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RESEARCH CONTRIBUTION

Counting Borel Orbits in Symmetric Spaces of Types *BI* and *CII*

Mahir Bilen Can[1] · **Özlem Uğurlu**[1]

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Abstract
This is a continuation of our combinatorial program on the enumeration of Borel orbits in symmetric spaces of classical types. Here, we determine the generating series the numbers of Borel orbits in $\mathbf{SO}_{2n+1}/\mathbf{S}(\mathbf{O}_{2p} \times \mathbf{O}_{2q+1})$ (type BI) and in $\mathbf{Sp}_n/\mathbf{Sp}_p \times \mathbf{Sp}_q$ (type CII). In addition, we explore relations to lattice path enumeration.

Keywords Borel orbits · Clans · Lattice paths · ODE's with irregular singular points

Mathematics Subject Classification 05A15 · 14M15

1 Introduction

The purpose of our paper is to continue the program that is initiated in our previous paper (Can and Uğurlu 2018), which is about finding generating functions and their combinatorial interpretations for certain families of involutions, called clans, in Weyl groups. There is an important motivation for undertaking such a task and it comes from a desire to better understand the cohomology rings of homogeneous spaces of the form G/K, where K is the fixed subgroup of an involutory automorphism of a complex reductive group G.

The study of symmetric spaces forms an integral part of geometry and many interesting manifolds are (locally) diffeomorphic to symmetric spaces. For example, $n-1$ dimensional sphere in $\mathbb{R}^n$ can be recognized as $G(\mathbb{R})/K(\mathbb{R})$, where G is $\mathbf{SO}_n$, the special orthogonal group of linear transformations with determinant 1, and K is its

✉ Mahir Bilen Can
mahirbilencan@gmail.com

Özlem Uğurlu
ougurlu@tulane.edu

1 Tulane University, New Orleans, USA

subgroup $\mathbf{S}(\mathbf{O_{n-1}} \times \mathbf{O_1})$ consisting of block matrices of the form $\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}$ where A is an orthogonal matrix of order $n-1$.

Among the important properties of symmetric spaces are the following:

(i) K is reductive, hence G/K is affine as an algebraic variety.
(ii) G/K has finitely many orbits under the left translation action of a Borel subgroup of G, see Brion (1986).

Note that the B-orbits in G/K are in 1–1 correspondence with the K-orbits in G/B and the topology of the latter (flag) variety is completely determined by the inclusion order on the set of Borel orbit closures.

By a *classical symmetric space* we simply mean a symmetric space G/K with G a classical group. In this manuscript, somewhat loosely following our previous work, where we studied the combinatorics of the generating functions regarding the number of B-orbits in type **AIII**, we will give a count of the B-orbits for the cases of **BI** and **CII**. These two types correspond to the decompositions of the vector spaces $\mathbb{C}^{2n+1}$ and $\mathbb{C}^{2n}$, respectively, into two orthogonally complementary subspaces with respect to a symmetric and a skew-symmetric bilinear form, see Howe (1995).

We know from (Wyser 2012) that, for a classical symmetric space G/K, the combinatorial objects parameterizing K-orbits in G/B have a rather concrete description; they are called "clans" with suitable adjectives. This nomenclature has first appeared in a paper of Matsuki and Oshima (1990). Yamamoto (1997) used these objects to determine the image of the moment map of the conormal bundle of K-orbits in G/B. (She worked with types **AIII** and **CII** only.) As far as we are aware of, after Yamamoto's work on clans, there was a long pause on the study of these combinatorial objects until McGovern's work in McGovern (2009) and Wyser's 2012 thesis (Wyser 2012), where Wyser clarified many obscurities around the definition of clans. In Wyser (2012), he used them to study degeneracy loci and in Wyser (2016) he used them to study the weak and strong Bruhat orders on K-orbit closures in G/B (in type **AIII**). More recent work on the combinatorics of type **AIII** clans can be found in Can et al. (2016). Finally, let us mention that in Woo et al. (2018) by studying the pattern avoidance properties of type **AIII** clans Woo et al. have characterized the singularities of the closures of $\mathbf{GL}_p \times \mathbf{GL}_q$-orbits in the flag variety.

We will refer to the clans corresponding to the Borel orbits in $\mathbf{Sp}_n/\mathbf{Sp}_p \times \mathbf{Sp}_q$ as ssymmetric $(2p, 2q)$ clans and we will call the clans corresponding to the Borel orbits of $\mathbf{SO}_{2n+1}/\mathbf{S}(\mathbf{O}_{2p} \times \mathbf{O}_{2q+1})$ the symmetric $(2p, 2q+1)$ clans. However, we should mention that these names are local to our paper. The definitions of symmetric and ssymmetric clans are rather lengthy, so, we postpone their precise definitions to the preliminaries section and introduce the notation for their collections and the corresponding cardinalities only.

$$
\begin{aligned}
BI(p,q) &:= \{\text{symmetric } (2p, 2q+1) \text{ clans}\}, \qquad b_{p,q} := \#BI(p,q);\\
CII(p,q) &:= \{\text{ssymmetric } (2p, 2q) \text{ clans}\}, \qquad c_{p,q} := \#CII(p,q).
\end{aligned}
$$

We know that the clans are in bijection with the 'signed' involutions; see the preliminaries section for the detailed proof. A signed involution is an involution in S_n, for some n, such that each fixed point of the involution is labeled with a $+$ sign or with a $-$ sign. Assuming the existence of a particular such bijection, which we will present in the sequel, we proceed to denote by $\beta_{k,p,q}$ the number of symmetric $(2p, 2q+1)$ clans whose corresponding involution has exactly k 2-cycles as a permutation. In a similar way, we denote by $\gamma_{k,p,q}$ the number of ssymmetric (p, q) clans whose corresponding involution has k 2-cycles. Clearly,

$$b_{p,q} = \sum_k \beta_{k,p,q} \text{ and } c_{p,q} = \sum_k \gamma_{k,p,q}.$$

Our goal in this manuscript is to present various formulas and combinatorial interpretations for $\beta_{k,p,q}$'s, $\gamma_{k,p,q}$'s, and foremost, for $b_{p,q}$'s and $c_{p,q}$'s.

Convention 1.1 If p and q are two nonnegative integers such that $q \geq p$, then we assume that $\beta_{k,p,q} = 0$ for all $0 \leq k \leq 2q+1$.

Now, we are ready to describe our results in more detail. First of all, by analyzing the structure of symmetric clans we prove the following result:

Theorem 1.2 *Let p and q be two nonnegative integers such that $p > q$. Then for every nonnegative integer k with $k \leq 2q+1$, we have*

$$\beta_{k,p,q} = \begin{cases} \binom{n-2l}{p-l}\binom{n}{2l} a_{2l} & \text{if } k = 2l; \\ \binom{n-(2l+1)}{p-(l+1)}\binom{n}{2l+1} a_{2l+1} & \text{if } k = 2l+1, \end{cases} \tag{1.1}$$

where

$$a_{2l} := \sum_{b=0}^{l} \binom{2l}{2b} \frac{(2b)!}{b!} \qquad \text{and} \qquad a_{2l+1} := \sum_{b=0}^{l} \binom{2l+1}{2b} \frac{(2b)!}{b!}. \tag{1.2}$$

In particular, we have

$$b_{p,q} = \sum_{l=0}^{q} \left(\binom{n-2l}{p-l}\binom{n}{2l} a_{2l} + \binom{n-(2l+1)}{p-(l+1)}\binom{n}{2l+1} a_{2l+1} \right).$$

The following formulas for the number of Borel orbits in $\mathbf{SO}_{2n+1}/\mathbf{S}(\mathbf{O}_{2p} \times \mathbf{O}_{2q+1})$ for $q = 0, 1, 2$ is now a simple consequence of our Theorem 1.2;

$$\begin{aligned} b_{p,0} &= p+1, \\ b_{p,1} &= (p+1)a_0 + p(p+1)a_1 + \frac{p(p+1)}{2}a_2 + \frac{p(p+1)(p-1)}{6}a_3 \\ &= \frac{7p^3 + 15p^2 + 14p + 6}{6}, \end{aligned}$$

$$\begin{aligned} b_{p,2} &= \binom{p+2}{2}\left(\frac{81p^3+22p^2+137p+60}{60}\right) \\ &= \frac{81p^5+265p^4+365p^3+515p^2+454p+120}{120}. \end{aligned}$$

Theorem 1.2 tells us that, for every fixed q, the integer $b_{p,q}$ can be viewed as a specific value of a polynomial function of p. However, it is already apparent from the case of $q = 1$ that this polynomial may have non-integer coefficients. We conjecture that $q = 0$ is the only case where $p \mapsto b_{p,q}$ is a polynomial function with integral coefficients. We conjecture also that for every nonnegative integer q, as a polynomial in p, $b_{p,q}$ is unimodal.

Note that the numbers a_{2l} and a_{2l+1} in Theorem 1.2 ($l = 0, 1, \ldots, q$) are special values of certain hypergeometric functions. More precisely,

$$\begin{aligned} a_{2l} &= \left(\frac{-1}{4}\right)^{-l} U\left(-l, \frac{1}{2}, -\frac{1}{4}\right), \\ a_{2l+1} &= \left(\frac{-1}{4}\right)^{-l} U\left(-l, \frac{3}{2}, -\frac{1}{4}\right), \end{aligned}$$

where $U(a, b, z)$ is the confluent hypergeometric function of the second kind. Such functions form one of the two distinct families of hypergeometric functions which solves the Kummer's differential equation

$$zy'' + (c - z)y' - ay = 0 \tag{1.3}$$

for some constants a and c. Kummer's ODE has a regular singular point at the origin and it has an irregular singularity at infinity.

The expressions in (1.1) are too complicated for practical purposes, therefore we seek for better expressions in the forms of recurrences and generating functions for $\beta_{k,p,q}$'s. It turns out that between various $\beta_{k,p,q}$'s there are four "easy-to-derive" recurrence relations as in (3.3), and there are four "somewhat easy-to-derive" recurrence relations as in (3.8), (3.9), (3.14), and (3.15). (We are avoiding showing these recurrences on purpose since they, especially the latter four, are rather lengthy.) The first four relations do not mix k's and they are linear. The second four recurrences are 3-term nonlinear recurrence relations and they do not mix p, q's. Moreover, the relations (3.8) and (3.9) are interwoven in the sense that both of them use consecutive terms in k's. The relations (3.14) and (3.15) maintain the parity of k, however, their coefficients are more complicated than the previous two.

It is not futile to expect that the eight recurrence relations we talked about lead to a manageable generating function, hence to a new formulation of the numbers $\beta_{k,p,q}$. We pursued this approach by studying the generating polynomial

$$h_{p,q}(x) := \sum_{k\geq 0} \beta_{k,p,q} x^k \tag{1.4}$$

and we run into some surprising complications. After pushing our computations as much as possible by using the easier, interrelated relations (3.8) and (3.9), we arrived at a 4×4 system of linear ODE's with an irregular singular point at the origin:

$$x^3 X' = \begin{bmatrix} (2p+2q-1)x^2+2 & x & -4pqx & -(2q+1) \\ x & (2p+2q-1)x^2+2 & -2p & -(4pq+2p-2q-1)x \\ x^3 & 0 & 0 & 0 \\ 0 & x^3 & 0 & 0 \end{bmatrix} X, \tag{1.5}$$

where

$$X = \begin{bmatrix} u(x) \\ v(x) \\ A_e(x) \\ A_o(x) \end{bmatrix} \quad \text{with} \quad \begin{aligned} u(x) &:= A_e^{'}(x), \\ v(x) &:= A_o^{'}(x), \\ A_e(x) &:= \sum_{l=0}^{q} \beta_{2l,p,q} x^{2l}, \\ A_o(x) &:= \sum_{l=0}^{q} \beta_{2l+1,p,q} x^{2l+1}. \end{aligned}$$

Without worrying about convergence, we are able to formally solve this system of ODE's however our method does not yield a satisfactorily clean formula. Instead, it provides us with a sequence of computational steps which eventually can be used for finding good approximations to $\beta_{k,p,q}$'s for any p, q. In order for not to break the flow of our exposition we decided to postpone the explanation of the intricacies of (1.5) to the appendix. Let us mention in passing that this type of ODE's (that is linear ODE's with irregular singular points) gave impetus to the development of reduction theory for connections where the structure group is an algebraic group, see Babbitt and Varadarajan (1983). Now, in a sense we are working our way back to such ODE's by trying to find the refined numbers of Borel orbits in a symmetric space.

