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ORIGINAL ARTICLE

Baire functional calculus for bounded-locally operators

Mohamed Mazighi[1] · **Abdellah El Kinani**[1]

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Abstract
We define and study a simultaneous Baire functional calculus for a commutative family of normal bounded-locally operators on a locally Hilbert space. The most significant properties of this calculus are presented. We also provide some applications dealing with locally Hilbert spaces, namely the existence of a particular orthonormal basis, the polar decomposition and the existence of proper hyper-invariant subspaces.

Keywords Locally Hilbert space · Bounded-locally operator · Simultaneous Baire functional calculus · Orthonormal basis · Hyper-invariant subspace · Polar decomposition

Mathematics Subject Classification 46J10 · 46H30

1 Preliminaries and introduction

Let Λ be a directed index set and let $\mathcal{H}_\lambda$ ($\lambda \in \Lambda$) be a Hilbert space with the inner product $\langle\ ,\ \rangle_\lambda$. Assume that the family $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ of Hilbert spaces satisfies $\mathcal{H}_\lambda \subset \mathcal{H}_\upsilon$ and $\langle\ ,\ \rangle_\lambda = \langle\ ,\ \rangle_{\upsilon|\mathcal{H}_\lambda}$ on $\mathcal{H}_\lambda$ if $\lambda \le \upsilon$. Let $i_{\upsilon\lambda} : \mathcal{H}_\lambda \longrightarrow \mathcal{H}_\upsilon$, $\lambda \le \upsilon$ in Λ, be the natural embedding of $\mathcal{H}_\lambda$ in $\mathcal{H}_\upsilon$. Then the family $(\mathcal{H}_\lambda, i_{\upsilon\lambda})$, $\lambda \le \upsilon$ in Λ, forms an inductive system of Hilbert spaces. Now consider:

$$\mathcal{H} = \underrightarrow{\lim}\mathcal{H}_\lambda = \bigcup_{\lambda\in\Lambda} \mathcal{H}_\lambda, \quad \lambda \in \Lambda.$$

Endow $\mathcal{H}$ with the inductive limit topology, that is the finest locally convex topology making the natural injections $i_\lambda : \mathcal{H}_\lambda \longrightarrow \mathcal{H}$, $\lambda \in \Lambda$, continuous. The space $\mathcal{H}$ endowed with this topology is called a locally Hilbert space ([5], Definition 5.2).

✉ Abdellah El Kinani
abdellah.elkinani@um5.ac.ma

Mohamed Mazighi
mohamedmazighi6@gmail.com

1 Université Mohammed V de Rabat, E.N.S de Rabat, B. P. 5118, 10105 Rabat, Morocco

Let $\mathcal{L}(\mathcal{H}_\lambda)$ be the C^*-algebra of all bounded linear operators on the Hilbert space $\mathcal{H}_\lambda,\ \lambda \in \Lambda$. For $T_i \in \mathcal{L}(\mathcal{H}_i)$, $i = \lambda, \upsilon$, one has:

$$T_\upsilon \mid_{\mathcal{H}_\lambda} = T_\lambda,\ \lambda \le \upsilon, \quad \text{if and only if} \quad i_{\upsilon\lambda} \circ T_\upsilon = T_\lambda \circ i_{\upsilon\lambda}.$$

Thus, under this last equality, we get a unique continuous linear map $T = \underrightarrow{\lim} T_\lambda : \mathcal{H} \longrightarrow \mathcal{H}$ such that:

$$T \mid_{\mathcal{H}_\lambda} = T_\lambda \in \mathcal{L}(\mathcal{H}_\lambda), \quad \text{for every } \lambda \in \Lambda.$$

In the sequel, an element T of $\mathcal{L}(\mathcal{H})$ is called bounded-locally operator on $\mathcal{H}$. Now, we consider the algebra $\mathcal{L}(\mathcal{H})$ of all continuous linear maps $T : \mathcal{H} \longrightarrow \mathcal{H}$ such that:

$$T = \underrightarrow{\lim} T_\lambda,\ T_\lambda \in \mathcal{L}(\mathcal{H}_\lambda), \quad \text{for every } \lambda \in \Lambda.$$

For every $\lambda \in \Lambda$, consider the function:

$$p_\lambda(T) = \|T_\lambda\|_\lambda, \quad \text{for every } T \in \mathcal{L}(\mathcal{H}), \quad \lambda \in \Lambda,$$

where $\|.\|_\lambda$ denotes the operator C^* -norm on $\mathcal{L}(\mathcal{H}_\lambda)$.Then, by Proposition 5.1, p. 232 of [5], the map $T \longmapsto T^*$ given by $T^* = \underrightarrow{\lim} T_\lambda^*$ defines an involution on $\mathcal{L}(\mathcal{H})$ such that $\left(\mathcal{L}(\mathcal{H}), (p_\lambda)_{\lambda\in\Lambda}\right)$ is a locally C^*-algebra. Whence

$$\mathcal{L}(\mathcal{H}) = \underleftarrow{\lim} \mathcal{L}(\mathcal{H}_\lambda), \quad \lambda \in \Lambda.$$

Let us remember that a locally C^*-algebra ([4], Definition 7.5, p. 102), is isomorphic to a closed $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ ([5], Theorem 5.1, p. 232). For a detailed account of the basic properties of the theory of locally Hilbert spaces, see [5].

