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ORIGINAL ARTICLE

Infinitely many solutions to a Kirchhoff-type equation involving logarithmic nonlinearity via Morse's theory

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Abstract

In the present paper, we study the existence of infinitely many solutions for $p(\mathrm{x}, \cdot)$-fractional Kirchhoff-type elliptic equation involving logarithmic-type nonlinearities. Our approach is based on the computation of the critical groups in the nonlinear fractional elliptic problem of type $p(\mathrm{x}, \cdot)$-Kirchhoff, the Morse relation combined with variational methods.

Keywords Fractional $p(\mathrm{x}, \cdot)$-Kirchhoff-type problem · Fractional Sobolev space · Existence of solutions · Infinitely many solutions · Morse's theory · Logarithmic nonlinearity · Local linking

Mathematics Subject Classification 14F35 · 35R11 · 58E05 · 49J35 · 35J65

1 Introduction

Let $\mathcal{U} \subset \mathbb{R}^N$ be an open-bounded set ($N \geq 2$). Our objective in this work is to discuss the existence of infinitely many solutions for $p(\mathrm{x}, \cdot)$-fractional Kirchhoff-type elliptic equation involving logarithmic-type nonlinearities. The approach is based on Morse's theory. More precisely, we combine Morse's relation with the computation of critical groups to study the following equation:

$$\begin{cases} M\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u})\right)\Delta^s_{p(\mathrm{x},\cdot)}\mathrm{u}(\mathrm{x}) = \lambda|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x})\log|\mathrm{u}(\mathrm{x})| + \lambda f(\mathrm{x},\mathrm{u}(\mathrm{x})) \text{ in } \mathcal{U}, \\ \mathrm{u} = 0 \text{ in } \mathbb{R}^N \backslash \mathcal{U}, \end{cases} \tag{1}$$

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where λ is a positive parameter, $r \in C(\mathcal{U}, (1, \infty))$, $p : \mathcal{U} \times \mathcal{U} \to (1, \infty)$ is a continuous function that verifies the following conditions:

$$p(y - m, z - m) = p(y, z), \quad \text{for all } (y, z, m) \in \mathbb{R}^N \times \times \mathbb{R}^N, \tag{2}$$

$$p(z, a) = p(a, z), \quad \text{for all } (z, a) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3}$$

$$1 < p^- = \min_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} p(y, z) \leq p(y, z) < p^+ = \sup_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} p(y, z), \tag{4}$$

$f : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function with $f(\mathrm{x}, 0) = 0$ and satisfies below conditions:

($\mathcal{B}_1$) There exist $\alpha > 0$ and a continuous function $q : \mathbb{R}^N \to (1, +\infty)$, such that

$$1 < q(\mathrm{x}) < p_s^\star(\mathrm{x}) = \frac{Np(\mathrm{x}, \mathrm{x})}{N - sp(\mathrm{x}, \mathrm{x})},$$

and

$$f(\mathrm{x}, \mathrm{y}) \leq \alpha \left(1 + |\mathrm{y}|^{q(\mathrm{x})-1}\right), \quad \text{a.e. } \mathrm{x} \in \mathbb{R}^N, \ \mathrm{y} \in \mathbb{R}.$$

($\mathcal{B}_2$) There exists $R > 0$, such that $\frac{f(\mathrm{x},t)}{|t|^{p(\mathrm{x},\mathrm{y})-2}t}$ is increasing for $t \geq R$ and is decreasing for $t \leq -R$ for all $\mathrm{x} \in \mathcal{U}$.

($\mathcal{B}_3$) $\lim_{t\to\infty} \frac{F(\mathrm{x},t)}{|t|^{r^+}} = +\infty$, where $F(\mathrm{x}, t) = \int_0^t f(\mathrm{x}, s)\mathrm{d}s$ is the primitive of function f, and $r^+ = \sup_{\mathrm{x}\in\mathbb{R}^N} r(\mathrm{x}) \leq q^- < p_s^\star(\mathrm{x})$.

($\mathcal{B}_4$) There are small constants and R with $0 < r < R$, such that

$$C_2|t|^{\alpha(\mathrm{x})} \leq \beta(\mathrm{x})F(\mathrm{x}, t) \leq C_3|t|^{\beta(\mathrm{x})} \text{ for all } r \leq t \leq R, \text{ a.e } \mathrm{x} \in \mathcal{U},$$

where C_2, C_3 are positive constants with $0 < C_2 < C_3 < 1$, and $\alpha, \beta \in C(\bar{\mathcal{U}})$ with $1 < \alpha(\mathrm{x}) < \beta(\mathrm{x}) < p_s^\star(\mathrm{x})$.

($\mathcal{B}_5$) There exist $\beta > p^+$ and some $I > 0$, such that, for each $|\alpha| > I$, we have

$$0 < \int_{\mathcal{U}} F(\mathrm{y}, \mathrm{x})\mathrm{d}\mathrm{y} \leq \int_{\mathcal{U}} f(\mathrm{y}, \mathrm{x})\frac{\alpha}{\beta}\mathrm{d}\mathrm{y},$$

$\Delta^s_{p(\mathrm{x},\cdot)}$ is the fractional $p(\mathrm{x}, .)$-Laplace operator which (up to normalization factors) may be defined as

$$\Delta^s_{p(\mathrm{x},\cdot)}\mathrm{u}(\mathrm{x}) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus\mathfrak{B}_\varepsilon(\mathrm{x})} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})-2}(\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y}))}{|\mathrm{x} - \mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}}\mathrm{d}\mathrm{x}, \tag{5}$$

for all $\mathrm{y} \in \mathbb{R}^N$, where $\mathfrak{B}_\varepsilon(\mathrm{x})$ denotes the Ball of center x, and radius ϵ, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function called Kirchhoff's function that satisfies the following conditions:

($\mathcal{B}_6$) There exists $m > 0$, such that

$$m \leq M(t), \text{ for all } t \in \mathbb{R}.$$

($\mathcal{B}_7$) There exists $\theta \in (0, 1)$, such that

$$\theta t M(t) \leq \widehat{M(t)} \text{ for all } t \in \mathbb{R},$$

where $\widehat{M(t)} = \int_0^t M(s)\mathrm{d}s$ is the primitive of function M, and

$$J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}) = \int_{\mathcal{U}\times\mathcal{U}} \frac{1}{p(\mathrm{x},\mathrm{y})} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}} \mathrm{dxdy},$$

for all $\mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$.

The operator defined in (5) is used in many branches of mathematics, including calculus of variations and partial differential equations. It has also been applied in a wide range of physical and engineering contexts, including fluid filtration in porous media, image processing, optimal control, constrained heating, elastoplasticity, image processing, financial mathematics, and elsewhere; for more details, see [7, 12, 31] and the references therein.

The Kirchhoff-type problem was primarily introduced in [23] to generalize the classical D'Alembert wave equation for free vibrations of elastic strings. Some interesting research by variational methods can be found in [13, 14, 24, 25, 28] for Kirchhoff-type problems. More precisely, Kirchhoff introduced a famous equation defined as

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{6}$$

that it is related to the problem (1). In (6), L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. See the paper [23] for more details.

Recently, results on fractional Sobolev spaces and fractional $p(\mathrm{x}, \cdot)$-Kirchhoff-type problem and their applications have received a lot of attention.

Kaufmann, Rossi, and Vidal [22] first introduced the new class $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ defined by

$$W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) = \left\{ \mathrm{u} \in L^{q(\mathrm{x})}(\mathcal{U}) : \int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{\mathcal{K}(\mathrm{x},\mathrm{y})} \mathrm{dxdy} < +\infty \right\},$$

where $q \in C(\overline{\mathcal{U}}, (1, \infty))$ and $\mathcal{K}(\mathrm{x}, \mathrm{y}) = |\mathrm{x} - \mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}$ and proved the existence of a compact embedding

$$W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) \hookrightarrow\hookrightarrow L^{r(\mathrm{x})}(\mathcal{U}), \quad \text{for all } r \in C(\mathcal{U}),$$

such that $1 < r(\mathrm{x}) < p_s^{\star}(\mathrm{x})$, for all $\mathrm{x} \in \overline{\mathcal{U}}$.

