Arnold Math J. (2016) 2:149–158
DOI 10.1007/s40598-015-0034-2

RESEARCH CONTRIBUTION

N-Division Points of Hypocycloids

Nitya Mani[1] · Simon Rubinstein-Salzedo[2]

Received: 4 May 2015 / Revised: 19 October 2015 / Accepted: 6 December 2015 /
Published online: 4 January 2016

Abstract We show that the n-division points of all rational hypocycloids are constructible with an unmarked straightedge and compass for all integers n, given a pre-drawn hypocycloid. We also consider the question of constructibility of n-division points of hypocycloids without a pre-drawn hypocycloid in the case of a tricuspoid, concluding that only the 1, 2, 3, and 6-division points of a tricuspoid are constructible in this manner.

Keywords Compass and straightedge construction · Hypocycloid · Division points

1 Introduction

One of the oldest classes of problems in mathematics is concerned with straightedge and compass constructions. Most famous in this collection are the three classical Greek geometrical challenges: trisecting an angle, squaring a circle, and duplicating the cube (i.e. constructing a segment of length $\sqrt[3]{2}$) with an unmarked straightedge and compass, all of which have been shown to be impossible. The first and third of these are elementary exercises in field theory, while the second requires the nontrivial fact that π is transcendental (or at least some similarly nontrivial information about π).

However, straightedge and compass constructions have by no means been completely resolved by the work of the ancient Greeks or later mathematicians. One of

✉ Nitya Mani
nityam@stanford.edu

Simon Rubinstein-Salzedo
simon@eulercircle.com

1 Department of Mathematics, Stanford University, Stanford, CA 94305, USA

2 Euler Circle, Palo Alto, CA 94306, USA

the most interesting straightedge and compass problems asks which regular polygons can be constructed with the aforementioned two tools. In 1796, Carl Friedrich Gauss demonstrated that the 17-gon was constructible with straightedge and compass, leading him in 1801 to arrive at a general, sufficient condition for polygon constructibility. Pierre Wantzel showed the necessity of this condition in 1837, leading to the final statement of the Gauss–Wantzel Theorem.

Theorem 1.1 (Gauss–Wantzel) Cox (2012)*, Theorem 10.2.1 A regular n-gon can be constructed with a straightedge and compass if and only if n is of the form*

$$n = 2^m p_1 \dots p_s \tag{1.1}$$

for distinct Fermat primes p_i. (A Fermat prime is a prime p of the form $p = 2^{2^a} + 1$, a is a nonnegative integer).

A reinterpretation of the Gauss–Wantzel Theorem is that it is possible to divide a circle into n arcs of equal length with a straightedge and compass if and only if n is of the form (1.1). It is natural to ask the corresponding question for other closed curves.

Definition 1.2 The n-division points of a closed curve C are a set of n points on C that divide C into pieces of equal arclength.

There is one other classical theorem similar to the Gauss–Wantzel Theorem: Abel's Theorem on the Lemniscate.

Theorem 1.3 (Abel) *The n-division points of the lemniscate defined by $(x^2 + y^2)^2 = x^2 - y^2$ can be constructed with a straightedge and compass if and only if n is of the form*

$$n = 2^m p_1 \dots p_s$$

for distinct Fermat primes p_i.

Detailed expositions of this theorem can be found in Cox (2012) and Stewart (2004).

Surprisingly, Theorem 1.3 arrives at the same closed-form expression (1.1) for the constructible n-division points of the lemniscate as Gauss and Wantzel did for the constructible regular n-gons.

More recently, Cox and Shurman in (2005) have investigated the n-division points of other closed curves, including two members of a class of clover-like curves (the cardioid and a specific three-leaved clover) that include the lemniscate and circle. We continue the study of n-division points on curves by investigating the class of rational hypocycloids, defined in Sect. 3.

Remark 1.4 For curves other than the circle, there are actually *two* possible problems regarding the constructibility of n-division points. First, for a closed curve C, we can ask whether, given the curve C already drawn on the plane, it is possible to construct the n-division points using a compass and a straightedge. However, we can also ask whether it is possible to construct the n-division points of C *without* C ever being drawn. To us, the former question feels more natural: it seems like a cruel prank to construct the n-division points of C without ever getting to see them proudly displayed on the curve to which they rightfully belong. However, we will also consider the second problem where we do not have a pre-drawn figure, in the case of the tricuspoid. We will consider the first of these questions in Sect. 4 and the second in Sect. 5. Abel's Theorem is an answer to a problem of the second type. To the best of our knowledge, the corresponding question with a drawn lemniscate has not yet been answered.

