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RESEARCH CONTRIBUTION

Integral Geometry of Euler Equations

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Abstract We develop an integral geometry of stationary Euler equations defining some function w on the Grassmannian of affine lines in $\mathbb{R}^3$ depending on a putative compactly supported solution $(v;\ p)$ of the system and deduce some linear differential equations for w. We conjecture that $w = 0$ everywhere and prove that this conjecture implies that $v = 0$.

Keywords Euler equations · Integral geometry · Tensor tomography

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1 Introduction

In the present paper we introduce and develop a version of integral geometry for the steady Euler system.

More precisely, the system which we consider is as follows

$$\sum_{j=1}^{3} \frac{\partial(v^i v^j)}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0 \quad \text{for } i = 1, 2, 3, \tag{1.1}$$

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for an unknown smooth vector field $v = v(x) = (v^1(x), v^2(x), v^3(x))$ and an unknown smooth scalar function $p = p(x)$, $x \in \mathbb{R}^3$; it expresses the conservation of fluid's momentum $v \otimes v + p\delta_{ij}$ and reads in a coordinate free form as follows

$$\mathrm{div}(v \otimes v) + \nabla p = 0. \tag{1.2}$$

Note that if we add to (1.1) the incompressibility condition

$$\mathrm{div}\, v = 0, \tag{1.3}$$

the system (1.2) and (1.3) describes a steady state flow of the ideal fluid.

A long-standing folklore conjecture states that a smooth compactly supported solution of (1.2) and (1.3) should be identically zero, and this is known for Beltrami flows; see Nadirashvili (2014) and also Chae and Constantin (2015). Let us state it explicitly:

Conjecture 1.1 *Let* $(v; p) \in C_0(\mathbb{R}^3)$ *be a solution of* (1.2)–(1.3). *Then* $v = 0,\ p = 0$.

Note, however, that there do exist nontrivial Beltrami flows slowly decaying at infinity; see Enciso and Peralta-Salas (2012). Note also that nontrivial compactly supported solutions of system (1.1) exist, e.g., any spherically symmetric vector field v is a solution of (1.1) for a suitable pressure p, but we do not know whether the system

$$\sum_{j=1}^{3} v^j \frac{\partial v^i}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2, 3$$

which is equivalent to (1.1) and to (1.2) under the additional condition (1.3), admits a non-trivial compactly supported solution not satisfying (1.3).

Our idea is to separate the study of system (1.1) from the study of Eq. (1.3) using an integral geometry technique which permits to encode (almost) all information about a compactly supported solution v of (1.1) in a single scalar function w defined on the Grassmannian M of affine lines in space. We conjecture that this function w is in fact identically zero, which can readily be verified for any spherically symmetric solution. If it were the case, then adding the incompressibility condition (1.3) to the condition $w = 0$ we were able to obtain that $v = 0$ everywhere. In order to proof that $w = 0$ we deduce a linear differential equation for w on M. Unfortunately, this equation is not elliptic and admits non-trivial solutions, and therefore, the purported annulation of w remains conjectural, despite some additional integral geometry arguments going in the same direction.

More concretely, below we characterize the kernel of (1.1) in terms of integral transforms of the quadratic forms $v^i v^j$. Thus, given any smooth compactly supported solution $(v; p)$ of (1.1), we define a smooth function w on the Grassmannian M classifying lines in space using the X-ray transforms of $v^i v^j$ and then derive a linear differential equation for w. Using a Radon plane transform of w we deduce one more linear homogeneous differential equation which suggests that $w = 0$ everywhere. However, we are not able to justify this claim rigorously and we formulate it as a conjecture; we show that assuming the conjecture and (1.3) one can deduce Conjecture 1.1. Therefore, we put forth

Conjecture 1.2 *Let w be the function on the Grassmannian $G_{1,3}$ of affine lines in $\mathbb{R}^3$, defined below in Sect. 3, which depends on a compactly supported solution $(v; p)$ of (1.1). Then $w = 0$ everywhere.*

Note, in particular, that this conjecture holds for any spherically symmetric compactly supported vector field v.

Note also that it is possible to prove that the general solution of Eq. (4.5) depends on two arbitrary (smooth compactly supported) functions (V.A. Sharafutdinov, personal communication) and therefore to prove Conjecture 1.2 one needs to use some nonlinear argument, e.g. using the fact that the tensor $v \otimes v$ is of rank 1 as opposed to the general symmetric tensor of rank 3 which also gives a solution of (4.5) by constructing the corresponding function w; thus, we can say that we have (at least) two more independent conditions on w. Unfortunately, we not know how to use this rank conditions to restrict possible functions w's. However, there do exist some natural linear conditions for a suitable tensor which are sufficient to prove Conjecture 1.2, for instance, system (5.4) which is partially supported by Eq. (5.3).

The rest of the paper is organized as follows: in Sect. 2 we recall some definitions and results from Sharafutdinov (1994) concerning the X-ray transform of symmetric tensor fields. In Sect. 3 we define and study a smooth function $w \in C^\infty(M)$ which depends on a smooth compactly supported solution (v, p) of (1.1). Section 4 contains a description of two invariant order 2 differential operators on $C^\infty(M)$ and a differential equation for w in terms of those operators. In Sect. 5 we define a plane Radon transform for quadratic tensor fields, prove that it vanishes for some quadratic tensor field connected with w and explain why this annulation partially confirms Conjecture 1.2. Finally, in Sect. 6 we deduce Conjecture 1.1 assuming Conjecture 1.2 together with (1.3).

2 Tensor X-ray Transform

We use throughout our paper the integral geometry of tensor fields developed in Sharafutdinov (1994) and discussed in Nadirashvili et al. (2016) in its three-dimensional form. Let us give some of its points in our simple situation. For details see Sharafutdinov (1994) and Nadirashvili et al. (2016).

In what follows we fix a positive scalar product $\langle x, y\rangle,\ x, y \in \mathbb{R}^n$. Let

$$T\mathbb{S}^{n-1} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \|\xi\| = 1,\ \langle x, \xi\rangle = 0\} \subset \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$$

be the tangent bundle of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

Given a continuous rank h symmetric tensor field f on $\mathbb{R}^n$, the *X-ray transform* of f is defined for $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ by

$$(If)(x, \xi) = \sum_{i_1,\dots,i_h=1}^{n} \int_{-\infty}^{\infty} f_{i_1\dots i_h}(x + t\xi)\, \xi_{i_1} \dots \xi_{i_h}\, dt \tag{2.1}$$

under the assumption that f decays at infinity so that the integral converges.

We denote by $\mathcal{S}(S^h;\mathbb{R}^n)$ the space of symmetric degree h tensor fields with all components lying in the Schwartz space, and denote by $\mathcal{S}(T\mathbb{S}^{n-1})$ the Schwartz space on $T\mathbb{S}^{n-1}$. Below we consider only tensors from $\mathcal{S}(S^h;\mathbb{R}^n)$ and functions from $\mathcal{S}(T\mathbb{S}^{n-1})$. For such $f \in \mathcal{S}(S^h;\mathbb{R}^n)$ we get a C^∞-smooth function $\psi(x,\xi) = (If)(x,\xi)$ on $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ satisfying the following conditions:

$$\psi(x,t\xi) = \operatorname{sgn}(t)t^{h-1}\psi(x,\xi) \quad (0 \neq t \in \mathbb{R}), \quad \psi(x+t\xi,\xi) = \psi(x,\xi), \tag{2.2}$$

which mean that $(If)(x,\xi)$ actually depends only on the line passing through the point x in direction ξ, and we parameterize the manifold of oriented lines in $\mathbb{R}^n$ by $T\mathbb{S}^{n-1}$. For $\chi(x,\xi) \in \mathcal{S}(T\mathbb{S}^{n-1})$ we can extend χ by homogeneity, setting $\chi(x,\xi) = \chi(x,\xi/\|\xi\|)$, to the open subset $\mathcal{Q} \cap \{\xi \neq 0\}$ of the quadric

$$\mathcal{Q} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \langle x,\xi\rangle = 0\} \supset T\mathbb{S}^{n-1}.$$

Conversely, for a tensor field $f \in \mathcal{S}(S^h;\mathbb{R}^n)$, the restriction $\chi = \psi|_{T\mathbb{S}^{n-1}}$ of the function $\psi = If$ to the manifold $T\mathbb{S}^{n-1}$ belongs to $\mathcal{S}(T\mathbb{S}^{n-1})$. Moreover, the function ψ is uniquely recovered from χ by the formula

$$\psi(x,\xi) = \|\xi\|^{h-1}\chi\left(x - \frac{\langle x,\xi\rangle}{\|\xi\|^2}\xi, \frac{\xi}{\|\xi\|}\right), \tag{2.3}$$

which follows from (2.2); note that $\left(x - \frac{\langle x,\xi\rangle}{\|\xi\|^2}\xi, \frac{\xi}{\|\xi\|}\right) \in T\mathbb{S}^{n-1} \subset \mathcal{Q}$, and thus the right-hand side of (2.3) is correctly defined. Therefore, the X-ray transform can be considered as a linear continuous operator $I\colon \mathcal{S}(S^h;\mathbb{R}^n) \longrightarrow \mathcal{S}(T\mathbb{S}^{n-1})$, and now we are going to describe its image and kernel.

The image of the operator I is described by Theorem 2.10.1 in Sharafutdinov (1994) as follows.