For more results on the functional framework, we refer to Bahrouni and Rădulescu [4, 5] who proved the solvability of the following problems:

$$\begin{cases} \left(-\Delta_{p(\mathrm{x})}\right)^s \mathrm{u}(\mathrm{x}) + |\mathrm{u}(\mathrm{x})|^{q(\mathrm{x})-2}\mathrm{u}(\mathrm{x}) = h(\mathrm{x}, \mathrm{u}(\cdot)) + \lambda|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x}) & \text{in } \mathcal{U}, \\ \mathrm{u} = 0 & \text{in } \mathbb{R}^N \backslash \mathcal{U}, \end{cases}$$

using Ekeland's variational method, and the sub-supersolution method.

For more results concerning the framework, we refer the readers to [1, 2, 4, 5, 21, 22]. The approaches for ensuring the existence of weak solutions for a class of nonlocal fractional problems with variable exponents were addressed in greater depth in [1–5, 8, 9, 11, 12, 21, 22, 25, 30, 32] and the references therein.

In recent years, wide research has been done on fractional $p(\mathrm{x}, \cdot)$-Kirchhoff-type problem with variable growth. In the case of the p-Laplacian operator, Li et al. used the concentration compactness principle and Ekelend's variational principle to study the existence of multiple solutions for the below equation

$$\begin{cases} M\left(\|\mathrm{u}\|^p\right) \Delta_p u(x) = \lambda u^{p^*} + \rho(x)u^{-\gamma} & \text{in } \Omega, \\ u \qquad = 0 & \text{in } \partial\Omega, \end{cases}$$

where $M(t) = a + bt^k$ and $0 < \gamma < 1 < p$. Recently, in the case $p = 2$, Cabanillas Lapa in [10] proved an existence result with exponential decay. In addition, the authors [18] studied the following problem:

$$\begin{cases} M\left(\displaystyle\int_\Omega A(x, \nabla u)dx\right) \operatorname{div}(a(x, \nabla u)) = \lambda h(x)\frac{\partial F}{\partial u}(x, u) & \text{in } \Omega, \\ u \qquad = 0 & \text{in } \partial\Omega. \end{cases}$$

In [16], the authors used the Nehari manifold method to prove the below singular Kirchhoff problem

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = \lambda f(x)u^{-\gamma} + g(x)u^{2*} - 1 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

For more recent works, we refer to [17, 19] and references therein.

Motivated by the above research, we prove the existence of infinitely many solutions of the generalized fractional $p(x, .)$-Kirchhoff-type problem (1) on the framework of fractional Sobolev spaces with variable exponent. Our approach uses the variational tools based on the critical point theory together with Morse theory (critical groups and local linking argument), in which we consider the energy functional ζ (9) satisfies the Cerami condition "(C) condition" (2), which leads to a deformation theorem, then we compute the critical groups at infinity and critical points 0 associated to ζ. Our first major result is the following theorem:

Theorem 1 *Under assumption* $(\mathcal{B}_1)-(\mathcal{B}_7)$. *Then, the problem* (1) *has a weak solution in* $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$.

Theorem 2 *Under assumption* $(\mathcal{B}_1)-(\mathcal{B}_7)$. *Then, the problem* (1) *has an infinitely many weak solutions in* $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$.

The paper is organized as follows: In Sect. 2, we collect the main definitions and properties of generalized Lebesgue spaces and generalized Sobolev spaces and provide crucial background on Morse's theory. In Sect. 3, we give the proofs of Theorem 1 by computing the critical groups at infinity and critical points 0 associated with the functional energy. Moreover, we use Morse's relation to establish the problem (1) has an infinitely many weak solutions.

2 Preliminaries

2.1 Fractional Sobolev space

This section contains results that will be used throughout the document concerning the Sobolev and generalized Lebesgue spaces. We consider the set

$$C^+(\bar{\mathcal{U}}) = \left\{m : \bar{\mathcal{U}} \to \mathbb{R}^+ : m \text{ is a continuous function and } 1 < m^- < m(y) < m^+ < +\infty\right\},$$

where $q^- = \min_{\mathrm{x}\in\bar{\mathcal{U}}} q(\mathrm{x}),\ q^+ = \max\limits_{\mathrm{x}\in\bar{\mathcal{U}}} q(\mathrm{x})$.

Definition 1 (see [15]) Let $q \in C^+(\bar{\mathcal{U}})$. We define the generalized Lebesgue space $L^{q(\mathrm{x})}(\mathcal{U})$ as usual

$$L^{q(\mathrm{x})}(\mathcal{U}) = \left\{u : \mathcal{U} \to \mathbb{R} \text{ is a measurable function } : \exists\lambda > 0 : \int_{\mathcal{U}} |\frac{u(\mathrm{x})}{\lambda}|^{q(\mathrm{x})}\mathrm{dx} < \infty\right\}.$$

We equip this space with the so-called Luxemburg norm defined as follows:

$$|\mathrm{u}|_{L^{q(\mathrm{x})}(\mathcal{U})} = \inf\left\{\xi > 0 : \int_{\mathcal{U}} |\frac{\mathrm{u}(\mathrm{x})}{\xi}|^{q(\mathrm{x})}\mathrm{dx} \le 1\right\}.$$

Lemma 1 (Hölder's inequality, see [15]) *For every* $q \in C^+(\mathbb{R}^N)$, *the following inequality holds:*

$$|\int_{\mathbb{R}^N} v(\mathrm{x})w(\mathrm{x})\mathrm{dx}| \le \left(\frac{1}{q^-} + \frac{1}{q'^-}\right)|v|_{L^{q(\mathrm{x})}(\mathbb{R}^N)}|w|_{L^{q'(x)}(\mathbb{R}^N)},$$

for all $(v, w) \in L^{q(\mathrm{x})}(\mathbb{R}^N) \times L^{q'(x)}(\mathbb{R}^N)$, *where* $\frac{1}{q(\mathrm{x})} + \frac{1}{q'(x)} = 1$.

Lemma 2 (see [15]) *Let* $\mathcal{U} \subset \mathbb{R}^N$ *be a Lipschitz-bounded domain, and* $q \in C^+(\mathbb{R}^N)$. *Then, we have the following statements:*

(i) *the space* $\left(L^{q(\mathrm{x})}\left(\mathbb{R}^N\right), |.|_{L^{q(\mathrm{x})}(\mathbb{R}^N)}\right)$ *is a separable, reflexive, and Banach space,*
(ii) *the space* $C^{\infty}(\mathcal{U})$ *is dense in the space* $\left(L^{q(\mathrm{x})}(\mathcal{U}), |.|_{L^{q(\mathrm{x})}(\mathcal{U})}\right)$.

We start by fixing the fractional exponent $s \in (0, 1)$. Let $\mathcal{U}$ be an open-bounded set of $\mathbb{R}^N$, $q \in C^+(\mathcal{U})$, and $p : \bar{\mathcal{U}} \times \bar{\mathcal{U}} \to (1, \infty)$ is a continuous function satisfies the conditions (2)–(4). We introduce the generalized fractional Sobolev space $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ as follows:

$$W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y}}(\mathcal{U}) = \left\{ \mathrm{w} \in L^{q(\mathrm{x})}(\mathcal{U}) : \frac{\mathrm{w}(\mathrm{x}) - \mathrm{w}(\mathrm{y})}{\beta |\mathrm{x}-\mathrm{y}|^{s+\frac{N}{p(\mathrm{x},\mathrm{y})}}} \in L^{p(\mathrm{x},\mathrm{y})}(\mathcal{U} \times \mathcal{U}) \text{ for some } \beta > 0 \right\}.$$

Let $[\mathrm{w}]^{s,p(\mathrm{x},\mathrm{y})} = \inf \left\{ \beta > 0 : \int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{w}(\mathrm{x})-\mathrm{w}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{\beta^{p(\mathrm{x},\mathrm{y})}|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}} \mathrm{dxdy} < 1 \right\}$ be the corresponding variable exponent Gagliardo semi-norm. We equip the space $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y}}(\mathcal{U})$ with the norm

$$\|\mathrm{w}\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y}}(\mathcal{U})} = [\mathrm{w}]^{s,p(\mathrm{x},\mathrm{y})} + |\mathrm{w}|_{q(\mathrm{x})},$$

where $(L^{q(\mathrm{x})}(\mathcal{U}), |.|_{q(\mathrm{x})}$ is the generalized Lebesgue space.

