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Open Problems on Configuration Spaces of Tensegrities

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Abstract In this small paper we bring together some open problems related to the study of the configuration spaces of tensegrities, i.e. graphs with stresses on edges. These problems were announced in Doray et al. (Discrete Comput Geom 43:436–466, 2010), Karpenkov et al. (ARS Math Contemp 6:305–322, 2013), Karpenkov (The combinatorial geometry of stresses in frameworks. arXiv:1512.02563 [math.MG], 2017), and Karpenkov (Geometric Conditions of Rigidity in Nongeneric settings, 2016) (by F. Doray, J. Schepers, B. Servatius, and the author), for more details we refer to the mentioned articles.

1 Notation and Definitions

1.1 Tensegrities

The subject of tensegrities was first considered by J.C. Maxwell in Maxwell (1864), who started to investigate first questions regarding force-loads for frameworks. Nowadays tensegrities are one of the leading directions of study in modern theory of rigidity (see, e.g., Connelly (1993) for further information). Let us recall several standard definitions.

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Definition 1.1 Fix a positive integer d. Let $G = (V, E)$ be an arbitrary graph without loops and multiple edges. Let it have n vertices $v_1, \ldots, v_n$.

- A *configuration* is a finite collection P of n labeled points $(p_1, p_2, \ldots, p_n)$, where each point p_i is in a fixed Euclidean space $\mathbb{R}^d$.
- The realization of G with straight edges, induced by mapping v_j to p_j is called a *tensegrity framework* and it is denoted as $G(P)$. (Here we allow the realization to have self-intersections).
- A *stress* w on a framework is an assignment of real scalars $w_{i,j}$ (called *tensions*) to its edges $p_i p_j$.
- A stress w is called a *self-stress* if at every vertex p_i we have

$$\sum_{\{j \mid j \neq i\}} w_{i,j}(p_j - p_i) = 0.$$

- A pair $(G(P), w)$ is a *tensegrity* if w is a self-stress for the framework $G(P)$.
- If $w_{i,j} < 0$ then we call the edge $p_i p_j$ a *cable*, if $w_{i,j} > 0$ we call it a *strut*.

1.2 Configuration Space of Tensegrities and its Stratification

Denote by $B_d(G) = (\mathbb{R}^d)^n$ the configuration space of all tensegrity frameworks. Let $W(n)$ denote the linear space with coordinates $w_{i,j}$ where $1 \leq i, j \leq n$. It is clear that $\dim W(n) = n^2$.

Definition 1.2 Consider a framework $G(P) \in B_d(G)$ and denote by $W(G, P)$ the linear subspace of $W(n)$ of all possible self-stresses for $G(P)$. The space $W(G, P)$ is the *fiber* at P.

Definition 1.3 Fibers $W(G, P_1)$ and $W(G, P_2)$ are said to be *equivalent* if there exists a homeomorphism $\xi : W(G, P_1) \to W(G, P_2)$, such that for any $w \in W(G, P_1)$ we have

$$\mathrm{sgn}\left(\xi(w_{i,j})\right) = \mathrm{sgn}\left(w_{i,j}\right)$$

for every coordinate $w_{i,j}$ of w. Here sgn denotes the standard sign function.

The described equivalence relation gives us a stratification of $B_d(G) = (\mathbb{R}^d)^n$. A *stratum* is by definition a maximal connected component of $B_d(G)$ with equivalent fibers. In Doray et al. (2010) we prove that all strata are semialgebraic sets.

Remark According to all known examples the majority of the strata of codimension k are intersections of the strata of codimension 1 see e.g. Doray et al. (2010), Karpenkov et al. (2013). So the most important case to study is the codimension 1 case.