To break free from the difficulties caused by complicated interactions between $\beta_{k,p,q}$'s, we consider the following alternative to $h_{p,q}(x)$:

$$b_{p,q}(x) := \sum_{l=0}^{q} (\beta_{2l,p,q} x^{q-l} + \beta_{2l+1,p,q} x^{q-l}).$$

Clearly, $b_{p,q}(x)$ is a polynomial of degree q and similarly to $h_{p,q}(x)$ its evaluation at $x = 1$ gives $b_{p,q}$. Let $B_p(x, y)$ denote the following generating function, which is actually a polynomial due to our convention (1.1):

$$B_p(x, y) := \sum_{q \geq 0} b_{p,q}(x) y^q.$$

Now, by using the previously mentioned recurrences, we observe that

$$b_{p,q}(x)' = (p+q)b_{p,q-1}(x).$$

From here it is not difficult to write down the governing partial differential equation for $B_p(x, y)$;

$$\frac{\partial}{\partial x} B_p(x, y) - y^2 \frac{\partial}{\partial y} B_p(x, y) = y(1+p)B_p(x, y). \tag{1.6}$$

By solving (1.6) we record a generating polynomial identity.

Theorem 1.3 *If $f_p(z)$ denotes the polynomial that is obtained from $B_p(1, y)$ by the transformation $y \leftrightarrow z/(1-z)$, then we have*

$$f_p(z) = (1+z)^{p+1}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,p,q} + \beta_{2q+1,p,q})z^q\right), \tag{1.7}$$

where $\beta_{k,p,q}$'s are as in Theorem 1.2.

Next, we proceed to explain our results on the number of Borel orbits in $\mathbf{Sp}_n/\mathbf{Sp}_p \times \mathbf{Sp}_q$. Recall that the notation $\gamma_{k,p,q}$ stands for the number of ssymmetric $(2p, 2q)$ clans whose corresponding involutions have exactly k 2-cycles. By counting the number of possible choices for the 2-cycles and the fixed points in an involution corresponding to a ssymmetric $(2p, 2q)$ clan, we obtain the following symmetric expression:

$$\gamma_{k,p,q} = \frac{(q+p)!}{(q-k)!(p-k)!k!}. \tag{1.8}$$

Note that the formula in (1.8) is defined independently of the inequality $q < p$, therefore, $\gamma_{k,p,q}$'s are defined for all nonnegative integers p, q, and k with the convention that $\gamma_{0,0,0} = 1$. As we show in the sequel (Lemma 4.3) $\gamma_{k,p,q}$'s satisfy a 3-term recurrence,

$$\gamma_{k,p,q} = \gamma_{k,p-1,q} + \gamma_{k,p,q-1} + 2(q+p-1)\gamma_{k-1,p-1,q-1} \quad \text{and}$$
$$\gamma_{0,p,q} = \binom{p+q}{p}. \tag{1.9}$$

Consequently, we obtain our first result on the number of ssymmetric $(2p, 2q)$ clans.

Proposition 1.4 *If p and q are positive integers, then the number of ssymmetric $(2p, 2q)$ clans satisfies the following recurrence:*

$$c_{p,q} = c_{p-1,q} + c_{p,q-1} + 2(p+q-1)c_{p-1,q-1}. \tag{1.10}$$

At this point we start to notice some similarities between the combinatorics of ssymmetric clans and our work in Can and Uğurlu (2018), where we studied the generating functions and combinatorial interpretations of the numbers of Borel orbits in symmetric spaces of type **AIII**. Following the notation from the cited reference, let us denote by $\alpha_{p,q}$ the number of Borel orbits in $\mathbf{SL}_n/\mathbf{S}(\mathbf{GL}_p \times \mathbf{GL}_q)$, where $p+q = n$. Then the identities

$$\alpha_{p,q} = \alpha_{p-1,q} + \alpha_{p,q-1} + (p+q-1)\alpha_{p-1,q-1} \tag{1.11}$$

hold true for all $p, q \geq 1$. From these relations, we obtained many combinatorial results on $\alpha_{p,q}$'s in Can and Uğurlu (2018). Here, by exploiting similarities between the two recurrences (1.10) and (1.11), we are able to follow the same route and obtain analogues of all of the results of Can and Uğurlu (2018). To avoid too much repetition, we will focus only on the selected analogues of our results from the previous paper. First we have a result on the generating function for $c_{p,q}$'s.

Let $v(x, y)$ denote the bivariate generating function

$$v(x, y) = \sum_{p,q \geq 0} c_{p,q} \frac{(2x)^q y^p}{p!}. \tag{1.12}$$

As we show that in the sequel, $v(x, y)$ obeys a first order linear partial differential equation of the form

$$(-2x^2)\frac{\partial v(x, y)}{\partial x} + (1 - 2x - 4xy)\frac{\partial v(x, y)}{\partial y} = (1 + 2x)\, v(x, y) \tag{1.13}$$

with initial conditions

$$v(0, y) = e^y \ \text{ and } \ v(x, 0) = \frac{1}{1 - 2x}.$$

The solution of (1.13) gives us a remarkable expression for the generating function (1.12) in suitably transformed coordinates.

Theorem 1.5 *Let r and s be two algebraically independent variables that are related to x and y by the relations*

$$x(r, s) = \frac{r}{2rs + 1} \ \textit{ and } \ y(r, s) = \frac{3s + 4r^2s^3 - 6r^2s^2 + 6rs^2 - 6rs}{3(2rs + 1)^2}.$$

In this case, the generating function $v(x, y)$ of $c_{p,q}$'s in r, s-coordinates is given by

$$v(r, s) = \frac{e^s(2rs + 1)}{1 - 2r}.$$

Next, we explain the most combinatorial results of our paper. The (p, q)*th Delannoy number*, denoted by $D(p, q)$, is defined via the recurrence relation

$$D(p, q) = D(p-1, q) + D(p, q-1) + D(p-1, q-1) \tag{1.14}$$

with respect to the initial conditions $D(p, 0) = D(0, q) = D(0, 0) = 1$. It is due to the linear nature of (1.14) that the generating function for $D(p, q)$'s is relatively simple;

$$\sum_{\substack{p+q\geq 0 \\ p,q\in\mathbb{N}}} D(p, q)x^i y^j = \frac{1}{1-x-y-xy}.$$

One of the most appealing properties of the Delannoy numbers is that they give the count of lattice paths that move with unit steps $E := (1, 0)$, $N := (0, 1)$, and $D := (1, 1)$ in the plane. More precisely, $D(p, q)$ gives the number of lattice paths that starts at the origin $(0, 0) \in \mathbb{N}^2$ and ends at $(p, q) \in \mathbb{N}^2$ moving with E, N, and D steps only. We will refer to such paths as the Delannoy paths and denote the set of them by $\mathcal{D}(p, q)$. For example, if $(p, q) = (2, 2)$, then $D(2, 2) = 13$ (see Fig. 1).

Let L be a Delannoy path that ends at the lattice point $(p, q) \in \mathbb{N}$. We agree to represent L as a word $L_1 L_2 \dots L_r$, where each L_i $(i = 1, \dots, r)$ is a pair of lattice points, say $L_i = ((a, b), (c, d))$, and $(c-a, d-b) \in \{N, E, D\}$. In this notation, we define the weight of the ith step as

$$weight(L_i) = \begin{cases} 1 & \text{if } L_i = ((a, b), (a+1, b)); \\ 1 & \text{if } L_i = ((a, b), (a, b+1)); \\ 2(a+b+1) & \text{if } L_i = ((a, b), (a+1, b+1)). \end{cases}$$

Finally, we define the weight of L, denoted by $\omega(L)$ as the product of the weights of its steps:

$$\omega(L) = weight(L_1)weight(L_2)\cdots weight(L_r). \tag{1.15}$$

Proposition 1.6 *Let p and q be two nonnegative integers and let $\mathcal{D}(p, q)$ denote the corresponding set of Delannoy paths. In this case, we have*

$$c_{p,q} = \sum_{L\in\mathcal{D}(p,q)} \omega(L).$$

Although Proposition 1.6 expresses $c_{p,q}$ as a combinatorial summation it does not give a combinatorial set of objects whose cardinality is given by $c_{p,q}$. The last result our paper offers such an interpretation.

Definition 1.7 A k-diagonal step (in $\mathbb{N}^2$) is a diagonal step L of the form $L = ((a, b), (a+1, b+1))$, where $a, b \in \mathbb{N}$ and $k = a + b + 1$.

Next, we define the "weighted Delannoy paths."

Definition 1.8 By a labelled step we mean a pair (K, m), where $K \in \{N, E, D\}$ and m is a positive integer such that $m = 1$ if $K = N$ or $K = E$. A weighted (p, q) Delannoy path is a word of the form $W := K_1 \cdots K_r$, where K_i's $(i = 1, \dots, r)$ are labeled steps $K_i = (L_i, m_i)$ such that

- $L_1 \cdots L_r$ is a Delannoy path from $\mathcal{D}(p, q)$;
- if L_i $(1 \le i \le r)$ is a k-diagonal step, then $2 \le m_i \le 2k - 1$.

The set of all weighted (p, q) Delannoy paths is denoted by $\mathcal{D}^w(p, q)$.

Theorem 1.9 *There is a bijection between the set of weighted* (p, q) *Delannoy paths and the set of ssymmetric* $(2p, 2q)$ *clans. In particular, we have*

$$c_{p,q} = \sum_{W \in \mathcal{D}^w(p,q)} 1.$$

There is much more to be said about the lattice path interpretation of the number of Borel orbits in $\mathbf{Sp}_n/\mathbf{Sp}_p \times \mathbf{Sp}_q$ but we postpone them to a future paper. We finish our introduction by giving a brief outline of our paper. We divided our paper into two parts. Before starting the first part, in Sect. 2 we introduce the background material and notation that we use in the sequel. In particular, we review a bijection between clans and involutions and we introduce the symmetric $(2p, 2q + 1)$ clans as well as ssymmetric $(2p, 2q)$ clans. We start Part I by analyzing the numbers $\beta_{k,p,q}$. In Sect. 3.1, we prove our Theorem 1.2 and in the following Sect. 3.2 we derive aforementioned recurrences for $\beta_{k,p,q}$'s. We devoted Sect. 3.3 to the proof of Theorem 1.3. The Part II of our paper starts with an analysis of the numbers $\gamma_{k,p,q}$. In Sect. 4.1, we prove the formula of $\gamma_{k,p,q}$'s as given in (1.8). In the following Sect. 4.2, by developing the recurrences for these numbers we prove Proposition 1.4. Section 4.3 is devoted to the proof of Theorem 1.5. In particular, we point out in Remark 4.7 that it is possible to find a formula for the generating function of $c_{p,q}$'s at the expense of a very complicated expression. In the remaining of Part II, we investigate the combinatorial interpretations of $c_{p,q}$'s. In Sect. 5, we prove Proposition 1.6 and Theorem 1.9. Finally, in the appendix, which is Sect. 1, we present our methods for solving ODE (1.5).

2 Notation and Preliminaries

The notation $\mathbb{N}$ stands for the set of natural numbers, which includes 0. Let us treat $+$ and $-$ as two symbols rather than viewing them as arithmetic operations. Throughout our paper the notation $\mathbb{P}$ stands for the set $\{+, -\} \cup \mathbb{N}$. The elements of $\mathbb{P}$ are called symbols. When we want to make a distinction between the symbols $\pm$ and the elements of $\mathbb{N}$, we call the latter by numbers, following their usual trait.

Let n be a positive integer. The symmetric group of permutations on $[n] := \{1, \dots, n\}$ is denoted by S_n. If $\pi \in S_n$, then its *one-line* notation is the string $\pi_1\pi_2 \cdots \pi_n$, where $\pi_i = \pi(i)$ $(i = 1, \dots, n)$. We trust that our reader is familiar with the most basic terminology about permutations such as their cycle decomposition, cycle type, etc.. However, just in case, let us mention that the cycle decomposition $C_1 \cdots C_r$ of a permutation is called standard if the entries of C_i's are arranged in such a way that $c_1 < c_2 < \cdots < c_r$, where c_i is the smallest number that appears in C_i ($i = 1, \ldots, r$). Here, we followed the common assumption that each cycle has at least two entries. Since we need the data of fixed points of a permutation, we will append to the cycle decomposition $C_1 \cdots C_r$ the one-cycles in an increasing order without using parentheses as indicated in the following example.

Example 2.1 $(2, 6, 8)(4, 5, 7, 9)13$ is the standard cycle decomposition of the permutation π from S_9 whose one-line notation is given by $\pi = 163578924$.

Clearly, a permutation π is an involution, that is to say $\pi^2 = id$, if and only if every cycle of π is of length at most 2.

Definition 2.2 Let p and q be two positive integers and set

$$n := p + q.$$

Suppose that $p > q$. A (p, q) preclan, denoted by γ, is a string of symbols such that

1. there are $p - q$ more $+$'s than $-$'s;
2. if a number appears in γ, then it appears exactly twice.