Lemma 3 (see [5]) *Let* $\mathcal{U} \subset \mathbb{R}^N$ *be a Lipschitz-bounded domain,* $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ *be a continuous function that satisfies conditions (2)–(4), and* $q \in C^+(\bar{\mathcal{U}})$. *Then,* $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y}}(\mathcal{U})$ *is a separable and reflexive Banach space.*

Theorem 3 (see [5]) *Let* $\mathcal{U} \subset \mathbb{R}^N$ *be a Lipschitz-bounded domain,* $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ *be a continuous function satisfies conditions (2)–(4),* $q \in C^+(\mathcal{U})$, *and*

$$sp(\mathrm{x}, \mathrm{y}) < N, \;\; p(\mathrm{x}, \mathrm{x}) < q(\mathrm{x}), \;\; \textit{for all } (\mathrm{x}, \mathrm{y}) \in \mathcal{U}^2,$$

and $\ell : \overline{\mathcal{U}} \to (1, +\infty)$ *is a continuous variable exponent, such that*

$$p_s^*(\mathrm{x}) = \frac{Np(\mathrm{x}, \mathrm{x})}{N - sp(\mathrm{x}, \mathrm{x})} > \ell(\mathrm{x}) \geq \ell^- = \min_{\mathrm{x}\in\overline{\mathcal{U}}} \ell(\mathrm{x}) > 1.$$

Then, the space $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y}}(\mathcal{U})$ *is continuously embedded in* $L^{\ell(y)}(\mathcal{U})$ *and there exists a positive constant* $C = C(N, s, p, q, \mathcal{U})$, *such that*

$$|\mathrm{w}|_{L^{l(\mathrm{x})}(\mathcal{U})} \leq C \|\mathrm{w}\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y}}(\mathcal{U})}, \; \textit{for all} \;\; w \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y}}(\mathcal{U}).$$

Moreover, this embedding is compact.

Definition 2 [26] Let X be a Banach space and $J \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$, we say that Φ satisfies the Cerami c condition (we denote condition (C_c)), if

(C_1): any bounded sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ has a convergent subsequence,

(C_2): there exist constants $\delta, R, \beta > 0$, such that

$$\left\|\Phi'(u)\right\| \|\mathrm{u}\| \geq \beta \;\; \forall u \in \Phi^{-1}([c-\delta, c+\delta]) \quad \text{with} \quad \|\mathrm{u}\| \geq R.$$

2.2 Critical groups

In this paragraph, we briefly give the basic properties and notions of Morse theory. Let W be a real Banach space, $\psi \in C^1(W, \mathbb{R})$, satisfies the Palais–Smale condition, and $c \in \mathbb{R}$. We consider the following sets:

$$\psi^c = \{\mathrm{w} \in W : \psi(\mathrm{w}) \leq c\}$$

and

$$K_\psi = \left\{\mathrm{w} \in W : \psi^{'}(\mathrm{w}) = 0\right\}.$$

The critical groups of ψ at w are defined by

$$C_k(\psi, \mathrm{w}) = H_k\left(\psi^c \cap U, \psi^c \cap U \backslash \{\mathrm{w}\}\right),$$

where $k \in \mathbb{N}$, U is a neighborhood of w, such that $K_\psi \cap U = \{\mathrm{w}\}$, and H_k is the singular relative homology with coefficient in an Abelian group G; see [26] for more details.

Definition 3 (*see* [6]) If ϕ satisfies the condition (C) and the critical values of ϕ are bounded from below by some $a < \inf \phi(K)$, then the critical groups of ϕ at infinity as

$$C_k(\phi, \infty) := H_k(W, \phi^a), \text{ for all } k \in \mathbb{N}.$$

Theorem 4 (see [27]) *Given W is a real Banach space, $\phi \in C^1(W, \mathbb{R})$ satisfies the Palais–Smale condition and is bounded from below. If at least one of its critical groups is nontrivial, then ϕ has at least three critical points.*

Definition 4 (*see* [27]) Given Y is a Banach space, $\psi \in C(Y, \mathbb{R})$, and 0 is an isolated critical point of ψ such that $\psi(0) = 0$. We say that ψ has a local linking at 0 with respect to $Y = V \bigoplus W$, $k = \dim V < \infty$, if there exists $\rho > 0$ small, such that

$$\begin{cases} \psi(\mathrm{u}) \leq 0, & \mathrm{u} \in V; \quad \||\mathrm{u}|\| \leq \rho; \\ \psi(\mathrm{u}) > 0, & \mathrm{u} \in W; \quad 0 < \||\mathrm{u}|\| \leq \rho. \end{cases}$$

Theorem 5 (see [27]) *Given Y is a Banach space, $\psi \in C(Y, \mathbb{R})$. If ψ has a local linking at 0 with respect to Y. Then, we get $C_k(\psi, 0) \neq 0$.*

Lemma 4 (Morse's relation) (see [26]) *If Y is a Banach space, $\psi \in C^1(Y, \mathbb{R})$, $a, b \in \mathbb{R} \backslash \psi\left(\left\{K_\psi\right), a < b, \psi^{-1}((a, b))$ contains a finite number of critical points $\{\mathrm{w}_i\}_{i=1}^n$ and ψ satisfies the Palais–Smale condition, then*

(1) *for all $k \in \mathbb{N}_0$, we have $\sum_{i=1}^n \operatorname{rank} C_k(\psi, u_i) \geq \operatorname{rank} H_k\left(\psi^b, \psi^a\right)$;*

(2) *if the Morse-type numbers* $\sum_{i=1}^{n} \operatorname{rank} C_k(\psi, u_i)$ *are finite for all* $k \in \mathbb{N}_0$ *and vanish for all large* $k \in \mathbb{N}_0$, *then so do the Betti numbers* $\operatorname{rank} H_k\left(\psi^b, \psi^a\right)$ *and we have*

$$\sum_{k\geq 0}\sum_{i=1}^{n} \operatorname{rank} C_k(\psi, u_i)\, t^k = \sum_{k\geq 0} \operatorname{rank} H_k\left(\psi^b, \psi^a\right) t^k + (1+t)Q(t) \text{ for all } t \in \mathbb{R},$$

where $Q(t)$ *is a polynomial in* $t \in \mathbb{R}$ *with non-negative integer coefficients.*

3 Main results

Lemma 5 *For every* $a > 0$. *Then, we have*

(1) $t^a |\log(t)| \leq \frac{1}{a\exp(1)}$, *for all* $t \in (0, 1]$;
(2) $\log(t) \leq \frac{t^a}{a\exp(1)}$, *for all* $t > 1$.