2 An Overview of Constructibility

Let $\mathbb{Q}^c$ denote the smallest extension of $\mathbb{Q}$ such that all quadratic polynomials with coefficients in $\mathbb{Q}^c$ are reducible. If K is an intermediate field between $\mathbb{Q}$ and $\mathbb{Q}^c$ which has finite degree over $\mathbb{Q}$, then $[K : \mathbb{Q}]$ is a power of 2. This field $\mathbb{Q}^c$ is important in constructibility thanks to the following definition and theorem:

Definition 2.1 A complex number $z = x + iy$ is constructible if and only if the point (x, y) can be constructed with a compass and straightedge on the Cartesian plane.

Theorem 2.2 Cox (2012) *Theorem 10.1.6 A complex number z is constructible if and only if $z \in \mathbb{Q}^c$.*

In the presence of a pre-drawn curve C, the problem of determining the constructible points is more delicate. However, a sufficient, but not necessary, condition for a point to be constructible is the following statement: a point P is constructible, given curve C already drawn, if it is a point of intersection of C, and a circle with constructible radius $r \in \mathbb{Q}^c \cap \mathbb{R}_{>0}$ and constructible center which intersects C at finitely many points, or if P is constructible as per Theorem 2.2.

3 The a/b Hypocycloid

The hypocycloid is a curve obtained by rolling a circle of radius b inside the circumference of a larger circle of radius a and tracing the path of a fixed point P on the smaller circle of radius b (a visual depiction of this rolling can be found in Fig. 1 for the tricuspoid where $a/b = 3$). These curves can be represented in parametric form as

$$x(\phi) = (a - b)\cos\phi + b\cos\left(\frac{a-b}{b}\phi\right),$$
$$y(\phi) = (a - b)\sin\phi - b\sin\left(\frac{a-b}{b}\phi\right).$$

Here, ϕ denotes the angle between the positive x-axis and the segment connecting the centers of the two circles, as in shown in Fig. 4.

It is natural to think about integer hypocycloids, where $a/b = c \in \mathbb{Z}_{>1}$, which have c cusps. However, none of our arguments are specific to the integral case, except in Sect. 5, where we assume that $c = 3$. The geometric picture is slightly different in the case that c is not an integer: if $c = a/b$ where $(a, b) = 1$, then the c-hypocycloid will have a cusps, and the small circle must travel around the large circle b times before the marked point completes its trajectory.

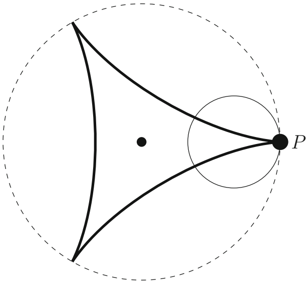

Fig. 1 The tricuspoid curve is the hypocycloid with three cusps obtained by rolling a *circle* of radius b (the *small circle*) counterclockwise inside a *circle* of radius $a = 3b$ (the *dashed outer circle*) and tracing the path of a point (the *large marked point* labeled P) on the *smaller circle*

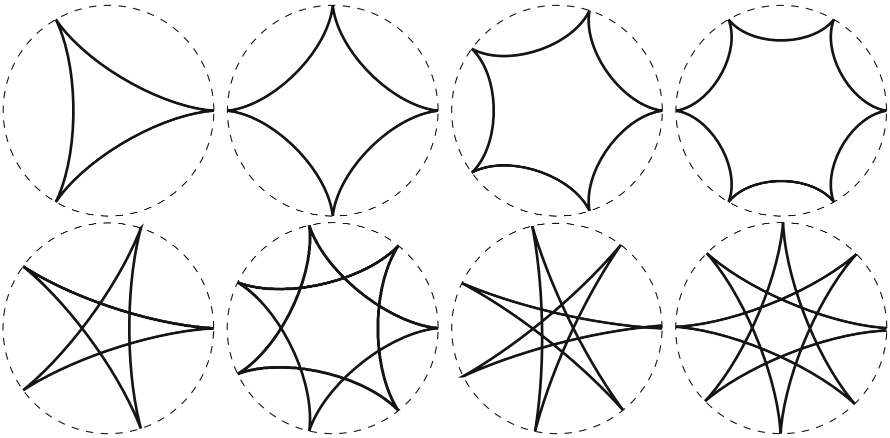

Fig. 2 The above diagram illustrates several examples of hypocycloids obtained by varying the value of c. The *top row*shows integral hypocycloids, including the tricuspoid and astroid, and the *bottom row* provides some examples of nonintegral rational hypocycloids, with $c = 5/2, 7/2, 7/3, 8/3$, respectively

