(0)}, \cdots, \left[f_{\alpha_{i_{\ell-1}}^\vee}^{(0)}, f_{\alpha_{i_\ell}^\vee}^{(r)}\right]\cdots\right]\right], \end{aligned} \tag{A.11}$$

where $\alpha^\vee = \alpha_{i_1}^\vee + \alpha_{i_2}^\vee + \cdots + \alpha_{i_{\ell-1}}^\vee + \alpha_{i_\ell}^\vee$ is an (ordered) decomposition into simple roots such that the element $[f_{\alpha_{i_1}^\vee}, [f_{\alpha_{i_2}^\vee}, \cdots, [f_{\alpha_{i_{\ell-1}}^\vee}, f_{\alpha_{i_\ell}^\vee}]\cdots]]$ is a non-zero element of $\mathfrak{g}$ (here $\{f_{\alpha_i^\vee}\}_{i\in I}$ denote the standard Chevalley generators of $\mathfrak{g}$). We will refer to these elements, together with $\{h_i^{(r)}\}_{i\in I}^{r\in\mathbb{N}}$, as the Yangian *PBW generators*. Throughout this appendix (and the next one), we fix some total ordering on the set of all PBW generators. It is well-known that $Y_\hbar$ is free over $\mathbb{C}[\hbar]$ with a basis given by the PBW monomials, as was proven in Levendorskii (1993). Since the original proof of Levendorskii (1993) contains a significant gap, we give an alternative short proof in Appendix B.

$Y_\hbar$ is a graded Hopf algebra, with $\deg(\hbar) = 1$ and $\deg(x^{(r)}) = r$ for $x = e_{\alpha^\vee}, h_i, f_{\alpha^\vee}$. Its coproduct is uniquely determined by

$$\begin{aligned} \Delta(x^{(0)}) &= x^{(0)} \otimes 1 + 1 \otimes x^{(0)} \text{ for } x = e_{\alpha^\vee}, h_i, f_{\alpha^\vee}, \\ \Delta(s_i) &= s_i \otimes 1 + 1 \otimes s_i - \hbar \sum_{\gamma^\vee > 0} \langle \alpha_i, \gamma^\vee \rangle f_{\gamma^\vee}^{(0)} \otimes e_{\gamma^\vee}^{(0)}. \end{aligned} \tag{A.12}$$

A proof of these formulas appears in Guay et al. (2018). Meanwhile, the counit of $Y_\hbar$ is given simply by

$$\epsilon\left(e_{\alpha^\vee}^{(0)}\right) = \epsilon\left(f_{\alpha^\vee}^{(0)}\right) = \epsilon\left(h_i^{(0)}\right) = \epsilon(s_i) = 0. \tag{A.13}$$

Finally, we note that in the classical limit there is an isomorphism of graded Hopf algebras

$$Y_\hbar / \hbar Y_\hbar \simeq U(\mathfrak{g}[t]), \tag{A.14}$$

where $U(\mathfrak{g}[t])$ carries the loop grading.

A.5 The Drinfeld–Gavarini Dual of the Yangian

In this subsection, we describe the Drinfeld–Gavarini dual $Y'_\hbar$, by applying the results of the previous subsections. To this end, we will verify that Assumptions (As1)–(As4) hold for $Y_\hbar$ and its PBW generators. Note that only assumption (As3) remains; the others hold as discussed in Sect. A.4.

Using Eqs. (A.12, A.13), a straightforward calculation shows that:

Lemma A.5 (1) $\delta_n(x^{(0)}) = \begin{cases} x, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}$, *for any* $x = e_{\alpha^\vee}, h_i, f_{\alpha^\vee}$.

(2) $\delta_n(s_i) = \begin{cases} s_i, & \text{if } n = 1 \\ -\hbar \sum_{\gamma^\vee > 0} \langle \alpha_i, \gamma^\vee \rangle f^{(0)}_{\gamma^\vee} \otimes e^{(0)}_{\gamma^\vee}, & \text{if } n = 2 \\ 0, & \text{otherwise} \end{cases}$.