John's Conditions. *A function $\chi \in \mathcal{S}(T\mathbb{S}^{n-1})$ $(n \geq 3)$ belongs to the range of the operator I if and only if the following two conditions hold*:

1. $\chi(x,-\xi) = (-1)^h\chi(x,\xi)$;
2. *The function $\psi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}))$ defined by* (2.3) *satisfies the equations*

$$\left(\frac{\partial^2}{\partial x_{i_1}\partial\xi_{j_1}} - \frac{\partial^2}{\partial x_{j_1}\partial\xi_{i_1}}\right)\cdots\left(\frac{\partial^2}{\partial x_{i_{h+1}}\partial\xi_{j_{h+1}}} - \frac{\partial^2}{\partial x_{j_{h+1}}\partial\xi_{i_{h+1}}}\right)\psi = 0 \tag{2.4}$$

written for all indices $1 \leq i_1, j_1, \ldots, i_{h+1}, j_{h+1} \leq n$.

Define the symmetric inner differentiation operator $d_s = \sigma\nabla$ by symmetrization of the covariant differentiation operator $\nabla\colon C^\infty(S^h) \longrightarrow C^\infty(T^{h+1})$,

$$(\nabla u)_{i_1,\ldots i_{h+1}} = u_{i_1\ldots i_h;i_{h+1}} = \frac{\partial u_{i_1\ldots i_h}}{\partial x_{i_{h+1}}};$$

it does not depend on the choice of a coordinate system.

The kernel of the operator I is given by (Theorem 2.2.1, (1),(2) in Sharafutdinov 1994).

Kernel of the Ray Transform. *Let $n \geq 2$ and $h \geq 1$ be integers. For a compactly-supported field $f \in C_0^\infty(S^h; \mathbb{R}^n)$ the following statements are equivalent*:

1. $If = 0$;
2. *There exists a compactly-supported field $v \in C_0^\infty(S^{h-1}; \mathbb{R}^n)$ such that its support is contained in the convex hull of the support of f and*

$$d_s v = f. \tag{2.5}$$

Note also that an inversion formula for the operator I given by Theorem 2.10.2 in Sharafutdinov (1994) implies that it is injective on the subspace of divergence-free (=solenoidal) tensor fields.

The 3-Dimensional Case For $n = 3$ one notes that the tangent bundle $T\mathbb{S}^2$ over $\mathbb{S}^2$ coincides with the homogeneous space $M = G/(\mathbb{R}\times \mathrm{SO}(2)) = G'/(\mathbb{R}\times \mathrm{O}(2))$, where $G = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$ is the group of proper rigid motions of $\mathbb{R}^3$, while $G' = G \cdot \{\pm I_3\}$ is the isometry group of $\mathbb{R}^3$. Therefore the operator I for $n = 3$ can be written as $I : \mathcal{S}(S^h; \mathbb{R}^3) \longrightarrow \mathcal{S}(M)$.

Let us define coordinates on the open subset M_{nh} of M consisting of non-horizontal affine lines. Namely, $m = m\,(y_1, y_2, \alpha_1, \alpha_2)$ is given by a parametric equation for a current point A on m,

$$A = (y_1, y_2, 0) + t\alpha = (y_1 + \alpha_1 t, y_2 + \alpha_2 t, t),$$

where t grows in the positive direction of m and thus the vector $\alpha = (\alpha_1, \alpha_2, 1)$ defines the positive direction of m.

We can now rewrite the above general formulas using the coordinates $(y_1, y_2, \alpha_1, \alpha_2)$. First we fix the following notation:

$$k = k(\alpha_1, \alpha_2) = \sqrt{1 + \alpha_1^2 + \alpha_2^2} = \sqrt{1 + \|\alpha\|^2}; \tag{2.6}$$

we will use this notation throughout the paper.

Define the diffeomorphism

$$\Phi : U \to \mathbb{R}^4, \quad (x, y, z, \xi) = (x, y, z, \xi_1, \xi_2, \xi_3) \mapsto (y, \alpha) = (y_1, y_2, \alpha_1, \alpha_2),$$

on the open set $U = T\mathbb{S}^2 \cap \{\xi_3 > 0\}$ by

$$y_1 = x - \frac{\xi_1}{\xi_3} z, \quad y_2 = y - \frac{\xi_2}{\xi_3} z, \quad \alpha_1 = \frac{\xi_1}{\xi_3}, \quad \alpha_2 = \frac{\xi_2}{\xi_3}. \tag{2.7}$$

Then (U, Φ) is a coordinate patch on M; this parametrization was used by F. John in his seminal paper (John 1938).

For a function $\chi \in C^\infty(U)$, we define $\varphi \in C^\infty(\mathbb{R}^4)$ by

$$\varphi = k^{h-1}\chi \circ \Phi^{-1}.$$

These two functions are expressed through each other by the formulas

$$\chi(x, y, z, \xi) = \xi_3^{h-1}\varphi\left(x - \frac{\xi_1 z}{\xi_3}, y - \frac{\xi_2 z}{\xi_3}, \frac{\xi_1}{\xi_3}, \frac{\xi_2}{\xi_3}\right),$$
$$\varphi(y, \alpha) = k^{h-1}\chi\left(y_1 - \frac{\langle y, \alpha\rangle\alpha_1}{k^2}, y_2 - \frac{\langle y, \alpha\rangle\alpha_2}{k^2}, -\frac{\langle y, \alpha\rangle}{k}, \frac{\alpha_1}{k}, \frac{\alpha_2}{k}, \frac{1}{k}\right). \quad (2.8)$$

If a function $\chi \in C^\infty(T\mathbb{S}^2)$ satisfies $\chi(-x, -\xi) = (-1)^h\chi(x, \xi)$, then it is uniquely determined by

$$\varphi = k^{h-1}\chi|_U \circ \Phi^{-1} \in C^\infty(\mathbb{R}^4).$$

For a tensor field $f \in \mathcal{S}(S^h; \mathbb{R}^3)$, the function

$$\varphi = k^{h-1}(If)|_U \circ \Phi^{-1} \in C^\infty(\mathbb{R}^4) \quad (2.9)$$

is expressed through f by the formula

$$\varphi(y, \alpha) = \sum_{i_1,\dots,i_h=1}^{3} \int_{-\infty}^{\infty} f_{i_1\dots i_h}(y_1 + \alpha_1 t, y_2 + \alpha_2 t, t)\,\alpha_{i_1}\dots\alpha_{i_h}\,dt, \quad (2.10)$$

with $\alpha_3 = 1$, which easily follows from (2.1).

Let

$$L =_{\text{def}} \frac{\partial^2}{\partial\alpha_2\partial y_1} - \frac{\partial^2}{\partial\alpha_1\partial y_2} \quad (2.11)$$

be the John operator. The main result of Nadirashvili et al. (2016) says that for $n = 3$, a function $\chi \in \mathcal{S}(T\mathbb{S}^2)$ belongs to the range of the operator I for a given $h \geq 0$ if and only if the following two conditions hold:

1. $\chi(-x, -\xi) = (-1)^h\chi(x, \xi)$;
2. The function $\varphi \in C^\infty(\mathbb{R}^4)$ defined by (2.8) solves the equation

$$L^{h+1}\varphi = 0. \quad (2.12)$$

Thus $\frac{h^2+5h+6}{2}$ Eq. (2.4) for $n = 3$ are equivalent to Eq. (2.12).

3 Function *w*

In what follows we fix a compactly supported smooth solution $(v, p) \in C_0^\infty(\mathbb{R}^3)$ of system and define a function $w \in C_0^\infty(M)$ using the following result.

Lemma 3.1 *Let L be an affine plane in $\mathbb{R}^3$ and let ν_L be its unit normal, then*

$$\int_L \langle v, z\rangle \langle v, \nu_L\rangle \, d\sigma_L = 0 \tag{3.1}$$

for any $z \in L$ where $d\sigma_L$ is the area element on L.

Proof We can assume without loss of generality that $L = \{(x_1, x_2, 0)\}$ and $\nu_L = (0, 0, 1) = e_3$, $z = z_1 e_1 + z_2 e_2$ for $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Then we have

$$\begin{aligned}\int_L \langle v, z\rangle \langle v, \nu\rangle \, d\sigma_L &= \int (z_1 v^1 + z_2 v^2) v^3 \, dx_1 dx_2 \\ &= z_1 \int v^1 v^3 \, dx_1 dx_2 + z_2 \int v^2 v^3 \, dx_1 dx_2.\end{aligned}$$

Note that by Eq. (1.1) with $i = 1$ and $i = 2$ there holds

$$\begin{aligned}\frac{\partial(v^1 v^3)}{\partial x_3} &= -\frac{\partial(v^1 v^2)}{\partial x_2} - \frac{\partial(v^1 v^1)}{\partial x_1} - \frac{\partial p}{\partial x_1}, \\ \frac{\partial(v^2 v^3)}{\partial x_3} &= -\frac{\partial(v^2 v^1)}{\partial x_1} - \frac{\partial(v^2 v^2)}{\partial x_2} - \frac{\partial p}{\partial x_2},\end{aligned}$$

and thus we get

$$\begin{aligned}\frac{\partial}{\partial x_3}\left(\int v^1 v^3 \, dx_1 dx_2\right) &= \int \frac{\partial(v^1 v^3)}{\partial x_3} \, dx_1 dx_2 = 0, \\ \frac{\partial}{\partial x_3}\left(\int v^2 v^3 \, dx_1 dx_2\right) &= \int \frac{\partial(v^2 v^3)}{\partial x_3} \, dx_1 dx_2 = 0.\end{aligned}$$

Therefore, the compactly supported functions

$$\int v^1 v^3 \, dx_1 dx_2 \quad \text{and} \quad \int v^2 v^3 \, dx_1 dx_2$$

of x_3 on $\mathbb{R}$ are constant and thus vanish everywhere which finishes the proof.