In this case, we call n the order of γ. For example, 1221 is a (2, 2) preclan of order 4 and $+1 + + + -1$ is a (5, 2) preclan of order 7. We call two (p, q) preclans γ and γ' equivalent if the positions of the matching numbers are the same in both of them. For example, $\gamma := 1221$ and $\gamma' := 2112$ are in the same equivalence class of (2, 2) preclans since both of γ and γ' have matching numbers in the positions $(1, 4)$ and $(2, 3)$. Finally, we call an equivalence class of (p, q) preclans a (p, q) clan.

In most places in our paper, we will abuse the notation and represent the equivalence class of a preclan γ by γ also, however, it is sometimes useful not to do that. When we need to distinguish between the preclan and its equivalence class we will use γ and $[\gamma]$, respectively, for the preclan and its equivalence class. The order of a clan is defined in the obvious way as the order of any preclan that it contains.

Lemma 2.3 *There exists a surjective map from the set of clans of order n to the set of involutions in S_n, the symmetric group of permutations on $\{1, \ldots, n\}$.*

Proof Let $\gamma = c_1 \cdots c_n$ be a (pre)clan of order n. For each pair of identical numbers (c_i, c_j) with $i < j$ we have a transposition in S_n which is defined by the indices, that is $(i, j) \in S_n$. Clearly, if (c_i, c_j) and $(c_{i'}, c_{j'})$ are two pairs of identical numbers from γ, then $\{i, j\} \neq \{i', j'\}$. Now, we define the involution $\pi = \pi(\gamma)$ corresponding to γ as the product of all transpositions that come from γ. Accordingly, the $\pm$'s in γ correspond to the fixed points of the involution π.

Conversely, if π is an involution from S_n, then we have a (p, q) preclan $\gamma = \gamma(\pi)$ that is defined as follows. We start with an empty string $\gamma = c_1 \cdots c_n$ of length n. If $\pi_1 \cdots \pi_n$ is the one-line notation for π, then for each pair of numbers (i, j) such that

$1 \leq i < j \leq n$ and $\pi_i = j, \pi_j = i$, we put $c_i = c_j = i$. Also, if $i_1, \ldots, i_m$ is the increasing list of indices such that $\pi_{i_j} = i_j$ $(j = 1, \ldots, m)$, then starting from i_1, we place $+$'s until the difference between the number of c_{i_j}'s with a + and the number of empty places is $p - q$. At this point, we place a $-$ in each of the empty places. It is easy to check that γ is a (p, q) preclan of order n, hence the proof follows. □

Definition 2.4 Let p and q be two positive integers with $p > q$ and let $n := p + q$. A signed (p, q) involution is an involution π from S_n whose fixed points are labeled either by $+$ or by $-$ in such a way that the number of $+$'s is $p - q$ more than the number of $-$'s.

Lemma 2.5 *There is a bijection between the set of all (p, q) clans and the set of all signed (p, q) involutions.*

Proof Let φ denote the surjection that is constructed in the proof of Lemma 2.3. We modify φ as follows. Let $\gamma = c_1 \cdots c_n$ be a (p, q) clan and let $\pi = \varphi(\gamma)$ denote involution that is obtained from γ via φ. If an entry c_i of γ is a $\pm$, then we know that i is a fixed point of π. We label i with $\pm$. Repeating this procedure for each $\pm$ that appear in γ we obtain a signed (p, q) involution $\widetilde{\pi}$. Clearly $\widetilde{\pi}$ is uniquely determined by γ. Therefore, the map defined by $\widetilde{\varphi}(\gamma) = \widetilde{\pi}$ is a bijection. □

Let γ be a preclan of the form $\gamma = c_1 \cdots c_n$. The *reverse* of γ, denoted by $rev(\gamma)$, is the preclan

$$rev(\gamma) = c_n c_{n-1} \cdots c_1.$$

Now, we are ready to define the notion of a symmetric clan.

Definition 2.6 A (p, q) clan γ is called symmetric if $[\gamma] = [rev(\gamma)]$.

Example 2.7 The (4, 3) clan $\gamma = (12 + - + 12)$ is symmetric since the clan $(21 + - + 21)$ which is obtained from γ by reversing its symbols is equal γ as a clan. More explicitly, they are the same since both of them have the same matching numbers in the positions 1, 6 and 2, 7.

In our next example, we list all symmetric (4, 3) clans.

Example 2.8 $\{1\,2\,3 + 3\,2\,1, 1\,2 + - + 2\,1, 1\,2\,3 + 3\,1\,2, 3\,1\,2 + 1\,2\,3, 1\,2 + - + 1\,2, 1 + 2 - 2 + 1, 1\,3\,2 + 1\,3\,2, 3\,1\,1 + 2\,2\,3, 1 + 2 - 1 + 2, + 1\,2 - 2\,1 +, 1 + - + - + 1, 1\,3\,1 + 2\,3\,2, 1 - + + + - 1, 1 + 1 - 2 + 2, + 1\,2 - 1\,2 +, + 1 - + - 1 +, 1\,1\,3 + 3\,2\,2, - 1 + + + 1 -, + 1\,1 - 2\,2 +, 1\,1 + - + 2\,2, + - 1 + 1 - +, - + 1 + 1 + -, - + + - + + -, + - + - + - +, + + - - - + +\}$.

Definition 2.9 An ssymmetric $(2p, 2q)$ clan $\gamma = c_1 \cdots c_{2n}$ is a symmetric clan such that $c_i \neq c_{2n+1-i}$ whenever c_i is a number (that is c_i is not a sign). The cardinality of the set of all ssymmetric $(2p, 2q)$ clans is denoted by $c_{p,q}$.

Example 2.10 The set of all ssymmetric (4, 2)-clans is given by
$\{1\,2 + + 1\,2, 1 + 2\,1 + 2, 1 + 1\,2 + 2, + 1\,2\,1\,2 +, + 1\,1\,2\,2 +, 1\,1 + + 2\,2, - + + + + -, + - + + - +, + + - - + +\}$.

We finish our preliminaries section by a remark/definition.

Remark 2.11 Let π denote the signed (p, q) involution corresponding to a (p, q) clan γ under the bijection $\widetilde{\varphi}$ that is defined in the proof of Lemma 2.5. A matching pair of numbers in any preclan that represents γ corresponds to a 2-cycle of π. We will call γ a (p, q) clan with k pairs if π has exactly k 2-cycles.

3 Part I: Counting Symmetric Clans

3.1 Symmetric $(2p, 2q + 1)$-Clans with k Pairs

Let p and q be two positive integers such that $0 \leq q < p$. Although we will be dealing with $(2p, 2q + 1)$ clans, we denote $p + q$ by n. Accordingly the number of symmetric $(2p, 2q + 1)$ clans is denoted by $b_{p,q}$.

By the proof of Lemma 2.5 we know that there is a bijection, denoted by $\widetilde{\varphi}$, between the set of all $(2p, 2q + 1)$ clans of order $2n + 1$ and the set of all signed $(2p, 2q + 1)$ involutions in S_{2n+1}. We have a number of simple observations regarding this bijection.

First of all, we observe that if π is a signed $(2p, 2q + 1)$ involution such that $\widetilde{\varphi}(\gamma) = \pi$, where γ is a symmetric $(2p, 2q + 1)$ clan, then the following holds true:

- if (i, j) with $1 \leq i < j \leq 2n + 1$ is a 2-cycle of π, then $n + 1 \notin \{i, j\}$ and $(2n + 2 - j, 2n + 2 - i)$ is a 2-cycle of π also.

Secondly, we see from its construction that $\widetilde{\varphi}$ maps a $(2p, 2q + 1)$ clan with k pairs to a signed $(2p, 2q + 1)$ involution with k 2-cycles. Let us denote the set of all such involutions by $I^{\text{ort}}_{k,p,q}$ and we define $\beta_{k,p,q}$ as the cardinality

$$\beta_{k,p,q} := |I^{\text{ort}}_{k,p,q}|.$$

Remark 3.1 If $\pi \in I^{\text{ort}}_{k,p,q}$, then in the corresponding clan there are $2p - 2q - 1$ more $+$'s than $-$'s. Notice that the inequality $2p - 2q - 1 \leq 2p + 2q + 1 - 2k$ implies that $0 \leq k \leq 2q + 1$.

It follows from the note in Remark 3.1 and the fact that $\widetilde{\varphi}$ is a bijection, the number of symmetric $(2p, 2q + 1)$ clans is given by

$$b_{p,q} = \sum_{l=0}^{q} (\beta_{2l,p,q} + \beta_{2l+1,p,q}).$$

Our goal in this section is to record a formula for $b_{p,q}$ that depends only on p and q. To this end, first we determine the number of $\pm$'s in a symmetric $(2p, 2q + 1)$ clan.

Lemma 3.2 *If* $\gamma = c_1 \cdots c_{2n+1}$ *is a symmetric* $(2p, 2q+1)$ *clan, then either* $c_{n+1} = +$ *or* $c_{n+1} = -$.

Proof First, assume that γ has even number of pairs. Let k denote this number, $k = 2l$. Let α, β, respectively, denote the number of $+$'s and $-$'s in γ. Then we have

$$\alpha + \beta = 2p + 2q + 1 - 4l \qquad \text{and} \qquad \alpha - \beta = 2p - 2q - 1.$$

It follows that

$$\alpha = 2p - 2l \qquad \text{and} \qquad \beta = 2q - 2l + 1,$$

so, in γ there are odd number of $-$'s and there are even number of $+$'s. As a consequence we see that c_{n+1} is a $-$.

Next, assume that γ has an odd number of pairs, that is $k = 2l + 1$. Arguing as in the previous case we see that there is an odd number of $+$'s, hence c_{n+1} is a +. This finishes the proof. □

We learn from the proof of Lemma 3.2 that it is important to analyze the parity of pairs, so we record the following corollary of the proof for a future reference.

Corollary 3.3 *Let k denote the number of pairs in a symmetric $(2p, 2q+1)$ clan γ. If $k = 2l (0 \le l \le q)$, then the number of $+$'s in γ is $2(p-l)$. If $k = 2l+1 (0 \le l \le q)$, then the number of $-$'s in γ is $2(q-l)$.*

Our next task is determining the number of possible ways of placing k pairs to build from scratch a symmetric $(2p, 2q+1)$ clan

$$\gamma = c_1 \cdots c_n c_{n+1} c_{n+2} \cdots c_{2n+1} \qquad (\text{with } c_{n+1} = \pm).$$

To this end we start with defining some interrelated sets.

$$\begin{aligned}
I_{1,1} &:= \{((i, j), (2n+2-j, 2n+2-i)) \mid 1 \le i < j \le n\}, \\
I_{1,2} &:= \{((i, j), (2n+2-j, 2n+2-i)) \mid 1 \le i < n+1 < j \le 2n+1\}, \\
I_1 &:= I_{1,1} \cup I_{1,2}, \\
I_2 &:= \{(i, j) \mid 1 \le i < n+1 < j \le 2n+1,\ i + j = 2n+2\}.
\end{aligned}$$

We view I_1 as the set of placeholders for two distinct pairs that determine each other in γ. The set I_2 corresponds to the list of stand alone pairs in γ. In other words, if $(i, j) \in I_2$, then $c_i = c_j$ and $j = 2n + 1 - i + 1$.

Example 3.4 Let us show what I_1 and I_2 correspond to with a concrete example. If γ is the symmetric $(8, 7)$ clan

$$\gamma = (7\ 2 +\ 0\ 8 +\ 9 -\ 8 +\ 9\ 0 +\ 7\ 2),$$

then $I_{1,1} = \{((1, 2), (14, 15))\}$, $I_{1,2} = \{((5, 9), (7, 11))\}$, $I_2 = \{(4, 12)\}$.

If (c_i, c_j) is a pair in the symmetric clan γ and if (i, j) is an element of I_2, then we call (c_i, c_j) a pair of type I_2. If x is a pair of pairs of the form $((c_i, c_j), (c_{2n+2-j}, c_{2n+2-i}))$ in a symmetric clan γ and if $((i, j), (2n+2-j, 2n+2-i)) \in I_{1,s}$ $(s \in \{1, 2\})$, then we call x a pair of pairs of type $I_{1,s}$. If there is no need for precision, then we will call x a pair of pairs of type I_1.

Clearly, if $|I_1| = b$ and $|I_2| = a$, then $2b + a = k$ is the total number of pairs in our symmetric clan γ. To see in how many different ways these pairs of indices can be situated in γ, we start with choosing k spots from the first n positions in $\gamma = c_1 \cdots c_{2n+1}$. Obviously this can be done in $\binom{n}{k}$ many different ways. Next, we count different ways of choosing b pairs within the k spots to place the b pairs of pairs of type I_1. This number of possibilities for this count is $\binom{k}{2b}$. Observe that choosing a pair from I_1 is equivalent to choosing (i, j) for the pairs of pairs in $I_{1,1}$ and choosing $(i, 2n+2-j)$ for the pairs of pairs in $I_{1,2}$. More explicitly, we first choose b pairs among the $2b$ elements and then place them on b spots; this can be done in $\binom{2b}{b}b!$ different ways. Once this is done, finally, the remaining spots will be filled by the a pairs of type I_2. This can be done in only one way. Therefore, in summary, the number of different ways of placing k pairs to build a symmetric $(2p, 2q+1)$ clan γ is given by

$$\binom{n}{k}\sum_{b=0}^{\lfloor \frac{k}{2} \rfloor}\binom{k}{2b}\binom{2b}{b}b!, \qquad \text{or equivalently,} \qquad \binom{n}{k}\sum_{b=0}^{\lfloor \frac{k}{2} \rfloor}\binom{k}{2b}\frac{(2b)!}{b!}.$$

In conclusion, we have the following preparatory result.

