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ORIGINAL ARTICLE

New rigidity results for complete LW submanifolds immersed in a Riemannian space form via certain maximum principles

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Abstract

In this paper, we establish new rigidity results concerning n-dimensional linear Weingarten (LW) submanifolds immersed in an $(n+p)$-dimensional Riemannian space form $\mathbb{Q}_c^{n+p}$ with constant sectional curvature $c \in \{-1, 0, 1\}$. Under the assumption that a complete LW submanifold has polynomial volume growth, we prove that it must be isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$. When the ambient space is the hyperbolic space $\mathbb{H}^{n+p}$, we suppose that the norm of the total umbilicity tensor converges to zero at infinity to show that a complete noncompact LW submanifold of $\mathbb{H}^{n+p}$ must be isometric to a horosphere of $\mathbb{H}^{n+1}$. Our approach is based on suitable maximum principles recently due to Alías, Caminha and do Nascimento (Alías et al. in J Math Anal Appl 474:242–247, 2019; Alías et al. in Ann Mat Pura Appl 200:1637–1650, 2021)[1, 2] related to complete noncompact Riemannian manifolds.

Keywords Riemannian space forms · Complete linear Weingarten hypersurfaces · Totally umbilical hypersurfaces · Convergence to zero at infinity · Polynomial volume growth

Mathematics Subject Classification Primary 53C42 · Secondary 53C20

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1 Introduction

A classical but still profuse thematic in Differential Geometry and, in particular, into the theory of isometric immersions, is the study of the rigidity of n-dimensional submanifolds immersed in a Riemannian space form $\mathbb{Q}_c^{n+p}$ with constant sectional curvature $c \in \{-1, 0, 1\}$. An analytical tool that has become fruitful for this research branch is a self-adjoint differential operator acting on smooth functions defined on a Riemannian manifold, known as *square operator*, which was introduced by Cheng and Yau in their remarkable paper [15]. In this work, they used the square operator to classify n-dimensional compact (without boundary) hypersurfaces with constant normalized scalar curvature R satisfying $R \geq c$ and nonnegative sectional curvature immersed in $\mathbb{Q}_c^{n+1}$. Posteriorly, Li [20] extended the results of Cheng and Yau in terms of the squared norm of the second fundamental form of the hypersurface. Next, Li [21] studied the rigidity of compact hypersurfaces with nonnegative sectional curvature immersed in a unit Euclidean sphere $\mathbb{S}^{n+1}$ under the assumption that the scalar and mean curvatures are proportional.

Proceeding with the picture described above, relevant results have appeared during the last decades. In 2009, for instance, Li, Suh and Wei [22] extended the results of [15] and [21] by considering *linear Weingarten* (LW) hypersurfaces immersed in $\mathbb{S}^{n+1}$ whose normalized scalar curvature R and mean curvature H satisfy a linear relation of the type $R = aH + b$, for some constants $a, b \in \mathbb{R}$. In this context, they obtained that if M^n is a compact LW-hypersurface with nonnegative sectional curvature immersed in $\mathbb{S}^{n+1}$, then M^n must be isometric to either a totally umbilical Euclidean sphere $\mathbb{S}^n(r)$ with radius $0 < r \leq 1$ or to a Clifford torus $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$ with $1 \leq k \leq n-1$ and $0 < r < 1$. Afterward, Shu [26] demonstrated some rigidity theorems concerning LW-hypersurfaces with two distinct principal curvatures immersed in $\mathbb{Q}_c^{n+1}$. Also working in this context and resorting to a suitable Cheng–Yau's modified operator, the first and third authors jointly with Aquino [5, 6] used suitable maximum principles to extend the results of [22] for complete LW-hypersurfaces immersed in $\mathbb{Q}_c^{n+1}$.

Regarding immersed submanifolds with (possibly) high codimension $p \geq 1$ and whose normalized mean curvature vector is parallel as a section of the normal bundle, we also have in the current literature several works addressing characterization results. In this setting, we can highlight the papers of Cheng [14] and Guo and Li [19]. In the first one, the author applied the generalized maximum principle of Omori–Yau [24, 27] to show that the totally umbilical sphere $\mathbb{S}^n(r)$, the totally geodesic Euclidean space $\mathbb{R}^n$ and the generalized cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$ are the only n-dimensional complete submanifolds with constant scalar curvature and parallel normalized mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$ satisfying a suitable constraint on the norm of the second fundamental form. In the second one, the authors investigated the problem of generalize the previous results of [20]. So, they proved that the only n-dimensional compact (without boundary) submanifolds immersed in $\mathbb{S}^{n+p}$ with constant scalar curvature, parallel normalized mean curvature vector and such that the second fundamental form satisfies an appropriate boundedness are the totally umbilical spheres $\mathbb{S}^n(r)$ and the Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$, where $r > 0$ stands for the positive radius.