Let K be a non-empty compact space and $ba(K)$ be the Baire σ-field on K, that is the σ-field generated by the collection of all the compact, G_δ-subsets of K. Notice that $ba(K)$ is smaller than $bo(K)$, the Borel σ-field on K (the σ-field generated by the topology of K). If K is a compact metric space, then every closed subset of K is a G_δ-set, and so $ba(K)$ is exactly the Borel σ-field on K.

Let $\mathcal{M}$ be a subset of $\mathbb{R}^K$. We say that $\mathcal{M}$ is a *bounded-pointwise-class* of K if it is closed under uniformly bounded pointwise-limits i.e. if $(f_n)_n$ is a uniformly bounded sequence contained in $\mathcal{M}$ and if f_n simply converges to $f \in \mathbb{R}^K$, then also $f \in \mathcal{M}$. Let $\mathcal{C}(K)$ be the algebra of continuous functions on K. Then, the set of Baire functions on K, denoted by $Ba(K)$, is defined to be the bounded-pointwise-class of K generated by $\mathcal{C}(K)$. It's clear that $Ba(K)$ is a sub C^*-algebra of $B(K)$, the algebra of bounded functions on K. Note that every $f \in B(K)$ is Baire-measurable ([2], Proposition 2.1.7) in the sense that $f : (K, ba(K)) \longrightarrow (\mathbb{C}, bo(\mathbb{C}))$ is measurable. For basic properties of Baire functions, we refer to [2].

In [1] and [2], the continuous functional calculus, for single normal bounded operators in a Hilbert space, is generalized to Baire functions. This makes it possible to obtain what is called the Baire functional calculus. In [6], we define a simultaneous continuous functional calculus for a commutative family of normal of C^*-algebras.

In this paper, we consider locally Hilbert spaces which are not necessarily Hilbert spaces. Using roughly the approach of [2], we build the Baire functional calculus for a commutative family of normal bounded-locally operators. The purpose of this functional calculus is to give meaning to $f(\mathbf{T})$ whenever $\mathbf{T} = (T_i)_{i \in I}$ is a commutative family of normal bounded-locally operators on $\mathcal{H}$ and f is a Baire complex-valued function on the simultaneous spectrum $Sp(\mathbf{T})$ of $\mathbf{T}$. To do this, we start first with simultaneous continuous functional calculus, that we extend to Baire functions, to finally obtain a Baire functional for $\mathbf{T}$. We treat the fundamental properties of this functional calculus such as its continuity, its uniqueness, the spectral mapping theorem and many other properties. Once this functional calculus is defined and studied, the task is then to give some applications. Using Baire functional calculus, we obtain a particular orthonormal basis on a locally Hilbert space. We show that for normal bounded-locally operators there are proper hyper-invariant subspaces. We also give a polar decomposition of a bounded-locally operator.

2 Simultaneous continuous functional calculus for bounded-locally operators

Let $\mathcal{H} = \underrightarrow{\lim} \mathcal{H}_\lambda$, $\lambda \in \Lambda$, be a locally Hilbert space. Then,

$$\mathcal{L}(\mathcal{H}) = \underleftarrow{\lim} \mathcal{L}(\mathcal{H}_\lambda), \quad \lambda \in \Lambda.$$

We will build a simultaneous continuous functional calculus in $\mathcal{L}(\mathcal{H})$. Our approach consists in using simultaneous continuous functionals of each $\mathcal{L}(\mathcal{H}_\lambda)$, $\lambda \in \Lambda$.

Let $\mathcal{H}$ be a Hilbert space and let $\mathbf{T} = (\mathbf{T}_i)_{i \in I}$ be a commutative family of normal operators on $\mathcal{H}$, i.e., $T_i T_j = T_j T_i$, for every $i, j \in I$. By the analog of a result of Fuglede, Putnam and Rosenblum ([7], Theorem 12.16, p. 315), each T_i commutes with T_j^*. It follows that the full subalgebra B generated by $\mathbf{T}$ is a unital commutative C^*-subalgebra of $\mathcal{L}(\mathcal{H})$. Let $\widehat{\mathbf{T}}$ denote the generalized Gelfand transformation defined by:

$$\widehat{\mathbf{T}}(\chi) = (\chi(T_i))_{i \in I} \in \mathbb{C}^I, \quad \text{for every } \chi \in Sp(B),$$

where $Sp(B)$ denotes the Gelfand spectrum of B, that is the set of non-zero characters of B. Then, $\widehat{\mathbf{T}} : Sp(B) \longrightarrow \mathbb{C}^I$ is continuous and injective. The image $\widehat{\mathbf{T}}(Sp(B)) \subset \mathbb{C}^I$ is therefore a non-empty compact subset of $\mathbb{C}^I$, which is homeomorphic to $Sp(B)$, and it is called the *simultaneous spectrum* of $\mathbf{T}$ and it is denoted by $Sp(\mathbf{T})$. The fact that $Sp(\mathbf{T})$ is homeomorphic to $Sp(B)$, induces an isomorphism:

$$\theta : \mathcal{C}[Sp(\mathbf{T})] \longrightarrow \mathcal{C}[Sp(B)] : \quad \theta(f) = f \circ \widehat{\mathbf{T}}.$$

Let $\mathcal{G} : B \to \mathcal{C}\,[Sp(B)]$ be the Gelfand transformation and consider:

$$i \circ \mathcal{G}^{-1} \circ \theta : \quad \mathcal{C}\,[Sp\,(\mathbf{T})] \longrightarrow \mathcal{L}\,(\mathcal{H}),$$

where $i : B \longrightarrow \mathcal{L}\,(\mathcal{H})$ is the canonical injection of B into $\mathcal{L}\,(\mathcal{H})$. Then, we obtain a morphism $\Phi_{\mathbf{T}} : \mathcal{C}\,[Sp\,(\mathbf{T})] \longrightarrow \mathcal{L}\,(\mathcal{H})$ which is defined by the equality:

$$\widehat{\Phi_{\mathbf{T}}\,(f)} = f \circ \widehat{\mathbf{T}}, \quad \text{for every } f \in \mathcal{C}\,[Sp\,(\mathbf{T})] \qquad \textbf{(1)}$$

either again by:

$$\chi\,(\Phi_{\mathbf{T}}\,(f)) = f\,(\chi\,(\mathbf{T}))\,, \quad \text{for every } \chi \in Sp(B),$$

where $\chi\,(\mathbf{T}) = (\chi\,(T_i))_i$. In particular, if $\mathbf{z}_i$ denotes the function $z \longmapsto z_i$ on $Sp\,(\mathbf{T})$, we obtain $\Phi_{\mathbf{T}}\,(\mathbf{z}_i) = T_i$ and $\Phi_{\mathbf{T}}\,(\overline{\mathbf{z}_i}) = T_i^*$. Moreover, since each morphism of $\mathcal{C}\,(Sp\,(\mathbf{T}))$ into $\mathcal{L}\,(\mathcal{H})$ is continuous and $\Phi_{\mathbf{T}}$ is known on all the polynomials P in z_i and $\overline{z_i}$, this implies the uniqueness of $\Phi_{\mathbf{T}}$. Moreover, $\Phi_{\mathbf{T}}$ is an $*$-isometry. For any polynomial P with respect to the variables z_i and $\overline{z_i}$, one has $\Phi_{\mathbf{T}}\,(P) = P\left(T_i, T_i^*\right)$, where $P = P\,(z_i, \overline{z_i})$.

The fundamental properties of this simultaneous continuous functional calculus are contained in the following result. The proof, being straightforward, is omitted.

Theorem 2.1 (1) *The mapping $\Phi_{\mathbf{T}} : \mathcal{C}\,[Sp\,(\mathbf{T})] \longrightarrow \mathcal{L}\,(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is a unique continuous unitary $*$-morphism from $\mathcal{C}\,(Sp\,(\mathbf{T}))$ into $\mathcal{L}\,(\mathcal{H})$ such that*

$$\Phi_{T}\,(z_i) = T_i, \quad \textit{for every } i \in I,$$

where $\mathbf{z}_i$ denotes the function $z \longmapsto z_i$ on $Sp\,(\mathbf{T})$.

(2) *$\Phi_{\mathbf{T}}$ is isometric and its image $\Phi_{\mathbf{T}}\,(\mathcal{C}\,(Sp\,(\mathbf{T})))$ is the full subalgebra of $\mathcal{L}\,(\mathcal{H})$ generated by $(T_i)_{i\in I}$, that is the sub-C^*-algebra of $\mathcal{L}\,(\mathcal{H})$ generated by $I_{\mathcal{H}}$ and $(T_i)_{i\in I}$ and, therefore, consists entirely of normal operators.*

(3) *$\Phi_{\mathbf{T}}$ satisfies the spectral mapping theorem, that is*

$$Sp\,(\Phi_{\mathbf{T}}\,(f)) = f(Sp\,(\mathbf{T})\,), \quad \textit{for every } f \in \mathcal{C}\,(Sp\,(\mathbf{T}))\,.$$