Using this lemma, we can now verify assumption (As3):

Lemma A.6 *For any PBW generator $x^{(r)}$ of $Y_\hbar$, the element $X^{(r+1)} := \hbar x^{(r)}$ belongs to the Drinfeld–Gavarini dual $Y'_\hbar$.*

Proof By the previous lemma, we have $\hbar e^{(0)}_{\alpha^\vee}, \hbar h^{(0)}_i, \hbar f^{(0)}_{\alpha^\vee}, \hbar s_i \in Y'_\hbar$. All the PBW generators $x^{(r)}$ can be obtained by taking repeated commutators of these elements, and $X^{(r+1)}$ by repeated application of the operation $a, b \mapsto \frac{1}{\hbar}[a, b]$ to the elements $\hbar e^{(0)}_{\alpha^\vee}, \hbar h^{(0)}_i, \hbar f^{(0)}_{\alpha^\vee}$, and $\hbar s_i$. Since $Y'_\hbar$ is closed under this operation, due to Lemma A.1, the claim follows. □

Thus Proposition A.2 applies providing a complete proof of the description of $Y'_\hbar$ given just before Kamnitzer et al. (2014, Theorem 3.5). Note that, as mentioned above, the relations given in Kamnitzer et al. (2014, Theorem 3.5) are incomplete (with the exception of $\mathfrak{g} = \mathfrak{sl}_2$). We do not address this issue here, as our methods do not provide a complete set of relations.

Theorem A.7 *The Drinfeld–Gavarini dual $Y'_\hbar$ is free over $\mathbb{C}[\hbar]$, with a basis given by the PBW monomials in the elements $X^{(r+1)}$. In particular, $Y'_\hbar \subset Y_\hbar$ is the $\mathbb{C}[\hbar]$-subalgebra generated by the elements $X^{(r+1)}$.*

Applying Proposition A.4, we also obtain the Rees algebra description of $Y'_\hbar$ of Finkelberg et al. (2018). In the case of the Yangian, the filtration $F'_\bullet Y_{\hbar=1}$ from (A.10) is known as the *Kazhdan filtration*.

Corollary A.8 *There is a canonical $\mathbb{C}[\hbar]$-algebra isomorphism*

$$Y'_\hbar \simeq \mathrm{Rees}^{F'_\bullet}(Y_{\hbar=1})$$

with the Rees algebra of $Y_{\hbar=1}$ with respect to the Kazhdan filtration.

A.6 The RTT Version

Recall the RTT Yangian $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of Sect. 2.1. We refer to the elements $\{t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 1}$ as the *PBW generators* of $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$. Fix some total ordering on the set of all PBW generators; this gives rise to the notion of the PBW monomials in $\{t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 1}$.

$Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ is an $\mathbb{N}$-graded Hopf algebra with $\deg(\hbar)=1$, $\deg(t_{ij}^{(r)})=r-1$. Its coproduct Δ^{rtt} and counit ϵ^{rtt} are determined explicitly by

$$\Delta^{\mathrm{rtt}}(T(z))=T(z)\otimes T(z),\ \epsilon^{\mathrm{rtt}}(T(z))=I_n. \tag{A.15}$$

Moreover, according to Remark 2.2, we have an isomorphism of graded Hopf algebras

$$Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)/\hbar Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)\simeq U(\mathfrak{gl}_n[t]). \tag{A.16}$$

Proposition A.9 *The PBW monomials in* $\{t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 1}$ *form a basis of a free* $\mathbb{C}[\hbar]$*-module* $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$.

Proof The proof is similar to that of Theorem B.3 below. First, combining the isomorphism (A.16) with the PBW theorem for $U(\mathfrak{gl}_n[t])$, we immediately see that the PBW monomials span $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ over $\mathbb{C}[\hbar]$. To prove the linear independence of the PBW monomials over $\mathbb{C}[\hbar]$, it suffices to verify that their images are linearly independent over $\mathbb{C}$ when we specialize $\hbar$ to any nonzero complex number. The latter holds for $\hbar=1$ (and thus for any $\hbar\neq 0$, since all such specializations are isomorphic), due to (Molev 2007, Theorem 1.4.1).

This completes our proof of Proposition A.9. □

The following result provides a new viewpoint towards $\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of Definition 2.3:

Theorem A.10 *The Drinfeld–Gavarini dual* $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)'$ *is free over* $\mathbb{C}[\hbar]$*, with a basis given by the PBW monomials in the elements* $\hbar t_{ij}^{(r)}$*. In particular,* $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)'=\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$.

Proof This follows from Proposition A.2 once we verify that Assumptions (As1)–(As4) hold for $Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$. Note that only assumption (As3) remains; the others hold as discussed above. The desired inclusion $\hbar t_{ij}^{(r)}\in Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)'$ follows immediately from (A.15), due to $(\mathrm{id}-\epsilon^{\mathrm{rtt}})(\hbar t_{ij}^{(r)})=\hbar t_{ij}^{(r)}\in\hbar Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$. □

Since the $\mathbb{C}[\hbar]$-algebra isomorphism $\Upsilon: Y_\hbar(\mathfrak{gl}_n)\xrightarrow{\sim} Y_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of Theorem 2.18 is actually an isomorphism of Hopf algebras, we conclude that it gives rise to an isomorphism of the corresponding Drinfeld–Gavarini duals $\Upsilon:\mathbf{Y}_\hbar(\mathfrak{gl}_n)\xrightarrow{\sim}\mathbf{Y}_\hbar^{\mathrm{rtt}}(\mathfrak{gl}_n)$. This provides an alternative computation-free proof of Proposition 2.21.