Later on, the first and third author jointly with Araújo and dos Santos [11] obtained an Omori-type maximum principle for the Cheng–Yau's operator and applied it to establish an extension of the results of [14, 19] for n-dimensional complete submanifolds immersed with parallel normalized mean curvature vector in $\mathbb{Q}_c^{n+p}$, with constant normalized scalar curvature. Next, these same authors [18] used the Hopf's strong maximum principle and a maximum principle at infinity due to Caminha [13] to obtain versions of the results of [11, 14, 19] for the context of n-dimensional complete LW submanifolds immersed with parallel normalized mean curvature vector in $\mathbb{Q}_c^{n+p}$. In [9], the third author jointly with Araújo established a version of the classical Liebmann's rigidity theorem showing that a compact LW-surface immersed with flat normal bundle and parallel normalized mean curvature vector with nonnegative Gaussian curvature in $\mathbb{Q}_c^{2+p}$ must be isometric to a totally umbilical round sphere. They also obtained in [10] another version of this Liebmann's result assuming that the ambient is the hyperbolic space (for other characterizations concerning complete LW submanifolds in the hyperbolic space we refer the reader to [4, 7, 8, 17]).

Motivated by these works above mentioned, our aim in this paper is to obtain new rigidity results concerning n-dimensional LW submanifolds immersed in $\mathbb{Q}_c^{n+p}$. First, under the assumption that a complete LW submanifold has polynomial volume growth, we establish sufficient conditions to guarantee that it must be isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$ (see Theorems 1, 2 and 3, and Corollaries 1 and 2). Afterward, when the ambient space is the hyperbolic space $\mathbb{H}^{n+p}$, supposing that the norm of the total umbilicity tensor converges to zero at infinity, we are able to show that a complete noncompact LW submanifold of $\mathbb{H}^{n+p}$ must be isometric to a horosphere of $\mathbb{H}^{n+1}$ (see Theorems 4, 5, 6 and 7). Our approach is based on suitable maximum principles recently due to Alías, Caminha and do Nascimento [1, 2] related to complete noncompact Riemannian manifolds (see Lemmas 1 and 2).

2 Preliminaries

Let us denote by $\mathbb{Q}_c^{n+p}$ the standard model of an $(n+p)$-dimensional Riemannian space form with constant sectional curvature $c \in \{0, 1, -1\}$. Actually, $\mathbb{Q}_c^{n+p}$ denotes the Euclidean $(n+p)$-space $\mathbb{R}^{n+p}$ when $c = 0$, the $(n+p)$-dimensional Euclidean sphere $\mathbb{S}^{n+p}$ when $c = 1$ and the $(n+p)$-dimensional hyperbolic space $\mathbb{H}^{n+p}$ when $c = -1$. We also denote by $\langle , \rangle$ the corresponding Riemannian metric induced on $\mathbb{Q}_c^{n+p} \hookrightarrow \mathbb{R}^{n+p+1}$.

Let M^n be an n-dimensional connected submanifold immersed in $\mathbb{Q}_c^{n+p}$. We choose a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $\mathbb{Q}_c^{n+p}$ with dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$ such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n and $e_n, \dots, e_{n+p}$ are normal to M^n. Moreover, let $\{\omega_{BC}\}$ denote the connection 1-forms on $\mathbb{Q}_c^{n+p}$. In what follows, we will use the following convention for the indices:

$$1 \leq A, B, C, \dots \leq n+p, \quad 1 \leq i, j, k, \dots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p.$$

The second fundamental form A, the curvature tensor R and the normal curvature tensor $R^{\perp}$ of M^n are given by

$$\omega_{i\alpha} = \sum_j h^{\alpha}_{ij}\omega_j, \quad A = \sum_{i,j,\alpha} h^{\alpha}_{ij}\omega_i \otimes \omega_j e_\alpha,$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$
$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2}\sum_{k,l} R^{\perp}_{\alpha\beta kl}\omega_k \wedge \omega_l.$$

It is not difficult to see that the components $h^{\alpha}_{ijk}\omega_k$ of the covariant derivate ∇A satisfy

$$\sum_k h^{\alpha}_{ijk}\omega_k = dh^{\alpha}_i j + \sum_k h^{\alpha}_{ki}\omega_{kj} + \sum_k h^{\alpha}_{kj}\omega_{ki} + \sum_\beta h^{\beta}_{kj}\omega_{ki}. \tag{2.1}$$