Remark 3.1 From now on, we will consider the scaled hypocycloid, in which $a = c$ and $b = 1$.

Note that it is necessary that $c > 1$, and when $c = 2$, the c-hypocycloid is simply the line segment $[-2, 2]$, traced out twice.

We characterize the constructible n-division points for all hypocycloids where $a/b = c \in \mathbb{Q}$. Of particular mathematical importance among this family of hypocycloids are the hypocycloids with $c = 3$ and 4, called respectively the tricuspoid (also known as the deltoid) and the astroid. The tricuspoid arises naturally in many areas of mathematics, and it was studied in depth by Euler and Steiner. For some more recent appearances, see Dunkl and Życzkowski (2009) and Sliepčević and Božić (2012). The astroid is also mathematically noteworthy in its relation to the tricuspoid, and because it is the envelope of a set of segments of constant length whose ends can be found on mutually perpendicular straight lines (see Weisstein 2015). Several examples of hypocycloids are illustrated in Fig. 2.

4 n-Division Points of the c-Hypocycloid

Definition 4.1 If C is a closed curve (possibly with self-intersections, but implicitly considered with a parametrization), P a point on C, and n a positive integer, then the n-division points of C relative to P are the n points $P = P_1, P_2, \ldots, P_n$ so that the arclength along C from P_i to P_{i+1} and from P_n to P_1 are all equal.

In the case of the c-hypocycloid, we will implicitly assume that $P = (c, 0)$, the rightmost point on the x-axis.

We say that the n-division points of C are constructible with compass and straightedge if *all* of them can be constructed with a compass and straightedge.

Theorem 4.2 *For all rational $c > 1$ and positive integers n, the n-division points of the c-hypocycloid are constructible with a straightedge and compass when a c-hypocycloid has been drawn in advance.*

See Fig. 3 for an example.

Proof The c-cusped hypocycloid can be parametrized in terms of the tangential angle ϕ with respect to the x axis, as shown in Fig. 4, to arrive at the following rectangular parametric equations:

$$x(\phi) = (c-1)\cos\phi + \cos((c-1)\phi),$$
$$y(\phi) = (c-1)\sin\phi - \sin((c-1)\phi).$$

With this parametrization, we can obtain equations for the polar radius r and arclength s up to the first cusp, as a function of ϕ using the aforementioned rectangular equations and a line integral: the polar radius is given by

$$r(\phi) = \sqrt{x(\phi)^2 + y(\phi)^2} = \sqrt{c^2 - 2c + 2 + 2(c-1)\cos(c\phi)},$$

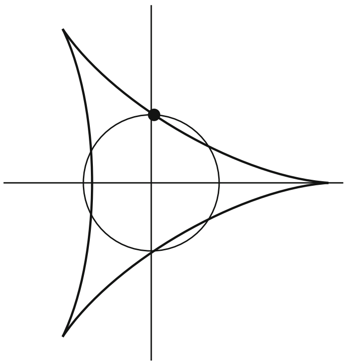

Fig. 3 The above diagram illustrates a circle of radius $\frac{\sqrt{33}}{5}$ intersecting a tricuspoid ($c = 3$), to form one of the 5-division points of the tricuspoid (the second intersection of the circle with the tricuspoid in the first quadrant is at the first 5-division point) (Note that this point does not lie on the y-axis)