Remark A.11 Let us compare the above exposition with that of Molev (2007), where an opposite order of reasoning is used. In *loc.cit.*, the author works with the $\mathbb{C}$-algebra $Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$ defined as a common specialization $Y^{\mathrm{rtt}}(\mathfrak{gl}_n)=Y_{\hbar=1}^{\mathrm{rtt}}(\mathfrak{gl}_n)=\mathbf{Y}_{\hbar=1}^{\mathrm{rtt}}(\mathfrak{gl}_n)$, endowed with two different filtrations $F_\bullet Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$, $F'_\bullet Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$ determined

by $\deg^{F_\bullet}(t_{ij}^{(r)}) = r, \deg^{F'_\bullet}(t_{ij}^{(r)}) = r-1$ (these notations are opposite to those we used in Sect. A.3). First, in Molev (2007, Corollary 1.4.2), an algebra isomorphism $\mathrm{gr}^{F_\bullet} Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \simeq \mathbb{C}[\{t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\ge 1}]$ is proven, and only then an algebra isomorphism $\mathrm{gr}^{F'_\bullet} Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \simeq U(\mathfrak{gl}_n[t])$ is deduced in Molev (2007, Proposition 1.5.2).

A.7 The Shifted Yangian

In Sects. 2.6 and 2.7 , respectively, the algebras $\mathbf{Y}_\mu$ (for any coweight μ) and $\mathbf{Y}'_\mu$ (only for a dominant coweight μ) are defined. In this section, we show that these two definitions are equivalent when μ is dominant. Note that although these definitions were only given in the case of $\mathfrak{g} = \mathfrak{sl}_n$, they can be easily extended to any semisimple Lie algebra $\mathfrak{g}$ (cf. Kamnitzer et al. 2014; Finkelberg et al. 2018). Till the end of this subsection, we assume that μ is dominant.

We first recall two auxiliary algebras. The first is the $\mathbb{C}$-algebra Y_μ defined in Sect. 2.6, and the second is the $\mathbb{C}[\hbar]$-algebra $Y_{\mu,\hbar}$ introduced in Sect. 2.7. Both have PBW bases in the corresponding generators over their respective ground rings, by Theorems 2.26 and 2.29 , respectively.

Fixing a splitting $\mu = \mu_1 + \mu_2$, recall that Y_μ has a corresponding filtration $F^\bullet_{\mu_1,\mu_2} Y_\mu$, see (2.23). Similarly $Y_{\mu,\hbar}$ has a corresponding grading, defined by setting $\deg(\hbar) = 1$ and

$$\deg(e_{\alpha^\vee}^{(r)}) = \alpha^\vee(\mu_1) + r, \quad \deg(f_{\alpha^\vee}^{(r)}) = \alpha^\vee(\mu_2) + r, \quad \deg(h_i^{(r)}) = \alpha_i^\vee(\mu) + r.$$

Thus for $x = e_{\alpha^\vee}, f_{\alpha^\vee}, h_i$, we have $\deg(x^{(r)}) = \deg(x) + r$, where the internal grading $\deg(x)$ is defined via $\deg(e_{\alpha^\vee}) = \alpha^\vee(\mu_1)$, $\deg(f_{\alpha^\vee}) = \alpha^\vee(\mu_2)$, $\deg(h_i) = \alpha_i^\vee(\mu)$.

By comparing their defining relations, it is clear that there is a $\mathbb{C}$-algebra isomorphism

$$Y_{\mu,\hbar}/(\hbar - 1)Y_{\mu,\hbar} \xrightarrow{\sim} Y_\mu. \tag{A.17}$$

On the PBW generators, this isomorphism involves a shift of labels: $x^{(r)} \mapsto X^{(r+1)}$ for $x = e_{\alpha^\vee}, f_{\alpha^\vee}, h_i$. It follows that Y_μ inherits a second filtration $G^\bullet_{\mu_1,\mu_2} Y_\mu$, coming from the above grading on $Y_{\mu,\hbar}$. This is analogous to the situation in Sect. A.3: if by abuse of notation we denote the PBW generators of Y_μ by $x^{(r)}$, then $G^k_{\mu_1,\mu_2} Y_\mu$ is the span of all PBW monomials

$$x_1^{(r_1)} \cdots x_\ell^{(r_\ell)} \tag{A.18}$$

with $(\deg(x_1) + r_1) + \cdots + (\deg(x_\ell) + r_\ell) \le k$. Meanwhile, $F^k_{\mu_1,\mu_2} Y_\mu$ is the span of those monomials (A.18) with $(\deg(x_1) + r_1 + 1) + \cdots + (\deg(x_\ell) + r_\ell + 1) \le k$.