Moreover, the Gauss equation is given by

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}).$$

In particular, the components of the Ricci tensor R_{ik} and the normalized scalar curvature R are given, respectively, by

$$R_{ik} = (n-1)\delta_{ik} + n\sum_\alpha H^\alpha h^{\alpha}_{ik} - \sum_{\alpha,j} h^{\alpha}_{ij}h^{\alpha}_{jk} \tag{2.2}$$

and

$$R = \frac{1}{n-1}\sum_i R_{ii}. \tag{2.3}$$

From (2.2) and (2.3), we get the following relation

$$n(n-1)R = n(n-1)c + n^2H^2 - |A|^2, \tag{2.4}$$

where $|A|^2 = \sum_{\alpha,i,j}(h^{\alpha}_{ij})^2$ is the squared norm of the second fundamental form A and $H = |\mathbf{H}|$ is the mean curvature function related to the mean curvature vector field $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha = \frac{1}{n}\sum_\alpha(\sum_k h^{\alpha}_{kk})e_\alpha$ of M^n.

Furthermore, Codazzi equation is given by

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj} = h^{\alpha}_{jik}. \tag{2.5}$$

We will also consider the symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi^{\alpha}_{ij}\omega_i \otimes \omega_j e_\alpha, \tag{2.6}$$

where $\Phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$. Consequently, we have that

$$\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^{\alpha}_{ij} = h^{\alpha}_{ij},$$

for $n+2 \leq \alpha \leq n+p$.

Let $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi^{\alpha}_{ij})^2$ be the squared norm of Φ. It is not difficult to check that Φ is traceless with

$$|\Phi|^2 = |A|^2 - nH^2. \tag{2.7}$$

In addition, from (2.4) we obtain

$$n(n-1)R = n(n-1)(c+H^2) - |\Phi|^2. \tag{2.8}$$

3 Key lemmas

Let $(M^n, \langle,\rangle)$ be a connected, oriented, complete Riemannian manifold. We denote by $B(p,t)$ the geodesic ball centered at p with radius t. Given a polynomial function $\sigma : (0,+\infty) \to (0,+\infty)$, we say that M^n has *polynomial volume growth* like $\sigma(t)$ if there exists $p \in M^n$ such that

$$\mathrm{vol}(B(p,t)) = \mathcal{O}(\sigma(t)),$$

as $t \to +\infty$, where vol denotes the standard Riemannian volume. As it was already observed in the beginning of Section 2 of [2], if $p, q \in M^n$ are at distance d from each other, we can verify that

$$\frac{\mathrm{vol}(B(p,t))}{\sigma(t)} \geq \frac{\mathrm{vol}(B(q,t-d))}{\sigma(t-d)} \cdot \frac{\sigma(t-d)}{\sigma(t)}.$$

Consequently, the choice of p in the notion of volume growth is immaterial. For this reason, we will just say that M^n has polynomial volume growth.

Keeping in mind this previous digression and denoting by $\mathrm{div}X$ the divergence of a smooth vector field $X \in \mathfrak{X}(M)$ in the metric $\langle,\rangle$, we quote the following key lemma which corresponds to a particular case of a new maximum principle due to Alías, Caminha and do Nascimento (see [2, Theorem 2.1]).

Lemma 1 *Let $(M^n, \langle,\rangle)$ be a connected, oriented, complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a bounded smooth vector field on M^n. Assume that $f \in C^{\infty}(M)$ is a smooth function on M^n such that $\langle \nabla f, X\rangle \geq 0$ and $\mathrm{div}X \geq \alpha f$, for some positive constant α. If M^n has polynomial volume growth, then $f \leq 0$ on M^n.*

Now, let us recall a notion of convergence to zero at infinity established in [1, Section 2]: If M^n is a connected, complete noncompact Riemannian manifold, we let

$d(\cdot, o) : M \to [0, +\infty)$ stand for the Riemannian distance of M^n, measured from a fixed point $o \in M^n$. Thus, if $f \in C^0(M^n)$ satisfies

$$\lim_{d(x,o)\to+\infty} f(x) = 0,$$

we say that f converges to zero at infinity. So, we quote a maximum principle presented in [1, Theorem 2.2(a)].

Lemma 2 *Let $(M^n, \langle,\rangle)$ be a connected, oriented, complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M^n)$ be a smooth vector field on M^n. Assume that there exists a nonnegative, non-identically vanishing function $f \in C^\infty(M)$ which converges to zero at infinity and such that $\langle \nabla f, X\rangle \geq 0$. If $\operatorname{div} X \geq 0$ on M^n, then $\langle \nabla f, X\rangle \equiv 0$ on M^n.*

We will also need the next key lemma, which is due to Barros et al. (see [12, Lemma 1]).