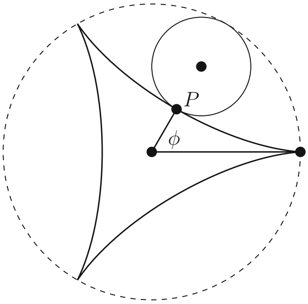

Fig. 4 Hypocycloids are parametrized in terms of the tangential angle ϕ, the angle formed by the intersection of the x axis and the line connecting the centers of the two circles, or equivalently to the segment connecting the origin and P. If $c \in \mathbb{N}$, then as ϕ goes from 0 to 2π, the entire hypocycloid is traced out at least once, and more generally as ϕ goes from 0 to $2\pi b$, where b is the denominator of c in lowest terms

and the arclength by

$$s(\phi) = \int_0^{\phi} r(\psi)\, d\psi = \frac{8(c-1)}{c} \sin^2\left(\frac{c\phi}{4}\right), \quad 0 \le \phi \le \frac{2\pi}{c}.$$

We write $\varpi_c = 8b(c-1)$ for the total arclength of the hypocycloid, obtained by multiplying the result of evaluating the arclength line integral $s\left(\frac{2\pi}{c}\right)$ from 0 to $\frac{2\pi}{c}$, which gives the arclength of a single cusp, by the number of cusps of the hypocycloid, a. An important ingredient in the proof is that $\varpi_c = 8b(c-1)$ is rational, or at least constructible. We will take advantage of this fact later on.

Using this information, we can express the polar radius r in terms of the arclength s, assuming that the tangential angle ϕ lies in the range $[0, 2\pi/c]$ (or, the piece of the hypocycloid up to the first cusp):

$$r = \sqrt{c^2 - 2c + 2 + 2(c-1)\cos\left(2\cos^{-1}\left(1 + \frac{cs}{4-4c}\right)\right)}. \tag{4.1}$$

Simplifying our expression for polar radius r, we obtain

$$r = \sqrt{c^2 - 2c + 2(c-1)\left(2\left(1 + \frac{cs}{4-4c}\right)^2 - 1\right)}. \tag{4.2}$$

Observe that, for each fixed c, r^2 is a polynomial in s with rational coefficients, whereas a priori (4.1) might have been a transcendental function. The fact that r^2 is a polynomial in s is what drives the proof.

This given expression (4.2) can be used to determine the radius of a circle that, when intersected with a drawn c-cusped hypocycloid, creates an arc on the hypocycloid of some length $\frac{\varpi_c}{n}$. That is, we input some arclength $s = \frac{\varpi_c}{n}$ into (4.2). Since ϖ_c is constructible, $s = \varpi_c/n$ is constructible for all n. Furthermore, the expression (4.2) for r only involves rational expressions and square roots. The combination of these two facts implies that for all $n \in \mathbb{Z}^+$, $r \in \mathbb{Q}^c$, and is constructible by Theorem 2.2.

From this observation, we can quickly conclude that all positive integer n-division points of all c-hypocycloids are constructible. The above Eq. (4.2) shows that we can construct a circle centered at the origin of radius r, and one of its intersections with the c-cusped hypocycloid in question will cut off a portion of the arclength of length s ($1/n$th of the total arclength), one of the n-division points of the curve. While this argument only shows that the *first* n-division point is constructible, the argument also holds if we replace n above by n/d, which gives us the dth n-division point. Thus, we know that the arclengths formed by finding the n-division points of the c-cusped hypocycloid for $n \in \mathbb{Z}^+$ are always constructible.

Thus, for all c, we can construct the n-division points of the c-cusped hypocycloid for all positive integers n. □

5 *n*-Division Points of the Tricuspoid

In the previous section, we assumed that we had a rational hypocycloid already drawn on the page, and that we could use this hypocycloid to aid our construction. In this section, we will consider the other problem mentioned in Remark 1.4, namely that of constructing the n-division points of a hypocycloid C without C being drawn in advance. In doing so, we restrict to the case of a tricuspoid (hypocycloid where $a/b = c = 3$). This necessitates being able to ascertain the location of the n-division points of the tricuspoid without intersecting lines or arcs with a tricuspoid; in other words, we need to be able to construct the rectangular coordinates of the n-division points of the tricuspoid solely with a straightedge and compass. It turns out that in this case, the answer is very different from before.

Theorem 5.1 *The n-division points of the tricuspoid can be constructed with a straightedge and compass and no pre-drawn tricuspoid, if and only if $n \mid 6$.*

Our technique will be to show that, for $n \nmid 6$, the x-coordinate of one of the n-division points is a root of an irreducible cubic polynomial. The theory of Newton polygons, which we briefly review below, is what allows us to prove that the cubic polynomials we obtain are actually irreducible.