In particular, there is an inclusion $F^k_{\mu_1,\mu_2} Y_\mu \subset G^k_{\mu_1,\mu_2} Y_\mu$, hence, an embedding of the Rees algebras

$$\mathrm{Rees}^{F^\bullet_{\mu_1,\mu_2}}(Y_\mu) \subset \mathrm{Rees}^{G^\bullet_{\mu_1,\mu_2}}(Y_\mu). \tag{A.19}$$

Now on the one hand, $\mathbf{Y}_\mu = \mathrm{Rees}^{F^\bullet_{\mu_1,\mu_2}}(Y_\mu)$ by Definition 2.27. On the other hand, since $Y_{\mu,\hbar}$ is free over $\mathbb{C}[\hbar]$, we have $Y_{\mu,\hbar} \simeq \mathrm{Rees}^{G^\bullet_{\mu_1,\mu_2}}(Y_\mu)$. Explicitly, on PBW

generators this isomorphism is defined by

$$Y_{\mu,\hbar} \ni x^{(k)} \mapsto \hbar^{\deg(x)+k} x^{(k)} \in \hbar^{\deg(x)+k} G^{\deg(x)+k}_{\mu_1,\mu_2} Y_\mu \subset \mathrm{Rees}^{G^\bullet_{\mu_1,\mu_2}}(Y_\mu). \quad \text{(A.20)}$$

Altogether, we obtain an injective homomorphism of graded $\mathbb{C}[\hbar]$-algebras $\mathbf{Y}_\mu \hookrightarrow Y_{\mu,\hbar}$.

We can now prove a generalization of Theorem 2.31 for an arbitrary $\mathfrak{g}$:

Theorem A.12 *For any dominant coweight μ, there is a canonical $\mathbb{C}[\hbar]$-algebra isomorphism $\mathbf{Y}_\mu \simeq \mathbf{Y}'_\mu$. For any splitting $\mu = \mu_1 + \mu_2$, this isomorphism is compatible with the associated gradings on $\mathbf{Y}_\mu$ (from the filtration $F^\bullet_{\mu_1,\mu_2} Y_\mu$) and $\mathbf{Y}'_\mu$ (as a graded subalgebra of $Y_{\mu,\hbar}$).*

Proof All that remains is to check that the image of $\mathbf{Y}_\mu \hookrightarrow Y_{\mu,\hbar}$ is precisely $\mathbf{Y}'_\mu$. This follows from the "shift" that distinguishes the filtrations $F^k_{\mu_1,\mu_2} Y_\mu$ and $G^k_{\mu_1,\mu_2} Y_\mu$. Indeed, for a monomial $x_1^{(r_1)} \cdots x_\ell^{(r_\ell)} \in F^k_{\mu_1,\mu_2} Y_\mu$, the corresponding element in the Rees algebra is

$$\hbar^k x_1^{(r_1)} \cdots x_\ell^{(r_\ell)} \in \hbar^k F^k_{\mu_1,\mu_2} Y_\mu \subset \mathrm{Rees}^{F^\bullet_{\mu_1,\mu_2}}(Y_\mu).$$

But in the Rees algebra $\mathrm{Rees}^{G^\bullet_{\mu_1,\mu_2}}(Y_\mu) \simeq Y_{\mu,\hbar}$, by inverting (A.20) this element gets sent to

$$\hbar^{k-(\deg(x_1)+r_1)-\cdots-(\deg(x_\ell)+r_\ell)} x_1^{(r_1)} \cdots x_\ell^{(r_\ell)} \in Y_{\mu,\hbar}.$$

Since $(\deg(x_1) + r_1 + 1) + \cdots + (\deg(x_\ell) + r_\ell + 1) \leq k$, we can rewrite this as

$$\hbar^{k-(\deg(x_1)+r_1+1)-\cdots-(\deg(x_\ell)+r_\ell+1)} (\hbar x_1^{(r_1)}) \cdots (\hbar x_\ell^{(r_\ell)}),$$

which lies in $\mathbf{Y}'_\mu$. Taking spans of such monomials, we see that $\mathbf{Y}_\mu = \mathrm{Rees}^{F^\bullet_{\mu_1,\mu_2}}(Y_\mu) \subset \mathbf{Y}'_\mu$. But it is easy to see that the generators of $\mathbf{Y}'_\mu$ lie in $\mathrm{Rees}^{F^\bullet_{\mu_1,\mu_2}}(Y_\mu)$, so actually $\mathbf{Y}_\mu = \mathbf{Y}'_\mu$. □