Theorem 3.5 (Theorem 1.2) *The number symmetric* $(2p, 2q+1)$ *clans with k pairs is given by*

$$\beta_{k,p,q} = \begin{cases} \binom{n-2l}{p-l}\binom{n}{2l}a_{2l} & \text{if } k = 2l; \\ \binom{n-(2l+1)}{p-(l+1)}\binom{n}{2l+1}a_{2l+1} & \text{if } k = 2l+1, \end{cases} \tag{3.1}$$

where

$$a_{2l} := \sum_{b=0}^{l}\binom{2l}{2b}\frac{(2b)!}{b!} \qquad \text{and} \qquad a_{2l+1} := \sum_{b=0}^{l}\binom{2l+1}{2b}\frac{(2b)!}{b!}. \tag{3.2}$$

Consequently, the total number of symmetric $(2p, 2q+1)$ *clans is given by*

$$b_{p,q} = \sum_{l=0}^{q}\left[\binom{n-2l}{p-l}\binom{n}{2l}a_{2l} + \binom{n-(2l+1)}{p-(l+1)}\binom{n}{2l+1}a_{2l+1}\right].$$

Proof As clear from the statement of our theorem, we will consider the two cases where k is even and where k is odd separately. We already computed the numbers of possibilities for placing k pairs, which are given by a_{2l} and a_{2l+1}, but we did not finish counting the number of possibilities for placing the signs.

1. $k = 2l$ for $0 \leq l \leq q$. In this case, by Lemma 3.2, we see that the number of $+$ signs is $\alpha := 2p - 2l = 2(p - l)$. Notice that because of symmetry condition it is enough to focus on the first n spots to place $\pm$ signs. Thus, there are $\binom{n-2l}{p-l}$ possibilities to place $\pm$ signs.
2. $k = 2l + 1$ for $0 \leq l \leq q$. In this case, it follows from Lemma 3.2 that the entry in the $(n + 1)$th place is $+$. By using an argument as before, we see that there are $\binom{n-(2l+1)}{p-(l+1)}$ possibilities to place $\pm$ signs.

This finishes the proof. □

The formula for $b_{p,q}$ that is derived in Theorem 1.2 is not optimal in the sense that it is hard to write down a closed form of its generating function this way. Of course, the complication is due to the form of $\beta_{k,p,q}$, where k is even or odd. Both of the cruces are resolved by considering the recurrences; we will present our results in the next subsection.

3.2 Recurrences for $\beta_{k,p,q}$'s

We start with some easy recurrences.

Lemma 3.6 *Let p and q be two positive integers, and l be a nonnegative integer. In this case, whenever both sides of the following equations are defined, they hold true:*

$$\beta_{2l,p-1,q} = \frac{p-l}{p+q}\beta_{2l,p,q}, \tag{3.3}$$

$$\beta_{2l,p,q-1} = \frac{q-l}{p+q}\beta_{2l,p,q}, \tag{3.4}$$

$$\beta_{2l+1,p-1,q} = \frac{p-l-1}{p+q}\beta_{2l+1,p,q}, \tag{3.5}$$

$$\beta_{2l+1,p,q-1} = \frac{q-l}{p+q}\beta_{2l+1,p,q}. \tag{3.6}$$

The proofs of the identities in Lemma 3.6 follow from obvious binomial identities and our formulas in Theorem 1.2. But note that l does not change in them. In the sequel, we will find other recurrences that run over l's. Towards this end, the following lemma, whose proof is simple, will be useful.

Lemma 3.7 *Let a_k denote the numbers as in (3.2). If $k \geq 2$, then we have*

$$a_k = a_{k-1} + 2(k-1)a_{k-2}. \tag{3.7}$$

By using (3.7) we find relations between $\beta_{k,p,q}$'s. Let k be an even number of the form $k = 2l$. Then we find that

$$\begin{aligned}
\beta_{2l,p,q} &= \binom{n-2l}{p-l}\binom{n}{2l}a_{2l} \\
&= \binom{n-2l}{p-l}\binom{n}{2l}(a_{2l-1}+2(2l-1)a_{2l-2}) \\
&= \binom{n-2l}{p-l}\binom{n}{2l}a_{2l-1}+2(2l-1)\binom{n-2l}{p-l}\binom{n}{2l}a_{2l-2} \\
&= \frac{n-2l+1-p+l}{n-2l+1}\binom{n-(2l-1)}{p-l}\frac{n+1-2l}{2l}\binom{n}{2l-1}a_{2l-1} \\
&\quad +2(2l-1)\frac{(p-l+1)(n-l+1-p)}{(n-2l+2)(n-2l+1)}\binom{n-(2l-2)}{p-l} \\
&\quad \times\frac{(n-2l+2)(n+1-2l)}{(2l)(2l-1)}\binom{n}{2l-2}a_{2l-2} \\
&= \frac{n-l-p+1}{2l}\beta_{2l-1,p,q}+2\frac{(p-l+1)(n-l+1-p)}{2l}\beta_{2l-2,p,q} \\
&= \frac{q-l+1}{2l}\beta_{2l-1,p,q}+2\frac{(p-l+1)(q-l+1)}{2l}\beta_{2l-2,p,q}. \qquad (3.8)
\end{aligned}$$

In a similar manner, for an odd number of the form $k = 2l+1$, we find that

$$\begin{aligned}
\beta_{2l+1,p,q} &= \binom{n-2l-1}{p-l-1}\binom{n}{2l+1}a_{2l+1} \\
&= \binom{n-2l-1}{p-l-1}\binom{n}{2l+1}(a_{2l}+2(2l)a_{2l-1}) \\
&= \binom{n-2l-1}{p-l-1}\binom{n}{2l+1}a_{2l}+2(2l)\binom{n-2l-1}{p-l-1}\binom{n}{2l+1}a_{2l-1} \\
&= \frac{p-l}{n-2l}\binom{n-2l}{p-l}\frac{n+1-(2l+1)}{2l+1}\binom{n}{2l}a_{2l} \\
&\quad +2(2l)\frac{(p-l)(p-l+1)}{(n-2l)(n-2l+1)}\binom{n-2l+1}{p-l+1} \\
&\quad \times\frac{(n-2l)(n+1-2l)}{(2l)(2l+1)}\binom{n}{2l-1}a_{2l-1} \\
&= \frac{p-l}{2l+1}\beta_{2l,p,q}+2\frac{(p-l)(q-l+1)}{2l+1}\beta_{2l-1,p,q}. \qquad (3.9)
\end{aligned}$$

Now, we have two recurrences (3.8) and (3.9) mixing the terms $\beta_{k,p,q}$ for even and odd k. To separate the parity, we rework on our initial recurrence (3.7).

Lemma 3.8 *For all* $1 \le l \le q-1$, *the following recurrences:*

$$a_{2l+2} = (8l+3)a_{2l} + 4(2l)(2l-1)a_{2l-2} \qquad (3.10)$$
$$a_{2l+3} = (8l+7)a_{2l+1} + 4(2l+1)(2l)a_{2l-1} \qquad (3.11)$$

with $a_0 = 1, a_1 = 1$ *are satisfied.*

Proof We will give a proof for the former equation here. The latter can be proved in a similar way.

We start with splitting (3.7) into two recurrences:

$$a_{2l+1} = a_{2l} + 2(2l)a_{2l-1} \tag{3.12}$$

$$a_{2l} = a_{2l-1} + 2(2l-1)a_{2l-2}. \tag{3.13}$$

On one hand it follows from Eq. (3.13) that we have

$$a_{2l-1} = a_{2l} - 2(2l-1)a_{2l-2}.$$

Plugging this into Eq. (3.12) yields

$$\begin{aligned} a_{2l+1} &= a_{2l} + 2(2l)(a_{2l} - 2(2l-1)a_{2l-2}) \text{ or} \\ a_{2l+1} &= (1 + 2(2l))a_{2l} - 4(2l)(2l-1)a_{2l-2}). \end{aligned}$$

On the other hand, we know that

$$a_{2l+2} = a_{2l+1} + 2(2l+1)a_{2l}.$$

If we plug this into the previous equation, then we obtain

$$\begin{aligned} a_{2l+2} &= (1 + 2(2l))a_{2l} - 4(2l)(2l-1)a_{2l-2} + 2(2l+1)a_{2l} \\ &= (8l+3)a_{2l} - 4(2l)(2l-1)a_{2l-2} \qquad (1 \le l \le q-1), \end{aligned}$$

which finishes the proof of our claim. □

Next, by the help of Lemma 3.8, we obtain a recurrence relation for $\beta_{k,p,q}$'s where all of k's are even numbers.

$$\begin{aligned} \beta_{2l+2,p,q} &= \binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l+2} \\ &= \binom{n-2l-2}{p-l-1}\binom{n}{2l+2}((8l+3)a_{2l} - 4(2l)(2l-1)a_{2l-2}) \\ &= (8l+3)\binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l} \\ &\quad +4(2l)(2l-1)\binom{n-2l-2}{p-l-1}\binom{n}{2l+2}a_{2l-2} \\ &= (8l+3)\frac{(p-l)(q-l)}{(n-2l)(n-2l-1)}\binom{n-2l}{p-l}\frac{(n-2l)(n-2l-1)}{(2l+2)(2l+1)}\binom{n}{2l}a_{2l} \\ &\quad -4(2l)(2l-1)\frac{(p-l)(p-l+1)(q-l)(q-l+1)}{(n-2l+2)(n-2l+1)(n-2l)(n-2l-1)} \\ &\quad \times\binom{n-2l}{p-l+1} \end{aligned}$$

$$\times\frac{(n-2l+2)(n+1-2l)(n-2l)(n-2l-1)}{(2l+2)(2l+1)(2l)(2l-1)}\binom{n}{2l-2}a_{2l-2}$$
$$=(8l+3)\frac{(p-l)(q-l)}{(2l+2)(2l+1)}\beta_{2l,p,q}$$
$$-4\frac{(p-l)(p-l+1)(q-l)(q-l+1)}{(2l+2)(2l+1)}\beta_{2l-2,p,q}. \tag{3.14}$$

The proof of the following recurrence follows from similar arguments.

$$\beta_{2l+3,p,q}=(8l+7)\frac{(q-l)(p-l-1)}{(2l+3)(2l+2)}\beta_{2l+1,p,q}$$
$$-4\frac{(p-l)(p-l-1)(q-l)(q-l+1)}{(2l+3)(2l+2)}\beta_{2l-1,p,q}. \tag{3.15}$$

3.3 The Proof of Theorem 1.3

As we mentioned in the introduction, we are looking for the closed form of the generating function

$$B(y,z)=\sum_{p\geq 0}B_p(1,y)z^p,$$

where

$$b_{p,q}(x)=\sum_{l=0}^{q}(\beta_{2l,p,q}x^{q-l}+\beta_{2l+1,p,q}x^{q-l})\quad\text{and}\quad B_p(x,y)=\sum_q b_{p,q}(x)y^q.$$

In particular, we are looking for an expression of $b_{p,q}(1)$ which is simpler than the one that is given in Theorem 1.2.