For any fixed value of x_3, we define the vector field $v^\perp v^3 = (-v^2 v^3,\ v^1 v^3)$ on the plane (x_1, x_2, x_3) with coordinates $\{x_1, x_2\}$ depending on x_3 as on a parameter, where $u^\perp = (-u_2, u_1)$ for a vector field $u = (u_1, u_2)$ on $\mathbb{R}^2$; note that below we use this notation for vector fields on various planes in $\mathbb{R}^3$. Then let us set

$$F = \int_{-\infty}^{\infty} (-v^2 v^3,\ v^1 v^3) \, dx_3; \tag{3.2}$$

note that F is a compactly supported vector field on the plane $\Pi_{12} = \{(x_1, x_2, 0)\}$ with coordinates $\{x_1, x_2\}$ and (3.1) implies that $IF = 0$. Indeed, choose $x^0 = (x_1^0, x_2^0) \in$

$\mathbb{R}^2$, $0 \neq \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, and let L be the 2-plane through the point x^0 parallel to the vectors ξ and $(0, 0, 1)$. Then the vector $\nu = (-\xi_1, \xi_2, 0)$ is orthogonal to L and the vector $\tilde{\xi}_a = (-\xi_1, \xi_2, a)$ is parallel to L for every a. By (3.1) we have

$$\int_L \langle v, \tilde{\xi}_a \rangle \langle v, \nu \rangle \, d\sigma_L = 0$$

and thus we get

$$\int_L (\xi_1 v^1 + \xi_2 v^2 + a v^3)(-\xi_2 v^1 + \xi_1 v^2) \, d\sigma_L = 0.$$

Substituting the values $a = 0$, $a = 1$ and taking the difference we get the equation $IF = 0$.

Therefore, by (2.5) we have

$$d_s w_0 = \nabla w_0 = -F \tag{3.3}$$

for a unique compactly supported smooth scalar function $w_0 = w_0(x)$.

Let us fix for a moment a point $P^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$, let $r \subset \Pi_{12} = \mathbb{R}^2$ be a ray emanating from P^0 and let e_r be a unit directional vector of r, then in virtue of (3.3) we have

$$\int_r \langle e_r, F \rangle \, ds_r = w_0(x_1^0, x_2^0) \tag{3.4}$$

for the line element ds_r of r. Let $H \subset \mathbb{R}^3$ be a half-plane perpendicular to Π_{12} with $\partial H = m(x_1^0, x_2^0, 0, 0)$, where $m(x_1^0, x_2^0, 0, 0)$ is the vertical line passing through the point $(x_1^0, x_2^0, 0) \in \Pi_{12}$; therefore, H orthogonally projects onto some ray r emanating from P_0. Let us consider the integral

$$\int_H v^3 \langle \nu_H, v \rangle \, d\sigma_H = -\int_r \langle e_r, F \rangle \, ds_r$$

for the area element $d\sigma_H$ of H and a suitable unit normal ν_H to H, then by (3.4) it does not depend on H for a fixed point $P^0 = (x_1^0, x_2^0)$ and a fixed line $\partial H = m(x_1^0, x_2^0, 0, 0)$. Since the choice of a vertical line in $\mathbb{R}^3$ is arbitrary we see that the following definition is correct:

Definition 3.2 Define

$$w = -\int_{H(m)} \langle e_m, v \rangle \langle \nu_{H(m)}, v \rangle \, d\sigma_{H(m)} \tag{3.5}$$

where $H(m)$ is a half-plane with $\partial H(m) = m$ and $\nu_{H(m)}$ is the unit normal to $H(m)$ such that the basis $(e_m, \nu_m, \nu_{H(m)})$ is positively oriented for the interior unit normal ν_m to m lying in $H(m)$.

Therefore, w is a compactly supported smooth function on M and it can be written as $w = w(y_1, y_2, \alpha_1, \alpha_2)$ on M_{nh}; moreover, we get

Lemma 3.3 *We have*

$$w(y_1, y_2, 0, 0) = w_0(y_1, y_2).$$

Proof Indeed, it is sufficient to verify that

$$\frac{\partial w}{\partial y_1}(0) = \int_{-\infty}^{\infty} v^2 v^3 \, dx_3, \quad \frac{\partial w}{\partial y_2}(0) = -\int_{-\infty}^{\infty} v^1 v^3 \, dx_3, \tag{3.6}$$

which is clear, since

$$w(\delta, 0, 0, 0) = -\int_{H_{1,\delta}} v^2 v^3 \, dx_1 dx_3 = -\int_{\delta}^{\infty} dx_1 \int_{-\infty}^{\infty} v^2 v^3 \, dx_3,$$
$$w(0, \mu, 0, 0) = \int_{H_{2,\mu}} v^1 v^3 \, dx_2 dx_3 = \int_{\mu}^{\infty} dx_2 \int_{-\infty}^{\infty} v^1 v^3 \, dx_3$$

for the half-planes

$$H_{1,\delta} = \{(x_1 > \delta, 0, x_3)\}, \quad H_{2,\mu} = \{(0, x_2 > \mu, x_3)\}.$$

Now we give two explicit formulas for w which use two specific choices of $H(m)$. We begin by putting

$$k_1 = \sqrt{1 + \alpha_1^2}, \quad k_2 = \sqrt{1 + \alpha_2^2}; \tag{3.7}$$

recall also that $k = \sqrt{1 + \alpha_1^2 + \alpha_2^2}$.

Given a line $m \in M_{nh}$, let $H(m)_1$ and $H(m)_2$ be the half-planes with the border-line m which are determined by the following conditions:

(i) $H(m)_1$ is parallel to x_1-axis, $H(m)_2$ is parallel to x_2-axis;
(ii) $\langle \nu_i, e_i \rangle > 0, \; i = 1, 2,$

for $e_1 = (1, 0, 0), \; e_2 = (0, 1, 0)$ and the internal normals $\nu_i \in H(m)_i, i = 1, 2.$

We have then

$$\nu_{H(m)_1} = \left(0, \frac{1}{k_2}, -\frac{\alpha_2}{k_2}\right), \quad \nu_{H(m)_2} = \left(-\frac{1}{k_1}, 0, \frac{\alpha_1}{k_1}\right),$$

and the plane $H(m)_1$ forms angle β_1 with the coordinate plane $\Pi_{13} = \{x_2 = 0\}$ where $\cos \beta_1 = 1/k_2$, while the plane $H(m)_2$ forms angle β_2 with the coordinate plane $\Pi_{23} = \{x_1 = 0\}$, $\cos \beta_2 = 1/k_1$. Note also that we have

$$e_m = \frac{1}{k}(\alpha_1, \alpha_2, 1) = \left(\frac{\alpha_1}{k}, \frac{\alpha_2}{k}, \frac{1}{k}\right)$$

for the positive unit directional vector e_m of m.

Proposition 3.4 *Let $d\sigma_i$ be the surface area element on $H(m)_i$ and let $l_i = y_i + x_3\alpha_i$ for $i = 1, 2$. Then in the introduced notation we have*

$$\begin{aligned}
\text{(i)}\quad w &= -\int_{H(m)_2} \langle e_m, v\rangle\langle \nu_{H(m)_2}, v\rangle\, d\sigma_2 \\
&= -\int_{-\infty}^{\infty}\int_{l_2}^{\infty} (\langle e_m, v\rangle\langle \nu_{H(m)_2}, v\rangle)|_{(l_1,x_2,x_3)} k_1 dx_2 dx_3 \\
&= \int_{-\infty}^{\infty}\int_{l_2}^{\infty} \frac{1}{k}((\alpha_1 v^1 + \alpha_2 v^2 + v^3)(v^1 - \alpha_1 v^3))|_{(l_1,x_2,x_3)} dx_2 dx_3 \qquad (3.8)
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad w &= -\int_{H(m)_1} \langle e_m, v\rangle\langle \nu_{H(m)_1}, v\rangle\, d\sigma_1 \\
&= -\int_{-\infty}^{\infty}\int_{l_1}^{\infty} (\langle e_m, v\rangle\langle \nu_{H(m)_1}, v\rangle)|_{(x_1,l_2,x_3)} k_2\, dx_1 dx_3 \\
&= -\int_{-\infty}^{\infty}\int_{l_1}^{\infty} \frac{1}{k}((\alpha_1 v^1 + \alpha_2 v^2 + v^3)(v^2 - \alpha_2 v^3))|_{(x_1,l_2,x_3)}\, dx_1 dx_3.
\end{aligned} \qquad (3.9)$$

Proof This is an elementary calculation which we give only for $H(m)_1$, since the case of $H(m)_2$ is completely similar; note only that the choice of normals $\nu_{H(m)_1}$ and $\nu_{H(m)_2}$ comes from the orientation condition. Let us fix the values of y_1, y_2, α_1, and α_2, and let $H_i \supset H(m)_i$ be an affine plane containing $H(m)_i$ for $i = 1, 2$. Then an equation of H_1 is of the form $ax_2 + bx_3 + c = 0$, and therefore $c = -ay_2$. Since $e_m \in \bar{H}_1$ for the vector plane $\bar{H}_1$ parallel to H_1, we get $a\alpha_2 + b = 0$, and we can choose $a = 1$, $b = -\alpha_2$, so the equation takes the form

$$x_2 - x_3\alpha_2 - y_2 = 0,$$

and therefore $x_2 = x_3\alpha_2 + y_2$ on the half-plane $H(m)_1$. Since $\cos\beta_1 = \cos\arctan\alpha_2 = \frac{1}{k_2}$, we see that $d\sigma_1 = k_2\, dx_1 dx_3$. Then one notes that the orthogonal projection $\pi_{13}(m)$ of m on the coordinate plane $\Pi_{13} = \{x_2 = 0\}$ is given by

$$\pi_{13}(m) = \Pi_{13} \cap H_2 = \{x_1 = y_1 + \alpha_1 x_3\},$$

and thus $H(m)_1$ projects onto

$$\{x_1 > y_1 + x_3\alpha_1\} \subset \Pi_{13},$$

since $\langle \nu_2, e_2\rangle > 0$, which completes the proof.

