Arnold Math J. (2017) 3:251–280
DOI 10.1007/s40598-017-0065-y

RESEARCH CONTRIBUTION

Vanishing Cycles and Cartan Eigenvectors

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To the memory of Bertram Kostant

Received: 19 December 2015 / Revised: 11 July 2016 / Accepted: 20 July 2016 /
Published online: 13 April 2017

Abstract Using the ideas coming from the singularity theory, we study the eigenvectors of the Cartan matrices of finite root systems, and of q-deformations of these matrices

Keywords Vanishing cycles · Sebastiani–Thom theorem · Ising model in a magnetic field · Integrable field theory · Purely elastic scattering theory

1 Introduction

Let $A(R)$ be the Cartan matrix of a finite root system R. The coordinates of its eigenvectors have an important meaning in the physics of integrable systems; we will say more on this below.

Revaz Ramazashvili contributed only to writing the Sect. 6.

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The aim of this note is to study these numbers and their q-deformations, using some results coming from the singularity theory.

We discuss three ideas:

(a) Cartan/Coxeter correspondence;
(b) Sebastiani–Thom product;
(c) Givental's q-deformations.

Let us explain what we are talking about.

Let us suppose that R is simply laced, i.e. of type A, D, or E. These root systems are in one-to-one correspondence with (classes of) simple singularities

$f : \mathbb{C}^N \to \mathbb{C}$, cf. Arnold et al. (1988). Under this correspondence, the root lattice $Q(R)$ is identified with the lattice of vanishing cycles, and the Cartan matrix $A(R)$ is the intersection matrix with respect to a *distinguished base*. The action of the Weyl group on $Q(R)$ is realized by Gauss–Manin monodromies—this is the Picard–Lefschetz theory (for some details see Sect. 2 below).

Remarkably, this geometric picture provides a finer structure: namely, the symmetric matrix $A = A(R)$ comes equipped with a decomposition

$$A = L + L^t \tag{1}$$

where L is a nondegenerate triangular "Seifert form", or "variation matrix". The matrix

$$C = -L^{-1}L^t \tag{2}$$

represents a Coxeter element of R; geometrically it is the operator of "classical monodromy".

We call the relation (1)–(2) between the Cartan matrix and the Coxeter element the *Cartan/Coxeter correspondence*. It works more generally for non-symmetric A (in this case (1) should be replaced by

$$A = L + U \tag{3}$$

where L is lower triangular and U is upper triangular), and is due to Coxeter, cf. Coxeter (1951), no. 1, p. 767, see Sect. 3 below.

In a particular case (corresponding to a bipartition of the Dynkin graph) this relation is equivalent to an observation by Steinberg, cf. Steinberg (1985), cf. Sect. 3.3 below.

This correspondence allows one to relate the eigenvectors of A and C, cf. Theorem 1.

A decomposition (1) will be called *a polarization* of the Cartan matrix A. In 4.1 below we introduce an operation of *Sebastiani–Thom*, or *joint* product $A * B$ of Cartan matrices (or of polarized lattices) A and B. The root lattice of $A * B$ is the tensor product of the root lattice of A and the root lattice of B. With respect to this operation the Coxeter eigenvectors factorize very simply.

For example, the lattices E_6 and E_8 decompose into three "quarks":

$$E_6 = A_3 * A_2 * A_1 \tag{4}$$

$$E_8 = A_4 * A_2 * A_1 \tag{5}$$

These decompositions are the main message from the singularity theory, and we discuss them in detail in this note.

We use (4), (5), and the Cartan/Coxeter correspondence to get expressions for all Cartan eigenvectors of E_6 and E_8; this is the first main result of this note, see Sects. 4.9, 4.11 below.

(An elegant expression for all the Cartan eigenvectors of all finite root systems was given by Dorey, cf. Dorey (1990, 1991) (a), Table 2 on p. 659.)

In the paper Givental (1988), A. Givental has proposed a q-twisted version of the Picard–Lefschetz theory, which gave rise to a q-deformation of A,

$$A(q) = L + qL^t. \tag{6}$$

Again, as Givental remarked, the decomposition (3) allows us to drop the assumption of symmetry in the definition above. In the last section, Sect. 5, we calculate the eigenvalues and eigenvectors of $A(q)$ in terms of the eigenvalues and eigenvectors of A. This is the second main result of this note.

It turns out that if λ is an eigenvalue of A then

$$\lambda(q) = 1 + (\lambda - 2)\sqrt{q} + q \tag{7}$$

will be an eigenvalue of $A(q)$. The coordinates of the corresponding eigenvector $v(q)$ are obtained from the coordinates of $v = v(1)$ by multiplication by appropriate powers of q; this is related to the fact that the Dynkin graph of A is a tree, cf. Sect. 5.2. For an example of E_8, see (26).

In physics, the coordinates of the Perron–Frobenius Cartan eigenvectors appear as particle masses in affine Toda field theories, cf. Dorey (1991), Mikhailov et al. (1981).

In a pioneering paper Zamolodchikov (1989a, b), has discovered an octuplet of particles of E_8 symmetry in the two-dimensional critical Ising model in a magnetic field, and calculated their masses, see Sect. 6.

The Appendix outlines some of the results of a neutron scattering experiment (Coldea et al. 2010), where the two lowest-mass E_8 particles of the Zamolodchikov's theory may have been observed. Some of us first learned about this experiment from a beautiful paper Kostant (2010).

2 Recollections from the Singularity Theory

Here we recall some classical constructions and statements, cf. Arnold et al. (1988).

2.1 Lattice of Vanishing Cycles

Let $f : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point at 0, with $f(0) = 0$. We will be interested only in polynomial functions (from the list below, cf. Sect. 2.4), so $f \in \mathbb{C}[x_1, \ldots, x_N]$. The *Milnor ring* of f is defined by

$$\mathrm{Miln}(f,0)=\mathbb{C}[[x_1,\ldots,x_N]]/(\partial_1 f,\ldots,\partial_N f)$$

where $\partial_i := \partial/\partial x_i$; it is a finite-dimensional commutative $\mathbb{C}$-algebra. (In fact, it is a Frobenius, or, equivalently, a Gorenstein algebra.) The number

$$\mu := \dim_{\mathbb{C}} \mathrm{Miln}(f,0)$$

is called the multiplicity or Milnor number of $(f,0)$.

A *Milnor fiber* is

$$V_z = f^{-1}(z)\cap \bar{B}_\rho$$

where

$$\bar{B}_\rho = \{(x_1,\ldots,x_N)|\ \sum |x_i|^2 \le \rho\}$$

for $1 \gg \rho \gg |z| > 0$.

For z belonging to a small disc $D_\epsilon = \{z\in\mathbb{C}|\ |z|<\epsilon\}$, the space V_z is a complex manifold with boundary, homotopically equivalent to a bouquet $\vee S^{N-1}$ of μ spheres, Milnor (2016).

The family of free abelian groups

$$Q(f;z) := \tilde{H}_{N-1}(V_z;\mathbb{Z}) \overset{\sim}{=} \mathbb{Z}^\mu,\ z\in \overset{\bullet}{D}_\epsilon := D_\epsilon\backslash\{0\}, \tag{8}$$

($\tilde{H}$ means that we take the reduced homology for $N=1$), carries a flat Gauss–Manin connection.

Take $t\in\mathbb{R}_{>0}\cap \overset{\bullet}{D}_\epsilon$; the lattice $Q(f;t)$ does not depend, up to a canonical isomorphism, on the choice of t. Let us call this lattice $Q(f)$. The linear operator

$$T(f): Q(f) \overset{\sim}{\longrightarrow} Q(f) \tag{9}$$

induced by the path $p(\theta)=e^{i\theta}t,\ 0\le\theta\le 2\pi$, is called the classical monodromy of the germ $(f,0)$.

In all the examples below $T(f)$ has finite order h. The eigenvalues of $T(f)$ have the form $e^{2\pi i k/h},\ k\in\mathbb{Z}$. The set of suitably chosen k's for each eigenvalue are called the *spectrum* of our singularity.