A.8 The Shifted Yangian, Construction III

Motivated by the discussion of the previous subsection, we provide one more alternative definition of the shifted Yangians. Fix a coweight μ of $\mathfrak{g}$ and set $b_i := \alpha_i^\vee(\mu)$, where $\{\alpha_i^\vee\}_{i\in I}$ are the simple roots of $\mathfrak{g}$. Let $\mathcal{Y}_{\mu,\hbar}$ be the associative $\mathbb{C}[\hbar, \hbar^{-1}]$-algebra generated by $\{e_i^{(r)}, f_i^{(r)}, h_i^{(s_i)}\}_{i\in I}^{r\geq 0, s_i \geq -b_i}$ with the defining relations similar to those of (2.24) (but generalized to any $\mathfrak{g}$) with the only exception:

$$[e_i^{(r)}, f_j^{(r')}] = \begin{cases} h_i^{(r+r')}, & \text{if } i = j \text{ and } r + r' \geq -b_i \\ \hbar^{-1}, & \text{if } i = j \text{ and } r + r' = -b_i - 1 \\ 0, & \text{otherwise} \end{cases}. \quad \text{(A.21)}$$

Remark A.13 If μ is dominant, the equality $r+r' = -b_i - 1$ never occurs for $r, r' \geq 0$. Thus, $\mathcal{Y}_{\mu,\hbar}$ is the $\mathbb{C}[\hbar, \hbar^{-1}]$-extension of scalars of $Y_{\mu,\hbar}$ of Appendix A.7 for dominant μ.

Define the elements $\{e_{\alpha^\vee}^{(r)}, f_{\alpha^\vee}^{(r)}\}_{\alpha^\vee \in \Delta^+}^{r \geq 0}$ (here Δ^+ denotes the set of positive roots of $\mathfrak{g}$) of $\mathcal{Y}_{\mu,\hbar}$ following (2.25) (but generalized from type A to any $\mathfrak{g}$, cf. (Finkelberg et al. 2018, (3.1))). Choose any total ordering on the set of *PBW generators* as in (2.26) (but generalized from type A to any $\mathfrak{g}$, cf. Finkelberg et al. (2018, (3.4))). The following is analogous to Finkelberg et al. (2018, Corollary 3.15) (cf. Theorem 2.26 in type A):

Theorem A.14 *For any coweight μ, the PBW monomials form a basis of a free $\mathbb{C}[\hbar, \hbar^{-1}]$-module $\mathcal{Y}_{\mu,\hbar}$.*

This follows immediately from Finkelberg et al. (2018, Corollary 3.15) and the following simple result:

Lemma A.15 *Fix a pair of coweights μ_1, μ_2 such that $\mu_1 + \mu_2 = \mu$.*
(a) There is an isomorphism of $\mathbb{C}[\hbar, \hbar^{-1}]$-algebras $\mathcal{Y}_{\mu,\hbar} \xrightarrow{\sim} Y_\mu[\hbar, \hbar^{-1}]$ defined by

$$h_i^{(r)} \mapsto \hbar^{\alpha_i^\vee(\mu)+r} H_i^{(r+1)}, \ e_i^{(r)} \mapsto \hbar^{\alpha_i^\vee(\mu_1)+r} E_i^{(r+1)}, \ f_i^{(r)} \mapsto \hbar^{\alpha_i^\vee(\mu_2)+r} F_i^{(r+1)}.$$

(b) The above isomorphism sends the PBW generators as follows:

$$e_{\alpha^\vee}^{(r)} \mapsto \hbar^{\alpha^\vee(\mu_1)+r} E_{\alpha^\vee}^{(r+1)}, \ f_{\alpha^\vee}^{(r)} \mapsto \hbar^{\alpha^\vee(\mu_2)+r} F_{\alpha^\vee}^{(r+1)}.$$

Proof Part (a) is straightforward. Part (b) follows by comparing (2.21) and (2.25). □

Following Definition 2.30, we introduce:

Definition A.16 Let $\mathcal{Y}_\mu$ be the $\mathbb{C}[\hbar]$-subalgebra of $\mathcal{Y}_{\mu,\hbar}$ generated by

$$\{\hbar e_{\alpha^\vee}^{(r)}\}_{\alpha^\vee \in \Delta^+}^{r \geq 0} \cup \{\hbar f_{\alpha^\vee}^{(r)}\}_{\alpha^\vee \in \Delta^+}^{r \geq 0} \cup \{\hbar h_i^{(s_i)}\}_{i \in I}^{s_i \geq -b_i}.$$

The following is the main result of this subsection:

Theorem A.17 *For any coweight μ, there is a canonical $\mathbb{C}[\hbar]$-algebra isomorphism $\mathbf{Y}_\mu \simeq \mathcal{Y}_\mu$.*

Remark A.18 We note that Theorem A.17 does not imply Theorem A.12, since the algebra $\mathbf{Y}'_\mu$ could a priori have an $\hbar$-torsion.