Obviously,

$$b_{p,q-1}(x)=\sum_{l=0}^{q-1}(\beta_{2l,p,q-1}x^{q-l-1}+\beta_{2l+1,p,q-1}x^{q-l-1}).$$

It follows from Lemma 3.6 that

$$b_{p,q}(x)=(\beta_{2q,p,q}+\beta_{2q+1,p,q})x^0+\sum_{l=0}^{q-1}(\beta_{2l,p,q}x^{q-l}+\beta_{2l+1,p,q}x^{q-l})$$
$$=(\beta_{2q,p,q+1}+\beta_{2q+1,p,q+1})$$
$$+\sum_{l=0}^{q-1}(p+q)\left(\frac{\beta_{2l,p,q-1}}{q-l}x^{q-l}+\frac{\beta_{2l+1,p,q-1}}{q-l}x^{q-l}\right).$$

Taking the derivative of both sides of the above equation gives us that

$$b'_{p,q}(x) = \sum_{l=0}^{q-1}(p+q)\left(\beta_{2l,p,q-1}x^{q-l-1} + \beta_{2l+1,p,q-1}x^{q-l-1}\right),$$

or, equivalently, gives that

$$b'_{p,q}(x) = (p+q)b_{p,q-1}(x). \tag{3.16}$$

The differential equation (3.16) leads to a PDE for our initial generating function $B_p(x, y)$:

$$\begin{aligned}
\frac{\partial}{\partial x}(B_p(x, y)) &= \frac{\partial}{\partial x}\left[\sum_{q\geq 0} b_{p,q}(x)y^q\right] = b'_{p,0}y^0 + \sum_{q\geq 1} b'_{p,q}(x)y^q \qquad (b'_{p,0} = 0)\\
&= \sum_{q\geq 1}(p+q)b_{p,q-1}(x)y^q = py\sum_{q\geq 1} b_{p,q-1}(x)y^{q-1}\\
&\quad + y\sum_{q\geq 1} qb_{p,q-1}(x)y^{q-1}\\
&= pyB_p(x, y) + y\left(\frac{\partial}{\partial y}(y\cdot B_p(x, y))\right)\\
&= y^2\frac{\partial}{\partial y}B_p(x, y) + yB_p(x, y) + pyB_p(x, y).
\end{aligned}$$

By the last equation we obtain the PDE that we mentioned in the introduction:

$$\frac{\partial}{\partial x}B_p(x, y) - y^2\frac{\partial}{\partial y}B_p(x, y) = y(1+p)B_p(x, y). \tag{3.17}$$

The general solution $S(x, y)$ of (3.17) is given by

$$S(x, y) = \frac{1}{y^{p+1}}G\left(\frac{1-xy}{y}\right), \tag{3.18}$$

where $G(z)$ is some function in one-variable. We want to choose $G(z)$ in such a way that $S(x, y) = B_p(x, y)$ holds true. To do so, first, we look at some special values of $B_p(x, y)$.

If let $x = 0$, then $B_p(0, y) = \sum_{q\geq 0} b_{p,q}(0)y^p$ and $b_{p,q}(0) = 2(\beta_{2q,q,p} + \beta_{2q+1,q,p})$ for all $q > 0$. Also, recall from the introduction that if $q = 0$, then $b_{p,q} = p + 1$. Thus, we ask from $G(z)$ that it satisfies the following equation

$$\frac{1}{y^{p+1}}G\left(\frac{1}{y}\right) = (p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})y^q,$$

or that

$$G\left(\frac{1}{y}\right) = y^{p+1}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})y^q\right). \tag{3.19}$$

Therefore, we see that our generating function is given by

$$\begin{aligned} B_p(x,y) &= \frac{1}{y^{p+1}} G\left(\frac{1}{y/(1-xy)}\right) \\ &= \frac{1}{y^{p+1}}\left(\frac{y}{1-xy}\right)^{p+1}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})\left(\frac{y}{1-xy}\right)^q\right) \\ &= \left(\frac{1}{1-xy}\right)^{p+1}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})\left(\frac{y}{1-xy}\right)^q\right). \end{aligned} \tag{3.20}$$

To get a more precise information about $b_{p,q}$'s we substitute $x = 1$ in (3.20):

$$B_p(1,y) = \frac{1}{(1-y)^{p+1}}\left((p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})\left(\frac{y}{1-y}\right)^q\right),$$

or

$$(1-y)^{p+1} B_p(1,y) = (p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})\left(\frac{y}{1-y}\right)^q. \tag{3.21}$$

Now, we apply the transformation $y \mapsto z = y/(1-y)$ in (3.21):

$$\frac{1}{(1+z)^{p+1}} B_p\left(1, \frac{z}{1+z}\right) = (p+1) + 2\sum_{q\geq 1}(\beta_{2q,q,p} + \beta_{2q+1,q,p})z^q. \tag{3.22}$$

This finishes the proof of Theorem 1.3 since $B_p\left(1, \frac{z}{1+z}\right) = f_p(z)$.

4 Part II: Counting Ssymmetric Clans

Convention 4.1 For this part of our paper, without loss of generality, we assume that p and q are nonnegative integers such that $p \geq q$.

4.1 Ssymmetric Clans with *k*-Pairs

Recall that a ssymmetric $(2p, 2q)$ clan $\gamma = c_1 \cdots c_{2n}$ is a symmetric clan such that $c_i \neq c_{2n+1-i}$ whenever c_i is a number. In this second part of our paper, we are going to find various generating functions and combinatorial interpretations for the number $c_{p,q}$ of ssymmetric $(2p, 2q)$ clans. We start by stating a simple lemma that tells about the involutions corresponding to ssymmetric clans.

Lemma 4.2 *Let $\gamma = c_1 c_2 \cdots c_{2n}$ be a ssymmetric $(2p, 2q)$ clan. If $\pi \in S_{2n}$ is the associated involution with γ, then there are even number of 2-cycles in π.*

Proof First, notice that if for some $1 \leq i < j \leq 2n$ the numbers c_i and c_j form a pair, that is to say a 2-cycle in π, then by symmetry c_{2n+1-i} and c_{2n+1-j} form a pair in π as well. In addition, by the condition that is requiring for all natural c_i's that $c_i \neq c_{2n+1-i}$, c_i and c_{2n+1-j} cannot form a pair in π. Therefore, if we have a pair (c_i, c_j) in π, then we must also have another pair (c_{2n+1-j}, c_{2n+1-i}) which is different from (c_i, c_j). Said differently, the number of 2-cycles in π must be even. □

In the light of Lemma 4.2, we will focus on the subset $I^{sp}_{k,p,q} \subset S_{2n}$ consisting of involutions π whose standard cycle decomposition is of the form

$$\pi = (i_1 j_1) \cdots (i_{2k} j_{2k}) d_1 \cdots d_{2n-4k}.$$

Furthermore, we assume that the fixed points of π are labeled by the elements of $\{+, -\}$ in such a way that there are $2p - 2q$ more $+$'s than $-$'s and we want the following conditions be satisfied:

1. $k \leq q$ (this is because there are $2p - 2q$ more $+$'s than $-$'s, hence $2q + 2p - 4k \geq 2p - 2q$);
2. if (i, j) is a 2-cycle such that $1 \leq i < j \leq n$, then $(2n + 1 - j, 2n + 1 - i)$ is a 2-cycle also;
3. if (i, j) is a 2-cycle such that $1 \leq i < n+1 \leq j \leq 2n$, then $(2n+1-j, 2n+1-i)$ is a 2-cycle as well.

The (signed) involutions in $I^{sp}_{k,p,q}$ are precisely the involutions that correspond to the ssymmetric $(2p, 2q)$ clans under the bijection of Lemma 2.5, so, $\gamma_{k,p,q}$ stands for the cardinality of $I^{sp}_{k,p,q}$. To find a formula for $\gamma_{k,p,q}$'s we argue similarly to the case of $\beta_{k,p,q}$, by counting the number of possible ways of placing pairs and by counting the number of possible ways of placing $\pm$'s on the fixed points. Also, we make use of the bijection $\widetilde{\varphi}$ of Lemma 2.5 to switch between the involution notation and the clan notation.

First of all, an involution π from $I^{sp}_{k,p,q}$ has $2k$ 2-cycles and $2n - 4k$ fixed points. The $2k$ 2-cycles, by using numbers from $\{1, \ldots, 2n\}$ can be chosen in $\binom{n}{2k}$; the number of rearrangements of these $2k$ pairs and their entries, to obtain the standard form of an involution, requires $\frac{(2k)!}{k!}$ steps. In other words, the 2-cycles of π are found and placed in the standard ordering in $\binom{n}{2k}\frac{(2k)!}{k!}$ possible ways. Once we have the 2-cycles of the involution, we easily see that the numbers and their positions in the corresponding ssymmetric clan are uniquely determined.

Next, we determine the number of ways to place ±'s. This amounts to finding the number of ways of placing 2α +'s and 2β -'s on the string $d_1 \cdots d_{2n-4k}$ so that there are exactly $2p - 2q = 2\alpha - 2\beta$ +'s more than −'s. By applying the inverse of the bijection $\widetilde{\varphi}$ of Lemma 2.5, we will use the symmetry condition on the corresponding clan. Thus, we observe that it is enough to focus on the first n places of the clan only. Now, the number of +'s in the first n places can be chosen in $\binom{n-2k}{\alpha}$ different ways. Once we place the +'s, the remaining entries will be filled with −'s. Clearly there is now only one way of doing this since we placed the numbers and the + signs already. Therefore, to finish our counting, we need to find what that α is. Since $\alpha + \beta = n - 2k = q + p - 2k$ and since $\alpha - \beta = p - q$, we see that $\alpha = p - k$.

In summary, the number of possible ways of constructing a signed involution corresponding to a ssymmetric $(2p, 2q)$ clan is given by

$$\gamma_{k,p,q} = \binom{q+p}{2k}\frac{(2k)!}{k!}\binom{q+p-2k}{p-k}. \tag{4.1}$$

Note here that we are using $n = p + q$. The right-hand side of (4.1) can be expressed more symmetrically as follows:

$$\gamma_{k,p,q} = \frac{(q+p)!}{(q-k)!(p-k)!k!}. \tag{4.2}$$

4.2 Recurrences for $\gamma_{k,p,q}$'s

Observe that the formula (4.7) is defined independently of the inequality $q \leq p$. From now on, for our combinatorial purposes, we skip mentioning this comparison between p and q and use the equality $\gamma_{k,p,q} = \gamma_{k,q,p}$ whenever it is needed. Also, we record the following obvious recurrences for a future reference:

$$\gamma_{k,p,q} = \frac{(p-k+1)(q-k+1)}{k}\gamma_{k-1,p,q}, \tag{4.3}$$

$$\gamma_{k,p-1,q} = \frac{p-k}{p+q}\gamma_{k,p,q}, \tag{4.4}$$

$$\gamma_{k,p,q-1} = \frac{q-k}{p+q}\gamma_{k,p,q}. \tag{4.5}$$

These recurrences hold true whenever both sides of the equations are defined. Notice that in (4.3)–(4.5) the parity, namely k does not change. Next, we will show that $\gamma_{k,p,q}$'s obey a 3-term recurrence once we allow change in all three numbers p, q, and k.

Lemma 4.3 *Let p and q be two positive integers. If $k \geq 1$, then we have*

$$\gamma_{k,p,q} = \gamma_{k,p-1,q} + \gamma_{k,p,q-1} + 2(q+p-1)\gamma_{k-1,p-1,q-1} \quad \textit{and}$$
$$\gamma_{0,p,q} = \binom{p+q}{p}. \tag{4.6}$$

Proof Instead of proving our result directly, we will make use of a similar result that we proved before. Let $\widetilde{\gamma}_{k,p,q}$ denote the number

$$\widetilde{\gamma}_{k,p,q} = \frac{(q+p)!}{2^k(q-k)!(p-k)!k!}. \tag{4.7}$$

In Can and Uğurlu (2018), it is proven that

$$\widetilde{\gamma}_{k,p,q} = \widetilde{\gamma}_{k,p-1,q} + \widetilde{\gamma}_{k,p,q-1} + (p+q-1)\widetilde{\gamma}_{k-1,p-1,q-1} \tag{4.8}$$

holds true for all $p, q, k \geq 1$. Note that $\widetilde{\gamma}_{0,p,q} = \binom{p+q}{p}$, which is our initial condition for $\gamma_{k,p,q}$'s. Therefore, combining (4.8) with the fact that $\gamma_{k,p,q} = 2^k\widetilde{\gamma}_{k,p,q}$ finishes our proof. □

Convention 4.4 From now on we will assume that $c_{p,q} = 1$ whenever one or both of p and q are zero.