2.2 Morse Deformations

The $\mathbb{C}$-vector space $\mathrm{Miln}(f,0)$ may be identified with the tangent space to the base B of the miniversal deformation of f. For

$$\lambda\in B^0 = B\backslash\Delta$$

where $\Delta \subset B$ is an analytic subset of codimension 1, the corresponding function $f_\lambda : \mathbb{C}^N \to \mathbb{C}$ has μ nondegenerate Morse critical points with distinct critical values, and the algebra $\mathrm{Miln}(f_\lambda)$ is semisimple, isomorphic to $\mathbb{C}^\mu$.

Let $0 \in B$ denote the point corresponding to f itself, so that $f = f_0$, and pick $t \in \mathbb{R}_{>0} \cap \overset{\bullet}{D}_\epsilon$ as in Sect. 2.1.

Afterwards pick $\lambda \in B^0$ close to 0 in such a way that the critical values $z_1, \dots z_\mu$ of f_λ have absolute values $\ll t$.

As in Sect. 2.1, for each

$$z \in \tilde{D}_\epsilon := D_\epsilon \backslash \{z_1, \dots z_\mu\}$$

the Milnor fiber V_z has the homotopy type of a bouquet $\vee S^{N-1}$ of μ spheres, and we will be interested in the middle homology

$$Q(f_\lambda; z) = \tilde{H}_{N-1}(V_z; \mathbb{Z}) \overset{\sim}{=} \mathbb{Z}^\mu$$

The lattices $Q(f_\lambda; z)$ carry a natural bilinear product induced by the cup product in the homology which is symmetric (resp. skew-symmetric) when N is odd (resp. even).

The collection of these lattices, when $z \in \tilde{D}_\epsilon$ varies, carries a flat Gauss–Manin connection.

Consider an "octopus"

$$Oct(t) \subset \mathbb{C}$$

with the head at t: a collection of non-intersecting paths p_i ("tentacles") connecting t with z_i and not meeting the critical values z_j otherwise. It gives rise to a base

$$\{b_1, \dots, b_\mu\} \subset Q(f_\lambda) := Q(f_\lambda; t)$$

(called "distinguished") where b_i is the cycle vanishing when being transferred from t to z_i along the tentacle p_i, cf. Gabrielov (1973), Arnold et al. (1988).

The Picard–Lefschetz formula describes the action of the fundamental group $\pi_1(\tilde{D}_\epsilon; t)$ on $Q(f_\lambda)$ with respect to this basis. Namely, consider a loop γ_i which turns around z_i along the tentacle p_i, then the corresponding transformation of $Q(f_\lambda)$ is the reflection (or transvection) $s_i := s_{b_i}$, cf. Lefschetz (1950), Théorème fondamental, Ch. II, p. 23.

The loops γ_i generate the fundamental group $\pi_1(\tilde{D}_\epsilon)$. Let

$$\rho : \pi_1(\tilde{D}_\epsilon; t) \to GL(Q(f_\lambda))$$

denote the monodromy representation. The image of ρ, denoted by $G(f_\lambda)$ and called the *monodromy group of* f_λ, lies inside the subgroup $O(Q(f_\lambda)) \subset GL(Q(f_\lambda))$ of linear transformations respecting the above mentioned bilinear form on $Q(f_\lambda)$.

The subgroup $G(f_\lambda)$ is generated by s_i, $1 \le i \le \mu$.

As in Sect. 2.1, we have the monodromy operator

$$T(f_\lambda) \in G(f_\lambda),$$

the image by ρ of the path $p \subset \tilde{D}_\epsilon$ starting at t and going around all points $z_1, \ldots, z_\mu$.

This operator $T(f_\lambda)$ is now a product of μ simple reflections

$$T(f_\lambda) = s_1 s_2 \ldots s_\mu,$$

this is because the only critical value 0 of f became μ critical values $z_1, \ldots, z_\mu$ of f_λ.

One can identify the relative (reduced) homology $\tilde{H}_{N-1}(V_t, \partial V_t; \mathbb{Z})$ with the dual group $\tilde{H}_{N-1}(V_t; \mathbb{Z})^*$, and one defines a map

$$\mathrm{var} : \tilde{H}_{N-1}(V_t, \partial V_t; \mathbb{Z}) \to \tilde{H}_{N-1}(V_t; \mathbb{Z}),$$

called a *variation operator*, which translates to a map

$$L : Q(f_\lambda)^* \xrightarrow{\sim} Q(f_\lambda)$$

("Seifert form") such that the matrix $A(f_\lambda)$ of the bilinear form in the distinguished basis is

$$A(f_\lambda) = L + (-1)^{N-1} L^t,$$

and

$$T(f_\lambda) = (-1)^{N-1} L L^{-t}.$$

Cf. Lamotke (1975).

A choice of a path q in B connecting 0 with λ, enables one to identify $Q(f)$ with $Q(f_\lambda)$, and $T(f)$ will be identified with $T(f_\lambda)$.

The image $G(f)$ of the monodromy group $G(f_\lambda)$ in $GL(Q(f)) \cong GL(Q(f_\lambda))$ is called the monodromy group of f; it does not depend on a choice of a path q.

2.3 Sebastiani–Thom Factorization

If $g \in \mathbb{C}[y_1, \ldots, y_M]$ is another function, the sum, or **join** of two singularities $f \oplus g : \mathbb{C}^{N+M} \to \mathbb{C}$ is defined by

$$(f \oplus g)(x, y) = f(x) + g(y)$$

Obviously we can identify

$$\mathrm{Miln}(f \oplus g) \cong \mathrm{Miln}(f) \otimes \mathrm{Miln}(g)$$

Note that the function $g(y) = y^2$ is a unit for this operation.

It follows that the singularities $f(x_1, \ldots, x_N)$ and

$$f(x_1, \ldots, x_N) + x_{M+1}^2 + \cdots + x_{N+M}^2$$

are "almost the same". In order to have good signs (and for other purposes) it is convenient to add some squares to a given f to get $N \equiv 3 \mod (4)$.

The fundamental Sebastiani–Thom theorem, Sebastiani (1971), says that there exists a natural isomorphism of lattices

$$Q(f \oplus g) \stackrel{\sim}{=} Q(f) \otimes_{\mathbb{Z}} Q(g),$$

and under this identification the full monodromy decomposes as

$$T_{f \oplus g} = T_f \otimes T_g$$

Thus, if

$$\mathrm{Spec}(T_f) = \{e^{\mu_p \cdot 2\pi i/h_1}\}, \quad \mathrm{Spec}(T_f) = \{e^{\nu_q \cdot 2\pi i/h_2}\}$$

then

$$\mathrm{Spec}(T_{f \oplus g}) = \{e^{(\mu_p h_2 + \nu_q h_1) \cdot 2\pi i/h_1 h_2}\}$$

2.4 Simple Singularities

Cf. Arnold et al. (1988) (a), 15.1. They are:

$$\begin{gathered} x^{n+1}, \quad n \geq 1, \qquad (\mathrm{A_n}) \\ x^2 y + y^{n-1}, \quad n \geq 4 \qquad (\mathrm{D_n}) \\ x^4 + y^3 \qquad (\mathrm{E_6}) \\ x y^3 + x^3 \qquad (\mathrm{E_7}) \\ x^5 + y^3 \qquad (\mathrm{E_8}) \end{gathered}$$

Their names come from the following facts:

- their lattices of vanishing cycles may be identified with the corresponding root lattices;
- the monodromy group is identified with the corresponding Weyl group;
- the classical monodromy T_f is a Coxeter element, therefore its order h is equal to the Coxeter number, and

$$\mathrm{Spec}(T_f) = \{e^{2\pi i k_1/h}, \ldots, e^{2\pi i k_r/h}\}$$

where the integers

$$1 = k_1 < k_2 < \cdots < k_r = h - 1,$$

are the exponents of our root system.

We will discuss the case of E_8 in some details below.