The formulas (3.8) and (3.9) are somewhat cumbersome and use below only the following simple consequence.

Corollary 3.5 *In the first order of* (α_1, α_2), *ignoring terms with total* $deg_\alpha \geq 2$, *we have the following expressions*:

$$w = \int_{-\infty}^{\infty} \int_{l_2}^{\infty} (\alpha_2 v^1 v^2 + v^1 v^3 + \alpha_1 (v^1)^2 - \alpha_1 (v^3)^2)|_{(l_1, x_2, x_3)}\, dx_2 dx_3, \quad (3.10)$$

$$w = \int_{-\infty}^{\infty} \int_{l_1}^{\infty} (\alpha_2 (v^3)^2 - \alpha_1 v^1 v^2 - v^2 v^3 - \alpha_2 (v^2)^2)|_{(x_1, l_2, x_3)} dx_1 dx_3. \quad (3.11)$$

This corollary permits to calculate the quantities

$$\frac{\partial^m w}{\partial y_1^i \partial y_2^j \partial \alpha_1^k \partial \alpha_2^l}(0), \quad i + j + k + l = m,$$

for $k + l \leq 1$, and in particular, implies the following.

Corollary 3.6 *We have*

$$\frac{\partial^2 w}{\partial y_2 \partial \alpha_1}(0) = \int_{-\infty}^{\infty} \left((v^3)^2 - (v^1)^2 - x_3 \frac{\partial (v^3 v^1)}{\partial x_1} \right) \Bigg|_{(0,0,x_3)} dx_3, \quad (3.12)$$

$$\frac{\partial^2 w}{\partial y_1 \partial \alpha_2}(0) = \int_{-\infty}^{\infty} \left((v^2)^2 - (v^3)^2 + x_3 \frac{\partial (v^3 v^2)}{\partial x_2} \right) \Bigg|_{(0,0,x_3)} dx_3, \quad (3.13)$$

$$\frac{\partial^2 w}{\partial y_1^2}(0) + \frac{\partial^2 w}{\partial y_2^2}(0) = \int_{-\infty}^{\infty} \left(\frac{\partial (v^2 v^3)}{\partial x_1} - \frac{\partial \left(v^1 v^3\right)}{\partial x_2} \right) \Bigg|_{(0,0,x_3)} dx_3. \quad (3.14)$$

Proof of (3.12) From (3.10) we have

$$w(0, y_2, \alpha_1, 0) = \int_{-\infty}^{\infty} \int_{y_2}^{\infty} (v^1 v^3 + (v^1)^2 \alpha_1 - (v^3)^2 \alpha_1)|_{(x_3 \alpha_1, x_2, x_3)}\, dx_2 dx_3,$$

whence we get

$$\frac{\partial w(0, 0, \alpha_1, 0)}{\partial y_2} = - \int_{-\infty}^{\infty} (v^1 v^3 + (v^1)^2 \alpha_1 - (v^3)^2 \alpha_1)|_{(\alpha_1 x_3, 0, x_3)}\, dx_3$$

and, finally,

$$\frac{\partial^2 w}{\partial y_2 \partial \alpha_1}(0) = \int_{-\infty}^{\infty} \left((v^3)^2 - (v^1)^2 - x_3 \frac{\partial (v^3 v^1)}{\partial x_1} \right) |_{(0,0,x_3)}\, dx_3.$$

The proof of (3.13) is completely similar and that of (3.14) is even simpler.

Taking then the difference of (3.12) and (3.13) we get the following formula:

$$\frac{\partial^2 w}{\partial y_1 \partial \alpha_2}(0) - \frac{\partial^2 w}{\partial y_2 \partial \alpha_1}(0) = \int_{-\infty}^{\infty} (p + (v^1)^2 + (v^2)^2 - (v^3)^2)|_{(0,0,x_3)}\, dx_3. \quad (3.15)$$

Indeed, we have $\frac{\partial(v^3v^1)}{\partial x_1} + \frac{\partial(v^3v^2)}{\partial x_2} = -\frac{\partial(p+(v^3)^2)}{\partial x_3}$ by (1.1) and integrating

$$\int_{-\infty}^{\infty} x_3 \left(\frac{\partial(v^3v^1)}{\partial x_1} + \frac{\partial(v^3v^2)}{\partial x_2} \right)\Bigg|_{(0,0,x_3)} dx_3 = -\int_{-\infty}^{\infty} \frac{x_3 \partial(p+(v^3)^2)}{\partial x_3}\Bigg|_{(0,0,x_3)} dx_3$$

by parts we get (3.15).

4 Operators P and Δ_M

Let us define first an order 2 differential operator P on the space $C^2(M)$.

Recall that $M = G/(\mathbb{R} \times \mathrm{SO}(2)) = G'/(\mathbb{R} \times \mathrm{O}(2))$ for $G = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$ and $G' = G \cdot \{\pm I_3\}$.

Definition 4.1 Let $f \in C^2(M)$, $m_0 \in M$, and let $g(m_0) = 0 = (0,0,0,0)$ for $g \in G$. Then

$$Pf(m_0) =_{\mathrm{def}} Lf_g(0),$$

where $f_g(m) = f(g^{-1}(m))$ for any $m \in M$ and L is defined by (2.11).

Lemma 4.2 *This definition is correct.*

Proof We must verify that $Lf_g(0) = Lf_h(0)$ for any $g, h \in G$ such that $g(m_0) = h(m_0) = (0)$.

We put $u = g^{-1}h,\ F = f_u$, and thus we have to verify that $LF(0) = LF_u(0)$ for $u \in \mathbb{R} \times SO(2) = St_0$, $St_0 < G$ being the stabilizer of the vertical line. It is sufficient to verify the equality separately for $u \in \mathbb{R}$ and $u \in SO(2)$. It is clear for a vertical shift $u \in \mathbb{R}$ since L has constant coefficients; for a rotation $u \in SO(2)$ by angle θ in the horizontal plane one easily calculates

$$\begin{aligned} F_u(y_1, y_2, \alpha_1, \alpha_2) = F(&y_1 \cos\theta - y_2 \sin\theta, y_2 \cos\theta + y_1 \sin\theta, \alpha_1 \cos\theta \\ &- \alpha_2 \sin\theta, \alpha_2 \cos\theta + \alpha_1 \sin\theta), \end{aligned}$$

and a simple calculation shows the necessary equation, since we get

$$\begin{aligned} \frac{\partial^2 F_u(0)}{\partial\alpha_2\partial y_1} &= \cos^2\theta \frac{\partial^2 F(0)}{\partial\alpha_2\partial y_1} - \sin^2\theta \frac{\partial^2 F(0)}{\partial\alpha_1\partial y_2} + \cos\theta\sin\theta \left(\frac{\partial^2 F(0)}{\partial\alpha_1\partial y_1} - \frac{\partial^2 F(0)}{\partial\alpha_2\partial y_2} \right), \\ \frac{\partial^2 F_u(0)}{\partial\alpha_1\partial y_2} &= \cos^2\theta \frac{\partial^2 F(0)}{\partial\alpha_1\partial y_2} - \sin^2\theta \frac{\partial^2 F(0)}{\partial\alpha_2\partial y_1} + \cos\theta\sin\theta \left(\frac{\partial^2 F(0)}{\partial\alpha_1\partial y_1} - \frac{\partial^2 F(0)}{\partial\alpha_2\partial y_2} \right). \end{aligned}$$

The proof is finished.

We can now rewrite (3.15) as follows

$$
\begin{aligned}
P_0 w &= \int_{-\infty}^{\infty} (p + (v^1)^2 + (v^2)^2 - (v^3)^2)\, dx_3 \\
&= \int_{-\infty}^{\infty} (p + |v|^2 - 2(v^3)^2)\, dx_3
\end{aligned} \tag{4.1}
$$

for the operator P_0 being P evaluated at 0, which implies that

$$
\begin{aligned}
P w &= \int_m (p + |v|^2 - 2\langle v, e_m\rangle^2)\, ds \\
&= \int_m (p + |v|^2 - 2v \otimes v)\, ds = I Q_0(m)
\end{aligned} \tag{4.2}
$$

for the quadratic tensor field $Q_0 = (p + |v|^2)\delta_{ij} - 2v \otimes v$ and any $m \in M$, since P is G-invariant; therefore $Pw = I Q_0$ as functions on M.