Lemma 3 *Let M^n be a Riemannian manifold isometrically immersed into a Riemannian manifold N^{n+p}. Consider $\Psi = \sum_{\alpha,i,j} \Psi^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha$ a traceless symmetric tensor satisfying Codazzi equation. Then the following inequality holds*

$$|\nabla|\Psi|^2|^2 \leq \frac{4n}{n+2}|\Psi|^2|\nabla\Psi|^2,$$

where $|\Psi|^2 = \sum_{\alpha,i,j}(\Psi^\alpha_{ij})^2$ and $|\nabla\Psi|^2 = \sum_{\alpha,i,j,k}(\Psi^\alpha_{ijk})^2$. In particular, the conclusion holds for the tensor Φ defined in (2.6).

4 Main results

We recall once more that a submanifold is said to be *linear Weingarten* (LW) when its first mean and normalized scalar curvatures are linearly related, that is, when they satisfy the following relation

$$R = aH + b, \tag{4.1}$$

for constants $a, b \in \mathbb{R}$. We observe that when $a = 0$, (4.1) reduces to R constant.

In this setting, Eq. (2.7) becomes

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 - n(n-1)aH - n(n-1)(b-c). \tag{4.2}$$

For a LW submanifold M^n satisfying (4.1), we introduce the second-order linear differential operator $\mathcal{L} : C^\infty(M) \to C^\infty(M)$ defined by

$$\mathcal{L} = L - \frac{n-1}{2}a\Delta, \tag{4.3}$$

where Δ is the Laplacian operator on M^n and $L : C^\infty(M) \to C^\infty(M)$ denotes the Cheng–Yau's operator, which is given by

$$Lu = \mathrm{tr}(P \circ \mathrm{Hess}\,(u)), \tag{4.4}$$

for every $u \in C^\infty(M)$, where Hess is the self-adjoint linear tensor metrically equivalent to the Hessian of u and $P : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the first Newton transformation of M^n which is given by $P = nHI - A$. So, from (4.3) and (4.4), we have that

$$\mathcal{L}u = \mathrm{tr}(\mathcal{P} \circ \mathrm{Hess}\,(u)),$$

with

$$\mathcal{P} = \left(nH - \frac{n-1}{2}a\right) I - A \tag{4.5}$$

and it is not difficult to verify that $\mathcal{L}$ can be rewritten in the following divergence form (see, for instance, [25, Section 4])

$$\mathcal{L}u = \mathrm{div}(\mathcal{P}(\nabla u)). \tag{4.6}$$

In the next subsections, we will use the modified Cheng–Yau's operator $\mathcal{L}$ jointly with the lemmas quoted in the previous section to establish our rigidity results concerning LW submanifolds in a Riemannian space form.

4.1 Rigidity results for complete LW submanifolds in space forms

Before to present our results, we need to collect some properties related to the following one-parameter family of real functions

$$Q_t(x) = -(n-2)x^2 - (n-2)x\sqrt{x^2 + n(n-1)(t-c)} + n(n-1)t, \tag{4.7}$$

where $t \in \mathbb{R}$ corresponds to the real parameter, while n and c are real constants. We note that Alías, García-Martínez and Rigoli introduced in [3] the definition of the function $Q_R(x)$ when they were studying hypersurfaces with constant normalized scalar curvature R in an $(n+1)$-dimensional Riemannian space form of constant sectional curvature $c \in \{-1, 0, 1\}$.

For each nonnegative (positive) parameter t, we have that $Q_t(0) = n(n-1)t$ is also nonnegative (positive). When $n \geq 3$, each function Q_t is (strictly) decreasing for $x \geq 0$, with $Q_t(x_t^*) = 0$ only at

$$x_t^* = t\sqrt{\frac{n(n-1)}{(n-2)(nt-(n-2)c)}}. \tag{4.8}$$

Moreover, in the case $n = 2$, we have that $Q_t(x) = 2t$.

Now, we are in position to present our first rigidity result concerning a complete LW-hypersurface M^n immersed in $\mathbb{Q}_c^{n+1}$.