3 Cartan–Coxeter correspondence

3.1 Lattices, Polarization, Coxeter Elements

Let us call *a lattice* a pair (Q, A) where Q is a free abelian group, and

$$A : Q \times Q \to \mathbb{Z}$$

a symmetric bilinear map ("Cartan matrix"). We shall identify A with a map

$$A : Q \to Q^\vee := Hom(Q, \mathbb{Z}).$$

A polarized lattice is a triple (Q, A, L) where (Q, A) is a lattice, and

$$L : \ Q \xrightarrow{\sim} Q^\vee$$

("variation", or "Seifert matrix") is an isomorphism such that

$$A = A(L) := L + L^\vee \tag{10}$$

where

$$L^\vee : Q = Q^{\vee\vee} \xrightarrow{\sim} Q^\vee$$

is the conjugate to L.

The *Coxeter automorphism* of a polarized lattice is defined by

$$C = C(L) = -L^{-1}L^\vee \in GL(Q). \tag{11}$$

We shall say that the operators A and C are in a *Cartan–Coxeter correspondence*.

Example Let (Q, A) be a lattice, and $\{e_1, \ldots, e_n\}$ an ordered $\mathbb{Z}$-base of Q. With respect to this base A is expressed as a symmetric matrix $A = (a_{ij}) = A(e_i, e_j) \in \mathfrak{gl}_n(\mathbb{Z})$. Let us suppose that all a_{ii} are even. We define the matrix of L to be the unique upper triangular matrix (ℓ_{ij}) such that $A = L + L^t$ (in particular $\ell_{ii} = a_{ii}/2$; in our examples we will have $a_{ii} = 2$.) We will call L the *standard polarization* associated to an ordered base. □

Polarized lattices form a groupoid:

an isomorphism of polarized lattices $f : \ (Q_1, A_1, L_1) \xrightarrow{\sim} (Q_2, A_2, L_2)$ is by definition an isomorphism of abelian groups $f : Q_1 \xrightarrow{\sim} Q_2$ such that

$$L_1(x, y) = L_2(f(x), f(y))$$

(and whence $A_1(x, y) = A_2(f(x), f(y))$).

3.2 Orthogonality

Lemma 1 (i) *(orthogonality)*

$$A(x, y) = A(Cx, Cy).$$

(ii) *(gauge transformations) For any $P \in GL(Q)$*

$$A(P^\vee LP) = P^\vee A(L)P,\ C(P^\vee LP) = P^{-1}C(L)P.$$

□

3.3 Black/white Decomposition and a Steinberg'S Theorem

Cf. Steinberg (1985), Casselman (2017). Let $\alpha_1, \dots, \alpha_r$ be a base of simple roots of a finite reduced irreducible root system R (not necessarily simply laced). Let

$$A = (a_{ij}) = (\langle \alpha_i, \alpha_j^\vee \rangle)$$

be the Cartan matrix.

Choose a black/white coloring of the set of vertices of the corresponding Dynkin graph $\Gamma(R)$ in such a way that any two neighbouring vertices have different colours; this is possible since $\Gamma(R)$ is a tree (cf. Sect. 5.2).

Let us choose an ordering of simple roots in such a way that the first p roots are black, and the last $r - p$ roots are white. In this base A has a block form

$$A = \begin{pmatrix} 2I_p & X \\ Y & 2I_{r-p} \end{pmatrix}$$

Consider a Coxeter element

$$C = s_1 s_2 \cdots s_r = C_B C_W, \tag{12}$$

where

$$C_B = \prod_{i=1}^{p} s_i, \quad C_W = \prod_{i=p+1}^{r} s_i.$$

Here s_i denotes the simple reflection corresponding to the root α_i.

The matrices of C_B, C_W with respect to the base $\{\alpha_i\}$ are

$$C_B = \begin{pmatrix} -I & -X \\ 0 & I \end{pmatrix}, \quad C_W = \begin{pmatrix} I & 0 \\ -Y & -I \end{pmatrix},$$

so that

$$C_B + C_W = 2I - A. \tag{13}$$

This is an observation due to R.Steinberg, cf. Steinberg (1985), p. 591.

We can also rewrite this as follows. Set

$$L = \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}, \quad U = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Then $A = L + U$, and one checks easily that

$$C = -U^{-1}L, \tag{14}$$

so we are in the situation Sect. 3.1. This explains the name "Cartan–Coxeter correspondence".

3.4 Eigenvectors' Correspondence

Theorem 1 *Let*

$$L = \begin{pmatrix} I_p & 0 \\ Y & I_{r-p} \end{pmatrix}, \quad U = \begin{pmatrix} I_p & X \\ 0 & I_{r-p} \end{pmatrix}$$

be block matrices. Set

$$A = L + U, \ C = -U^{-1}L.$$

Let $\mu \neq 0$ *be a complex number,* $\sqrt{\mu}$ *be any of its square roots, and*

$$\lambda = 2 - \sqrt{\mu} - 1/\sqrt{\mu}. \tag{15}$$

Then a vector $v_C = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *is an eigenvector of* C *with eigenvalue* μ *if and only if*

$$v_A = \begin{pmatrix} v_1 \\ \sqrt{\mu} v_2 \end{pmatrix}$$

is an eigenvector of A *with the eigenvalue* λ.[1]

Proof A direct check. □

[1] This formulation has been suggested by A. Givental.

3.4.1 Remark

Note that the formula (15) gives two possible values of λ corresponding to $\pm\sqrt{\mu}$. On the other hand, λ does not change if we replace μ by μ^{-1}.

In the simplest case of 2×2 matrices the eigenvalues of A are $2 \pm (\sqrt{\mu} + \sqrt{\mu^{-1}})$, whereas the eigenvalues of C are $\mu^{\pm 1}$.

Corollary 1 *In the notations of Sect. 3.1, a vector*

$$x = \sum x_j \alpha_j$$

is an eigenvector of A with the eigenvalue $2(1 - \cos\theta)$ *iff the vector*

$$x_c := \sum e^{\pm i\theta/2} x_j \alpha_j$$

where the sign in $e^{\pm i\theta/2}$ *is plus if* i *is a white vertex, and minus otherwise, is an eigenvector ofC with eigenvalue* $e^{2i\theta}$.

Cf. [Brillon and Schechtman (2016), Kostant (1959)].

Proof Without loss of generality, we can suppose that A is expressed in a basis of simple roots such that the first $r - p$ ones are white, and the last p roots are black.

Then A has a block form

$$A = \begin{pmatrix} 2I_{r-p} & X \\ Y & 2I_p \end{pmatrix} = \begin{pmatrix} I_{r-p} & 0 \\ Y & I_p \end{pmatrix} + \begin{pmatrix} I_{r-p} & X \\ 0 & I_p \end{pmatrix} = L + U$$

Applying Theorem 1 with

$$v_1 = \begin{pmatrix} e^{i\theta/2} x_1 \\ .. \\ e^{i\theta/2} x_{r-p} \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} e^{-i\theta/2} x_{r-p+1} \\ .. \\ e^{-i\theta/2} x_r \end{pmatrix}$$

and the well-known eigenvalues of the Cartan matrix A,

$$\lambda = 2 - 2\cos\theta_k, \quad \text{with} \quad \theta_k = 2\pi k/h, k \in \mathrm{Exp}(R)$$

we obtain : $x_c := \sum e^{\pm i\theta/2} x_j \alpha_j$ is an eigenvector of C with the eigenvalue $e^{2i\theta_k}$ iff $e^{i\theta_k} x = e^{i\theta_k} \sum x_j \alpha_j$ is an eigenvector of A with the eigenvalue $2 - 2\cos\theta_k$. □

3.5 Example: The Root Systems A_n

We consider the Dynkin graph of A_n with the obvious numbering of the vertices.

The Coxeter number $h = n + 1$, the set of exponents:

$$\mathrm{Exp}(A_n) = \{1, 2, \dots, n\}$$

The eigenvalues of any Coxeter element are $e^{i\theta_k}$, and the eigenvalues of the Cartan matrix $A(A_n)$ are $2 - 2\cos\theta_k$, $\theta_k = 2\pi k/h$, $k \in \mathrm{Exp}(A_n)$.