As $\mathbf{Y}_\mu = \mathrm{Rees}^{F^\bullet_{\mu_1,\mu_2}}(Y_\mu)$, by the very definition of the Rees algebra we have a natural embedding $\mathbf{Y}_\mu \subset Y_\mu[\hbar, \hbar^{-1}]$. Applying Lemma A.15 with the same decomposition $\mu = \mu_1 + \mu_2$, we also obtain an embedding $\mathcal{Y}_\mu \subset \mathcal{Y}_{\mu,\hbar} \xrightarrow{\sim} Y_\mu[\hbar, \hbar^{-1}]$. Therefore, Theorem A.17 follows from:

Lemma A.19 *The images of $\mathbf{Y}_\mu$ and $\mathcal{Y}_\mu$ in $Y_\mu[\hbar, \hbar^{-1}]$ are equal.*

Proof The filtration $F^{\bullet}_{\mu_1,\mu_2} Y_\mu$ is defined by the degrees of PBW monomials as in (Finkelberg et al. 2018, (5.1)) (cf. (2.23) in type A). In particular, $\mathbf{Y}_\mu$ is the $\mathbb{C}[\hbar]$-subalgebra of $Y_\mu[\hbar, \hbar^{-1}]$ generated by the elements

$$\hbar^{\alpha^\vee(\mu_1)+r} E^{(r)}_{\alpha^\vee}, \ \hbar^{\alpha_i^\vee(\mu)+r} H_i^{(r)}, \ \hbar^{\alpha^\vee(\mu_2)+r} F^{(r)}_{\alpha^\vee}.$$

Note that these are precisely the images of the generators $\hbar e^{(r-1)}_{\alpha^\vee}$, $\hbar h_i^{(r-1)}$, $\hbar f^{(r-1)}_{\alpha^\vee}$ of $\mathcal{Y}_\mu$ under the isomorphism of Lemma A.15. The claim follows. □

Remark A.20 We note that Lemma A.19 provides another proof of the fact that the Rees algebras $\mathrm{Rees}^{F^{\bullet}_{\mu_1,\mu_2}}(Y_\mu)$ are canonically isomorphic for any choice of a splitting $\mu = \mu_1 + \mu_2$.

Appendix B: A Short Proof of the PBW Theorem for the Yangians

The PBW theorem for the Yangians is well-known and was first proven by Levendorskii in Levendorskii (1993). However, we feel that the proof of Levendorskii (1993) contains a gap: in Levendorskii (1993, p. 40) it is stated that certain exponents $m^{\pm}(i, j), m^0(r, j)$ are independent of j without any hint (actually, this seems to be wrong), and this fact plays a crucial role in the proof. For this reason, we present here a short proof of the PBW theorem for the Yangians, which is inspired by Levendorskii's, but which avoids the aforementioned gap.[8]

B.1 Useful Lemma

Let $A = \bigoplus_{k\in\mathbb{Z}} A_k$ be a graded algebra over $\mathbb{C}[\hbar]$, with $1 \in A_0$ and $\hbar \in A_1$. Consider its two specializations $A_{\hbar=0} = A/\hbar A$ and $A_{\hbar=1} = A/(\hbar - 1)A$. The former is naturally graded via $A_{\hbar=0} = \bigoplus_{k\in\mathbb{Z}} A_k/\hbar A_{k-1}$, while the latter inherits a natural filtration $F_\bullet A_{\hbar=1}$ with $F_k A_{\hbar=1}$ denoting the image of $\bigoplus_{\ell\le k} A_\ell \subset A$, giving rise to a graded algebra $\mathrm{gr}\, A_{\hbar=1} = \mathrm{gr}^{F_\bullet} A_{\hbar=1}$.

An explicit relation between the resulting graded $\mathbb{C}$-algebras $A_{\hbar=0}$ and $\mathrm{gr}\, A_{\hbar=1}$ is presented in the following result:

Lemma B.1 *(a) There is a canonical epimorphism of graded $\mathbb{C}$-algebras $\vartheta : A_{\hbar=0} \twoheadrightarrow \mathrm{gr}\, A_{\hbar=1}$.*
(b) The kernel of ϑ is the image of the $\hbar$–torsion[9] of A in $A_{\hbar=0}$.

Proof The proof is straightforward. □

[8] A similar proof appears in Wendlandt (2018), while a completely different proof of the PBW theorem for the Yangian defined in its J-realization was recently presented in Guay et al. (2019, Proposition 2.2).