We will also use the fiber-wise Laplacian $\Delta_M = \Delta_{y_1,y_2}$ acting in tangent planes to $\mathbb{S}^2$; it is defined by the usual formula

$$
\Delta_M f(m) = \frac{\partial^2 f(m)}{\partial {y_1}^2} + \frac{\partial^2 f(m)}{\partial {y_2}^2}
$$

for $f \in C^2(M)$ and a vertical line $m = m(y_1, y_2, 0, 0)$. For any $m \in M$ the value $\Delta_M f(m)$ is determined by the G-invariance condition as for the operator P above, and the rotational symmetry of Δ_{y_1,y_2} guarantees the correctness of that definition. Note that the operators P, Δ_M commute and note also that (2.10) implies that for $Q \in C_0^\infty(S^h, \mathbb{R}^3)$ there holds a commutation rule

$$
I(\Delta Q) = \Delta_M (I Q). \tag{4.3}
$$

Remark 4.1 The algebra $D_{G'}(M)$ of the G'-invariant differential operators on M is freely generated by Δ_M and P^2 as a commutative algebra, see Gonzalez and Helgason (1986).

One can also give explicit formulas for P and Δ_M in our coordinates, namely,

$$
P = k^2 L + \alpha_1 \frac{\partial}{\partial y_2} - \alpha_2 \frac{\partial}{\partial y_1}, \quad \Delta_M = k_1^2 \frac{\partial^2}{\partial y_1^2} + k_2^2 \frac{\partial^2}{\partial y_2^2} + 2\alpha_1\alpha_2 \frac{\partial^2}{\partial y_1 \partial y_2}. \tag{4.4}
$$

Now we deduce the principal linear differential equation for w.

Proposition 4.3 *We have*

$$
P^2 w = -4\Delta_M w. \tag{4.5}
$$

Proof We begin with the following simple result.

Lemma 4.4 *If $f \in C_0^\infty(\mathbb{R}^3)$ is a scalar function then $P(If) = 0$.*

Indeed, since P is G-invariant, it is sufficient to verify the equation at a single point $0 \in M$ which follows from (2.12) with $h = 0$.

Lemma 4.4 implies by (4.2) that

$$P^2 w = PIQ_0 = PI((p + |v|^2)\delta_{ij} - 2v \otimes v) = -2PI(v \otimes v) \tag{4.6}$$

for a compactly supported vector field v solving (1.1). Moreover, we have

$$I(v \otimes v)(y_1, y_2, \alpha_1, \alpha_2) = \int_{-\infty}^{\infty} ((v^3)^2 + 2v^1 v^3 \alpha_1 + 2v^2 v^3 \alpha_2)\, dx_3 + O(|\alpha|^2)$$

and thus by (3.14) we get

$$PI(v \otimes v)(0) = P_0 I(v \otimes v) = 2\int_{-\infty}^{\infty} \left(\frac{\partial\left(v^2 v^3\right)}{\partial x_1} - \frac{\partial\left(v^1 v^3\right)}{\partial x_2} \right) dx_3 = 2\Delta_M w(0),$$

hence $PI(v \otimes v) = 2\Delta_M w$ everywhere and $P^2 w = -4\Delta_M w$ by (4.6).

Corollary 4.5 *The equation*

$$IQ_0(m) = 0, \quad \forall m \in M \tag{4.7}$$

implies Conjecture 1.2.

Indeed, if $IQ_0(m) = 0$ then $\Delta_M w(m) = -\frac{1}{4}P^2 w(m) = -\frac{1}{4}PIQ_0(m) = 0$ and thus $w = 0$, since w is compactly supported.

Invariant definitions and the second proof of (4.5) Now let us give a description of P and Δ_M in terms of the Lie algebra $\mathfrak{g}$ of G. We have $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{t}(3) = \mathbb{R}^3 \oplus \mathbb{R}^3$ as vector spaces, where $\mathfrak{t}(3)$ is 3-dimensional and abelian. Thus, we can write any $g \in \mathfrak{g}$ as $g = (r; s) \in \mathfrak{so}(3) \oplus \mathfrak{t}(3)$, and the commutators in $\mathfrak{g}$ are given by

$$[(r_1; 0), (r_2; 0)] = (r_1 \times r_2; 0), \quad [(0; s_1), (0; s_2)] = 0, \quad [(r; 0), (0; s)] = (0; r \times s).$$

Let (R_1, R_2, R_3) be the standard basis of $\mathfrak{so}(3)$, and (S_1, S_2, S_3) be that of $\mathfrak{t}(3)$. Consider the following operators on M:

$$\tilde{\Delta}_M = S_1^2 + S_2^2 + S_3^2, \quad \tilde{P} = S_1 R_1 + S_2 R_2 + S_3 R_3, \tag{4.8}$$

where we denote simply by g the action on M of an element $g \in U(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$; therefore, S_i acts as the infinitesimal shift in the x_i-direction, and R_i as the infinitesimal rotation about x_i-axis.

Proposition 4.6 *We have $\tilde{\Delta}_M = \Delta_M$ and $\tilde{P} = P$.*

Proof First, the operators $\tilde{\Delta}_M$ and $\tilde{P}$ are G-invariant. Indeed, it follows from the rotational invariance of the quadratic form $x_1^2 + x_2^2 + x_3^2$ that $\tilde{\Delta}_M$ is rotationally invariant; for translations, the same follows from the commutation rule $S_i S_j = S_j S_i$ for $i, j = 1, 2, 3$.

To prove the invariance of $\tilde{P}$ under the x_3-axis rotation we verify that $\tilde{P}$ and R_3 commute which can be shown as follows:

$$[S_1 R_1, R_3] = -S_2 R_1 - S_1 R_2, \quad [S_2 R_2, R_3] = S_1 R_2 + S_2 R_1, \quad [S_3 R_3, R_3] = 0.$$

Similarly we get its invariance under the x_1- and x_2-axis rotations and thus its SO(3)-invariance, while its x_3-translations invariance follows from

$$[S_1 R_1, S_3] = -S_1 S_2, \quad [S_2 R_2, S_3] = S_1 S_2, \quad [S_3 R_3, S_3] = 0.$$

Since any line is $SO(3)$-conjugate to a vertical one, $\tilde{P}$ is G-invariant. Finally, we have $\tilde{\Delta}_M(m_0) = \Delta_M(m_0)$, $\tilde{P}(m_0) = P(m_0)$ for $m_0 = m(0, 0, 0, 0)$, which finishes the proof, since Δ_M and P are both G-invariant. Indeed, e.g., in $\tilde{P}(m_0)$ the terms $S_1 R_1 + S_2 R_2$ give $L(0)$, while $S_3 R_3$ vanishes since m_0 is invariant under both S_3 and R_3.

Second Proof of Proposition 4.3 Let $t \in \mathbb{R}$, and let $l_t = m(t, 0, 0, 0)$ be a vertical line in the plane Π_{13}; note that $\Pi_{13} = \bigcup_{t \in \mathbb{R}} l_t$.

Lemma 4.7 *For $f \in C_0^\infty(M)$ we have*

$$\int_{\mathbb{R}} P^2 f(l_t)\, dt = \int_{\mathbb{R}} S_2^2 R_2^2 f(l_t)\, dt. \tag{4.9}$$

Proof Define the operators A and B by $A = S_1 R_1 P,\ B = S_3 R_3 P$; then

$$P^2 = A + B + S_2 R_2 P = A + B + S_2 R_2 \left(S_1 R_1 + S_2 R_2 + S_3 R_3\right).$$

Since $Af(l_t)$ is a derivative of a function of t, while B vanishes identically on Π_{13}, we get that

$$\int_{\mathbb{R}} (Af(l_t) + Bf(l_t))dt = 0.$$

We have also

$$S_2 R_2 S_1 R_1 (f(l_t)) = S_1 S_2 R_2 R_1 (f(l_t)) - S_3 S_2 R_1 (f(l_t)),$$

thus the integral of the left-hand side is zero and the same is true for

$$S_2 R_2 S_3 R_3 (f(l_t)) = S_3 S_2 R_2 R_3 (f(l_t)) + S_1 S_2 R_3 (f(l_t));$$

since $S_2 R_2 S_2 R_2 = S_2^2 R_2^2$ we get the conclusion.