An eigenvector of $A(A_n)$ with the eigenvalue $2 - 2\cos\theta$ has the form

$$x(\theta) = \left(\sum_{k=0}^{n-1} e^{i(n-1-2k)\theta}, \sum_{k=0}^{n-2} e^{i(n-2-2k)\theta}, \ldots, 1\right) \tag{16}$$

Denote by $C(A_n)$ the Coxeter element

$$C(A_n) = s_1 s_2 \cdots s_n$$

Its eigenvector with the eigenvalue $e^{2i\theta}$ is:

$$X_{C(A_n)} = (\sum_{k=0}^{n-j} e^{2ik\theta})_{1 \le j \le n}$$

For example, for $n = 4$:

$$C_{A_4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad X_{C(A_4)} = \begin{pmatrix} 1 + e^{2i\theta} + e^{4i\theta} + e^{6i\theta} \\ 1 + e^{2i\theta} + e^{4i\theta} \\ 1 + e^{2i\theta} \\ 1 \end{pmatrix}$$

is an eigenvector with eigenvalue $e^{2i\theta}$.

Similarly, for $n = 2$: s

$$C_{A_2} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad X_{C(A_2)} = \begin{pmatrix} 1 + e^{2i\gamma} \\ 1 \end{pmatrix}$$

□

4 Sebastiani–Thom Product; Factorization of E_8 and E_6

4.1 Join Product

Suppose we are given two polarized lattices (Q_i, A_i, L_i), $i = 1, 2$.

Set $Q = Q_1 \otimes Q_2$, whence

$$L := L_1 \otimes L_2 : Q \xrightarrow{\sim} Q^\vee,$$

and define

$$A := A_1 * A_2 := L + L^\vee : Q \xrightarrow{\sim} Q^\vee$$

The triple (Q, A, L) will be called the **join**, or **Sebastiani–Thom**, product of the polarized lattices Q_1 and Q_2, and denoted by $Q_1 * Q_2$.

Obviously

$$C(L) = -C(L_1) \otimes C(L_2) \in GL(Q_1 \otimes Q_2).$$

It follows that if $\mathrm{Spec}(C(L_i)) = \{e^{2\pi i k_i/h_i},\ k_i \in K_i\}$ then

$$\mathrm{Spec}(C(L)) = \left\{-e^{2\pi i(k_1/h_1+k_2/h_2)},\ (k_1, k_2) \in K_1 \times K_2\right\} \tag{17}$$

4.2 E_8 Versus $A_4 * A_2 * A_1$: Elementary Analysis

The ranks:

$$r(E_8) = 8 = r(A_4)r(A_2)r(A_1);$$

the Coxeter numbers:

$$h(E_8) = h(A_4)h(A_2)h(A_1) = 5 \cdot 3 \cdot 2 = 30.$$

It follows that

$$|R(E_8)| = 240 = |R(A_4)||R(A_2)||R(A_1)|.$$

The exponents of E_8 are:

$$1, 7, 13, 19, 11, 17, 23, 29.$$

All these numbers, except 1, are primes, and these are all primes ≤ 30, not dividing 30.

They may be determined from the formula

$$\frac{i}{5} + \frac{j}{3} + \frac{1}{2} = \frac{30 + k(i, j)}{30}, \quad 1 \leq i \leq 4, \quad 1 \leq j \leq 2,$$

so

$$k(i, 1) = 1 + 6(i - 1) = 1, 7, 13, 19;$$
$$k(i, 2) = 1 + 10 + 6(i - 1) = 11, 17, 23, 29.$$

This shows that the exponents of E_8 are the same as the exponents of $A_4 * A_2 * A_1$.

The following theorem is more delicate.

4.3 Decomposition of $Q(E_8)$

Theorem 2 (Gabrielov, cf. Gabrielov (1973), Sect. 6, Example 3). *There exists a polarization of the root lattice $Q(E_8)$ and an isomorphism of polarized lattices*

$$\Gamma : Q(A_4) * Q(A_2) * Q(A_1) \xrightarrow{\sim} Q(E_8). \tag{18}$$

In the left hand side $Q(A_n)$ means the root lattice of A_n with the standard Cartan matrix and the standard polarization

$$A(A_n) = L(A_n) + L(A_n)^t$$

where the Seifert matrix $L(A_n)$ is upper triangular.

In the process of the proof, given in Sects. 4.4–4.6 below, the isomorphism Γ will be written down explicitly. Cf. Arnold et al. (1988), Chapter I, Sect. 4 (especially Fig. 39), and references to the articles by A'Campo and Gusein-Zade therein.

4.4 Beginning of the Proof

For $n = 4, 2, 1$, we consider the bases of simple roots $e_1, \ldots, e_n$ in $Q(A_n)$, with scalar products given by the Cartan matrices $A(A_n)$.

The tensor product of three lattices

$$Q_* = Q(A_4) \otimes Q(A_2) \otimes Q(A_1)$$

will be equipped with the "factorizable" basis in the lexicographic order:

$$(f_1, \ldots, f_8) := (e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_2 \otimes e_1, e_2 \otimes e_1 \otimes e_1, e_2 \otimes e_2 \otimes e_1,$$
$$e_3 \otimes e_1 \otimes e_1, e_3 \otimes e_2 \otimes e_1, e_4 \otimes e_1 \otimes e_1, e_4 \otimes e_2 \otimes e_1).$$

Introduce a scalar product (x, y) on Q_* given, in the basis $\{f_i\}$, by the matrix

$$A_* = A_4 * A_2 * A_1.$$

4.5 Gabrielov–Picard–Lefschetz Transformations α_m, β_m

Let $(Q, (,))$ be a lattice of rank r. We introduce the following two sets of transformations $\{\alpha_m\}$, $\{\beta_m\}$ on the set $Bases - cycl(Q)$ of cyclically ordered bases of Q.

If $x = (x_i)_{i \in \mathbb{Z}/r\mathbb{Z}}$ is a base, and $m \in \mathbb{Z}/r\mathbb{Z}$, we set

$$(\alpha_m(x))_i = \begin{cases} x_{m+1} + (x_{m+1}, x_m)x_m & \text{if } i = m \\ x_m & \text{if } i = m+1 \\ x_i & \text{otherwise} \end{cases}$$

Fig. 1 Gabrielov's ordering of E_8

$$\begin{array}{ccccccccccccc} 1 & - & 2 & - & 3 & - & 5 & - & 6 & - & 7 & - & 8 \\ & & & & | & & & & & & & & \\ & & & & 4 & & & & & & & & \end{array}$$

and

$$(\beta_m(x))_i = \begin{cases} x_m & \text{if } i = m-1 \\ x_{m-1} + (x_{m-1}, x_m)x_m & \text{if } i = m \\ x_i & \text{otherwise} \end{cases}$$

We define also a transformation γ_m by

$$(\gamma_m(x))_i = \begin{cases} -x_m & \text{if } i = m \\ x_i & \text{otherwise} \end{cases}$$

4.6 Passage from $A_4 * A_2 * A_1$ to E_8

Consider the base $f = \{f_1, \dots f_8\}$ of the lattice $Q_* := Q(A_4) \otimes Q(A_2) \otimes Q(A_1)$ described in Sect. 4.4, and apply to it the following transformation

$$G' = \gamma_2\gamma_1\beta_4\beta_3\alpha_3\alpha_4\beta_4\alpha_5\alpha_6\alpha_7\alpha_1\alpha_2\alpha_3\alpha_4\beta_6\beta_3\alpha_1, \tag{19}$$

cf. Gabrielov (1973), Example 3. Note that

$$\gamma_2\gamma_1 = \alpha_1^6, \tag{20}$$

cf. Brieskorn (1988).

Then the base $G'(f)$ has the intersection matrix given by the Dynkin graph of E_8, with the ordering indicated in Fig. 1 below.

This concludes the proof of Theorem 2. □

4.7 The Induced Map of Root Sets

By definition, the isomorphism of lattices Γ, (22), induces a bijection between the bases

$$g: \{f_1, \dots, f_8\} \xrightarrow{\sim} \{\alpha_1, \dots, \alpha_8\} \subset R(E_8).$$

where in the right hand side we have the base of simple roots, and a map

$$G: R(A_4) \times R(A_2) \times R(A_1) \to R(E_8), \; G(x, y, z) = \Gamma(x \otimes y \otimes z)$$

of sets of the same cardinality 240 which is not a bijection however: its image consists of 60 elements.