Theorem 1 *Let M^n be a complete LW-hypersurface immersed into a Riemannian space form $\mathbb{Q}_c^{n+1}$ with $n \geq 3$, such that $R = aH + b$ with $b \geq c$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^n, for some positive constant β, and that $R > \frac{n-2}{n}$ for $c = 1$ and $R > 0$ for $c = 0$ or $c = -1$. Assume in addition that $|\nabla\Phi|$ is bounded and $\sup_M |\Phi| \leq \gamma < x_R^*$, for some constant γ and x_R^* defined in (4.8). If M^n has polynomial volume growth and $\inf_R(Q_R(\gamma)) > 0$, then M^n is isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Proof Taking the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$, it will fulfill the required conditions to apply Lemma 1. Indeed, by hypothesis we have that $|\Phi|$ is bounded on M^n and, by Eq. (4.2), H and $|A|$ are also bounded on M^n. Consequently, from definition (4.5), we get

$$|X| = |\mathcal{P}(\nabla|\Phi|^2)| \leq |\mathcal{P}||\nabla|\Phi|^2| \leq \Bigg(n\sqrt{n}|H| + \frac{(n-1)\sqrt{n}}{2}|a| + |A|\Bigg)|\nabla|\Phi|^2| \leq k|\nabla|\Phi|^2|,$$

for some positive constant k. But, since we are supposing the boundedness of $|\Phi|$ and $|\nabla\Phi|$, Lemma 3 guarantees that $\nabla|\Phi|^2$ is also bounded. Thus, we have that

$$|X| \leq C < +\infty,$$

for some positive constant C.

On the other hand, the condition

$$\langle\nabla f, X\rangle = \langle\nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0$$

is also verified because [18, Lemma 4.4] gives that $\mathcal{P}$ is positive semi-definite for $b \geq c$.

Now, we must obtain $\operatorname{div}X \geq \alpha f$ on M^n, for some positive constant α. For this, we will find a suitable lower bound for $\mathcal{L}(|\Phi|^2)$. Applying $\mathcal{L}$ in (4.2), we get that

$$\begin{aligned}\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) &= \frac{1}{2}\mathcal{L}(nH^2) - \frac{a}{2}\mathcal{L}(nH)\\ &= H\mathcal{L}(nH) + n\langle\mathcal{P}\nabla H, \nabla H\rangle - \frac{a}{2}\mathcal{L}(nH).\end{aligned} \tag{4.9}$$

In particular, since $\mathcal{P}$ is positive semi-definite, from (4.9) we obtain

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq \left(H - \frac{a}{2}\right)\mathcal{L}(nH). \tag{4.10}$$

Let us choose a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$. Since $R = aH + b$, from [6, Equation (2.19)] jointly with the definition of $\mathcal{L}$ and with $R_{ijij} = \lambda_i \lambda_j + c$, we get

$$\mathcal{L}(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + nc(|A|^2 - nH^2) - |A|^4 + nH\sum_i \lambda_i^3. \tag{4.11}$$

Moreover, we have $\Phi_{i,j} = \mu_i \lambda_{ij}$ and, with straightforward computation, we verify that

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{4.12}$$

Thus, using Gauss equation jointly with (4.11) and (4.12), we get

$$\mathcal{L}(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + nH\sum_i \mu_i^3 + |\Phi|^2(-|\Phi|^2 + nH^2 + nc). \tag{4.13}$$

We can apply [18, Lemma 4.1] jointly with the classical lemma due to Okumura [23] for $n \geq 3$, to obtain from (4.13) that

$$\mathcal{L}(nH) \geq |\Phi|^2 \left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| + nH^2 + nc\right). \tag{4.14}$$

Furthermore, from (2.8) we obtain

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (R - c). \tag{4.15}$$

Thus, from (4.14) and (4.15), we achieve in

$$\mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|), \tag{4.16}$$

where Q_R is defined in (4.7). Hence, using (4.10) jointly with (4.16), from (4.6) we conclude that

$$\text{div}X = \text{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2. \tag{4.17}$$

Since we have $\left(H - \frac{a}{2}\right) \geq \beta > 0$ by hypothesis and from the behavior of $Q_R(x)$ for $0 \leq |\Phi| \leq \sup_M |\Phi| \leq \gamma < x_R^*$, we have that

$$Q_R(|\Phi|) \geq Q_R(\gamma) > \inf_R (Q_R(\gamma)) > 0. \tag{4.18}$$

Then, from (4.17) and (4.18), we obtain

$$\text{div}X \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2 \geq \alpha|\Phi|^2, \tag{4.19}$$

and $\operatorname{div} X \geq \alpha f$ for $\alpha = 2\beta \inf_R(Q_R(\gamma)) > 0$.

Consequently, supposing that M^n is noncompact and with polynomial volume growth, we are able to apply Lemma 1 obtaining that $|\Phi|^2 \leq 0$ on M^n. Then, $|\Phi| \equiv 0$, which means that M^n is a totally umbilical hypersurface. But, from the characterizations of the totally umbilical hypersurfaces of the Riemannian space forms, we conclude that M^n must be isometric to $\mathbb{R}^n$, which corresponds to a contradiction with the hypothesis that $R > 0$.