Proposition 4.5 (Proposition 1.4) *For all positive integers p and q, the following recurrence relation holds true:*

$$c_{p,q} = c_{p-1,q} + c_{p,q-1} + 2(p+q-1)c_{p-1,q-1}. \tag{4.9}$$

Proof Recall that $c_{p,q} = \sum_k \gamma_{k,p,q}$. Thus, summing both sides of Eq. (4.6) over k with $1 \leq k \leq p-1$ gives

$$\begin{aligned}
&c_{p,q} - c_{p-1,q} - c_{p,q-1} - 2(p+q-1)c_{p-1,q-1} \\
&\quad = \gamma_{0,p,q} - \gamma_{0,p-1,q} - \gamma_{0,p,q-1} \\
&\qquad + \gamma_{p,p,q} - \gamma_{p,p,q-1} - 2(p+q-1)\gamma_{p-1,p-1,q-1} \\
&\quad = 0.
\end{aligned}$$

□

4.3 Proof of Theorem 1.5

One of the many options for a bivariate generating function for $c_{p,q}$'s is the following

$$v(x,y) := \sum_{p,q \geq 0} c_{p,q} \frac{(2x)^q y^p}{p!}. \tag{4.10}$$

Let us tabulate first few terms of $v(x, y)$:

$$\begin{aligned}
\sum_{p,q \geq 0} c_{p,q} \frac{(2x)^q y^p}{p!} &= c_{0,0} + c_{0,1}2x + \cdots + c_{0,q}(2x)^q + \cdots \\
&\quad + \frac{c_{1,0}}{1!}y + \cdots + \frac{c_{p,0}}{p!}y^p + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c_{1,1}}{1!}(2x)y+\cdots+\frac{c_{p,1}}{p!}(2x)y^p+\cdots\\
&+\frac{c_{1,2}}{1!}(2x)^2y+\cdots+\frac{c_{p,2}}{p!}(2x)^2y^p+\cdots \qquad (4.11)
\end{aligned}
$$

It follows from our Convention 4.4 and Eq. (4.11) that

$$
v(x,y)=\frac{1}{1-2x}+e^y-1+\sum_{p,q\geq 1}c_{p,q}\frac{(2x)^q y^p}{p!}. \qquad (4.12)
$$

We feed this observation into our recurrence (4.9) and use similar arguments for the right hand side of it:

$$
\begin{aligned}
v(x,y)-\frac{1}{1-2x}-e^y+1 &= \int \sum_{p\geq 1,q\geq 0}\frac{c_{p-1,q}}{(p-1)!}(2x)^q y^{p-1}dy - e^y\\
&\quad +2x\left(\sum_{p,q\geq 0}\frac{c_{p,q}}{p!}(2x)^q y^p-\frac{1}{1-2x}\right)\\
&\quad +2\sum_{p,q\geq 1}p\frac{c_{p-1,q-1}}{p!}(2x)^q y^p\\
&\quad +2\sum_{p,q\geq 1}q\frac{c_{p-1,q-1}}{p!}(2x)^q y^p\\
&\quad -2\sum_{p,q\geq 1}\frac{c_{p-1,q-1}}{p!}(2x)^q y^p\\
&= \int \sum_{p\geq 1,q\geq 0}\frac{c_{p-1,q}}{(p-1)!}(2x)^q y^{p-1}dy - e^y\\
&\quad +2x\left(\sum_{p,q\geq 0}\frac{c_{p,q}}{p!}(2x)^q y^p-\frac{1}{1-2x}\right)\\
&\quad +4xy\sum_{p,q\geq 0}\frac{c_{p-1,q-1}}{(p-1)!}(2x)^{q-1}y^{q-1}\\
&\quad +4x\int\sum_{p,q\geq 1}\frac{qc_{p-1,q-1}}{(p-1)!}(2x)^{q-1}y^{p-1}dy\\
&\quad -4x\int\sum_{p,q\geq 1}\frac{c_{p-1,q-1}}{(p-1)!}(2x)^{q-1}y^{p-1}dy.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&v(x,y)-\frac{1}{1-2x}-e^y+1\\
&\quad=\int v(x,y)dy-e^y+2x\,v(x,y)-\frac{2x}{1-2x}+4xy\,v(x,y)
\end{aligned}
$$

$$+ x \int \left(\frac{\partial}{\partial x} (2x\ v(x, y)) \right) dy - 4x \int v(x, y) dy,$$

or equivalently,

$$(1 - 2x - 4xy) v(x, y) = (1 - 4x) \int v(x, y) dy + x \int \left(\frac{\partial}{\partial x} (2x v(x, y)) \right) dy.$$

Now, differentiating with respect to y gives us a PDE:

$$\begin{aligned} &- 4x\ v(x, y) + (1 - 2x - 4xy) \frac{\partial v(x, y)}{\partial y} \\ &= (1 - 4x)\ v(x, y) + x \left(2v(x, y) + 2x \frac{\partial v(x, y)}{\partial x} \right), \end{aligned}$$

which we re-organize as in

$$(-2x^2) \frac{\partial v(x, y)}{\partial x} + (1 - 2x - 4xy) \frac{\partial v(x, y)}{\partial y} = (1 + 2x)\ v(x, y). \tag{4.13}$$

Here, we have the obvious initial conditions

$$v(0, y) = e^y \text{ and } v(x, 0) = \frac{1}{1 - 2x}.$$

Solutions of such PDE's are easily obtained by applying the method of "characteristic curves." Our characteristic curves are $x(r, s)$, $y(r, s)$, and $v(r, s)$. Their tangents are equal to

$$\frac{\partial x}{\partial s} = -2x^2, \qquad \frac{\partial y}{\partial s} = 1 - 2x - 4xy, \qquad \frac{\partial v}{\partial s} = (1 + 2x)v, \tag{4.14}$$

with the initial conditions

$$x(r, 0) = r, \quad y(r, 0) = 0, \quad \text{and} \quad v(r, 0) = \frac{1}{1 - 2r}.$$

From the first equation given in (4.14) and its initial condition underneath, we have

$$x(r, s) = \frac{r}{2rs + 1}. \tag{4.15}$$

Plugging this into the second equation gives us $\frac{\partial y}{\partial r} = 1 - \frac{2}{2rs+1}(1 + 2y)$, which is a first order linear ODE. The general solution for this ODE is

$$y(r, s) = \frac{3s + 4r^2s^3 - 6r^2s^2 + 6rs^2 - 6rs}{3(2rs + 1)^2}. \tag{4.16}$$

Finally, from the last equation in (4.14) together with its initial condition we conclude that

$$v(r,s) = \frac{e^s(2rs+1)}{1-2r}.$$

In summary, we outlined the proof of our next result.

Theorem 4.6 *Let $v(x,y)$ denote the power series $\sum_{p,q\geq 0} c_{p,q}(2x)^q \frac{y^p}{p!}$. If r and s are the variables related to x and y as in Eqs. (4.16) and (4.15), then we*

$$v(r,s) = \frac{e^s(2rs+1)}{1-2r}. \tag{4.17}$$

We finish this section with a remark.

Remark 4.7 Although we solved our PDE by using the useful method of characteristic curves, the answer is given as a function of transformed coordinates r and s. Actually, we can find the solution in x and y. Indeed, it is clear from the outset that the general solution $\tilde{S}(x,y)$ of (4.13) is given by

$$\tilde{S}(x,y) = \frac{e^{1/(2x)}F(\frac{6xy+3x-1}{6x^3})}{x}, \tag{4.18}$$

where $F(z)$ is some function in one-variable. (This can easily be verified by substituting $\tilde{S}(x,y)$ into the PDE.) Let us find a concrete expression for $F(z)$ here so that the initial condition $\tilde{S}(x,y) = v(x,y)$ holds true. To this end, we set $y=0$. In this case, we know that $v(x,0) = \frac{1}{1-2x}$. Therefore, $F(z)$ satisfies the following equation:

$$\frac{e^{1/(2x)}F(\frac{3x-1}{6x^3})}{x} = \frac{1}{1-2x} \quad \text{or} \quad F\left(\frac{3x-1}{6x^3}\right) = \frac{xe^{-1/(2x)}}{1-2x}. \tag{4.19}$$

The inverse of the transformation $z = \frac{3x-1}{6x^3}$ which appears in (4.19) is given by

$$x = \frac{1}{6^{1/3}(-3z^2+\sqrt{3}\sqrt{-2z^3+3z^4})^{1/3}} + \frac{(-3z^2+\sqrt{3}\sqrt{-2z^3+3z^4})^{1/3}}{6^{2/3}z}. \tag{4.20}$$

By back substitution of (4.20) into (4.19), we find an expression for $F(z)$, which in turn will be evaluated at $\frac{6xy+3x-1}{6x^3}$ [as in (4.18)]. Obviously, the resulting expression is very complicated, however, this way we can write the solution of our PDE in x and y only.

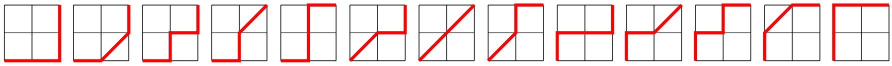

Fig. 1 Delannoy paths

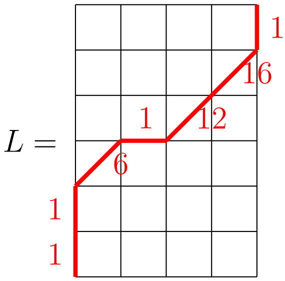

Fig. 2 A weighted Delannoy path

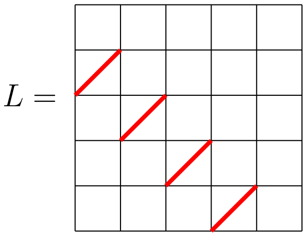

Fig. 3 All of the 4-diagonal steps in $\mathbb{N}^2$

5 A Combinatorial Interpretation

In this section, we will use Delannoy paths for our combinatorial interpretation of the number of ssymmetric clans. In Fig. 1 we listed the Delannoy paths that are contained in the 2×2-grid.

Recall our claim (Proposition 1.6) from Introduction that one can compute the numbers $c_{p,q}$ as a sum $\sum_{L \in \mathcal{D}(p,q)} \omega(L)$, where $\mathcal{D}(p, q)$ denotes the set of all Delannoy paths that ends at (p, q). Here $\omega(L)$ is the weight of the Delannoy path L, which is defined in (1.15).

Example 5.1 Let L denote the Delannoy path that is depicted in Fig. 2. In this case, the weight of L is $\omega(L) = 6 \cdot 12 \cdot 16 = 1152$.

Recall also that a weighted (p, q) Delannoy path is a word of the form $W := K_1 \cdots K_r$, where K_i's $(i = 1, \dots, r)$ are labeled steps $K_i = (L_i, m_i)$ such that

- $L_1 \cdots L_r$ is a Delannoy path from $\mathcal{D}(p, q)$;
- if L_i $(1 \le i \le r)$ is a k-diagonal step, then $2 \le m_i \le 2k - 1$. (We depicted all of the 4-diagonal steps in $\mathbb{N}^2$ in Fig. 3.)

Theorem 1.9 states that there is a bijection between the set of weighted (p, q) Delannoy paths and the set of ssymmetric $(2p, 2q)$ clans. Our goal in this section is to prove these statements.

Proof of Proposition 1.6 Let $c'_{p,q}$ denote the sum $\sum_{L \in \mathcal{D}(p,q)} \omega(L)$. As a convention we define $c'_{0,0} = 1$. Recall that n stands for $p + q$. We prove our claim $c'_{p,q} = c_{p,q}$ by induction on n. Obviously, if $n = 1$, then (p, q) is either $(0, 1)$ or $(1, 0)$, and in both of these cases, there is only one step which either N or E. Therefore, $c'_{p,q} = 1$ in this case. Now, let n be a positive integer and we assume that our claim is true for all (p, q) with $(p, q) = n$. We will prove that $c_{p,q} = c'_{p,q}$, whenever $p + q = n + 1$. To this end, we look at the possibilities for the ending step of a Delannoy path $L = L_1 \cdots L_r \in \mathcal{D}(p, q)$. If L_r is a diagonal step, then

$$\omega(L) = (2(p + q) - 1)\omega(L_1 \cdots L_{r-1}).$$

In particular, $L_1 \cdots L_{r-1} \in \mathcal{D}(p - 1, q - 1)$. If L_r is from $\{N, E\}$, then

$$\omega(L) = \omega(L_1 \cdots L_{r-1}).$$

In particular, $L_1 \cdots L_{r-1} \in \mathcal{D}(p - 1, q)$ or $L_1 \cdots L_{r-1} \in \mathcal{D}(p, q - 1)$, depending on $L_r = E$ or $L_r = N$. We conclude from these observations that

$$\begin{aligned} c'_{p,q} &= c'_{p-1,q} + c'_{p,q-1} + 2(p + q - 1)c'_{p-1,q-1} \\ &= c_{p-1,q} + c_{p,q-1} + 2(p + q - 1)c_{p-1,q-1} \qquad \text{(by induction hypothesis)} \\ &= c_{p,q}. \end{aligned}$$

This finishes the proof of our claim. □

The proof of Theorem 1.9 is based on the same idea however it requires more attention in some of the constructions that are involved.

Proof of Theorem 1.9 Let $d_{p,q}$ denote the cardinality of $\mathcal{D}^w(p, q)$. We will prove that $d_{p,q}$ obeys the same recurrence as $c_{p,q}$'s and it satisfies the same initial conditions.

Let $\gamma = c_1 \cdots c_{2n}$ be a ssymmetric $(2p, 2q)$ clan and let $\pi = \pi_\gamma$ denote the signed involution

$$\pi = (i_1, j_1) \cdots (i_{2k}, j_{2k}) l_1^{s_1} \cdots l_{2n-4k}^{s_{2n-4k}}, \quad \text{where } s_1, \ldots, s_{2n-4k} \in \{+, -\},$$

which is given by $\pi = \widetilde{\varphi}(\gamma)$. Here, $\widetilde{\varphi}$ is the map that is constructed in the proof of Lemma 2.5. We will construct a weighted (p, q) Delannoy path $W = W_\gamma$ which is uniquely determined by π.