Proof For (1). We consider the function by $g : (0, 1] \to \mathbb{R}$ as $g(t) = t^a|\log(t)|$. The function is continuous on $(0, 1]$, and $\lim_{t\to 0} t^a |\log(t)| = 0$. Using a direct computation, we show that the function g achieves the maximum at $t_0 = \exp(\frac{-1}{a})$. Finally, we have $t^a|\log(t)| \leq \frac{1}{a\exp(1)}$, for all $t \in (0, 1]$. Now, we prove (2). We construct the following function:

$$f(t) = \log(t) - \frac{1}{a\exp(1)} t^a, \text{ for all } t \in [1, \infty).$$

Obvious, we prove that the function f achieves the maximum at $t^* = \exp(\frac{1}{a})$, for all $t \in [1, \infty)$. Therefore, we get $f(t) \leq f(t^*)$. □

Lemma 6 *Let* $r : \mathcal{U} \to (1, \infty)$ *be a continuous function, such that* $1 < r^- \leq r(\mathrm{x}) \leq r^+ < p_s^*(\mathrm{x})$, *for each* $\mathrm{x} \in \mathcal{U}$. *Then, we have the following estimate:*

$$\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \log|\mathrm{u}(\mathrm{x})| d\mathrm{x} \leq C \max\left\{|\mathrm{u}|^{r^-}, |\mathrm{u}|^{r^+}\right\} + \log|\mathrm{u}| \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})} |\mathrm{u}|^{r(\mathrm{x})} dx,$$

for all $\mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) \backslash \{0\}$, *where* $C = C\left(|\mathcal{U}|, r, p_s^*(\mathrm{x})\right)$ *is a suitable constant.*

Proof Let $\mathcal{U}_1 = \{\mathrm{x} \in \mathcal{U} : |\mathrm{u}(\mathrm{x})| \leq \|\mathrm{u}\|\}$, and $\mathcal{U}_2 = \{\mathrm{x} \in \mathcal{U} : |\mathrm{u}(\mathrm{x})| \geq \|\mathrm{u}\|\}$. From Lemma 5 (1) with $a = r^-$, we obtain that

$$\begin{aligned} \int_{\mathcal{U}_1} \frac{1}{r(\mathrm{x})} |\mathrm{u}|^{r(\mathrm{x})} \log\frac{|\mathrm{u}|}{\|\mathrm{u}\|} d\mathrm{x} &\leq \frac{1}{r^-} \max\left\{\|\mathrm{u}\|^{r^-}, \|\mathrm{u}\|^{r^+}\right\} \int_{\mathcal{U}_1} \left(\frac{|\mathrm{u}|}{\|\mathrm{u}\|}\right)^{r(\mathrm{x})} \left|\log\frac{|\mathrm{u}|}{\|\mathrm{u}\|}\right| d\mathrm{x} \\ &\leq \frac{|\mathcal{U}|}{\left(r^-\right)^2 \exp(1)} \max\left\{\|\mathrm{u}\|^{r^-}, \|\mathrm{u}\|^{r^+}\right\}. \end{aligned} \tag{7}$$

Estimating the second integral expression. Combining Lemma 3 (2) with Lemma 5, such that $a = \left(p_s^*\right)^- - \epsilon - r^+$, for some sufficiently small $\epsilon > 0$, we have that

$$\begin{aligned}
&\int_{\mathcal{U}_2} \frac{1}{r(\mathrm{x})} |\mathrm{u}|^{r(\mathrm{x})} \log \frac{|u|}{\|\mathrm{u}\|} \mathrm{dx} \leq \frac{1}{r^-} \int_{\mathcal{U}_2} |\mathrm{u}|^{r(\mathrm{x})} \log \frac{|\mathrm{u}|}{\|\mathrm{u}\|} \mathrm{dx} \\
&\leq \frac{1}{r^- \exp\left(\left(p_s^*\right)^- - \epsilon - r^+\right)} \int_{\mathcal{U}_2} |\mathrm{u}|^{r(\mathrm{x})} \left(\frac{|\mathrm{u}|}{\|\mathrm{u}\|}\right)^{\left(p_s^*\right)^- - \epsilon - r^+} \mathrm{dx} \\
&\leq \frac{1}{\exp\left(\left(p_s^*\right)^- - \epsilon - r^+\right) r^-} \int_{\mathcal{U}_2} |\mathrm{u}|^{r(x)} \left(\frac{|\mathrm{u}|}{\|\mathrm{u}\|}\right)^{\left(p_s^*\right)^- - \epsilon - r(\mathrm{x})} \mathrm{dx} \\
&\leq \frac{1}{\min\left(\|\mathrm{u}\|^{\left(p_s^*\right)^- - \epsilon - r^-}, \|\mathrm{u}\|^{\left(p_s^*\right)^- - \epsilon - r^+}\right) \exp\left(\left(p_s^*\right)^- - \epsilon - r^+\right) r^-} \int_{\mathcal{U}_2} |\mathrm{u}|^{\left(p_s^*\right)^- - \epsilon} \mathrm{dx} \\
&\leq \frac{C_{p_s^{*-} - \epsilon}}{\min\left(\|\mathrm{u}\|^{\left(p_s^*\right)^- - \epsilon - r^-}, \|\mathrm{u}\|^{\left(p_s^*\right)^- - \epsilon - r^+}\right) \exp\left(\left(p_s^*\right)^- - \epsilon - r^+\right) r^-} \|\mathrm{u}\|^{\left(p_s^*\right)^- - \epsilon} \\
&= \frac{C_{p_s^{*-} - \epsilon}}{\exp\left(\left(p_s^*\right)^- - \epsilon - r^+\right) r^-} \min\left(\|\mathrm{u}\|^{r^-}, \|\mathrm{u}\|^{r^+}\right) \\
&\leq \frac{C_{p_s^{*-} - \epsilon}}{\exp\left(\left(p_s^*\right)^- - \epsilon - r^+\right) r^-} \max\left(\|\mathrm{u}\|^{r^-}, \|\mathrm{u}\|^{r^+}\right),
\end{aligned} \tag{8}$$

where $C_{p_s^{*-} - \epsilon} > 0$ is constant. From (7), and (8, we deduce that

$$\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \log |\mathrm{u}(\mathrm{x})| \mathrm{dx} \leq C \max\left\{\|\mathrm{u}\|^{r^-}, \|\mathrm{u}\|^{r^+}\right\} + \log \|\mathrm{u}\| \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})} |\mathrm{u}|^{r(\mathrm{x})} \mathrm{dx}.$$

□

Definition 5 A measurable function $\mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ is said to be a weak solution of (1) if

$$\begin{aligned}
&M\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u})\right) \int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})-2}(\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y}))(\mathrm{v}(\mathrm{x}) - \mathrm{v}(\mathrm{y}))}{|\mathrm{x} - \mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}} \mathrm{dxdy} \\
&\quad = \lambda \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2} \log |\mathrm{u}(\mathrm{x})| \mathrm{u}(\mathrm{x}) \mathrm{v}(\mathrm{x}) \mathrm{dx} + \lambda \int_{\mathcal{U}} f(\mathrm{x}, \mathrm{u}(\mathrm{x})) \mathrm{v}(\mathrm{x}) \mathrm{dx},
\end{aligned}$$

for all $\mathrm{v} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$.

We consider the functional $\zeta : W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) \to \mathbb{R}$ defined by

$$\begin{aligned}\zeta(\mathrm{u}) &:= \widehat{M}\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u})\right) - \lambda \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}\left(|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}(\mathrm{x})| - \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\right)\mathrm{dx} \\ &\quad - \lambda \int_{\mathcal{U}} F(\mathrm{x},\mathrm{u}(\mathrm{x}))\mathrm{dx} \\ &= L_1(\mathrm{u}) - L_2(\mathrm{u}) - L_3(\mathrm{u}).\end{aligned} \tag{9}$$

Then, it follows from [5, 22] that $L_1 - L_3 \in C^1\left(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}),\mathbb{R}\right)$ and:

$$\begin{aligned}&\left\langle (L_1 - L_3)'(\mathrm{u}), \mathrm{v}\right\rangle \\ &= M\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u})\right)\int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}(\mathrm{x})-\mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})-2}(\mathrm{u}(\mathrm{x})-\mathrm{u}(\mathrm{y}))(\mathrm{v}(\mathrm{x})-\mathrm{v}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{x})}}\mathrm{dxdy} \\ &\quad - \lambda \int_{\mathcal{U}} f(\mathrm{x},\mathrm{u}(\mathrm{x}))\mathrm{v}(\mathrm{x})\mathrm{dx}.\end{aligned}$$

Our first result is the following Lemma.