Note that the set of vectors $\alpha \in Q(E_8)$ with $(\alpha, \alpha) = 2$ coincides with the root system $R(E_8)$, cf. Serre (1970), Première Partie, Ch. 5, 1.4.3.

4.8 Passage to Bourbaki Ordering

The isomorphism G' (19) is given by a matrix $G' \in GL_8(\mathbb{Z})$ such that

$$A_G(E_8) = G'^t A_* G'$$

where we denoted

$$A_* = A(A_4) * A(A_2) * A(A_1),$$

the factorized Cartan matrix, and A_G denotes the Cartan matrix of E_8 with respect to the numbering of roots indicated on Fig. 1.

Now let us pass to the numbering of vertices of the Dynkin graph of type E_8 indicated in Bourbaki (2007) (the difference with Gabrielov's numeration is in three vertices 2, 3, and 4).

The Gabrielov's Coxeter element (the full monodromy) in the Bourbaki numbering looks as follows:

$$C_G(E_8) = s_1 \circ s_3 \circ s_4 \circ s_2 \circ s_5 \circ s_6 \circ s_7 \circ s_8$$

Lemma 2 *Let $A(E_8)$ be the standard Cartan matrix of E_8 from [B]:*

$$A(E_8) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Then

$$A(E_8) = G^t A_* G$$

and

$$C_G(E_8) = G^{-1} C_* G$$

where

$$C_* = C(Q(A_4) * Q(A_2) * Q(A_1)) = C(A_4) \otimes C(A_2) \otimes C(A_1),$$

Fig. 2 Bourbaki ordering of E_8

$$\begin{array}{ccccccccccccc} 1 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 & - & 8 \\ & & & & | & & & & & & & & \\ & & & & 2 & & & & & & & & \end{array}$$

is the factorized Coxeter element, and

$$G = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here

$$G = G'P$$

where P is the permutation matrix of passage from the Gabrielov's ordering in Fig. 1 to the Bourbaki ordering in Fig. 2.

4.9 Cartan Eigenvectors of E_8

To obtain the Cartan eigenvectors of E_8, one should pass from $C_G(E_8)$ to the "black/white" Coxeter element (as in Sect. 3.3)

$$C_{BW}(E_8) = s_1 \circ s_4 \circ s_6 \circ s_8 \circ s_2 \circ s_3 \circ s_5 \circ s_7$$

Any two Coxeter elements are conjugate in the Weyl group $W(E_8)$.

The elements $C_G(E_8)$ and $C_{BW}(E_8)$ are conjugate by the following element of $W(E_8)$:

$$C_G(E_8) = w^{-1} C_{BW}(E_8) w$$

where

$$w = s_7 \circ s_5 \circ s_3 \circ s_2 \circ s_6 \circ s_4 \circ s_5 \circ s_1 \circ s_3 \circ s_2 \circ s_4 \circ s_1 \circ s_3 \circ s_2 \circ s_1 \circ s_2$$

This expression for w can be obtained using an algorithm described in Casselman (2017), cf. also Brieskorn (1988).

Thus, if x_* is an eigenvector of $C_*(E_8)$ then

$$x_{BW} = wG^{-1}x_*$$

is an eigenvector of $C_{BW}(E_8)$. But we know the eigenvectors of $C_*(E_8)$, they are all factorizable.

This provides the eigenvectors of $C_{BW}(E_8)$, which in turn have very simple relation to the eigenvectors of $A(E_8)$, due to Theorem 1.

Conclusion: an expression for the eigenvectors of $A(E_8)$.

Let $\theta = \frac{a\pi}{5},\ 1 \le a \le 4, \gamma = \frac{b\pi}{3},\ 1 \le b \le 2, \delta = \frac{\pi}{2}$,

$$\alpha = \theta + \gamma + \delta = \pi + \frac{k\pi}{30},$$
$$k \in \{1, 7, 11, 13, 17, 19, 23, 29\}.$$

The 8 eigenvalues of $A(E_8)$ have the form

$$\lambda(\alpha) = \lambda(\theta, \gamma) = 2 - 2\cos\alpha$$

An eigenvector of $A(E_8)$ with the eigenvalue $\lambda(\theta, \gamma)$ is

$$X_{E_8}(\theta, \gamma) = \begin{pmatrix} \cos(\gamma + \theta - \delta) + \cos(\gamma - 3\theta - \delta) + \cos(\gamma - \theta - \delta) \\ \cos(2\gamma + 2\theta) \\ \cos(2\gamma) + \cos(2\gamma + 2\theta) + \cos(2\gamma - 2\theta) + \cos(4\theta) + \cos(2\theta) \\ \cos(\gamma + 3\theta - \delta) + \cos(\gamma + \theta - \delta) + \cos(-\gamma + 3\theta - \delta) \\ 2\cos(2\gamma) + 2\cos(2\gamma + 2\theta) + \cos(2\gamma - 2\theta) + \cos(2\gamma + 4\theta) + \cos(4\theta) + 2\cos(2\theta) + 1 \\ \cos(\gamma + 3\theta - \delta) + \cos(\gamma + \theta - \delta) \\ \cos(2\gamma) + \cos(2\theta - 2\delta) \\ \cos(\gamma - \theta - \delta) \end{pmatrix}$$

One can simplify it as follows:

$$X_{E_8}(\theta, \gamma) = - \begin{pmatrix} 2\cos(4\theta)\cos(\gamma - \theta - \delta) \\ -\cos(2\gamma + 2\theta) \\ 2\cos^2(\theta) \\ -2\cos(\gamma)\cos(3\theta - \delta) - \cos(\gamma + \theta - \delta) \\ -2\cos(2\gamma + 3\theta)\cos(\theta) + \cos(2\gamma) \\ -2\cos\theta\cos(\gamma + 2\theta - \delta) \\ -2\cos(\gamma + \theta - \delta)\cos(\gamma - \theta + \delta) \\ -\cos(\gamma - \theta - \delta) \end{pmatrix} \tag{21}$$

4.10 Perron–Frobenius and All That

The Perron–Frobenius eigenvector corresponds to the eigenvalue

$$2 - 2\cos\frac{\pi}{30},$$

and may be chosen as

$$v_{PF} = \begin{pmatrix} 2\cos\frac{\pi}{5}\cos\frac{11\pi}{30} \\ \cos\frac{\pi}{15} \\ 2\cos^2\frac{\pi}{5} \\ 2\cos\frac{2\pi}{30}\cos\frac{\pi}{30} \\ 2\cos\frac{4\pi}{15}\cos\frac{\pi}{5}+\frac{1}{2} \\ 2\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 2\cos\frac{\pi}{30}\cos\frac{11\pi}{30} \\ \cos\frac{11\pi}{30} \end{pmatrix}$$

Ordering its coordinates in the increasing order, we obtain

$$v_{PF<} = \begin{pmatrix} \cos\frac{11\pi}{30} \\ 2\cos\frac{\pi}{5}\cos\frac{11\pi}{30} \\ 2\cos\frac{\pi}{30}\cos\frac{11\pi}{30} \\ \cos\frac{\pi}{15} \\ 2\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 2\cos^2\frac{\pi}{5} \\ 2\cos\frac{4\pi}{15}\cos\frac{\pi}{5}+\frac{1}{2} \\ 2\cos\frac{2\pi}{30}\cos\frac{\pi}{30} \end{pmatrix}$$

In the Ref. Zamolodchikov (1989a, b), obtains the following expression for the PF vector:

$$v_{Zam}(m) = \begin{pmatrix} m \\ 2m\cos\frac{\pi}{5} \\ 2m\cos\frac{\pi}{30} \\ 4m\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 4m\cos\frac{\pi}{5}\cos\frac{2\pi}{15} \\ 4m\cos\frac{\pi}{5}\cos\frac{\pi}{30} \\ 8m\cos^2\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 8m\cos^2\frac{\pi}{5}\cos\frac{2\pi}{15} \end{pmatrix}$$

Setting $m = \cos\frac{11\pi}{30}$, we find indeed :

$$v_{PF<} = v_{Zam}\left(\cos\frac{11\pi}{30}\right)$$

4.11 Factorization of E_6

Theorem 3 (Gabrielov, cf. 1973, Sect. 6, Example 2). *There exists a polarization of the root lattice $Q(E_6)$ and an isomorphism of polarized lattices*

$$\Gamma_{E_6} : Q(A_3) * Q(A_2) * Q(A_1) \xrightarrow{\sim} Q(E_6). \tag{22}$$

The proof is exactly the same as for $Q(E_8)$. The passage from $A_3 * A_2 * A_1$ to E_6 is obtained by the following transformation

$$G'_{E_6} = \gamma_4\gamma_1\alpha_1\alpha_2\alpha_3\alpha_4\beta_6\beta_3\alpha_1$$

cf. Gabrielov (1973), Example 2.