Thus, M^n must be compact. So, we can integrate both sides of (4.19) and use Divergence Theorem to get that

$$\int_M |\Phi|^2 \mathrm{d}M = 0.$$

Therefore, we have that $|\Phi| \equiv 0$ and, hence, M^n is a compact totally umbilical hypersurface of $\mathbb{Q}_c^{n+1}$. Hence, M^n must be isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$. □

Revisiting the proof of Theorem 1, we observe that if $n = 2$, then $\sum_i \mu_i^3 = 0$. Consequently, from (4.13) we get

$$\mathcal{L}(nH) \geq |\Phi|^2 \left(-|\Phi|^2 + 2H^2 + 2c\right),$$

and (4.14) is still true in this case. Hence, it is not difficult to verify that we also have the following rigidity result.

Theorem 2 *Let M^2 be a complete LW-surface immersed into a Riemannian space form $\mathbb{Q}_c^3$, such that $R = aH + b$ with $b \geq c$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^2, for some positive constant β, and that $\inf_M R > 0$. Assume in addition that $|\Phi|$ and $|\nabla\Phi|$ are bounded. If M^2 has polynomial volume growth, then M^2 is isometric to an Euclidean sphere $\mathbb{S}^2(r)$, with radius $r > 0$.*

Observing that, when $R > 0$ is constant, the hypothesis $\inf_R(Q_R(\gamma)) > 0$ is automatically satisfied, from Theorems 1 and 2 we obtain, respectively, the following consequences:

Corollary 1 *Let M^n be a complete hypersurface immersed into a Riemannian space form $\mathbb{Q}_c^{n+1}$ with $n \geq 3$, with constant normalized scalar curvature $R \geq 1$ for $c = 1$ and $R > 0$ when $c = -1$ or $c = 0$. Suppose that $H \geq \beta$ on M^n, for some positive constant β. Assume in addition that $|\nabla\Phi|$ is bounded and $\sup_M |\Phi| < x_R^*$, for x_R^* defined in (4.8). If M^n has polynomial volume growth, then M^n is isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Corollary 2 *Let M^2 be a complete surface immersed into a Riemannian space form $\mathbb{Q}_c^3$, with constant normalized scalar curvature $R \geq 1$ for $c = 1$ and $R > 0$ when $c = -1$ or $c = 0$. Suppose that $H \geq \beta$ on M^2, for some positive constant β. Assume in addition that $|\Phi|$ and $|\nabla\Phi|$ are bounded. If M^2 has polynomial volume growth, then M^2 is isometric to an Euclidean sphere $\mathbb{S}^2(r)$, with radius $r > 0$.*

Proceeding, we will deal with LW submanifolds M^n of $\mathbb{Q}_c^{n+p}$ having parallel normalized mean curvature vector field $\mathbf{H}$, which means that the mean curvature function H is positive and that the corresponding normalized mean curvature vector field $\frac{\mathbf{H}}{H}$ is parallel as a section of the normal bundle. In this context, we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ such that $e_{n+1} = \frac{\mathbf{H}}{H}$. Consequently, we have

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^\alpha = \frac{1}{n}\mathrm{tr}(h^\alpha) = 0, \quad \alpha \geq n+2. \tag{4.20}$$

Considering this previous context, we can state a version of Theorem 1 for higher codimension.

Theorem 3 *Let M^n be a complete LW submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ with $n \geq 4$, such that $R = aH + b$ with $a \geq 0$ and $b \geq c$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^n, for some positive constant β, and that $R > \frac{n-2}{n}$ for $c = 1$ and $R > 0$ when $c = -1$ or $c = 0$. Assume in addition that $|\nabla\Phi|$ is bounded and such that $\sup_M |\Phi| \leq \gamma < x_R^*$, for some constant γ and x_R^* defined in (4.8). If M^n has polynomial volume growth and $\inf_R(Q_R(\gamma)) > 0$, then M^n is isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Proof Reasoning as in the proof of Theorem 1, we take the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. So, we have that

$$|X| \leq C, \tag{4.21}$$

for some positive constant C, and

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0. \tag{4.22}$$

Moreover,

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq \left(H - \frac{a}{2}\right)\mathcal{L}(nH). \tag{4.23}$$

On the other hand, following the same initial steps of the proof of [18, Theorem 5.1], we can achieve in [18, Inequality (5.16)] which is given by

$$\mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|) + (|\Phi| - |\Phi^{n+1}|)\left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|.$$

Thus, since we are also assuming that $n \geq 4$, we get

$$\mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|). \tag{4.24}$$

So, using (4.23) jointly with (4.24), we conclude that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2. \tag{4.25}$$

But, since $\left(H - \frac{a}{2}\right) \geq \beta > 0$, taking into account once more the behavior of $Q_R(x)$, for $0 \leq |\Phi| \leq \sup_M |\Phi| < \gamma < x_R^*$, we have that

$$Q_R(|\Phi|) \geq Q_R(\gamma) > \inf_R(Q_R(\gamma)) > 0.$$

Hence, from (4.25), we obtain

$$\mathrm{div}X \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2 \geq \alpha f, \tag{4.26}$$

where $\alpha = 2\beta \inf_R(Q_R(\gamma)) > 0$.