Lemma 7 *Let $\mathcal{U} \subset \mathbb{R}^N$ be a Lipschitz-bounded domain, λ be a parameter positive, and $r : \mathcal{U} \to (1,\infty)$ be a continuous function. Then, we have $L_2 \in C^1\left(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}),\mathbb{R}\right)$, and*

$$\left\langle L_2'(\mathrm{u}), \mathrm{v}\right\rangle = \lambda \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x})\log|\mathrm{u}(\mathrm{x})|\mathrm{v}(\mathrm{x})\mathrm{dx},$$

for all $\mathrm{u},\mathrm{v} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$.

Proof Let $\mathrm{v},\mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$. For each $\mathrm{x} \in \mathcal{U}$, and $0 < t < 1$. By the definition of Gâteaux-differentiable, we get

$$\begin{aligned}&\left\langle L_2'(\mathrm{u}), \mathrm{v}\right\rangle \\ &= \lim_{t\to 0} \frac{L_2(\mathrm{u}+t\mathrm{v}) - L_2(\mathrm{u})}{t} \\ &= \lim_{t\to 0} \lambda \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})} \frac{\left(|\mathrm{u}+t\mathrm{v}|^{r(\mathrm{x})}\log|\mathrm{u}+t\mathrm{v}| - |\mathrm{u}|^{r(\mathrm{x})}\log|\mathrm{u}| + \frac{1}{r(\mathrm{x})}\left(|\mathrm{u}+t\mathrm{v}|^{r(\mathrm{x})} - |\mathrm{u}|^{r(\mathrm{x})}\right)\right)}{t}\mathrm{dx}.\end{aligned}$$

We consider the function defined by $K : [0,1] \to \mathbb{R}$ as

$$K(y) = \frac{|\mathrm{u}+yt\mathrm{v}|^{r(\mathrm{x})}\log|\mathrm{u}+yt\mathrm{v}|}{r(\mathrm{x})} - \frac{|\mathrm{u}+yt\mathrm{v}|^{r(\mathrm{x})}}{r^2(\mathrm{x})}.$$

According to the mean value Theorem, there exists $\theta \in (0,1)$, such that

$$K'(y)(\theta) = K(1) - K(0).$$

Combining the Lebesgue's dominated converge theorem with a direct computation, we have

$$\left\langle L_2^{'}(\mathrm{u}), \mathrm{v}\right\rangle = \lambda \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x}) \log|\mathrm{u}(\mathrm{x})|\mathrm{v}(\mathrm{x})d\mathrm{x}.$$

Using the same method as appear in paper [20], we can easily prove that $L_2 \in C^1\left(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}), \mathbb{R}\right)$. □

Lemma 8 *We assume that the conditions* $(\mathcal{B}_5)-(\mathcal{B}_7)$ *are fulfilled. Then, the functional* ζ *satisfies the Palais–Smale condition at level* $c \in \mathbb{R}$.

Proof Let $\{\mathrm{u}_n\}_{n\in\mathbb{N}} \subset W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ with $\zeta(\mathrm{u}_n) \to c$ as $n \to +\infty$ and $\zeta^{'}(\mathrm{u}_n) \to 0$ as $n \to +\infty$ in $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$. Without loss of generality, we assume that $\|\mathrm{u}_n\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \geq 1$. By contradiction, we prove the sequence $\{\mathrm{u}_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$. Therefore, there exists $C > 0$, such that

$$\langle \zeta \mathrm{u}_n, \mathrm{u}_n\rangle \leq C\|\mathrm{u}_n\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \text{ and } \zeta(\mathrm{u}_n) \leq C.$$

We combine condition $(\mathcal{B}_5)$ with condition $(\mathcal{B}_7)$, and we have

$$\begin{aligned}
&C + C\|\mathrm{u}_n\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})}\\
&\quad\geq \zeta(\mathrm{u}_n) - \frac{1}{r^-}\left\langle \zeta^{'}(\mathrm{u}_n), \mathrm{u}_n\right\rangle\\
&\quad= \widehat{M}\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}_n)\right) - \lambda\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}\left(|\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}_n(\mathrm{x})| - \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\right)d\mathrm{x}\\
&\qquad- \lambda\int_{\mathcal{U}} F(\mathrm{x}, \mathrm{u}_n(\mathrm{x}))d\mathrm{x} - \frac{1}{r^-}M\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}_n)\right)\int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}_n(\mathrm{x}) - \mathrm{u}_n(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{x})}}d\mathrm{x}d\mathrm{y}\\
&\qquad+ \frac{\lambda}{r^-}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}_n(\mathrm{x})|d\mathrm{x} + \frac{\lambda}{r^-}\int_{\mathcal{U}} f(\mathrm{x}, \mathrm{u}_n(\mathrm{x}))\mathrm{u}_n(\mathrm{x})d\mathrm{x}\\
&\quad\geq (1-\alpha)M\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}_n)\right)J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}_n) - \frac{1}{r^-}M\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}_n)\right)\\
&\qquad\times\int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}_n(\mathrm{x}) - \mathrm{u}_n(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{x})}}d\mathrm{x}d\mathrm{y}\\
&\qquad+ \frac{\lambda}{r^-}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}_n(\mathrm{x})|d\mathrm{x} - \lambda\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}_n(\mathrm{x})|d\mathrm{x}\\
&\qquad+ \frac{\lambda}{r^-}\int_{\mathcal{U}} f\left(\mathrm{x}, \mathrm{u}_n(\mathrm{x})\right)\mathrm{u}_n(\mathrm{x})d\mathrm{x} - \lambda\int_{\mathcal{U}} F(\mathrm{x}, \mathrm{u}_n(\mathrm{x}))d\mathrm{x}\\
&\quad\geq m\left[\int_{\mathcal{U}\times\mathcal{U}} \frac{1}{p(\mathrm{x},\mathrm{y})}\frac{|\mathrm{u}_n(\mathrm{x}) - \mathrm{u}_n(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{x})}}d\mathrm{x}d\mathrm{y}\right.\\
&\qquad\left.-\frac{1}{r^-}\int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}_n(\mathrm{x}) - \mathrm{u}_n(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{x})}}d\mathrm{x}d\mathrm{y}\right]
\end{aligned}$$

$$+\frac{\lambda}{r^-}\int_{\mathcal{U}}|u_n(x)|^{r(x)}\log|u_n(x)|dx-\frac{\lambda}{r^-}\int_{\mathcal{U}}|u_n(x)|^{r(x)}\log|u_n(x)|dx$$
$$\geq m\left(\frac{1}{p^-}-\frac{1}{r^-}\right)\|u_n\|^{p^+}_{W^{s,q(x),p(x,y)}(\mathcal{U})}.$$

This is a contradiction as $\|u_n\|_{W^{s,q(x),p(x,y)}(\mathcal{U})}\to\infty$. From Lemma 3, we get that there exists $u\in W^{s,q(x),p(x,y)}(\mathcal{U})$ and a subsequence of u_n still denoted by u_n that satisfies the following inequality:

$$\begin{cases} u_n\to u \text{ a.e in } \mathcal{U}, \\ u_n\rightharpoonup u \text{ weakly in } W^{s,q(x),p(x,y)}(\mathcal{U}), \\ u_n\to u \text{ strongly in } L^{\sigma(x)}(\mathcal{U}) \text{ for } 1\leq\sigma(x)<p_s^*(x). \end{cases} \tag{10}$$

Since $W^{s,q(x),p(x,y)}(\mathcal{U})$ is a reflexive space, we deduce that

$$\left\langle\zeta'(u_n),u_n-u\right\rangle\to 0 \quad \text{as } n\to\infty.$$

Thus, we get that

$$\begin{aligned}&\left\langle\zeta'(u_n),u_n-u\right\rangle\\ &\quad=-\lambda\int_{\mathcal{U}}|u_n(x)|^{r(x)-2}\log|u_n(x)|u_n(x)\left(u_n(x)-u(x)\right)dx\\ &\qquad-\lambda\int_{\mathcal{U}}f\left(x,u_n(x)\right)\left(u_n(x)-u(x)\right)dx++M\left(J_{s,p(x,\cdot)}(u_n)\right)\\ &\qquad\times\int_{\mathcal{U}\times\mathcal{U}}\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))(u_n(x)-u_n(y)-(u(x)-u(y))}{|x-y|^{N+sp(x,x)}}dxdy\end{aligned}$$

as $n\to\infty$.