After a passage from Gabrielov's ordering to Bourbaki's, we obtain a transformation

$$G_{E_6} = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in GL_6(\mathbb{Z})$$

such that

$$A(E_6) = G^t_{E_6} A_* G_{E_6} \quad \text{and} \quad C_G(E_6) = G^{-1}_{E_6} C_* G_{E_6}$$

where $A_* = A(A_3) * A(A_2) * A(A_1)$ and $C_* = C(A_3) \otimes C(A_2) \otimes C(A_1)$ and

$$C_G(E_6) = s_1 \circ s_3 \circ s_4 \circ s_2 \circ s_5 \circ s_6$$

$C_G(E_6)$ is the Gabrielov's Coxeter element in the Bourbaki numbering, cf. Bourbaki (2007).

Let $C_{BW}(E_6) = s_1 \circ s_4 \circ s_6 \circ s_2 \circ s_3 \circ s_5$ be the "black/white" Coxeter element. $C_G(E_6)$ and $C_{BW}(E_6)$ are conjugated by the following element of the Weyl group $W(E_6)$:

$$v = s_5 \circ s_3 \circ s_2 \circ s_4 \circ s_1 \circ s_3 \circ s_3 \circ s_1 \circ s_2$$

Thus, if x_* is an eigenvector of $C_*(E_6)$ then $x_{BW} = vG^{-1}_{E_6} x_*$ is an eigenvector of $C_{BW}(E_6)$.

Finally, let $\theta = \frac{a\pi}{4}$, $1 \le a \le 3$, $\gamma = \frac{b\pi}{3}$, $1 \le b \le 2$, $\delta = \frac{\pi}{2}$ and

$$\alpha = \theta + \gamma + \delta$$

The 6 eigenvalues of $A(E_6)$ have the form $\lambda(\alpha) = \lambda(\theta, \gamma) = 2 - 2\cos\alpha$. An eigenvector of $A(E_6)$ with the eigenvalue $\lambda(\alpha)$ is

$$X_{E_6}(\theta, \lambda) = \begin{pmatrix} \cos(3\gamma + 3\theta - \delta) \\ 2\cos^2\theta \\ -2\cos(3\gamma + 3\theta - \delta)\cos(\gamma + \theta - \delta) \\ -4\cos^2\theta\cos(\gamma + \theta - \delta) \\ 1 - 2\cos(2\gamma + 3\theta)\cos\theta \\ -2\cos(\gamma)\cos(\theta - \delta) \end{pmatrix}$$

5 Givental's q-Deformations

5.1 q-Deformations of Cartan Matrices

Let $A = (a_{ij})$ be a $n \times n$ complex matrix. We will say that A is a *generalized Cartan matrix* if

(i) for all $i \neq j$, $a_{ij} \neq 0$ implies $a_{ji} \neq 0$;
(ii) all $a_{ii} = 2$. If only (i) is fulfilled, we will say that A is a *pseudo-Cartan matrix*.

We associate to a pseudo-Cartan matrix A an unoriented graph $\Gamma(A)$ with vertices $1, \dots, n$, two vertices i and j being connected by an edge $e = (ij)$ iff $a_{ij} \neq 0$.

Let A be a generalized Cartan matrix. There is a unique decomposition

$$A = L + U$$

where $L = (\ell_{ij})$ (resp. $U = (u_{ij})$) is lower (resp. upper) triangular, with 1's on the diagonal.

We define a q-deformed Cartan matrix by

$$A(q) = qL + U$$

This definition is inspired by the q-deformed Picard–Lefschetz theory developed by Givental (1988).

Theorem 4 *Let A be a generalized Cartan matrix such that $\Gamma(A)$ is a tree.*

(i) *The eigenvalues of $A(q)$ have the form*

$$\lambda(q) = 1 + (\lambda - 2)\sqrt{q} + q \tag{23}$$

where λ is an eigenvalue of A.

(ii) *There exist integers $k_1, \dots, k_n$ such that if $x = (x_1, \dots, x_n)$ is an eigenvector of A for the eigenvalue λ then*

$$x(q) = (q^{k_1/2}x_1, \dots, q^{k_n/2}x_n) \tag{24}$$

is an eigenvector of $A(q)$ for the eigenvalue $\lambda(q)$.

The theorem will be proved after some preparations.

5.2 Let Γ be an unoriented tree with a finite set of vertices $I = V(\Gamma)$.

Let us pick a root of Γ, and partially order its vertices by taking the minimal vertex i_0 to be the bottom of the root, and then going "upstairs". This defines an orientation on Γ.

Lemma 3 *Suppose we are given a nonzero complex number b_{ij} for each edge $e = (ij), i < j$ of Γ. There exists a collection of nonzero complex numbers $\{c_i\}_{i \in I}$ such that*

$$b_{ij} = c_j / c_i, \ i < j.$$

for all edges (ij). We can choose the numbers c_i in such a way that they are products of some numbers b_{pq}.

Proof Set $c_{i_0} = 1$ for the unique minimal vertex i_0, and then define the other c_i one by one, by going upstairs, and using as a definition

$$c_j := b_{ij} c_i, \quad i < j.$$

Obviously, the numbers c_i defined in such a way, are products of b_{pq}. □

Lemma 4 *Let $A = (a_{ij})$ and $A' = (a'_{ij})$ be two pseudo-Cartan matrices with $\Gamma(A) = \Gamma(A')$. Set $b_{ij} := a'_{ij}/a_{ij}$. Suppose that*

$$b_{ij} = b_{ji}^{-1}. \tag{25}$$

for all $i \neq j$, and $a_{ii} = a'_{ii}$ for all i. Then there exists a diagonal matrix

$$D = \mathrm{Diag}(c_1, \ldots, c_r)$$

such that $A' = D^{-1} A D$.

Moreover, the numbers c_i may be chosen to be products of some b_{pq}.

Proof Let us choose a partial order $<_p$ on the set of vertices $V(\Gamma)$ as in Sect. 5.2.

Warning This partial order *differs* in general from the standard total order on $\{1, \ldots, n\}$.

Let us apply Lemma 3 to the collection of numbers $\{b_{ij}, \ i <_p j\}$. We get a sequence of numbers c_{ij} such that

$$b_{ij} = c_j / c_i$$

for all $i <_p j$. The condition (25) implies that this holds true for all $i \neq j$.

By definition, this is equivalent to

$$a'_{ij} = c_i^{-1} a_{ij} c_j,$$

i.e. to $A' = D^{-1} A D$. □

5.3 Proof of Theorem 4

Let us consider two matrices: $A(q) = (a(q)_{ij})$ with $a(q)_{ii} = 1 + q$

$$a(q)_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ qa_{ij} & \text{if } i > j \end{cases}$$

and

$$A'(q) = \sqrt{q}A + (1 - \sqrt{q})^2 I = (a(q)'_{ij})$$

with $a(q)'_{ii} = 1 + q$ and $a(q)'_{ij} = \sqrt{q}a(q)_{ij}$, $i \neq j$.