Remark A stratification of a subgraph is a substratification of the original graph (i.e., each stratum for a subgraph is the union of certain strata for the original graph). Hence the case of complete graphs K_n is universal. This is straightforward as each extra edge contributes at least as much to dimensions of the fibers of a stratum as to the dimensions of the fibers of adjacent strata (locally).
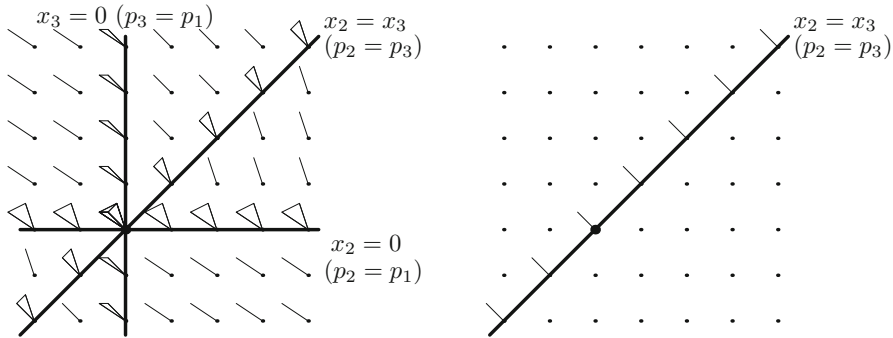
Fig. 1 Stratification of $B_1^0(K_3)$ (left) and $B_1^0(G_{1,2-3})$ (right)

Example 1.4 Let us consider a simple example of $B_1(K_3)$, namely we study tensegrities for a complete graph on three vertices and its realizations in the line. We assume that the line has a coordinate, so each point of $B_1(K_3)$ is associated with three coordinates (x_1, x_2, x_3).

First we study a particular case $x_1 = 0$, which we denote by $B_1^0(K_3)$. The set $B_1^0(K_3)$ has the following stratification (see Fig. 1, Left):

- 1 stratum of codimension two: the origin. Here all three vertices coincide and the dimension of the fiber is 3.
- 6 strata of codimension one: some pair of vertices coincide. The dimension of fiber is two.
- 6 connected components of full dimension correspond to triple of distinct vertices. The dimension of a fiber is one.

It is clear that the dimension of the fiber for (x_1, x_2, x_3) coincides with the dimension of the fiber $(0, x_2 - x_1, x_3 - x_1)$ for every x_1, x_2, x_3. Therefore, we have

$$B_1(K_3) = B_1^0(K_3) \times \mathbb{R}^1.$$

Example 1.5 Let now $G_{1,2-3}$ be the graph on three vertices v_1, v_2, v_3 with the only edge connecting v_2 and v_3. Then we have 1 stratum of codimension one defined by $p_2 = p_3$ (with fibers of dimension 1) and two connected components in the complement, i.e., where $p_2 \neq p_3$ (with fibers of dimension 0). As in previous example we have

$$B_1(G_{1,2-3}) = B_1^0(G_{1,2-3}) \times \mathbb{R}^1,$$

where $B_1^0(G_{1,2-3})$ is the stratification of the section $x_1 = 0$ (see Fig. 1, Right).

2 Combinatoric Properties of Stratification

In Example 1.4 above we discussed a stratification of $B_2(K_n)$. The most important information on the stratification contains its combinatorial structure, namely the list

of strata of different dimensions and the adjacency diagram for the strata. For the combinatorial description of $B_2(K_n)$ as in Example 1.4, for $n = 2, 3, 4, 5$ we refer to Karpenkov et al. (2013). There is not much known about more complicated configuration spaces. The next simplest and most interesting unstudied cases are listed in the following problem.

Problem 1 Describe the combinatorics of $B_2(K_6)$, $B_3(K_4)$ and $B_3(K_5)$.

As experiments show, for every codimension 1 stratum there exists a certain subgraph that *locally identifies* the stratum (i.e., for every point x of the stratum there exists a neighborhood $B(x)$ such that every configuration in $B(x)$ has a nonzero self-stress for the subgraph if and only if this point is on the stratum). This observation is valid for all dimensions d.

Remark 2.1 An illustration here are the cases $B_1(K_3)$ and $B_1(G_{1,2-3})$ of Examples 1.4 and 1.5. Here both codimension 1 strata of $B_1(K_3)$ corresponding to $p_2 = p_3$ are in the union of the codimension 1 stratum of $B_1(G_{1,2-3})$. This is a general situation. In particular for any graph G on n vertices the union of all strata for G is contained in the union of all strata for K_n.

Let us say a few words about two-dimensional tensegrities. In the paper Doray et al. (2010) one can find the classification of all strata of codimension 1 for $n \leq 8$ points. For further examples, see the papers of White and Whiteley (1983), and de Guzmán and Orden (2004), Guzmán and Orden (2006). In the paper Karpenkov (2017) it was shown how to approach every stratum for the case $n = 9$. The next case which contains unknown strata is $n = 10$ (see also Problem 8 below).