Now, we show that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathcal{U}}|u_n(x)|^{r(x)-2}u_n(x)\left(u_n(x)-u_n(x)\right)\log|u_n(x)|\,dx\\ &\quad=\lim_{n\to\infty}\int_{\mathcal{U}}|u(x)|^{r(x)-2}u(x)\left(u_n(x)-u(x)\right)\log|u(x)|dx.\end{aligned}$$

From (10), it is easy to see that

$$\lim_{n\to\infty}|u_n(x)|^{r(x)}\,|\log|u_n(x)||=|u(x)|^{r(x)}\,|\log|u(x)||\text{ a.e in }\mathcal{U}. \tag{11}$$

Let $\gamma \in (0, p_s^{*-} - r^+)$. From Lemma 5, and Theorem 3, we have that

$$\begin{aligned} \int_{\mathcal{U}} |\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})} \,|\log|\mathrm{u}_n(\mathrm{x})||\mathrm{dx} &= \int_{\mathcal{U}\cap\{|\mathrm{u}_n(\mathrm{x})|\le 1\}} |\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})} \,|\log|\mathrm{u}_n(\mathrm{x})||\mathrm{dx} \\ &\quad + \int_{\mathcal{U}\cap\{|\mathrm{u}_n(\mathrm{x})|> 1\}} |\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})} \,|\log|\mathrm{u}_n(\mathrm{x})||\mathrm{dx} \\ &\le \frac{|\mathcal{U}|}{r^-\exp(1)} + \frac{1}{\gamma\exp(1)}\int_{\mathcal{U}} |\mathrm{u}_n(\mathrm{x})|^{r^++\gamma}\,\mathrm{dx} \\ &\le \frac{|\mathcal{U}|}{r^-\exp(1)} + M\frac{|\mathcal{U}|\, C_{r^++\gamma}^{r^++\gamma}}{\gamma\exp(1)}, \end{aligned} \tag{12}$$

where $L = \sup \|\mathrm{u}_n\|^{r^++\gamma} < \infty$. Therefore, the sequence $\left\{|\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})}\,|\log|\mathrm{u}_n(\mathrm{x})||\right\}_{n\ge 1}$ is equi-integral in $L^1(\mathcal{U})$, and uniformly bounded. Combining (12), (11) with Vitali's convergence theorem, we have that

$$\lim_{n\to\infty}\int_{\mathcal{U}} |\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})}\,|\log|\mathrm{u}_n(\mathrm{x})||\mathrm{dx} = \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\,|\log|\mathrm{u}(\mathrm{x})\|\mathrm{dx}. \tag{13}$$

Similarly, we prove

$$\lim_{n\to\infty}\int_{\mathcal{U}} \mathrm{u}\,|\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})-2}\,\mathrm{u}_n(\mathrm{x})|\log|\mathrm{u}_n(\mathrm{x})||\mathrm{dx} = \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\,|\log|\mathrm{u}(\mathrm{x})||\mathrm{dx}, \tag{14}$$

and

$$\lim_{n\to\infty}\int_{\mathcal{U}} \mathrm{u}_n(\mathrm{x})\,|\mathrm{u}_n(\mathrm{x})|^{r(\mathrm{x})-2}\,|\log|\mathrm{u}(\mathrm{x})||\mathrm{u}(\mathrm{x})\mathrm{dx} = \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\,|\log|\mathrm{u}(\mathrm{x})||\mathrm{dx}. \tag{15}$$

From (13), (14), and (15), we have that

$$\begin{aligned} &\lim_{n\to\infty}\int_{\mathcal{U}} |\mathrm{u}_n(\mathrm{x})|^{r(x)-2}\,\mathrm{u}_n(\mathrm{x})\,(\mathrm{u}_n(\mathrm{x})-\mathrm{u}_n(\mathrm{x}))\log|\mathrm{u}_n(\mathrm{x})|\,\mathrm{dx} \\ &\quad= \lim_{n\to\infty}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x})\,(\mathrm{u}_n(\mathrm{x})-\mathrm{u}(\mathrm{x}))\log|\mathrm{u}(\mathrm{x})|\mathrm{dx}. \end{aligned} \tag{16}$$

Combining (16) with the same argument in Lemma 3.1 [1], we get that $\mathrm{u}_n \to \mathrm{u}$ strongly in $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$. □

Remark 1 We assume that the conditions $(\mathcal{B}_1)-(\mathcal{B}_7)$ are fulfilled. Then, the functional ζ satisfies the (C_c) condition.

Proof We use the same technical in Theorem 4 [29] and from Lemma 8, we deduce that the functional ζ satisfies the (C_c) condition. □

Now, we compute the critical groups. From Lemma 1, it follows that $C_k(\zeta, \infty)$ make sense.

Theorem 6 *We assume that the functional ζ satisfies the conditions $(\mathcal{B}_1)$, $(\mathcal{B}_7)$. Then, we get $C_k(\zeta, \infty) = 0$.*

Proof Let $G(\mathrm{x}, t) = f(\mathrm{x}, t)t - p^+ F(\mathrm{x}, t)$ and $c_1 = 1 + \sup_{\bar{\mathcal{U}}\times[-R;R]} G(\mathrm{x}, t) - \inf_{\bar{\mathcal{U}}\times[-R;R]} G(\mathrm{x}, t)$. From the condition (H_5), we get that

$$G(\mathrm{x}, s) \leq G(\mathrm{x}, t) + c_1 \text{ for all } \mathrm{x} \in \bar{\mathcal{U}} \text{ and } 0 \leq s < t \text{ or } t \leq s \leq 0. \tag{17}$$

By (17), we get

$$G(\mathrm{x}, t) \geq -c_1 \text{ when } s = 0. \tag{18}$$

Let $\mathrm{u} \in \mathcal{S}^1 = \left\{\mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) : \|\mathrm{u}\| = 1\right\}$ and $t \geq 1$. From Fatou's Lemma and condition $(\mathcal{B}_3)$, we get that

$$+\infty = \int_{\mathcal{U}} \lim_{t\to\infty} \frac{F(\mathrm{x}, t\mathrm{u})}{|t\mathrm{u}|^{r^+}} |\mathrm{u}|^{r^+} \mathrm{dx} \leq \lim_{t\to\infty} \int_{\mathcal{U}} \frac{F(\mathrm{x}, t\mathrm{u})}{|t|^{r^+}} \mathrm{dx}. \tag{19}$$

Using the condition $(\mathcal{B}_7)$, it is easy to see that

$$\widehat{M}(t) \leq c_1^{'} t. \tag{20}$$

By (20) and (19), we have

$$\begin{aligned}
\zeta(t\mathrm{u}) &= \widehat{M}\left(J_{s,p(\mathrm{x},\cdot)}(t\mathrm{u})\right) - \lambda \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})} |t|^{r(\mathrm{x})} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} |\log|t\mathrm{u}(\mathrm{x})|| \mathrm{dx} \\
&\quad + \lambda \int_{\mathcal{U}} \frac{1}{r^2(\mathrm{x})} |t|^{r(\mathrm{x})} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \mathrm{dx} - \lambda \int_{\mathcal{U}} F(\mathrm{x}, t\mathrm{u}(\mathrm{x})) \mathrm{dx} \\
&\leq c_1^{'} \frac{t^{p^+}}{p^+} \int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x} - \mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}} \mathrm{dxdy} + \frac{t^{r^+}}{(r^-)^2} \lambda \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \mathrm{dx} \\
&\quad - \lambda \int_{\mathcal{U}} F(\mathrm{x}, t\mathrm{u}(\mathrm{x})) \mathrm{dx} \\
&\leq t^{r^+} \left(t^{p^+ - r^+} \frac{c_1^{'}}{p^+} + \frac{\lambda}{(r^-)^2} \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \mathrm{dx} - \lambda \int_{\mathcal{U}} \frac{F(\mathrm{x}, t\mathrm{u}(\mathrm{x}))}{t^{r^+}} dx \right) \\
&\to -\infty \text{ as } t \to +\infty.
\end{aligned}$$