Thus, we can apply Lemma 4 to $A(q)$ and $A'(q)$. So, there exists a diagonal matrix D as above such that

$$A(q) = D^{-1}A'(q)D.$$

But the eigenvalues of $A'(q)$ are obviously

$$\lambda(q) = \sqrt{q}\lambda + (1 - \sqrt{q})^2 = 1 + (\lambda - 2)\sqrt{q} + q.$$

If v is an eigenvector of A for λ then v is an eigenvector of $A'(q)$ for $\lambda(q)$, and Dv will be an eigenvector of $A(q)$ for $\lambda(q)$. □

5.4 Remark (M. Finkelberg)

The expression (23) resembles the number of points of an elliptic curve X over a finite field $\mathbb{F}_q$. To appreciate better this resemblance, note that in all our examples λ has the form

$$\lambda = 2 - 2\cos\theta,$$

so if we set

$$\alpha = \sqrt{q}e^{i\theta}$$

("a Frobenius root") then $|\alpha| = \sqrt{q}$, and

$$\lambda(q) = 1 - \alpha - \bar{\alpha} + q,$$

cf. Ireland and Rosen (2013), Chapter 11, §1, Knapp (1992), Chapter 10, Theorem 10.5.

So, the Coxeter eigenvalues $e^{2i\theta}$ may be seen as analogs of "Frobenius roots of an elliptic curve over $\mathbb{F}_1$".

5.5 Examples

5.5.1 Standard Deformation for A_n

Let us consider the following q-deformation of $A = A(A_n)$:

$$A(q) = \begin{pmatrix} 1+q & -1 & 0 & \dots & 0 \\ -q & 1+q & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -q & 1+q \end{pmatrix}$$

Then

$$\mathrm{Spec}(A(q)) = \{\lambda(q) := 1 + (\lambda - 2)\sqrt{q} + q \mid \lambda \in \mathrm{Spec}(A(1))\}.$$

If $x = (x_1, \dots, x_n)$ is an eigenvector of $A = A(1)$ with eigenvalue λ then

$$x(q) = (x_1, q^{1/2}x_2, \dots, q^{(n-1)/2}x_n)$$

is an eigenvector of $A(q)$ with eigenvalue $\lambda(q)$.

5.5.2 Standard Deformation for E_8

A q-deformation:

$$A_{E_8}(q) = \begin{pmatrix} 1+q & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+q & 0 & -1 & 0 & 0 & 0 & 0 \\ -q & 0 & 1+q & -1 & 0 & 0 & 0 & 0 \\ 0 & -q & -q & 1+q & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q & 1+q & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 1+q & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q & 1+q & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q & 1+q \end{pmatrix}$$

Its eigenvalues are

$$\lambda(q) = 1 + q + (\lambda - 2)\sqrt{q} = 1 + q - 2\sqrt{q}\cos\theta$$

where $\lambda = 2 - 2\cos\theta$ is an eigenvalue of $A(E_8)$.

If $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is an eigenvector of $A(E_8)$ for the eigenvalue λ, then

$$X = (x_1, \sqrt{q}x_2, \sqrt{q}x_3, qx_4, q\sqrt{q}x_5, q^2x_6, q^2\sqrt{q}x_7, q^3x_8) \tag{26}$$

is an eigenvector of $A_{E_8}(q)$ for the eigenvalue $\lambda(q)$.

6 A Physicist's Appendix: Cobalt Niobate Producing an E_8 Chord

In this Section, we briefly describe the relation of Perron-Frobenius components, in the case of $R = E_8$, to the physics of certain magnetic systems as anticipated in a pioneering theoretical work (Zamolodchikov 1989a, b) and possibly observed in a beautiful neutron scattering experiment (Coldea et al. 2010).

6.1 One-Dimensional Ising Model in a Magnetic Field

(a) *The Ising Hamiltonian*

Let $W = \mathbb{C}^2$. Recall three Hermitian Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $\mathbb{C}$-span of $\sigma^x, \sigma^y, \sigma^z$ inside $End(W)$ is a complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$; the $\mathbb{R}$-span of the anti-Hermitian matrices $i\sigma^x, i\sigma^y, i\sigma^z$ is a real Lie subalgebra $\mathfrak{k} = \mathfrak{su}(2) \subset \mathfrak{g}$. The resulting representation of $\mathfrak{g}$ (or $\mathfrak{k}$) on W is what physicists refer to as the "spin-$\frac{1}{2}$ representation".

For a natural N, consider a 2^N-dimensional tensor product

$$V = \otimes_{n=1}^{N} W_n$$

with all $W_n = W$. We are interested in the spectrum of the following linear operator H acting on V:

$$H = H(J, h_z, h_x) = -J \sum_{n=1}^{N} \sigma_n^z \sigma_{n+1}^z - h_z \sum_{n=1}^{N} \sigma_n^z - h_x \sum_{n=1}^{N} \sigma_n^x, \tag{27}$$

where J, h_x, h_z are positive real numbers. Here for $A : W \longrightarrow W$, $A_n : V \longrightarrow V$ denotes an operator acting as A on the nth tensor factor and as the identity on all the other factors. By definition, $A_{N+1} := A_1$.

In keeping with the conditions of the experiment (Coldea et al. 2010), everywhere below we assume that N is very large ($N >> 1$), and that $0 < h_z << J$.

The space V arises as the space of states of a quantum-mechanical model describing a chain of N atoms on the plane $\mathbb{R}^2$ with coordinates (x, z). The chain is parallel to the z axis, and is subject to a magnetic field with a component h_z along the chain, and a component h_x along the x-axis. The W_n is the space of states of the n-th atom. Only the nearest-neighbor atoms interact, and the J parameterizes the strength of this interaction.

The operator H in the Eq. (27) is called the Hamiltonian, and its eigenvalues ϵ correspond to the energy of the system. It is a Hermitian operator (with respect to an obvious Hermitian scalar product on V), thus all its eigenvalues are real.

Consider also the *translation* operator T, acting as follows:

$$T(v_1 \otimes v_2 \otimes \ldots \otimes v_N) = v_2 \otimes v_3 \otimes \cdots \otimes v_N \otimes v_1, \tag{28}$$

The operator T is unitary, and commutes with H.

An eigenvector $v_0 \in V$ of H with the lowest energy eigenvalue ϵ_0 is called the ground state.

What happens as h_x varies, at fixed J and h_z? When $h_x << J$, the ground state v_0 is close to the ground state v_J of the operator $H_J = H(J, 0, 0)$:

$$v_J = \otimes_{n=1}^N v_n^z,$$

where v_n^z is an eigenvector of σ^z in W_i with eigenvalue 1. Thus, the state v_J is interpreted as "all the spins pointing along the z-axis".

In the opposite limit, when $h_x >> J$, the ground state v_0 is close to the ground state v_x of the operator $H_x = H(0, 0, h_x)$:

$$v_x = \otimes_{n=1}^N v_n^x,$$

where v_n^x is an eigenvector of σ^x in W_n with eigenvalue 1. Thus, the state v_x is interpreted as "all the spins pointing along the x-axis".

As a function of h_x at fixed J and h_z, the system has two phases. There is a critical value $h_x = h_c$, of the order of $J/2$: for $h_x < h_c$, the ground state v_0 is close to v_J, and one says that the chain is in the *ferromagnetic* phase. By contrast, for $h_x > h_c$, the ground state v_0 is close to v_x, and one says that the chain is in the *paramagnetic* phase. (The transition between the two phases is far less trivial than the spins simply turning to follow the field upon increasing h_x: to find out more, curious reader is encouraged to consult the Ref. Chakrabarti et al. (1996).)