First, we look at the position of $2n$ in π. If it appears as a fixed point with a + sign, then we draw an E step between (p, q) and $(p - 1, q)$. If it appears as a fixed point with a − sign, then we draw a an N-step between (p, q) and $(p, q - 1)$. We label both of these steps by 1 to turn them into labeled steps. Next, we remove the fixed points 1 and $2n$ from π and then subtract 1 from each remaining entry. The result is a either signed $(2(p - 1), 2q)$ involution or a signed $(2p, 2(q - 1))$ involution. Now, by our induction hypothesis, in the first case, there are $d_{p-1,q}$ possible ways of extending this path to a weighted (p, q) Delannoy path. In a similar manner, in the latter case there are $d_{p,q-1}$ possible ways of extending it to a weighted (p, q) Delannoy path.

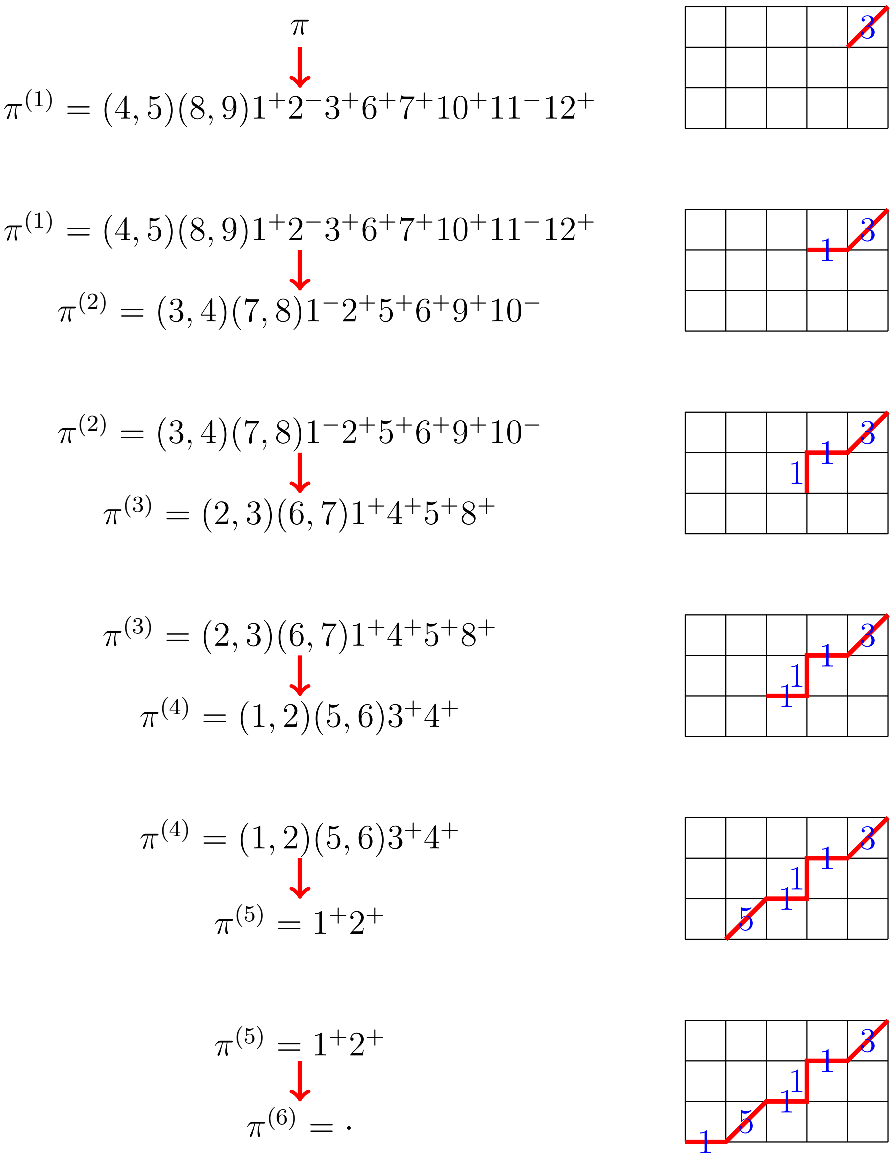

Fig. 4 Algorithmic construction of the bijection onto weighted Delannoy paths

Now, we assume that $2n$ appears in a 2-cycle in π, say (i_s, j_s), where $1 \leq s \leq k$. Then $(i_r, j_r) = (i, 2n)$, for some $i \in \{2, \ldots, 2n-1\}$. Then by the symmetry condition, there is a partnering 2-cycle, which is necessarily of the form $(1, i')$ for some i'. In this case, we draw a D-step between (p, q) and $(p-1, q-1)$ and we label this step by i. Then we remove the two cycle $(i, 2n)$ as well as its partner $(1, i')$ from π. Let us denote the resulting object by $\pi_0^{(1)}$. To get rid of the gaps created by the removal of two 2-cycles, we renormalize the remaining entries by appropriately subtracting numbers so that the resulting object, which we denote by $\pi^{(1)}$ has every number from $\{1, \ldots, 2n-4\}$ appears in it exactly once. It is easy to see that we have a signed $(2(p-1), 2(q-1))$ involution which corresponds to a ssymmetric $(2(p-1), 2(q-1))$ clan under $\widetilde{\varphi}^{-1}$. Now, the label of this diagonal step can be chosen as one the $2(p+q-1)$ numbers from $\{2, \ldots, 2n-1\}$. Finally, let us note that there are $d_{p-1,q-1}$ possible ways to extend this labeled diagonal step to a weighted (p, q) Delannoy path.

Combining our observations we see that, starting with a random ssymmetric $(2p, 2q)$ clan, there are exactly

$$d_{p-1,q} + d_{p,q-1} + 2(p+q-1)d_{p-1,q-1} \tag{5.1}$$

possible weight Delannoy paths that we can construct. By induction hypothesis the number (5.1) is equal to $c_{p,q}$. This finishes our proof. □

Let us illustrate our construction by an example.

Example 5.2 Let γ denote the ssymmetric (10, 6) clan

$$\gamma = 4 + 6 - +1\,1 + +2\,2 + -4 + 6,$$

and let π denote the corresponding signed involution

$$\pi = (1, 14)(3, 16)(6, 7)(10, 11)2^+4^-5^+8^+9^+12^+13^-15^+.$$

The steps of our construction are shown in Fig. 4.

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Appendix

In this appendix, as we promised in the introduction, we outline a method for approximating the number of symmetric $(2p, 2q+1)$ clans with k pairs, $\beta_{k,p,q}$. Recall our notation that $A_e(x) = \sum_{l=0}^{q} \beta_{2l,p,q} x^{2l}$ and $A_o(x) = \sum_{l=0}^{q} \beta_{2l+1,p,q} x^{2l+1}$.

First of all, by multiplying both sides of the recurrence relation (3.8) by x^{2l} and summing over l lead us to the following integral/differential equation

$$\begin{aligned} A_e(x) - \beta_{0,p,q} = (q+1) \int (A_o(x) - \beta_{2q+1,p,q} x^{2q+1}) dx \\ -\frac{x}{2}(A_o(x) - \beta_{2q+1,p,q} x^{2q+1}) \\ +2(pq+p+q+1) \int x(A_e(x) - \beta_{2q,p,q} x^{2q}) dx \end{aligned}$$

$$-(p+q+1)x^2(A_e(x)-\beta_{2q,p,q}x^{2q})$$
$$+\frac{x^3}{2}A_e^{'}(x)-q\beta_{2q,p,q}x^{2q+2}.$$

We get rid of the integrals by taking the derivative with respect to x and then we reorganize our equation which is now a second order ODE as in

$$x^3A_e^{''}(x)-\Big((2p+2q-1)x^2+2\Big)A_e^{'}(x)+4pqxA_e(x)$$
$$-xA_o^{'}(x)+(2q+1)A_0(x)=0.$$

By applying a similar procedure to the recurrence relation (3.9) and also by using the fact that $\beta_{1,p,q}=p\beta_{0,p,q}$, we obtain our second second order ODE:

$$x^3A_o^{''}(x)-\Big((2p+2q-1)x^2+2\Big)A_o^{'}(x)+(4pq+2p-2q-1)xA_o(x)$$
$$-xA_e^{'}(x)+2pA_e(x)=0.$$

Note that we the following initial conditions that follow from the definitions of $A_e(x)$ and $A_o(x)$:

$$A_e(0)=\beta_{0,p,q} \text{ and } A_o(0)=0,$$
$$A_e^{'}(0)=0 \text{ and } A_o^{'}(0)=p\beta_{0,p,q}=\beta_{1,p,q}.$$

We will reduce our second order system to a first order ODE by setting $u(x):=A_e^{'}(x)$ and $v(x):=A_o^{'}(x)$. Then

$$x^3u'(x)=((2p+2q-1)x^2+2)u(x)-4pqxA_e(x)-(2q+1)A_0(x)+xv(x)$$
$$x^3v'(x)=((2p+2q-1)x^2+2)v(x)-(4pq+2p-2q-1)xA_o(x)$$
$$-2pA_e(x)+xu(x)$$
$$x^3A_e^{'}(x)=x^3u(x)$$
$$x^3A_o^{'}(x)=x^3v(x).$$

We write this system in matrix form

$$x^3X'=A(x)X,$$

where $A(x)$ the 4×4 matrix as in (1.5). Note that our initial conditions become

$$X(0)=\begin{bmatrix}u(0)\\v(0)\\A(0)\\A(0)\end{bmatrix}=\begin{bmatrix}0\\p\beta_{0,p,q}\\\beta_{0,p,q}\\0\end{bmatrix}. \tag{6.1}$$

Once a system of first order ordinary differential equations of this type is given, formal series solutions can always be obtained by carrying out the computational procedure, which is outlined in Turrittin (1955). We will use those techniques to solve the above system of first order ordinary differential equations.

Before proceeding any further let us define the matrices $A_0, A_1, \ldots$ by decomposing the coefficient matrix $A(x)$:

$$A(x) = \sum_{k=0}^{\infty} A_k x^k = \begin{bmatrix} 2 & 0 & 0 & -(2q+1) \\ 0 & 2 & -2p & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^0 + \begin{bmatrix} 0 & 1 & -4pq & 0 \\ 1 & 0 & 0 & -(4pq+2q-2q-1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^1 + \begin{bmatrix} (2q+2q-1) & 0 & 0 & 0 \\ 0 & (2p+2q-1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^2 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x^3 + \mathbf{0} x^4 + \cdots$$

Since the eigenvalues of the leading matrix A_0 fall into two groups, namely $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = 2$, there exists a normalizing transformation matrix P obtained from the Jordan canonical form of A_0. More precisely, since

$$B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -p & 0 \\ 1 & 0 & 0 & -\frac{2q+1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & -(2q+1) \\ 0 & 2 & -2p & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2q+1}{2} & 0 & 0 & 1 \\ 0 & p & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = P^{-1} A_0 P.$$

the normalizing transformation $X = PY$ turns our system into

$$x^3 Y' = B(x) Y; \quad \text{with} \quad Y(0) = \begin{bmatrix} 0 \\ \beta_{0,p,q} \\ 0 \\ 0 \end{bmatrix} \tag{6.2}$$

where

$$\begin{aligned}
B(x) &= P^{-1}A(x)P \\
&= \begin{bmatrix} 0 & px^3 & x^3 & 0 \\ \frac{2q+1}{2}x^3 & 0 & 0 & x^3 \\ -\frac{2q+1}{2}(px^3+4px-3x) & p(2p+2q-1)x^2 & (2p+2q-1)x^2+2 & x-px^3 \\ \frac{(2q+1)(2p+2q-1)}{2}x^2 & (p-4pq)x-\frac{p(2q+1)}{2}x^2 & x-\frac{2q+1}{2}x^3 & (2p+2q-1)x^2+2 \end{bmatrix} \\
&= \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&2&0\\0&0&0&2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{(2q+1)(4p-3)}{2} & 0 & 0 & 1 \\ 0 & p(1-4q) & 1 & 0 \end{bmatrix} x \\
&\quad + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & p(2p+2q-1) & 2p+2q-1 & 0 \\ \frac{(2q+1)(2p+2q-1)}{2} & 0 & 0 & 2p+2q-1 \end{bmatrix} x^2 \\
&\quad + \begin{bmatrix} 0 & p & 1 & 0 \\ \frac{2q+1}{2} & 0 & 0 & 1 \\ -\frac{p(2q+1)}{2} & 0 & 0 & -p \\ 0 & -\frac{p(2q+1)}{2} & -\frac{2q+1}{2} & 0 \end{bmatrix} x^3 + \mathbf{0}x^4 + \mathbf{0}x^5 + \cdots
\end{aligned}$$

We denote the coefficient matrix of x^i $(i = 0, 1, 2, \ldots)$ in $B(x)$ by B_i. Thus,

$$B(x) = B_0 + B_1 x + B_2 x^2 + B_3 x^3.$$

We will work with a system that is obtained from $B(x)$ by a "shearing" transformation. Let Q be a formal power series of the form $Q = \sum Q_r x^r$ with Q_r's are some constant matrices of order 4. We assume that our desired solution $Y = Y(x)$ for $x^3 Y' = BY$ is of the form $Y = QZ$ for some 4×1 column matrix $Z = Z(x)$. Formally, substituting QZ into $x^3 Y' = B(x)Y$ will give us a new ODE:

$$x^3(QZ)' = BQZ \Rightarrow x^3(Q'Z + QZ') = BQZ \text{ or } x^3 Z' = (Q^{-1}BQ + x^3 Q^{-1}Q')Z.$$

Let C denote the formal power series $\sum C_r x^r$ that is defined by

$$Q^{-1}BQ + x^3 Q^{-1}Q' = C = \sum C_r x^r, \tag{6.3}$$

hence our ODE is equivalent to

$$x^3 Z' = CZ. \tag{6.4}$$

By multiplying both sides of (6.3) with Q and rearranging we obtain a new ODE whose solution will lead to a solution of (6.5):

$$x^3 Q' = QC - BQ. \tag{6.5}$$

To solve (6.5) we simply substitute $B = \sum B_r x^r$, $Q = \sum Q_r x^r$ and $C = \sum C_r x^r$ and equate the coefficients. Then we get the following relations which we call as our fundamental recurrences:

(i) $0 = Q_0 C_0 - B_0 Q_0$;
(ii) $0 = (Q_0 C_1 - B_1 Q_0) + (Q_1 C_0 - B_0 Q_1)$;
(iii) $(r-2)Q_{r-2} = \sum_{i=0}^{r} (Q_i C_{r-i} - B_{r-i} Q_i)$ for $r \geq 2$.