Choosing $a < \min\left\{\inf_{\|\mathrm{u}\|\leq 1} \zeta(\mathrm{u}); \frac{-\lambda|\mathcal{U}|c_1}{p^+}\right\}$, then for $\mathrm{u} \in \mathcal{S}^1$, there exists $t_0 > 1$, such that $\zeta(t_0\mathrm{u}) \leq a$. Therefore, if

$$\begin{aligned}\zeta(t\mathrm{u}) &= \widehat{M}\left(J_{s,p(\mathrm{x},\cdot)}(t\mathrm{u})\right) - \lambda \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}|\log|t\mathrm{u}(\mathrm{x})||\mathrm{dx}\\ &\quad + \lambda \int_{\mathcal{U}} \frac{1}{r^2(\mathrm{x})}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\mathrm{dx} - \lambda \int_{\mathcal{U}} F(\mathrm{x}, t\mathrm{u}(\mathrm{x}))\mathrm{dx}\\ &\le a.\end{aligned}$$

Then,

$$\begin{aligned}&M\left(J_{s,p(\mathrm{x},\cdot)}(t\mathrm{u})\right)\int_{\mathcal{U}\times\mathcal{U}} |t|^{p(\mathrm{x},\mathrm{y})}\frac{|\mathrm{u}(\mathrm{x})-\mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}}\mathrm{dxdy}\\ &\quad - \lambda \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}|\log|t\mathrm{u}(\mathrm{x})||\mathrm{dx}\\ &\quad \le \frac{p^+}{\theta}\left[a + \lambda\int_{\mathcal{U}} F(\mathrm{x}, t\mathrm{u}(\mathrm{x}))\mathrm{dx}\right].\end{aligned}$$

Using (18), we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t\mathrm{u}) &= \frac{1}{t}\left\langle \zeta'(t\mathrm{u}), t\mathrm{u}\right\rangle\\ &= \frac{1}{t}\left(M\left(J_{s,p(\mathrm{x},\cdot)}(t\mathrm{u})\right)\int_{\mathcal{U}\times\mathcal{U}} |t|^{p(\mathrm{x},\mathrm{y})}\frac{|\mathrm{u}(\mathrm{x})-\mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}}\mathrm{dxdy}\right)\\ &\quad - \frac{1}{t}\int_{\mathcal{U}} |t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\mathrm{dx} - \frac{1}{t}\lambda\int_{\mathcal{U}} f(\mathrm{x}, t\mathrm{u})t\mathrm{u})\mathrm{dx})\\ &\le \frac{1}{t}\left(ap^+ + p^+\lambda\int_{\mathcal{U}} F(\mathrm{x}, t\mathrm{u}(\mathrm{x}))\mathrm{dx} - \lambda\int_{\mathcal{U}} f(\mathrm{x}, t\mathrm{u})t\mathrm{u})\mathrm{dx}\right)\\ &\le \frac{1}{t}\left(ap^+ + \lambda c_1|\mathcal{U}|\right) < 0,\end{aligned}$$

where $|\mathcal{U}|$ denote the measure of the domain $\mathcal{U}$. Thanks to the implicit function theorem, there exists a unique $T \in C(\mathcal{S}^1, \mathbb{R})$, such that $\zeta(T(\mathrm{u})\mathrm{u}) = a$ for any $\mathrm{u} \in \mathcal{S}^1$. We extend T to all of $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ by

$$T_0(\mathrm{u}) = \frac{1}{\|\mathrm{u}\|}T\left(\frac{\mathrm{u}}{\|\mathrm{u}\|}\right) \text{ for all } \mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})\backslash\{0\}.$$

Then, $T_0 \in C^1(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}, \mathbb{R})\backslash\{0\}$, and $\zeta(T_0(\mathrm{u})\mathrm{u}) = a$. Also, if $\zeta(\mathrm{u}) = a$, then $T_0(\mathrm{u}) = 1$. We define a function $\widehat{T_0} : W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) \to \mathbb{R}$ as

$$\widehat{T_0}(\mathrm{u}) := \begin{cases} T_0(\mathrm{u}), & \text{if } \zeta(\mathrm{u}) \ge a,\\ 1, & \text{if } \zeta(\mathrm{u}) < a.\end{cases}$$

Clearly, $\widehat{T_0} \in C(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}), \mathbb{R})\backslash 0$. Let $h : [0,1] \times W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) \to W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ be the map defined as

$$h(t,\mathrm{u}) = (1-t)\mathrm{u} + t\mathrm{u}\widehat{T_0}(\mathrm{u}) \text{ for all } (t,\mathrm{u}) \in [0,1] \times W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})\backslash 0.$$

Evidently, we have

$$h(0,\mathrm{u}) = \mathrm{u}, \text{ and } h(1,\mathrm{u}) = \widehat{T_0}(\mathrm{u})\mathrm{u} \in \zeta^a. \tag{21}$$

From (25), we get

$$h(t,.)_{|\zeta^a} = id_{|\zeta^a} \text{ for all } t \in [0,1].$$

It follows that:

$$\zeta^a \text{ is a strongly deformation of } W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}). \tag{22}$$

We consider the radial retraction $\bar{T} : W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) \to \mathbb{R}$ defined by

$$\bar{T}(\mathrm{u}) = \frac{\mathrm{u}}{\|\mathrm{u}\|} \text{ for all } \mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})\backslash\{0\}.$$

This map is continuous and $\bar{T}_{|\mathcal{S}^1} = id_{|\mathcal{S}^1}$. Then, $\mathcal{S}^1$ is a retract of $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})\backslash\{0\}$. Let $\bar{h} : [0,1] \times W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) \to W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ be the map defined as

$$\bar{h}(t,\mathrm{u}) = (1-t)\mathrm{u} + t\bar{T}(\mathrm{u}).$$

Clearly, we have

$$\bar{h}(0,\mathrm{u}) = \mathrm{u}, \quad \bar{h}(0,\mathrm{u}) \quad \text{and} \quad \bar{h}(1,.)_{|\mathcal{S}^1} = id_{|\mathcal{S}^1}. \tag{23}$$

Hence, we refer that

$$\mathcal{S}^1 \text{ is a deformation retract of } W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})\backslash\{0\}. \tag{24}$$

Combining (24) with (22), it follows that:

$$\zeta^a \text{ and } \mathcal{S}^1 \text{ are homotopy equivalent .}$$

Therefore, we have

$$H_k\left(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}), \zeta^a\right) = H_k\left(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}), \mathcal{S}^1\right) \text{ for all } k \in \mathbb{N}.$$

We already know that the space $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ is an infinite-dimensional Banach space and $\mathcal{S}^1$ is a contractible space. See Remark 6.1.13 in [26]. Therefore, it follows that:

$$C_k(\zeta,\infty) = H_k\left(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}),\zeta^a\right) = H_k\left(W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}),\mathcal{S}^1\right)$$
$$= 0 \text{ for all } k \in \mathbb{N}. \tag{25}$$

□

Theorem 7 *We assume that the conditions* $(\mathcal{B}_1)-(\mathcal{B}_7)$ *are fulfilled. Then, there exists* $k_0 \in \mathbb{N}$*, such that* $C_{k_0}(\zeta, 0) \neq 0$

Proof Evidently, the zero function is a critical point of ζ. Since $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ is a separable and reflexive Banach space, from Theorems 2, 3 in [33], there exist $\{e_i\}_{i=1}^{\infty} \subset W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ and $\{f_i\}_{i=1}^{\infty} \subset W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})^*$, such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases}$$
$$W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) = \overline{\operatorname{span}\{e_j : j = 1, 2, \dots\}};\ W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})^* = \overline{\operatorname{span}\{f_j : j = 1, 2, \dots\}}.$$