(b) *Elementary excitations at $h_x = h_c$*

Zamolodchikov's theory (Zamolodchikov 1989a, b), says something spectacularly precise about the next few, after ϵ_0, eigenvalues ("energy levels") of a nearly-critical Hamiltonian $H_c := H(J, h_z << J, h_c)$. To see this, notice that the possible eigenvalues of the translation operator T have the form $e^{2\pi ik/N}$, with $-N/2 \leq k \leq N/2$; let us call the number

$$p = 2\pi k/N$$

the *momentum* of an eigenstate. Since H commutes with T, each eigenspace $V_\epsilon := \{v \in V |\ H_c v = \epsilon v\}$ decomposes further as per

$$V_\epsilon = \oplus_p V_{p,\epsilon},$$

where

$$V_{p,\epsilon} := \{v \in V |\ H_c v = \epsilon v, Tv = e^{ip} v\}$$

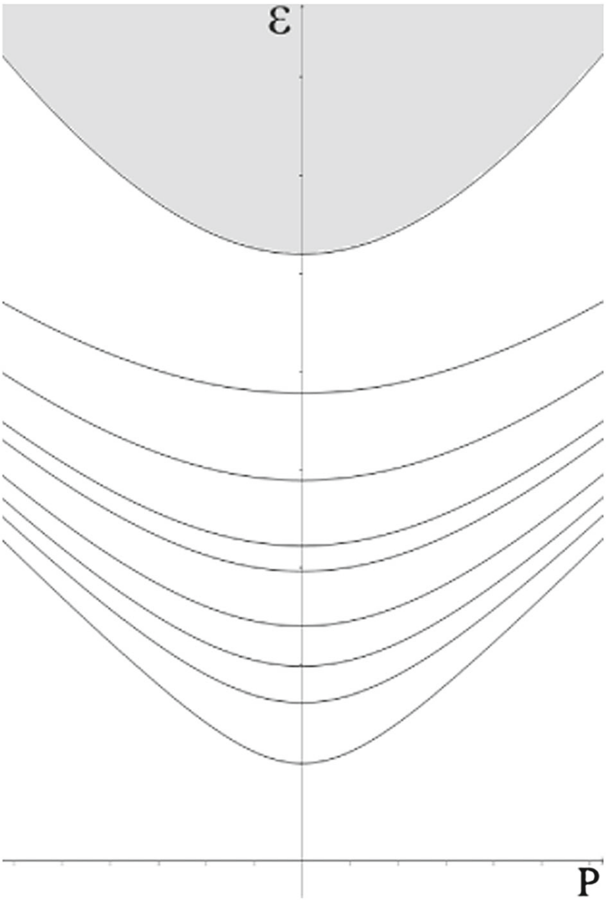

Fig. 3 The expected joint spectrum of the operators T, H

Let us add a constant to H_c in such a way that the ground state energy ϵ_0 becomes 0 and, on the plane P with coordinates (p, ϵ), let us mark all the points, for which $V_{p,\epsilon} \neq 0$.

Zamolodchikov predicted (Zamolodchikov 1989a, b), that there exist 8 numbers $0 < m_1 < \cdots < m_8$ with the following property. Let us draw on P eight hyperbolae

$$\mathrm{Hyp}_i : \epsilon = \sqrt{m_i^2 + p^2},\ 1 \leq i \leq 8. \tag{29}$$

All the marked points will be located:

- either in a vicinity of one of the hyperbolae Hyp_i (in the limit $N \longrightarrow \infty$ they will all lie on these hyperbolae).
- or in a shaded region separated from these hyperbolae as shown in the Fig. 3.

The states $v \in V_{p,\epsilon}$ with $(p, \epsilon) \in \mathrm{Hyp}_i$ are called *elementary excitations*. The numbers m_i are called their *masses*.

The vector

$$\mathbf{m} = (m_1, \ldots, m_8) \tag{30}$$

is proportional to the Perron–Frobenius $v_{PF<}$ for E_8 from Sect. 4.10, whose normalized approximate value is

$$v_{PF<} = (1, 1.62, 1.99, 2.40, 2.96, 3.22, 3.89, 4.78) \tag{31}$$

These low-lying excitations (hyperbolae) are observable: one may be able to see them

(a) in a computer simulation, or
(b) in a neutron scattering experiment.

6.2 Neutron Scattering Experiment

The paper (Coldea et al. 2010) reports the results of a magnetic neutron scattering experiment on cobalt niobate $CoNb_2O_6$, a material that can be pictured as a collection of parallel non-interacting one-dimensional chains of atoms. We depict such a chain as a straight line, parallel to the z-axis in our physical space $\mathbb{R}^3$ with coordinates x, y, z.

The sample, at low temperature $T < 2.95\,\text{K}$ (Kelvin), was subject to an external magnetic field with components (h_x, h_z), with the h_x at the critical value $h_x = h_c$, and with $h_z << h_c$. The system may be described as the Ising chain with a nearly-critical Hamiltonian $H = H(J, h_z << h_c, h_c)$ of the Eq. (27). The experiment (Coldea et al. 2010) may be interpreted with the help of the following (oversimplified) theoretical picture.

Consider a neutron scattering off the sample. If the incident neutron has energy ϵ and momentum p, and scatters off with energy ϵ' and momentum p', the energy and momentum conservation laws imply that the differences, called energy and momentum transfers $\omega = \epsilon - \epsilon'$, $q = p - p'$, are absorbed by the sample.

The energy transfer cannot be arbitrary. Suppose that, prior to scattering the neutron, the sample was in the ground state v_0; upon scattering the neutron, it undergoes a transition to a state that is a linear combination of the eight elementary excitations $v \in V_{p,\epsilon}$.

We will be interested in neutrons that scatter off with zero momentum transfer. The Zamolodchikov theory (Zamolodchikov 1989a, b) predicted, that the neutron scattering intensity $\mathcal{S}(0, \omega)$ should have peaks at $\omega = m_a$, $(a = 1, \ldots, 8)$ of the Eq. (31). At zero momentum transfer, a neutron scattering experiment would measure the proportion of neutrons that scattered off with the energies $m_1, \ldots, m_8$: the resulting $\mathcal{S}(0, \omega)$ would look as in the schematic Fig. 4. Metaphorically speaking, the crystal would thus "sound" as a "chord" of eight "notes": the eigenfrequencies m_i.

At the lowest temperatures, and in the immediate vicinity of $h_x = h_c$, the experiment (Coldea et al. 2010) succeeded to resolve the first two excitations, and to extract their masses m_1 and m_2. The mass ratio m_2/m_1 was found to be $\frac{m_2}{m_1} = 1.6 \pm 0.025$, consistent with $\frac{m_2}{m_1} = \frac{1+\sqrt{5}}{2} \approx 1.618$ of the expression for the $v_{Zam}(m)$ in the Sect.

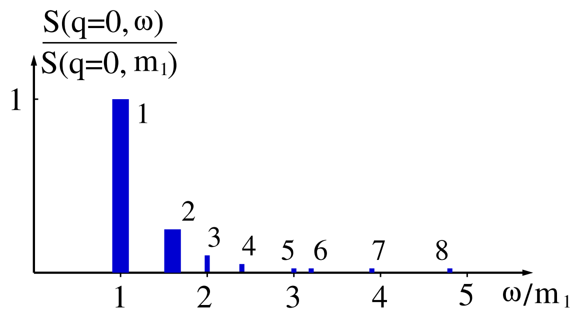

Fig. 4 A sketch of the scattering intensity $\mathcal{S}(0, \omega)$ at zero momentum peaks relative to $\mathcal{S}(0, m_1)$, against the ω/m_1 ratio. The two leftmost peaks shown by *thick lines* correspond to the excitations with the masses m_1 and m_2, that were resolved in the experiment (Coldea et al. 2010). The experimentally found mass ratio m_2/m_1 is consistent with $\frac{m_2}{m_1} = \frac{1+\sqrt{5}}{2}$, as per the expression for the $v_{Zam}(m)$ in the Sect. 4.10

4.10. In other words, the experimentalists were able to hear two of the eight notes of the Zamolodchikov E_8 chord.

A reader wishing to find out more about various facets of the story is invited to turn to the references (Rajaraman 1989; Delfino 2004; Gosslevi 2010; Borthwick and Garibaldi 2011).

Acknowledgements We are grateful to Misha Finkelberg, Andrei Gabrielov, and Sabir Gusein-Zade for the inspiring correspondence, and to Patrick Dorey for sending us his thesis. Our special gratitude goes to Sasha Givental whose remarks enabled us to generalize some statements and to simplify the exposition and to Maxim Kazarian for the enlightening comments on Gabrielov's theorem. A.V. thanks MPI in Bonn for hospitality; he was supported in part by NSF Grant DMS-1362924 and the Simons Foundation Grant No. 336826.

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