[9] Explicitly, the $\hbar$–torsion of A is given by $T_\hbar(A) = \{a \in A : \hbar^r a = 0 \text{ for some } r \ge 0\}$.

B.2 Setup

We follow Sect. A.4 for the conventions regarding the Yangian $Y_\hbar$. However, throughout this section we will work with its specialization $Y_{\hbar=1}$, which we denote simply by Y. Below, we prove the PBW theorem for Y over $\mathbb{C}$. We then give a simple argument extending the PBW theorem to the one for $Y_\hbar$ over $\mathbb{C}[\hbar]$. By abuse of notation, we denote the images of the Yangian's PBW generators in Y by $e^{(r)}_{\alpha^\vee}, h^{(r)}_i, f^{(r)}_{\alpha^\vee}$.

Let us recall a few basic facts about Y. First of all, there is a natural linear map $\mathfrak{g} \to Y$, defined on the Chevalley generators by $e_i \mapsto e^{(0)}_i, h_i \mapsto h^{(0)}_i, f_i \mapsto f^{(0)}_i$, cf. Lemma 2.12(a). This map is injective. Indeed, according to Drinfeld (1985, Theorem 8),[10] the faithful action of $\mathfrak{g}$ on $\mathfrak{g} \oplus \mathbb{C}$ (the direct sum of the adjoint representation and the trivial one-dimensional) can be extended to an action of Y, hence, any element in the kernel of the above map $\mathfrak{g} \to Y$ is zero.

Second, the grading on $Y_\hbar$ of Sect. A.4 gives rise to a filtration $F_\bullet Y$ as in Sect. B.1. In particular, every PBW generator $x^{(r)}$ belongs to $F_r Y$. We note that the coproduct $\Delta : Y \to Y \otimes Y$ satisfies

$$\text{Total filtered degree}\Big(\Delta(x^{(r)}) - x^{(r)} \otimes 1 - 1 \otimes x^{(r)}\Big) < r \tag{B.1}$$

for any PBW generator $x^{(r)}$, which follows from (A.12). Note that the aforementioned embedding $\mathfrak{g} \hookrightarrow Y$ yields a surjection $U(\mathfrak{g}) \twoheadrightarrow F_0 Y$. Moreover, combining the isomorphism (A.14) with Lemma B.1, we obtain a graded algebra epimorphism $\vartheta : U(\mathfrak{g}[t]) \twoheadrightarrow \bigoplus_{k\geq 0} F_k Y / F_{k-1} Y$, in particular, we get a surjective linear map from the degree k part of $U(\mathfrak{g}[t])$ to $F_k Y / F_{k-1} Y$.

Finally, we recall that there is a *translation homomorphism* $\tau_a : Y \to Y[a]$ (here a is a formal parameter) defined on the PBW generators by

$$\tau_a(x^{(r)}) = \sum_{s=0}^{r} \binom{r}{s} a^{r-s} x^{(s)} \tag{B.2}$$

for any PBW generator $x = e_{\alpha^\vee}, h_i, f_{\alpha^\vee}$ (note that this formula is valid for $e_{\alpha^\vee}, f_{\alpha^\vee}$ with $\alpha^\vee$ a non-simple root because of our choices (A.11)). In particular, it follows that the filtered degree of any PBW monomial $y \in Y$ is precisely the degree in a of $\tau_a(y)$.

Define a homomorphism $\Delta_n : Y \to Y[a_1] \otimes \cdots \otimes Y[a_n]$ as the composition

$$Y \xrightarrow{\Delta^n} Y^{\otimes n} \xrightarrow{\tau_{a_1} \otimes \cdots \otimes \tau_{a_n}} Y[a_1] \otimes \cdots \otimes Y[a_n]. \tag{B.3}$$

Here Δ^n is the n-th iterated coproduct as in Sect. A.1, and $\tau_{a_i} : Y \to Y[a_i]$ is the translation homomorphism. In particular, it follows from the above discussion that for any PBW generator $x^{(r)}$, we have

[10] The proof of this result is presented in Chari and Pressley (1991, Section 6).

$$\Delta_n(x^{(r)}) = a_1^r x \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes a_2^r x \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes a_n^r x \quad \text{(B.4)}$$

modulo terms of total degree $< r$ in $a_1, \ldots, a_n$.

B.3 The PBW Theorem for the Yangians

In this subsection, we prove the PBW theorems for Y and $Y_\hbar$.