Supposing that M^n is a noncompact submanifold, since (4.21), (4.22) and (4.26) were verified and M^n has polynomial volume growth, we are able to apply Lemma 1 to obtain that $|\Phi|^2 \leq 0$ on M^n. Then, $|\Phi| \equiv 0$ and M^n is totally umbilical submanifold. Consequently, taking into account (4.20), we get

$$h^\alpha = \langle H, e_\alpha \rangle I = H^\alpha I = 0,$$

for all $\alpha > n+1$. Thus, we have that the first normal subspace

$$N_1 = \left\{ e_\alpha \in \mathfrak{X}^\perp(M^n);\, h^\alpha = 0 \right\}^\perp$$

is parallel and it has dimension 1. Therefore, we can apply [16, Proposition 4.1] to reduce the codimension of M^n to 1. Hence, since M^n is, in fact, a totally umbilical noncompact hypersurface with polynomial volume growth, we infer that it is isometric to $\mathbb{R}^n$, which corresponds to a contradiction with the hypothesis $R > 0$.

At this point, we can reason as in the last part of the proof of Theorem 1 to conclude, reducing the codimension of M^n again, that M^n must be isometric to a totally umbilical Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$. □

4.2 Further rigidity results in the hyperbolic space

In what follows we will apply Lemma 2 to get further rigidity results concerning n-dimensional complete noncompact LW submanifolds in the $(n+p)$-dimensional hyperbolic space $\mathbb{H}^{n+p}$. So, we state and prove our first one related to the case $p = 1$.

Theorem 4 *Let M^n be a complete noncompact LW-hypersurface immersed into the hyperbolic space $\mathbb{H}^{n+1}$ with $n \geq 3$, such that $R = aH + b$ with $b > -1$. Suppose that $\left(H - \frac{a}{2}\right) \geq 0$ on M^n and that $R \geq 0$. Assume in addition that $|\Phi| \leq x_R^*$, for x_R^* defined in (4.8). If $|\Phi|$ converges to zero at infinity, then M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$.*

Proof Let us consider the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$ and let us suppose that M^n is not a umbilical hypersurface. So, f is non-identically vanishing function which converges to zero at infinity. Moreover, we have that

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0.$$

We claim that $\operatorname{div} X \geq 0$. Indeed, we already know that

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq \left(H - \frac{a}{2}\right)\mathcal{L}(nH) \quad \text{and} \quad \mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|), \tag{4.27}$$

where Q_R is the function given by (4.7). Thus, since $\left(H - \frac{a}{2}\right) \geq 0$, from (4.27) jointly with the behavior of $Q_R(x)$ for $0 \leq |\Phi| \leq x_R^*$, we conclude that

$$\operatorname{div} X = \operatorname{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2 \geq 0.$$

Hence, we can apply Lemma 2 to get that

$$\langle \nabla f, X\rangle = \langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2\rangle \equiv 0.$$

Consequently, since [18, Lemma 4.4] gives that $\mathcal{P}$ is positive definite, we have that $\nabla|\Phi|^2 \equiv 0$. Thus, $f = |\Phi|^2$ is constant. But, since f converges to zero at infinity, it must be identically zero, leading us to a contradiction. Therefore, M^n is a complete noncompact totally umbilical hypersurface of $\mathbb{H}^{n+1}$ with $R \geq 0$, which means that M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$. □

In the case $n = 2$, reasoning as in the proof of Theorem 4, we also obtain the following—

Theorem 5 *Let M^2 be a complete noncompact LW-surface immersed into the hyperbolic space $\mathbb{H}^3$, such that $R = aH + b$ with $b > -1$. Suppose that $\left(H - \frac{a}{2}\right) \geq 0$ on M^2 and that $R \geq 0$. If $|\Phi|$ converges to zero at infinity, then M^2 is isometric to a horosphere of $\mathbb{H}^3$.*

Applying again a codimension reduction process, we obtain our next rigidity result.