Lemma 4.8 *We have*

$$\int_{\mathbb{R}} R_2^2 w(l_t)\, dt = -4 \int_{\mathbb{R}} w(l_t)\, dt. \tag{4.10}$$

Proof Let us fix a positive constant $c < \frac{\pi}{2}$, and let $l_t^\theta = (\frac{t}{\cos(\theta)}, 0, \tan(\theta), 0)$ for any θ with $|\theta| < c$; therefore, l_t^θ is just the line l_t rotated (in the clockwise direction) through the angle θ about the origin in the plane Π_{13}, and for any t we have

$$R_2^2 w(l_t^\theta)|_{\theta=0} = \frac{\partial^2}{\partial\theta^2} w(l_t^\theta)|_{\theta=0}.$$

Let $e_1^\theta = (\cos\theta, 0, -\sin\theta),\ e_3^\theta = (\sin\theta, 0, \cos\theta)$ then we have

$$w(l_t^\theta) = \frac{1}{\cos\theta} \int_0^\infty \int_{l_t} \langle v, e_1^\theta\rangle \langle v, e_3^\theta\rangle|_{(\frac{t}{\cos\theta}+x_3\tan\theta, x_2, x_3)}\, dx_3 dx_2$$

by (3.8), and if we put $t = x_1\cos\theta - x_3\sin\theta$ we get that

$$\begin{aligned}\int_{\mathbb{R}} w(l_t^\theta)\, dt &= \frac{1}{\cos\theta}\int_0^\infty \int_{\mathbb{R}^2} \langle v, e_1^\theta\rangle\langle v, e_3^\theta\rangle\, dx_3 dx_2 dt \\ &= \int_{x_2>0} \langle v, e_1^\theta\rangle\langle v, e_3^\theta\rangle\, dx_1 dx_2 dx_3.\end{aligned}$$

Therefore, since $\langle v, e_1^\theta\rangle\langle v, e_3^\theta\rangle = v^1 v^3 \cos 2\theta + ((v^1)^2 - (v^3)^2)\frac{\sin 2\theta}{2}$ we have

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}\int_{\mathbb{R}} w(l_t^\theta)\, dt &= \int_{x_2>0} \frac{\partial^2}{\partial\theta^2}(\langle v, e_1^\theta\rangle\langle v, e_3^\theta\rangle)\, dx_1 dx_2 dx_3 \\ &= -4\int_{x_2>0} (\langle v, e_1^\theta\rangle\langle v, e_3^\theta\rangle)\, dx_1 dx_2 dx_3\end{aligned}$$

and evaluating at $\theta = 0$ we get a proof of Lemma 4.8.

We can finish now our second proof of (4.5). Indeed, (4.9) and (4.10) imply that

$$\int_{\mathbb{R}} P^2 w(l_t)\, dt = \int_{\mathbb{R}} S_2^2 R_2^2 w(l_t)\, dt = -4\int_{\mathbb{R}} S_2^2 w(l_t)\, dt = -4\int_{\mathbb{R}} \Delta_M w(l_t)\, dt. \tag{4.11}$$

If we define a function $F(x_1, x_2)$ on $\mathbb{R}^2$ by

$$F(x_1, x_2) = P^2 w(x_1, x_2, 0, 0) + 4\Delta_M w(x_1, x_2, 0, 0),$$

then the integral of F over the x_1-axis vanishes by (4.11). Changing the coordinate system x_1, x_2 in $\mathbb{R}^2$, we get the same for the integral of F over any line in the plane $\{x_1, x_2\}$. Thus $F = 0$ by Radon's theorem, and we get the conclusion.

Remark 4.2 One can compare (4.5) with results that can be deduced from (2.12) for $h = 2$. A simple direct calculation using (4.4) gives for $h = 2$

$$P^3\psi + 4P\Delta_M\psi = 0 \tag{4.12}$$

if $\psi = IQ$ for $Q \in C_0^\infty(S^2, \mathbb{R}^3)$. Applying then (4.12) to $\psi = \Delta_M w$ (which can be written as $\Delta_M w = IQ'$ for a certain Q' not given here) we obtain $P^3\Delta_M w + 4P\Delta_M^2 w = \Delta_M P(P^2 w + 4\Delta_M w) = 0$ and thus $P(P^2 w + 4\Delta_M w) = 0$ which is much weaker than (4.5) since the kernel of P is enormous.

However, it is possible to construct a function $u \in C_0^\infty(M)$ verifying

$$P\Delta_M u = -2\Delta_M w, \quad \Delta_M u = IQ_1$$

for some $Q_1 \in C_0^\infty(S^2, \mathbb{R}^3)$ and applying (4.12) to $\psi = \Delta_M u = IQ_1$ we get

$$0 = P^3\Delta_M u + 4\Delta_M^2 P u = -2\Delta_M(P^2 w + 4\Delta_M w),$$

and thus we reprove (4.5). We can define u similarly to (3.5) as follows

$$u = \int_{H(m)} \mathrm{dist}(P, m)(p + \langle \nu_{H(m)}, v\rangle^2) d\sigma_{H(m)},$$

where $\mathrm{dist}(P, m)$ is the distance from a point $P \in H(m)$ to m.

5 A Radon Plane Transform

Let us define a Radon tensor plane transform J as follows:

$$JQ(L) = \int_L \mathrm{tr}(Q_{|L})\, d\sigma_L \tag{5.1}$$

for an affine plane $L \subset \mathbb{R}^3$ and $Q \in C^\infty(S^2; \mathbb{R}^3)$ satisfying

$$|Q(x)| \le C(1 + |x|)^{-2-\varepsilon}, \tag{5.2}$$

for some $\varepsilon > 0$, where $Q_{|L}$ is the restriction onto L; then we get a bounded linear operator

$$J : \mathcal{S}(S^2; \mathbb{R}^3) \longrightarrow \mathcal{S}(\mathbb{R}P^3)$$

for the manifold $\mathbb{R}P^3$ parametrizing affine planes $L \subset \mathbb{R}^3$.

Proposition 5.1 *We have* $JQ_0(L) = 0$.

Proof We have for any affine plane $L \subset \mathbb{R}^3$ that

$$JQ_0(L) = 2\int_L (p + \langle v, \nu_L\rangle^2)\, d\sigma_L = 0.$$

Indeed, setting without loss of generality $L = \Pi_{12},\ \nu_L = e_3$ we get that

$$\begin{aligned}\frac{\partial JQ_0}{\partial x_3}(\Pi_{12}) &= \frac{\partial}{\partial x_3}\left(\int_L (p + \langle v, \nu_L\rangle^2)\, d\sigma_L\right)\\ &= \int_{\Pi_{12}} \frac{\partial (p + (v^3)^2)}{\partial x_3}\, dx_1 dx_2\\ &= -\int_{\Pi_{12}} \left(\frac{\partial (v^1 v^3)}{\partial x_1} + \frac{\partial (v^2 v^3)}{\partial x_2}\right) dx_1 dx_2 = 0.\end{aligned}$$

Therefore, $JQ_0(\Pi_{12})$ does not depend on x_3 and hence equals 0.

Let us explain in what Proposition 5.1 partially confirms (4.7) and thus Conjecture 1.2. One can verify that the condition $JQ_0(L) = 0, \forall L \in \mathbb{R}P^3$ is equivalent to the following equation for the components $\{q^{ij}\}$ of Q_0:

$$\sum_{i,j=1}^{3} \frac{\partial^2 q^{ij}}{\partial x_i \partial x_j} = \Delta \mathrm{tr}\, Q_0, \tag{5.3}$$

while $IQ_0(m) = 0,\ \forall m \in M$ is equivalent to the following system

$$\frac{2\,\partial q^{ij}}{\partial x_i \partial x_j} = \frac{\partial^2 q^{ii}}{\partial x_j^2} + \frac{\partial^2 q^{jj}}{\partial x_i^2}, \quad 1 \le i < j \le 3 \tag{5.4}$$

of three equations and their sum gives (5.3).

6 A Uniqueness Theorem

Now we can deduce Conjecture 1.1 from Conjecture 1.2.

Theorem 6.1 *Let* $(v, p) \in \mathbb{C}_0^\infty(\mathbb{R}^3)$ *be a solution of* (1.1)–(1.3) *and let the corresponding function* w *vanish everywhere on* M *then* $(v; p) = 0$ *everywhere.*

Proof For $m \in M$ denote by t a vector parallel to m and by n a vector perpendicular to m, then the equality $w = 0$ implies that

$$\int_m \langle v, t\rangle \langle v, n\rangle\, ds = 0. \tag{6.1}$$

Let $L \subset \mathbb{R}^3$ be an affine plane; let $v_n = \langle \nu_L, v\rangle$ and $v_t = v - v_n \nu_L$ be its components normal to L and tangent to L, respectively. Then by (6.1) we have $IV = 0$ for the vector field $V = v_n v_t$ on L, and hence $\mathrm{curl}_L\, V = 0$ for the curl operator curl_L on L which gives

$$v_n\, \mathrm{curl}_L\, v_t - \langle v_t^{\perp}, \nabla_L v_n\rangle = -v_n\omega_n - \langle v_t^{\perp}, \nabla_L v_n\rangle = 0 \tag{6.2}$$

for the normal component ω_n of $\omega = \mathrm{curl}\, v$ and the gradient operator ∇_L on L. We will apply (6.2) to various planes $L \subset \mathbb{R}^3$.

First take $L = \Pi_{12}$, then $v_t = (v^1, v^2, 0),\ v_n = v^3,\ V = v^3 v_t$ and we get

$$v^3\, \mathrm{curl}_L\, v_t - \langle v_t^{\perp}, \nabla v^3\rangle = v^3\left(\frac{\partial v^1}{\partial x_2} - \frac{\partial v^2}{\partial x_1}\right) - v^1\frac{\partial v^3}{\partial x_2} + v^2\frac{\partial v^3}{\partial x_1} = 0. \tag{6.3}$$

Let now

$$\Omega =_{\mathrm{def}} \{u \in \mathbb{R}^3 | v(u) \neq 0,\ \omega(u) \neq 0\}$$

and let $D =_{\mathrm{def}} \mathbb{R}^3\backslash\bar{\Omega}$. We can suppose that Ω is not empty, since otherwise in a neighborhood of a point x_0 where the maximum of $|v|$ is attained we have $\Delta v = -\,\mathrm{curl}\,\mathrm{curl}\, v + \nabla \mathrm{div}\, v = 0$ and thus v is harmonic which contradicts the maximum principle for harmonic fields. It follows then that $v = 0$ in this neighborhood and thus everywhere.