Definition 5.2 Let K be a valued field, i.e. a field K equipped with a homomorphism $v : K^\times \to \mathbb{R}$ that satisfies the ultrametric inequality, $v(a+b) \geq \min(v(a), v(b))$. Now, consider a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in K[x]$ where $a_0, a_n \neq 0$. Then, take the points $(i, v(a_i))$ in $\mathbb{R}^2$, disregarding points where a_i is 0. (Equivalently, take $v(0)$ to be ∞.) Then, the Newton polygon of $f(x)$ is the lower convex hull of the set of points $(i, v(a_i))$.

Theorem 5.3 *Consider the Newton polygon of a polynomial $f(x) \in K[x]$ as described above. Label the k vertices of the Newton polygon consecutively from $i = 0, \ldots, k$ as (x_i, y_i). Furthermore, for $i = 1, \ldots k$, let $\sigma_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$ be the slope of the ith segment. Then, for all i, there are $x_i - x_{i-1}$ roots of $f(x)$, counted with multiplicity, in the algebraic closure $\overline{K}$ of K with valuation $-\sigma_i$.*

Corollary 5.4 *If the Newton polygon of a polynomial* $f(x) \in \mathbb{Q}_p[x]$ *with degree* n *is comprised of a single segment of slope* a/b *with* $\gcd(a,b) = 1$, *then* $f(x)$ *is irreducible.*

Remark 5.5 Corollary 5.4 is a generalization of Eisenstein's Criterion. In the setting of Corollary 5.4, if $\alpha \in \overline{K}$ is any root of $f(x)$, then the extension $K(\alpha)/K$ is totally ramified.

For a more detailed exposition on Newton polygons, and proofs of Theorem 5.3 and Corollary 5.4, consult Barbeau (2003).

Proof of Theorem 5.1 We will prove Theorem 5.1 by showing that for $n \nmid 6$, there is some point P_i in the set $\{P = P_1, P_2, \ldots, P_n\}$ of n-division points of the tricuspoid that is not constructible with a compass and straightedge.

The tricuspoid can be parametrized in terms of the angle ϕ as

$$x(\phi) = 2\cos(\phi) + \cos(2\phi),$$
$$y(\phi) = 2\sin(\phi) - \sin(2\phi).$$

We can compute the polar radius r of any point on the curve in terms of ϕ using the above rectangular equations, and we can compute the polar radius r of the n-division points of the tricuspoid in terms of the desired arc length fraction s using the expression obtained in Sect. 3, with $c = 3$.

Equating these two expressions for the polar radius enables us to determine an expression for ϕ and subsequently for the x coordinate; the first expression comes from evaluating the final result of the previous section when $c = 3$ and the second is the parametric definition of x for the tricuspoid in terms of ϕ, the tangential angle:

$$r^2 = \frac{9n^2 - 96n + 288}{n^2} = x^2 + y^2 = (2\cos\phi + \cos 2\phi)^2 + (2\sin\phi - \sin 2\phi)^2,$$

$$x = 2\cos(\phi) + \cos(2\phi).$$

Since $y = \sqrt{r^2 - x^2}$, and the polar radius r is constructible for all n-division points of the tricuspoid, the x and y coordinates of the n-division points of a tricuspoid are constructible if and only if the x coordinate is constructible.

Using the above equations, we find that the x-coordinate of the tricuspoid's first n-division point in terms of n is a real root of the following cubic polynomial:

$$f_n(x) = (2^2n^4)x^3 - (3^3n^4 - 2^5 3^2 n^3 + 2^5 3^3 n^2)x - (3^3 n^4 - 2^4 3^3 n^3 + 2^4 3^2 17 n^2 - 2^8 3^3 n + 2^7 3^4).$$

Due to the parametrization described in Sect. 3, we require the first n-division point to fall within the first cusp of the tricuspoid, meaning that our polynomial is only valid for $n \geq 3$.

We will show that the n-division points are not constructible when $n \neq 3$ or 6 for $n \geq 3$.