Theorem 6 *Let M^n be a complete noncompact LW submanifold immersed with parallel normalized mean curvature vector field into the hyperbolic space $\mathbb{H}^{n+p}$ with $n \geq 4$, such that $R = aH + b$ with $a \geq 0$ and $b > -1$. Suppose that $\left(H - \frac{a}{2}\right) \geq 0$ on M^n and that $R \geq 0$. Assume in addition that $|\Phi| \leq x_R^*$, for x_R^* defined in (4.8). If $|\Phi|$ converges to zero at infinity, then M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$.*

Proof It is not difficult to verify that, using inequality (4.25) and following similar steps of the proof of Theorem 4, we can achieve in $\nabla|\Phi|^2 \equiv 0$. So, taking into account (4.20), we get

$$h^\alpha = \langle H, e_\alpha\rangle I = H^\alpha I = 0$$

for every $\alpha > n+1$. This implies that the first normal subspace

$$N_1 = \left\{ e_\alpha \in \mathfrak{X}^\perp(M^n);\, h^\alpha = 0 \right\}^\perp$$

is parallel and has dimension 1. Therefore, we are in position to apply once more [16, Proposition 4.1], reducing the codimension of M^n to 1 and concluding that it is a totally umbilical noncompact hypersurface of $\mathbb{H}^{n+1}$ with $R \geq 0$. Consequently, M^n must be a horosphere of $\mathbb{H}^{n+1}$. □

In our last rigidity result, we will deal with complete noncompact LW submanifolds having nonnegative sectional curvature, which are immersed with globally flat normal bundle in $\mathbb{H}^{n+p}$.

Theorem 7 *Let M^n be a complete noncompact LW submanifold with nonnegative sectional curvature immersed into the hyperbolic space $\mathbb{H}^{n+p}$, $n \geq 2$ with globally flat normal bundle and parallel normalized mean curvature vector field, such that $R = aH + b$ with $b > -1$ and $\left(H - \frac{a}{2}\right) \geq 0$. If the total umbilicity tensor of the immersion $|\Phi|$ converges to zero at infinity, then M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$.*

Proof As before, we take the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. Supposing that M^n is not a totally umbilical submanifold, reasoning as in the proof of Theorem 4 we obtain that f is non-identically vanishing function which converges to zero at infinity and such that $\langle \nabla f, X \rangle \geq 0$.

Now, let us verify that $\operatorname{div} X \geq 0$. Indeed, we have

$$\frac{1}{2}\Delta|A|^2 = \sum_{i,j,\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} + \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2. \tag{4.28}$$

Using Codazzi equation (2.5) into (4.28), we get

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_{i,j,k,\alpha} h^\alpha_{ij} h^\alpha_{kijk}. \tag{4.29}$$

On the other hand, by exterior differentiation of (2.1) and assuming that M^n has globally flat normal bundle (that is, $R^\perp = 0$), we obtain the following Ricci identity

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj} R_{mikl} + \sum_m h^\alpha_{im} R_{mjkl}. \tag{4.30}$$

Thus, from (4.20), (4.29) and (4.30), we reach at

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_{i,j} nH^{n+1}_{ij} h^{n+1}_{ij} + \sum_{i,j,m,k,\alpha} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} + \sum_{i,j,k,m,\alpha} h^\alpha_{ij} h^\alpha_{km} R_{mijk}. \tag{4.31}$$

Consequently, taking a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$, for every α, from (4.31), we obtain the following Simons-type formula

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_i \lambda_i^{n+1}(nH)_{ii} + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{4.32}$$

Moreover, using the definition (4.4), we obtain

$$\begin{aligned} L(nH) &= nH\Delta(nH) - \sum_i \lambda_i^{n+1}(nH)_{ii} \\ &= \frac{n(n-1)}{2}\Delta R + \frac{1}{2}\Delta|A|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i^{n+1}(nH)_{ii}. \end{aligned} \tag{4.33}$$

Thus, inserting (4.32) into (4.33), we get

$$L(nH) = \frac{n(n-1)}{2}\Delta R + |\nabla A|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{4.34}$$

Provided that $R = aH + b$, from (4.3) and (4.34), we have

$$\mathcal{L}(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{4.35}$$

Hence, since M^n is supposed to have nonnegative sectional curvature and using [18, Lemma 4.1], from (4.35), we get $\mathcal{L}(nH) \geq 0$. Thus, since $(H - \frac{a}{2}) \geq 0$, from (4.27), we finally deduce that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H - \frac{a}{2}\right)\mathcal{L}(nH) \geq 0.$$

Now, applying Lemma 2, we obtain

$$\langle \nabla f, X\rangle = \langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2\rangle \equiv 0.$$

So, since [18, Lemma 4.4] guarantees that $\mathcal{P}$ is positive definite, we get that $\nabla|\Phi|^2 \equiv 0$. Thus, as in the last part of the proof of Theorem 4, we will have that $f = |\Phi|^2$ is identically zero, leading us to a contradiction. Therefore, M^n must be totally umbilical and, reducing the codimension of M^n to 1, we conclude that M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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