For convenience, we write $X_j = \operatorname{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^{k} X_j$, and $Z_k = \bigoplus_{j=k}^{\infty} X_j$. Thus, we have $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}) = Y_k \bigoplus Z_k$. Let $\mathrm{u} \in Y_k$. Since Y_k is a finite-dimensional space, we get that for given $R > 0$, there exists $0 < \rho < 1$ small, such that

$$\mathrm{u} \in Y_k,\ \| |\mathrm{u}| \|_{Y_k} < \rho \Rightarrow |\mathrm{u}(\mathrm{x})| < R \text{ for all } \mathrm{x} \in \mathcal{U}.$$

Let $0 < r < R$. We consider the following sets: $\mathcal{U}_1 = \{\mathrm{x} \in \mathcal{U} : |\mathrm{u}(\mathrm{x})| < r\}$, $\mathcal{U}_2 = \{\mathrm{x} \in \mathcal{U} : r < |\mathrm{u}(\mathrm{x})| < R\}$, and $\mathcal{U}_3 = \{\mathrm{x} \in \mathcal{U} : |\mathrm{u}(\mathrm{x})| > R\}$. We put $G(\mathrm{x}, t) = F(\mathrm{x}, t) - \frac{C}{p^-}|\mathrm{u}|^{\alpha(\mathrm{x})}$. Obviously, we get that $\mathcal{U}_i \cap \mathcal{U}_j$ and $\mathcal{U} = \cup_{i=1}^{3}\mathcal{U}_i$. We combine condition $(\mathcal{B}_7)$ with condition $(\mathcal{B}_4)$, and we obtain that

$$\begin{aligned}
\zeta(\mathrm{u}) &= \widehat{M}\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u})\right) - \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\mathrm{dx} - \lambda \int_{\mathcal{U}} F(\mathrm{x}, \mathrm{u}(\mathrm{x}))\mathrm{dx} \\
&\leq C_1 J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}) - \frac{1}{r^-}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\mathrm{dx} - \lambda \int_{\mathcal{U}} F(\mathrm{x}, \mathrm{u}(\mathrm{x}))\mathrm{dx} \\
&\leq C_1 J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}) - \frac{1}{r^-}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\mathrm{dx} - \lambda \int_{\mathcal{U}} \frac{C_2}{p^+}|\mathrm{u}(\mathrm{x})|^{\alpha(\mathrm{x})}\mathrm{dx} \\
&\quad - \lambda \int_{\mathcal{U}_1} G(\mathrm{x}, \mathrm{u}(\mathrm{x}))\mathrm{dx} - \lambda \int_{\mathcal{U}_2} G(\mathrm{x}, \mathrm{u}(\mathrm{x}))\mathrm{dx} - \lambda \int_{\mathcal{U}_3} G(\mathrm{x}, \mathrm{u}(\mathrm{x}))\mathrm{dx}.
\end{aligned}$$

From Theorem 3, there exists a positive constant, such that

$$|\mathrm{u}|_{L^{\alpha(\mathrm{x})}(\mathcal{U})} \leq C\|\mathrm{u}\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \leq 2C\| |\mathrm{u}| \|_{Y_k} \leq 2C\rho.$$

If $\rho < \frac{1}{2C}$, then $|u|_{L^{\alpha(x)}(\mathcal{U})} \leq 1$. Since $\mathcal{U}$ is compact, there exist a finite sub-covering $\{\mathcal{Q}_j\}_{j=1}^m$, such that

$$\overline{\mathcal{U}} = \cup_{i=1}^m \mathcal{Q}_i.$$

Notice that $\int_{\mathcal{U}_1} G(x, u(x))dx \to 0$ as $r \to 0$. Therefore, we get

$$\begin{aligned}\zeta(u) &\leq \sum_{j=1}^{m}\left[|u|^{p^+}_{L^{r(x)}(\mathcal{Q}_j)} - \frac{1}{r^-}|u|^{r(x)}_{L^{r(x)}(\mathcal{Q}_j)} - \frac{1}{p^+}|u|^{\alpha(x)}_{L^{\alpha(x)}(\mathcal{Q}_j)}\right] - \int_{\mathcal{U}_1} G(x, u(x))dx \\ &\leq 0.\end{aligned}$$

Let $u \in Z_k$. Since $q(x),\ r(x) < p_s^*(x)$, from Theorem 3, we deduce that there exist constants c_1 and c_1, such that

$$|u|_{L^{r(x)}(\mathcal{U})} \leq c_1\|u\|_{W^{s,q(x),p(x,y)}(\mathcal{U})} \text{ and } |u|_{L^{\beta(x)}(\mathcal{U})} \leq c_2\|u\|_{W^{s,q(x),p(x,y)}(\mathcal{U})}. \tag{26}$$

Using (26), $(\mathcal{B}_4)$, and $(\mathcal{B}_7)$,, we deduce that

$$\begin{aligned}\zeta(u) &= \widehat{M}\left(J_{s,p(x,\cdot)}(u)\right) - \lambda\int_{\mathcal{U}}\frac{1}{r(x)}\left(|u(x)|^{r(x)}\log|u(x)| - \frac{1}{r(x)}|u(x)|^{r(x)}\right)dx \\ &\quad - \lambda\int_{\mathcal{U}} F(x, u(x))dx \\ &\geq \widehat{M}\left(J_{s,p(x,\cdot)}(u)\right) - \int_{\mathcal{U}}\frac{1}{r(x)}|u(x)|^{r(x)}dx - \lambda\int_{\mathcal{U}} F(x, u(x))dx \\ &\geq m_1 J_{s,p(x,\cdot)}(u) - \frac{1}{r^-}\int_{\mathcal{U}}|u(x)|^{r(x)}dx - \lambda\frac{C_3}{\beta^-}\int_{\mathcal{U}}|u(x)|^{\beta(x)}dx \\ &\geq \frac{m_1}{p^+}\|u\|^{p^+}_{W^{s,q(x),p(x,y)}(\mathcal{U})} - \frac{C_1}{r^-}\|u\|^{r^-}_{W^{s,q(x),p(x,y)}(\mathcal{U})} - \frac{C_3}{\beta^-}\|u\|^{\beta^-}_{W^{s,q(x),p(x,y)}(\mathcal{U})}.\end{aligned}$$

Since $r^-, \beta^- < p^+$, we deduce that

$$\zeta(u) > 0 \text{ for all } u \in Z_k.$$

Finally, from Theorem 5, there exists $k_0 \in \mathbb{N}$, such that $C_{k_0}(\zeta, 0) \neq 0$.

Conclusion

By Theorems 6 and 7, we deduce that our problem admits at least three solutions in $W^{s,q(x),p(x,y)}(\mathcal{U})$. □

Proof of Theorem 2

Proof We suppose that our problem admits three solutions $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$. That is, $K_\zeta = \{0, \mathrm{u}, \mathrm{v}\}$. From the Morse's relation, it follows that:

$$C_n(\zeta, 0) := \begin{cases} \mathbb{R}, & \text{if } n = m(0), \\ 0, & \text{otherwise }, \end{cases}$$

where $m(0)$ is a Morse index of 0. See [6] for more details. We use Morse's relation, and we get that

$$\begin{aligned}\sum_{k\geq 0} \operatorname{rank} C_k(\zeta, \infty)X^k + (1 + X)Q(X) =& \sum_{k\geq 0} \operatorname{rank} C_k(\zeta, 0)X^k + \sum_{k\geq 0} \operatorname{rank} C_k(\zeta, \mathrm{u})X^k \\ &+ \sum_{k\geq 0} \operatorname{rank} C_k(\zeta, \mathrm{v})X^k \\ =& X^{m(0)} + 2\sum_{k\geq 0} \beta_k X^k.\end{aligned}$$

From (25), it follows that:

$$(1 + X)Q(X) = X^{m(0)} + 2\sum_{k\geq 0} \beta_k X^k,$$

where β_k non-negative integer and Q is a polynomial with non-negative integer coefficient. In particular, for $X = 1$, we have $2a = 1 + 2\sum_{k\geq 0} \beta_k$. Since $\beta_k \in \mathbb{N}$, we have that $\sum_{k\geq 0} \beta_k = +\infty$ leads to a contradiction. Thus, there exist infinitely solutions to problems (1). □

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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