In orthonormal coordinates with $v^1(u_0) = v^2(u_0) = 0$ we get for $u_0 \in \Omega$ that

$$v^3\left(\frac{\partial v^1}{\partial x_2} - \frac{\partial v^2}{\partial x_1}\right)(u_0) = 0 \quad \text{and therefore} \quad \langle v, \omega\rangle(u_0) = 0. \tag{6.4}$$

Therefore $\langle v, \omega\rangle = 0$ holds everywhere on Ω, thus on $\mathbb{R}^3$ and differentiating this relation in the v-direction we obtain

$$\langle v\nabla v, \omega\rangle + \langle v\nabla\omega, v\rangle = 0; \tag{6.5}$$

using the commutation law

$$v\nabla\omega = \omega\nabla v \tag{6.6}$$

we get then from (6.5) that

$$\langle v\nabla v, \omega\rangle + \langle \omega\nabla v, v\rangle = 0. \tag{6.7}$$

In orthonormal coordinates x_1, x_2, x_3 with x_1 directed along v and x_2 directed along ω at u_0 we can rewrite (6.7) as follows

$$\frac{\partial v^1}{\partial x_2}(u_0) + \frac{\partial v^2}{\partial x_1}(u_0) = 0, \tag{6.8}$$

since $v(u_0) \neq 0$ and $\omega(u_0) \neq 0$. Below we always use that coordinate system.

Moreover, since the vector ω is directed along x_2, we get

$$\frac{\partial v^1}{\partial x_2}(u_0) = \frac{\partial v^2}{\partial x_1}(u_0) \quad \text{and therefore} \quad \frac{\partial v^1}{\partial x_2}(u_0) = \frac{\partial v^2}{\partial x_1}(u_0) = 0. \tag{6.9}$$

Let then $L = \Pi_{13}$, and thus $v_t = (v^1, 0, v^3)$, $v_n = v^2$, $V = v^2 v_t$. Since $v_n(u_0) = 0$, we get from (6.2) and (6.6) that

$$\frac{\partial v^2}{\partial x_3}(u_0) = 0 \quad \text{and hence} \quad \frac{\partial v^3}{\partial x_2}(u_0) = 0 \quad \text{and} \quad \frac{\partial \omega^3}{\partial x_1}(u_0) = 0. \tag{6.10}$$

Then, differentiating (6.4) with respect to x_1 and x_3 at u_0, we get

$$\frac{\partial \omega^1}{\partial x_1}(u_0) = 0 \quad \text{and} \quad \frac{\partial \omega^1}{\partial x_3}(u_0) = 0. \tag{6.11}$$

Now we take $L = \{x_2 + x_3 = 0\}$, therefore $v_n = \frac{v^2+v^3}{\sqrt{2}}$, $\nu_L = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $v_t = \left(v^1, \frac{v^2-v^3}{2}, \frac{v^3-v^2}{2}\right)$ and $V = \frac{v^2+v^3}{\sqrt{2}}\left(v^1, \frac{v^2-v^3}{\sqrt{2}}\right)_{\mathcal{B}}$ in the orthonormal basis $\mathcal{B} = \left\{e_1' = (1,0,0), e_2' = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}$. Since $v_n(u_0) = 0$ and the vector $v_t^{\perp}(u_0)$ is directed along e_1', we get from (6.2) that

$$v^1(u_0)\left(\frac{\partial v^2}{\partial x_2}(u_0) + \frac{\partial v^2}{\partial x_3}(u_0) - \frac{\partial v^3}{\partial x_3}(u_0) - \frac{\partial v^3}{\partial x_2}(u_0)\right) = 0$$

and thus

$$\frac{\partial v^2}{\partial x_2}(u_0) + \frac{\partial v^2}{\partial x_3}(u_0) - \frac{\partial v^3}{\partial x_3}(0) - \frac{\partial v^3}{\partial x_2}(u_0) = 0.$$

Therefore by (6.10) we get also

$$\frac{\partial v^2}{\partial x_2}(u_0) = \frac{\partial v^3}{\partial x_3}(u_0). \tag{6.12}$$

For $L = \{x_1 + x_2 = 0\}$ we then have $v_n = \frac{v^1+v^2}{\sqrt{2}}$, $\nu_L = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $v_t = \left(\frac{v^1-v^2}{2}, \frac{v^2-v^1}{2}, v^3\right)$ and thus $V = \frac{v^1+v^2}{\sqrt{2}}\left(\frac{v^1-v^2}{\sqrt{2}}, v^3\right)_{\mathcal{B}'}$ in the plane basis $\mathcal{B}' = \left\{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), (0,0,1)\right\}$. Since the vector $v_t^{\perp}(u_0)$ is directed along $(0,0,1)$ we get from (6.2) and (6.9) that

$$v^1(u_0)\left(\omega^2(u_0) + \frac{\partial v^1}{\partial x_3}(u_0) + \frac{\partial v^2}{\partial x_3}(u_0)\right) = 0;$$

therefore, we get by (6.10) that

$$\omega^2(u_0) = -\frac{\partial v^1}{\partial x_3}(u_0). \tag{6.13}$$

If a trajectory $\gamma = \gamma(t)$ of the flow v is parametrized by t, i.e. $\frac{d\gamma}{dt} = v$, we have in virtue of (6.6) a differential inequality

$$|q'(t)| \le C|q(t)|,$$

for the function $q(t) =_{\text{def}} \omega^2(\gamma(t))$ and a positive constant C. Therefore, if $q(0) \neq 0$ then $q(t) \neq 0$ for any $t \in \mathbb{R}$, thus any trajectory of v does not cross $\partial\Omega = \partial D$ and hence stays either in $\bar{\Omega}$ or in $\bar{D}$.

Using (6.9), we see that $|v|$ is constant on any trajectory Γ of the vector field ω and, conversely, (6.10) and (6.11) imply that ω has a constant direction on any trajectory γ of the flow v and therefore $\gamma \subset \Pi_\gamma$ is a plane curve for an affine plane Π_γ. Set $\xi = \omega/|\omega|$, then we can define the vector $\xi(\gamma)$ for any $\gamma \subset \Omega$ and $\xi(\gamma) \perp \Pi_\gamma$. For $z \in \Omega$ we get

$$\frac{\partial \xi}{\partial x_1}(z) = 0 \tag{6.14}$$

since $v(z)$ is parallel to x_1, $\omega(z)$ is parallel to x_2 and $\xi(z) = (0, 1, 0)$; thus (6.11) implies that

$$\frac{\partial \xi^1}{\partial x_3}(z) = 0. \tag{6.15}$$

Therefore ξ satisfies the Frobenius integrability condition $\langle \xi, \operatorname{curl} \xi \rangle = 0$ and hence in a neighborhood of z there exists a smooth function $U(x)$ with $\nabla U \neq 0$ parallel to ξ. Moreover, U is a first integral of the flow v since $\partial U/\partial v = 0$. Let then S be a level surface of U containing z, then S is foliated by the trajectories of v and the vector field ξ defines the Gauss map $\xi : S \to \mathbb{S}^2$. Since ξ is constant on the trajectories of v the image $\beta = \xi(S) \subset \mathbb{S}^2$ is a curve or a point. Moreover, β is orthogonal to the axis x_1 at $\xi(\gamma)$ by (6.14), (6.15) and (6.13) implies that γ is not a straight line. Thus we can choose $z' \in \gamma$, $z' \neq z$ and get that γ is orthogonal at $\xi(\gamma)$ to some line not parallel to x_1. Therefore rank $\xi(\gamma) = 0$, hence rank $\xi = 0$ on S, ξ is constant on S and thus S is a plane. We see that a neighborhood of z in $\mathbb{R}^3$ is foliated by planes invariant under the flow v.

Denote by $\gamma(s, t) \subset \Omega$ the trajectory of v passing through $z + (0, s, 0)$, and let L_s^z be the plane containing $\gamma(s, t)$. Then $L_s^z \perp \omega(z + (0, s, 0))$ and the planes L_s^z are invariant under the flow v and foliate a neighborhood of z in $\mathbb{R}^3$. Let $\lambda(s, y)$ for $y \in \Pi_{13}$ be an affine function on Π_{13} with the graph L_s^z and let $l_z(y) = \frac{\partial \lambda}{\partial s}(0, y)$ then l_z is an affine linear function. Denote by G a connected component of $\Pi_{13} \cap \Omega$, $z \in G$

then G is invariant under the flow v. We fix some $z' \in G$ and set

$$l = l_{z'},$$

then $l(z') = 1$.

Let $z_1, z_2 \in G$, then some neighborhoods of z_1 and z_2 in $\mathbb{R}^3$ are foliated by the same set of planes invariant under the flow v and thus the sets of planes $L_s^{z_1}$ and $L_s^{z_2}$ are the same after a reparametrization. Therefore l does not vanish in G and we have $l_{z_1} = l_{z_2}/l_{z_2}(z_1)$. Set now

$$h(t) = \frac{\partial \gamma(s,t)}{\partial s}|_{s=0}$$

then $h(0) = (0, 1, 0)$ and thus by (6.6) the vector field h is proportional to ω on $\gamma = \gamma(0, t)$. Since ω is orthogonal to Π_{13} on γ we get that $h(y) = (0, l(y), 0)$,

$$\frac{\partial v^2}{\partial x_2}(y) = \frac{\partial \ln l}{\partial v}(y) \tag{6.16}$$

for any $y \in \gamma$ and l does not vanish on γ.