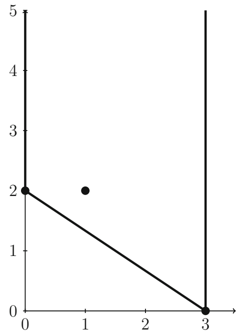

Fig. 5 The Newton polygon for $f_n(x)$ at $p = 3$, when $(n, 3) = 1$. As shown above, the polygon has a line with slope $-2/3$, indicating that $f_n(x)$ is irreducible

Let us first consider the case where $(n, 3) = 1$, $n > 2$. Then, we use the Newton polygon at $p = 3$ to show that $f_n(x)$ is irreducible. Let us write $f_n(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. (Here, $a_2 = 0$.) For $a \in \mathbb{Q}^\times$, let $v_3(a)$ denote the 3-adic valuation of a. Then

$$v_3(a_0) = 2, \quad v_3(a_1) = 2, \quad v_3(a_3) = 0.$$

Thus, the Newton polygon shown in Fig. 5 has a line with a slope of $-2/3$. This implies by Corollary 5.4 that $f_n(x)$ is irreducible for $(n, 3) = 1$, $n \geq 3$, as the x-coordinates of these n-division points of the tricuspoid are in a cubic field extension of $\mathbb{Q}$, and as a result not in $\mathbb{Q}^c$ by Theorem 2.2.

We now consider the case when n is a multiple of 3 and $n > 6$. Suppose that $n = 3^e m$ where $(m, 3) = 1$. Then the m-division points are contained in the n-division points, so it suffices to show that the m-division points are not constructible. Since m is not a multiple of 3, we see that unless $m = 1$ or 2, the m-division points are not constructible. In all remaining cases, n is a multiple of 9, so the 9-division points are included among the n-division points. However, the 9-division points of a tricuspoid are not constructible, since $f_9(x) = 81(324x^3 - 459x - 107)$ is irreducible, by the same reasoning used in the earlier case when $(n, 3) = 1$, $n > 2$.

Finally, $f_n(x)$ is, in fact, reducible for $n = 3$ and 6, so the coordinates of the 3- and 6-division points lie in a quadratic extension of $\mathbb{Q}$ (in fact, they lie in $\mathbb{Q}(\sqrt{3})$), which shows that they are constructible by Theorem 2.2 □

Remark 5.6 Note that since the value of x is the root of a cubic with rational coefficients, were we to allow ourselves an angle trisector or origami in addition to a compass or straightedge, we would be able to construct all of the n-division points of the tricuspoid even without it drawn.

Remark 5.7 One may be tempted to wonder which, if any, of the *individual* n-division points are constructible when $n \nmid 6$. It appears that no n-division points other than those contained among the 6-division points are constructible. The x-coordinates of n-division points are roots of the polynomials $f_r(x)$, where r is a rational number ≥ 3 whose numerator divides n. It appears that, except in the cases already discussed, $f_r(x)$ is indeed irreducible, and further, that the field $\mathbb{Q}(\alpha)$ obtained by adjoining a root of $f_r(x)$ is totally ramified at 3. However, the analysis using Newton polygons in a naïve manner as above fails, as the slope of the Newton polygon is sometimes integral. This is due to the fact that (assuming f_r is indeed irreducible) $\mathbb{Z}_3[\alpha]$ is not typically the full valuation ring of the field $\mathbb{Q}(\alpha) \otimes \mathbb{Q}_3$.

Acknowledgments We would like to thank the anonymous referee for helpful suggestions.

References

Barbeau, E.J.: Polynomials. Springer, New York (2003)

Cox, D.A.: Galois Theory: Pure and Applied Mathematics (Hoboken), 2nd edn. Wiley, Hoboken, NJ (2012)

Cox, D.A., Shurman, J.: Geometry and number theory on clovers. Am. Math. Monthly **112**(8), 682–704 (2005)

Dunkl, C., Życzkowski, K.: Volume of the set of unistochastic matrices of order 3 and the mean Jarlskog invariant. J. Math. Phys. **50**(12), 123521–123525 (2009)

Sliepčević, A., Božić, I.: Steiner curve in a pencil of parabolas. KoG **16**, 13–15 (2012)

Stewart, I.: Galois Theory, 3rd edn. Chapman & Hall/CRC Mathematics, Chapman & Hall/CRC, Boca Raton, FL (2004)

Weisstein, E.W.: Hypocycloid. Wolfram MathWorld. http://mathworld.wolfram.com/Hypocycloid.html. Accessed 1 Mar 2015