We will recursively assign specific values to the matrices $Q_i, i = 0, 1, 2, \ldots$ which will allow us to solve (6.5). Along the way we will determine the series $C(x) = \sum x^i C_i$, which is what we want to solve in the first place. Indeed, our goal is to choose Q_i's in such a way that C_i's become block diagonal. To this end, we assume that Q_i ($i = 0, 1, 2, \ldots$) is a block anti-diagonal matrix:

$$Q_i = \begin{bmatrix} 0 & Q_i^{12} \\ Q_i^{21} & 0 \end{bmatrix}$$

for some 2×2 matrices Q_i^{12}, Q_i^{21} ($i = 0, 1, 2, \ldots$).

Step 1. We choose $Q_0 = I_4$, the 4×4 identity matrix. It follows from (i) that

$$C_0 = B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

We have a remark in order.

Remark 6.1 Let us point out that, since

$$Q_i B_0 - C_0 C_i = \begin{bmatrix} 0 & 2Q_i^{12} \\ -2Q_i^{21} & 0 \end{bmatrix} \text{ for } i = 1, 2, \ldots \tag{6.6}$$

by using the fundamental recurrences (ii) and (iii) we will always be able to choose Q_i^{12} and Q_i^{21} so that C_i is of the form

$$C_i = \begin{bmatrix} C_i^{11} & 0 \\ 0 & C_i^{22} \end{bmatrix},$$

where C_i^{11} and C_i^{22} are 2×2 matrices.

Step 2. By (ii) and Step 1, $C_1 = B_1 - Q_1 C_0 + B_0 Q_1$, so we set

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{(2q+1)(4p-1)}{4} & 0 & 0 & 0 \\ 0 & \frac{p(4q-1)}{2} & 0 & 0 \end{bmatrix} \implies C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Step 3. By (iii) and Steps 1,2, $C_2 = B_2 - Q_1C_1 + B_1Q_1 - Q_2C_0 + B_0Q_2$, so we set

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{p(4p+8q-3)}{4} & 0 & 0 \\ -\frac{(2q+1)(4q-1)}{8} & 0 & 0 & 0 \end{bmatrix}$$

$$\implies C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2p+2q-1 & 0 \\ 0 & 0 & 0 & 2p+2q-1 \end{bmatrix}.$$

In a similar manner, we put

$$Q_3 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{(2q+1)(16p^2+16pq-1)}{16} & 0 & 0 & 0 \\ 0 & \frac{-p(16pq+16q^2-8p-16q+1)}{8} & 0 & 0 \end{bmatrix}$$

$$\implies C_3 = \begin{bmatrix} 0 & p & 0 & 0 \\ \frac{2q+1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & -\frac{2q+1}{2} & 0 \end{bmatrix}.$$

The above computations are in some sense are our initial conditions. To get a better understanding of the general case we make a few more preliminary observations and formal computations.

$$C_i^{jj} = B_i^{jj} \text{ for } i = 0, 1, 2, 3 \text{ and } j = 1, 2 \tag{6.7}$$

$$Q_iC_1 = \begin{bmatrix} 0 & Q_i^{12}C_1^{22} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_1Q_i = \begin{bmatrix} 0 & 0 \\ B_1^{22}Q_i^{21} & B_1^{21}Q_i^{12} \end{bmatrix} \tag{6.8}$$

$$Q_iC_2 = \begin{bmatrix} 0 & Q_i^{12}C_2^{22} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2Q_i = \begin{bmatrix} 0 & 0 \\ B_2^{22}Q_i^{21} & B_2^{21}Q_i^{12} \end{bmatrix} \tag{6.9}$$

$$Q_iC_3 = \begin{bmatrix} 0 & Q_i^{12}C_3^{22} \\ Q_i^{21}C_3^{11} & 0 \end{bmatrix} \quad \text{and} \quad B_3Q_i = \begin{bmatrix} B_3^{12}Q_i^{21} & B_3^{11}Q_i^{12} \\ B_3^{22}Q_i^{21} & B_3^{21}Q_i^{12} \end{bmatrix} \tag{6.10}$$

$$Q_iC_j = \begin{bmatrix} 0 & Q_i^{12}C_j^{22} \\ Q_i^{21}C_j^{11} & 0 \end{bmatrix}. \tag{6.11}$$

Finally, since $B_r = 0$, the fundamental recurrence (iii) simplifies to

$$C_r = (r-2)Q_{r-2} - \left(\sum_{i=0}^{3}(Q_{r-i}C_i - B_iQ_{r-i})\right) - \left(\sum_{i=4}^{r-1} Q_{r-i}C_i\right). \tag{6.12}$$

Recall that we started with the system $x^3X' = A(x)X$ which is transformed into $x^3Y' = B(x)Y$ by conjugating with a constant matrix, and the latter system is transformed into $x^3Z' = C(x)Z$ by the shearing transformation $Y = Q(x)Z$.

Proposition 6.2 *Let $C(x) = \sum_r C_r x^r$ and $Q(x) = \sum_r Q_r x^r$ be as in the previous paragraph. If $r \geq 4$, then we have*

$$C_r = \begin{bmatrix} Q^{21}_{r-3} & 0 \\ 0 & B^{21}_1 Q^{12}_{r-1} + B^{21}_2 Q^{12}_{r-2} + B^{21}_3 Q^{12}_{r-3} \end{bmatrix}.$$

In particular, the system $x^3Z' = C(x)Z$ decomposes into two 2×2 systems of ODE's

$$x^3K' = R(x)K, \tag{6.13}$$

$$x^3L' = T(x)L, \tag{6.14}$$

where

$$R(x) = C^{11}_3 x^3 + \sum_{r\geq 4} Q^{21}_{r-3} x^r;$$

$$T(x) = \sum_{i=0}^{3} C^{22}_i x^3 + \sum_{r\geq 4} (B^{21}_1 Q^{12}_{r-1} + B^{21}_2 Q^{12}_{r-2} + B^{21}_3 Q^{12}_{r-3}) x^r.$$

Proof Since C_r is a block diagonal matrix, recurrence (6.12) combined with equations (6.7)–(6.11) gives us the desired result. □

What remains is to solving the systems (6.13) and (6.14). The former ODE is relatively easy since it does not have a singularity anymore. However, the second ODE (6.14) does have a singularity. Moreover, we still do not know the exact forms of neither $Q^{12}(x)$ nor $Q^{21}(x)$. On the positive side, by taking advantage of the particular structure of $B(x)$'s we are able to find recurrences for $R(x)$ and $T(x)$.

To find a recurrence for the blocks of Q_r's, we use Proposition 6.2 as well as the simplified fundamental recurrence (6.12) as follows:

$$\begin{aligned} C_r &= \begin{bmatrix} Q^{21}_{r-3} & 0 \\ 0 & B^{21}_1 Q^{12}_{r-1} + B^{21}_2 Q^{12}_{r-2} + B^{21}_3 Q^{12}_{r-3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & (r-2)Q^{12}_{r-2} \\ (r-2)Q^{21}_{r-2} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2Q^{12}_r \\ -2Q^{21}_r & 0 \end{bmatrix} - \begin{bmatrix} 0 & Q^{12}_{r-1} B^{22}_1 \\ -B^{22}_1 Q^{21}_{r-1} & -B^{21}_1 Q^{12}_{r-1} \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & Q^{12}_{r-2} B^{22}_2 \\ -B^{22}_2 Q^{21}_{r-2} & -B^{21}_2 Q^{12}_{r-2} \end{bmatrix} - \begin{bmatrix} -B^{12}_3 Q^{21}_{r-3} & Q^{12}_{r-3} B^{22}_3 - B^{11}_3 Q^{12}_{r-3} \\ Q^{21}_{r-3} B^{11}_3 - B^{22}_3 Q^{21}_{r-3} & -B^{21}_3 Q^{12}_{r-3} \end{bmatrix} \\ &\quad - \sum_{i=4} \begin{bmatrix} 0 & Q^{12}_{r-i} \\ Q^{21}_{r-i} & 0 \end{bmatrix} \begin{bmatrix} Q^{21}_{i-3} & 0 \\ 0 & B^{21}_1 Q^{12}_{i-1} + B^{21}_2 Q^{12}_{i-2} + B^{21}_3 Q^{12}_{i-3} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} Q^{21}_{i-3} & \begin{array}{r}(r-2)Q^{12}_{r-2} - 2Q^{12}_r - Q^{12}_{r-1}B^{22}_1 \\ - Q^{12}_{r-2}B^{22}_2 - Q^{12}_{r-3}B^{22}_3 + B^{11}_3 Q^{12}_{r-3} \\ - \sum_{i=4} Q^{12}_{r-i}(B^{21}_1 Q^{12}_{i-1} + B^{21}_2 Q^{12}_{i-2} + B^{21}_3 Q^{12}_{i-3})\end{array} \\ \begin{array}{l}(r-2)Q^{21}_{r-2} + 2Q^{21}_r + B^{22}_1 Q^{21}_{r-1} + B^{22}_2 Q^{21}_{r-2} \\ - Q^{21}_{r-3}B^{11}_3 + B^{22}_3 Q^{21}_{r-3} - \sum_{i=4} Q^{21}_{r-i} Q^{21}_{i-3}\end{array} & B^{21}_1 Q^{12}_{i-3} + B^{21}_2 Q^{12}_{i-2} + B^{21}_3 Q^{12}_{i-3} \end{bmatrix}.$$

Observe that the diagonal blocks do not give us any new information, however, the anti-diagonal blocks do. By the equality of the bottom left blocks, we have

$$\begin{aligned} 2Q^{21}_r &= -(r-2)Q^{21}_{r-2} - B^{22}_1 Q^{21}_{r-1} - B^{22}_2 Q^{21}_{r-2} + Q^{21}_{r-3}B^{11}_3 - B^{22}_3 Q^{21}_{r-3} \\ &\quad + \sum_{i=4}^{r-1} Q^{21}_{r-i} Q^{21}_{i-3}. \end{aligned} \tag{6.15}$$

Similarly, the equality of the top right blocks give

$$\begin{aligned} 2Q^{12}_r =&(r-2)Q^{12}_{r-2} - Q^{12}_{r-1}B^{22}_1 - Q^{12}_{r-2}B^{22}_2 - Q^{12}_{r-3}B^{22}_3 + B^{11}_3 Q^{12}_{r-3} \\ &- \sum_{i=4}^{r-1} Q^{12}_{r-i}(B^{21}_1 Q^{12}_{i-1} + B^{21}_2 Q^{12}_{i-2} + B^{21}_3 Q^{12}_{i-3}). \end{aligned} \tag{6.16}$$

Obviously, these recurrences enable us to write the precise forms of the ODE's (6.13) and (6.14). Both of these ODE's can now be solved by applying suitable shearing transformations leading to a solution of our original equation $x^3 X' = A(x)X$. However, due to its high computational cost the result is still not better than the expressions for $\beta_{k,p,q}$'s that we recorded in Theorem 1.2.

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