It follows by (6.16) and (6.12) that

$$\frac{\partial v^3}{\partial x_3}(y) = \frac{\partial \ln l}{\partial v}(y) \tag{6.17}$$

and since $\operatorname{div} v = 0$ there holds (recall that x_1 is directed along $v(y)$)

$$\frac{\partial |v|}{\partial x_1}(y) = -2\frac{\partial \ln l}{\partial v}(y) = \frac{\partial \ln |v|}{\partial v}(y); \tag{6.18}$$

hence we get that

$$|v(y)| = \frac{C_\gamma}{l^2(y)} \tag{6.19}$$

along the trajectory γ for a positive constant C_γ depending on γ. Note also that equations (6.16)–(6.19) hold for any trajectory of v in G and hence by continuity in $\bar{G}$ outside the zero locus of l; in particular, we see that

$$v(y) \neq 0 \quad \text{for any } y \in \bar{G} \text{ with } l(y) \neq 0. \tag{6.20}$$

Let $z_0 \in \bar{G}$ be a point where the function $|v|$ attains its maximum on $\bar{G}$, then $z_0 \in \partial G$. Indeed, if it is not the case, we have

$$\frac{\partial v^1}{\partial x_3}(z_0) = 0,$$

and hence $\omega(z_0) = 0$ by (6.13) which implies $z_0 \in \partial G$.

Let γ_1 be the trajectory of v starting from $z_1 \in \partial G$ with $v(z_1) \neq 0$, then $\omega = 0$ on the whole trajectory γ_1 and $\gamma_1 \subset \partial\Omega$. Therefore $\nabla b = v \times \omega = 0$ on γ_1, where $b = p + \frac{1}{2}|v|^2$ is the Bernoulli function (see, e.g., Arnold and Khesin 1998) and we get

$$\nabla p = -\frac{1}{2}\nabla|v|^2 \quad \text{on } \gamma_1. \tag{6.21}$$

Let $y \in \gamma_1$ and let e be a unit vector in Π_{13} orthogonal to $v(y)$, then $\langle v_e(y), v(y)\rangle = 0$ for $v_e = (\nabla_e v^1, \nabla_e v^3)$ by (6.13) since $\omega(y) = 0$. Therefore $\langle \nabla|v|^2(y), e\rangle = 0$ and (6.21) implies that γ_1 is a straight line interval I which is finite since v has a compact support, v vanishes at its end points and thus $l|_I = 0$ by (6.20).

Let now $z_0 = 0$ and continue to assume that x_1 is directed along $v(0) \neq 0$ and x_2 is directed along $\omega(x) \neq 0$ for some $x \in G$ (the direction of $\omega(x)$ does not depend on x), then l is a linear function on Π_{13} vanishing on the x_1-axis: $l = Cx_3$ for $C \neq 0$. Denote now $D_\varepsilon^+ = B_\varepsilon \cap \{0 < x_3\}$ and $D_\varepsilon^- = B_\varepsilon \cap \{0 > x_3\}$, then we have $D_\varepsilon^+ \subset G$. Indeed, first note that $(D_\varepsilon^+ \cap \partial G) \cup (D_\varepsilon^- \cap \partial G) = \emptyset$ since otherwise the trajectory γ_1 starting from $z_1 \in (D_\varepsilon^+ \cap \partial G) \cup (D_\varepsilon^- \cap \partial G)$ leads to a contradiction since $l|_{\gamma_1} = 0$. Moreover, for the trajectory $\alpha_0(t)$ of $v^\perp = (-v^3, v^1)$ starting at 0 we have $\alpha_0(t) \in D_\varepsilon^+$ for a small $t > 0$ and $\alpha_0(t) \in D_\varepsilon^-$ for a small $t < 0$. Since $\omega \neq 0$ on G we get that $|v|$ strictly decreases along α_0 by (6.13) ($v(0)$ being parallel to x_1) while $\alpha_0(t)$ stays in G and since $|v|$ attains at 0 its maximum in $\bar{G}$ we get that $D_\varepsilon^- \bigcap G = \emptyset$; therefore, $D_\varepsilon^+ \subset G$.

Furthermore, any trajectory γ_s of v starting from the point $(0, 0, s) \in D_\varepsilon^+$ with $0 < s < \varepsilon$ and a sufficiently small $\varepsilon > 0$ is closed. Indeed, by (6.19) we may assume that C_{γ_s} strictly increases as a function of $s \in (0, \varepsilon)$ and thus γ_s with $s \in (0, \varepsilon)$ intersects the interval $(0, (0, \varepsilon))$ only once. By the Poincaré–Bendixson theorem we get that γ_s either

(i) tends to a limit, or
(ii) tends to a limit cycle $\rho \subset G$, or
(iii) is closed.

Since (i) contradicts (6.19) and (ii) implies that any trajectory γ_a with $s < a < \varepsilon$ tends to ρ which contradicts (6.19) as well, we get that (iii) holds. Moreover, any trajectory starting inside D_ε^+ enters the domain$\{x_3 > \delta\} \cap G$ for some fixed $\delta > 0$ and the union $A = \bigcup_{0<s<\varepsilon} \gamma_s \subset G$ is a topological annulus.

Note now that C_{γ_s} tends to zero for $s \to 0$ by (6.19). Any trajectory α of $v^\perp$ in A is orthogonal to the trajectories γ_s and thus intersects all γ_s while $|v|$ strictly decreases along α by (6.13) since $\omega^2 > 0$ on G. If $\lambda \in (0, \varepsilon)$ then

$$\inf_{\gamma_\lambda} |v| > |v(z)|$$

for a sufficiently small $s \in (0, \lambda)$ and some $z \in \gamma_s$, while the trajectory α_z of $v^\perp$ starting from z intersects γ_λ and $|v|$ strictly decreases along α_z which gives a contradiction and thus finishes the proof.

7 Vector Analysis' Framework

Let us briefly discuss Conjectures 1.1 and 1.2 in terms of the vector analysis for compactly supported tensor fields in $\mathbb{R}^3$. In this section we suppose that $v \in C_0^\infty(S^1, \mathbb{R}^3)$. We can rewrite (1.2) as follows

$$\operatorname{curl}(\operatorname{div}(v \otimes v)) = 0. \tag{7.1}$$

Proposition 7.1 *If* (7.1) *holds and the corresponding function* $w \in C_0^\infty(M)$ *is everywhere zero on* M *then*

$$\Psi = \Psi(v) =_{\text{def}} \sigma(\operatorname{curl}(v \otimes v)) = 0 \tag{7.2}$$

for the symmetrization Ψ *of the tensor field* $\operatorname{curl}(v \otimes v)$, *i.e.*

$$2\Psi^{ij} = \varepsilon_{ilm}\frac{\partial(v^j v^l)}{\partial x_m} + \varepsilon_{jlm}\frac{\partial(v^i v^l)}{\partial x_m},$$

where ε_{ijk} *is the standard permutation* (*pseudo-*)*tensor*, *giving the sign of the permutation* (ijk) *of* (123) *and the summation convention applies.*

Proof For any fixed value of x_1, we define the vector field

$$Z = v^1(v^2, v^3) = (v^1 v^2,\ v^1 v^3)$$

on the vertical plane $\Pi_{23}(x_1) = \{x_1, x_2, x_3\}$ with coordinates $\{x_2, x_3\}$ depending on x_1 as on a parameter.

We have then $IZ(m') = w_{\nu_m}(m) = \langle \nabla w, \nu_m \rangle$ for a line $m \subset \Pi_{23}(x_1)$, a normal ν_m to m and a line $m' \perp m \subset \Pi_{23}(x_1)$, thus $IZ = 0$ and hence the solenoidal component ${}^sZ$ equals zero, where $Z =^s Z + {}^pZ$ is the Helmholtz decomposition of the vector field Z. Therefore we have

$$\Psi^{11} = \operatorname{curl} Z = \operatorname{curl} {}^sZ = 0,$$

thus $\Psi^{ii} = 0$ for $i = 1, 2, 3$ and rotating the coordinate system in each plane $\{x_i, x_j\}$ through the angle $\frac{\pi}{4}$ we get $\Psi^{ij} = 0$ for all $1 \le i \le j \le 3$.

Moreover, the proof of Theorem 6.1 shows that the conditions $\Psi(v) = 0$ and $\operatorname{div} v = 0$ imply $v = 0$.

Therefore Conjectures 1.1 and 1.2 follow from

Conjecture 7.2 *If* $\operatorname{curl}(\operatorname{div}(v \otimes v)) = 0$ *then* $\sigma(\operatorname{curl}(v \otimes v)) = 0$.

Another equivalent statement can be formulated as follows

Conjecture 7.3 *If* $I(\operatorname{div}(v \otimes v)) = 0$ *then* $PI(v \otimes v) = 0$.

One can also ask whether the condition $\operatorname{curl}(\operatorname{div}(v \otimes v)) = 0$ implies that v is spherically symmetric, which would grant Conjectures 1.1, 1.2, 7.2 and 7.3.

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