Problem 2 Describe all the possible different types of strata for 10 points.

Problem 3 Compute the number of different types of strata for n points with arbitrary n.

In many cases the strata for different graphs coincide. So it is natural to ask the following question.

Problem 4 Which subgraphs of K_n define the same stratifications?

Finally the following question remains open.

Problem 5 Find all strata of codimension more than 1 that are not defined as an intersection of the closure of several codimension 1 strata.

Here we have only a trivial example of a graph on 2 vertices and one edge in the plane. It has one generic stratum of full dimension and one stratum of codimension 2, corresponding to two coinciding points.
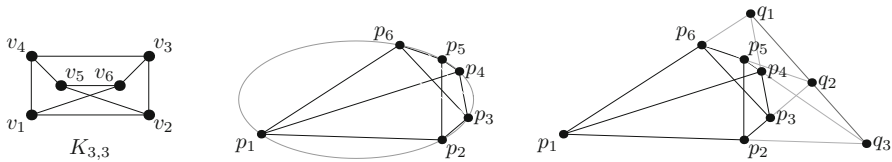
Fig. 2 The graph $K_{3,3}$ and the corresponding geometric conditions

3 Geometric Conditions Defining Strata in $\mathbb{R}^2$

Majority of known examples in the planar case are expressed in terms of Cayley algebra. Recall that the objects of (planar) Cayley algebra are points and lines in the plane. Cayley algebra has two major operations:

- *join operation* $\vee$ is defined for a pair of points. The resulting object is the line joining these points.
- *meet operation* $\wedge$ is defined for a pair of lines. The meet of two lines is the intersection point of two lines.

For more information on Cayley algebras we refer to Doubilet et al. (1974), White and McMillan (1988), Li (2008), and Karpenkov (2017).

It turns out that many geometric conditions can be expressed in terms of Cayley operations. Let us illustrate this with the following example.

Example Let $G = K_{3,3}$ be as in Fig. 2 (Left). A framework $P = (p_1, \ldots, p_6)$ admits a non-zero tensegrity in $\mathbb{R}^2$ if and only if six points $p_1, p_2, \ldots, p_6$ are on a conic (see Fig. 2, Middle). This condition is described via Cayley algebra as Pascal's theorem:

$$q_1, q_2 \quad \text{and} \quad q_3 \text{ are in a line for } \begin{cases} q_1 = (p_1 \vee p_2) \cap (p_4 \vee p_5) \\ q_2 = (p_2 \vee p_3) \cap (p_5 \vee p_6) \\ q_3 = (p_3 \vee p_4) \cap (p_6 \vee p_1) \end{cases},$$

(Fig. 2, Right). Similar to Remark 2.1 the set of points of $B(K_{3,3})$ that satisfy the above condition is the union of several strata for $B(K_{3,3})$. The statement of this example is an exercise on surgeries on graphs introduced in Doray et al. (2010).

In fact in the above example, the property of 6 points to lie on one conic does not depend on the order of these points. Therefore, there are 60 different Cayley algebra systems defining the same stratum. This lead to the following important open problem.

Problem 6 Which Cayley algebra systems define the same strata?

This problem is a kind of a question on finding generators and relations for the set of all conditions.

One of the main long-standing open problems on the Cayley strata description is as follows.

Problem 7 Given a graph G. Does there exist a Cayley algebra system (or several systems) describing the union of the codimension 1 tensegrity strata in the plane [i.e., the union of the codimension 1 strata of $B_2(K_n)$]?

Recently this problem was solved in a *weaker* settings of *extended Cayley algebra* in Karpenkov (2017). Nevertheless it is not clear if it is possible to avoid additional elements involved in the construction of Karpenkov (2017). Here is an example of a graph for which the systems describing codimension 1 strata are not known.

Problem 8 Write (if exist) Cayley algebra systems defining the strata for the following graph:
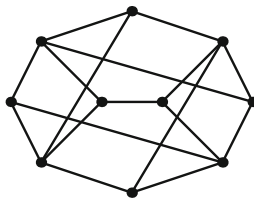
Currently this example is a strong candidate for a counterexample to Problem 7. There is almost nothing known in multidimensional case.

Problem 9 Develop theory of geometric conditions for strata in multidimensional case.

We refer to White and Whiteley (1983) for examples of geometric conditions in dimension 3.

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