Theorem B.2 *The PBW monomials in the generators $e_{\alpha^\vee}^{(r)}, h_i^{(r)}, f_{\alpha^\vee}^{(r)}$ form a $\mathbb{C}$-basis of Y.*

Proof First, we claim that the PBW monomials span Y. The proof is by induction in the filtered degree. For degree 0, we recall that there is an algebra epimorphism $U(\mathfrak{g}) \twoheadrightarrow F_0 Y$, so the usual PBW theorem for $U(\mathfrak{g})$ applies; in particular, the PBW monomials in $e_{\alpha^\vee}^{(0)}, h_i^{(0)}, f_{\alpha^\vee}^{(0)}$ span $F_0 Y$. For any $k > 0$, recall that the degree k part of $U(\mathfrak{g}[t])$ surjects onto $F_k Y / F_{k-1} Y$. Combining this with the PBW theorem for $U(\mathfrak{g}[t])$, we see that $F_k Y$ is spanned by the PBW monomials modulo terms of the lower filtered degree. By induction, the claim follows.

Next, suppose that we could find a relation R between some PBW monomials. Consider the set of the PBW monomials of the maximal filtered degree d that appear non-trivially in this relation. Since this is a finite set, we may find a list of PBW generators $x_1^{(\mathsf{d}_1)} \leq \ldots \leq x_n^{(\mathsf{d}_n)}$ (possibly with multiplicities) such that each of these maximal degree monomials has the form

$$(x_1^{(\mathsf{d}_1)})^{\varepsilon_1} \cdots (x_n^{(\mathsf{d}_n)})^{\varepsilon_n}, \quad \text{(B.5)}$$

with all $\varepsilon_i \in \{0, 1\}$ and $\sum_i \varepsilon_i \mathsf{d}_i = \mathsf{d}$. When multiplicities do occur, we take the convention that the $\varepsilon_i = 1$ appear to the left of $\varepsilon_i = 0$. With this convention, each tuple $(\varepsilon_1, \ldots, \varepsilon_n)$ corresponds uniquely to a PBW monomial.

By (B.4), we find that $\Delta_n(R)$ is a sum of expressions of the form

$$\left(\sum_{i=1}^n 1^{\otimes(i-1)} \otimes a_i^{\mathsf{d}_1} x_1^{(0)} \otimes 1^{\otimes(n-i)}\right)^{\varepsilon_1} \cdots \left(\sum_{i=1}^n 1^{\otimes(i-1)} \otimes a_i^{\mathsf{d}_n} x_n^{(0)} \otimes 1^{\otimes(n-i)}\right)^{\varepsilon_n}, \quad \text{(B.6)}$$

modulo terms of total degree $< \mathsf{d}$ in $a_1, \ldots, a_n$. In particular, in the expression (B.6) there is a summand

$$(a_1^{\mathsf{d}_1} x_1^{(0)})^{\varepsilon_1} \otimes (a_2^{\mathsf{d}_2} x_2^{(0)})^{\varepsilon_2} \otimes \cdots \otimes (a_n^{\mathsf{d}_n} x_n^{(0)})^{\varepsilon_n} \quad \text{(B.7)}$$

which appears with coefficient 1. Moreover, there is a unique PBW monomial for which (B.7) appears as a summand.

Since $x_r^{(0)}$ are in the image of the embedding $\mathfrak{g} \hookrightarrow Y$, the elements (B.7) are linearly independent in $Y[a_1] \otimes \cdots \otimes Y[a_n]$. Thus the expressions (B.6) are also linearly independent. This implies that the top total degree term in $\Delta_n(R)$ must be zero, a contradiction.

Hence no linear relations exist, proving the PBW theorem for Y. □

This PBW theorem can be easily generalized to $Y_\hbar$ over $\mathbb{C}[\hbar]$:

Theorem B.3 *$Y_\hbar$ is free over $\mathbb{C}[\hbar]$, with a basis of the PBW monomials in the generators $e_{\alpha^\vee}^{(r)}, h_i^{(r)}, f_{\alpha^\vee}^{(r)}$.*

Proof Similarly to the proof of Theorem B.2, we see that the PBW monomials span $Y_\hbar$ over $\mathbb{C}[\hbar]$. Moreover, if we specialize $\hbar$ to any complex number, the images of the PBW monomials form a basis. Indeed, the previous theorem proves this for $\hbar = 1$ (and thus for any $\hbar \neq 0$, since all such specializations are isomorphic), while the case $\hbar = 0$ follows from (A.14) and the PBW theorem for $U(\mathfrak{g}[t])$.

Suppose that there is some linear relation among the PBW monomials. Its coefficients are elements of $\mathbb{C}[\hbar]$. But they must vanish wherever $\hbar$ is specialized in $\mathbb{C}$, since the PBW monomials become a basis. Therefore, all the coefficients are zero. So there are no relations, and the theorem is proved. □

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