As in the classical case, we will note repeatedly $\Phi_{\mathbf{T}}\,(f) = f(\mathbf{T})$ which respects the multiplicative symbolism: $(fg)\,(\mathbf{T}) = f(\mathbf{T})g(\mathbf{T})$ as well as the equality: $\overline{f}(\mathbf{T}) = f(\mathbf{T})^*$. So, for every $f \in \mathcal{C}\,(Sp\,(\mathbf{T}))$, we also have

$$\|f(\mathbf{T})\| = \|f\| \text{ and } \left\|f(\mathbf{T})^*\right\| = \left\|\overline{f}\right\|.$$

We now come to the quite general case of bounded-locally operators. Let $\mathcal{H} = \underrightarrow{\lim}\mathcal{H}_\lambda$, $\lambda \in \Lambda$, be a locally Hilbert space. Let $\mathbf{T} = (T_i)_{i\in\Gamma}$ be a commutative family of normal locally bounded operators on $\mathcal{H}$. Then, for every $i \in I$, $\mathbf{T}_i = \left(T_{\lambda,i}\right)_{\lambda\in\Lambda}$, where $T_{\lambda,i} \in \mathcal{L}(\mathcal{H}_\lambda)$, for every $\lambda \in \Lambda$. Then, the full subalgebra B_λ, generated by $\mathbf{T}_\lambda$, is a C^*-subalgebra of the C^*-algebra $\mathcal{L}(\mathcal{H}_\lambda)$. Let $\lambda, \mu \in \Lambda$ such that $\lambda \leq \mu$. Then,

$$Sp(\mathbf{T}_\lambda) = \left\{ \left(\chi \left(T_{\lambda,i} \right) \right)_{i \in I} : \chi \in Sp(B_\lambda) \right\}.$$

Let $\mathbf{z} = (z_i)_{i\in I} \in Sp(\mathbf{T}_\lambda)$ with $\mathbf{z} \notin Sp\left(\mathbf{T}_\mu\right)$. Then,

$$\exists j \in I, \ \ \forall \chi \in Sp(B_\mu) : z_j \neq \chi\left(T_{\mu,i}\right).$$

So, $\left(z_j Id_\mu - T_{\mu,i}\right) \in G\left(B_\mu\right)$. Therefore, there exists $w_j \in B_\mu$ such that

$$w_j\left(z_j Id_\mu - T_{\mu,i}\right) = \left(z_j Id_\mu - T_{\mu,i}\right) w_j = Id_\mu.$$

Using the connecting morphism $\rho_{\lambda\mu} : B_\mu \longrightarrow B_\lambda$, $(\lambda \leq \mu)$, we obtain $\left(z_j Id_\lambda - T_{\lambda,i}\right) \in G(B_\lambda)$, where $G(B_\lambda)$ denotes the group of invertible elements of B_λ, and so $\mathbf{z} \notin Sp(\mathbf{T}_\lambda)$, which is a contradiction. Whence, for $\lambda, \mu \in \Lambda$ such that $\lambda \leq \mu$, one has $Sp(\mathbf{T}_\lambda) \subset Sp\left(\mathbf{T}_\mu\right)$. Now since Λ is saturated, it follows that every finite union of $Sp(\mathbf{T}_\lambda)$, $\lambda \in \Lambda$, is contained in some $Sp(\mathbf{T}_\upsilon)$, $\upsilon \in \Lambda$.

Let $\mathcal{C}(Sp(\mathbf{T}))$ be the algebra of all continuous functions on $Sp(\mathbf{T})$ endowed with the topology of uniform convergence on the compacts $Sp(\mathbf{T}_\lambda)$, $\lambda \in \Lambda$. Then $\mathcal{C}(Sp(\mathbf{T}))$ is a C^*-locally convex algebra with a defining family of C^*-seminorms given by:

$$\|f\|_\lambda = \sup_{t \in Sp(\mathbf{T}_\lambda)} |f(t)|, \quad \text{for every } \ f \in \mathcal{C}(Sp(\mathbf{T})) \text{ and } \lambda \in \Lambda. \qquad (*)$$

If $N_\lambda = \ker(\|.\|_\lambda)$, for $\lambda \in \Lambda$, denote by $\mathcal{C}(Sp(\mathbf{T}))_\lambda$ the Banach algebra, completion of $(\mathcal{C}(Sp(\mathbf{T})), \|.\|_\lambda)/N_\lambda$, $\lambda \in \Lambda$. Then, the map

$$(\mathcal{C}(Sp(\mathbf{T})), \|.\|_\lambda)/N_\lambda \longrightarrow \mathcal{C}(Sp(\mathbf{T}_\lambda)) : \quad \mathrm{f} + N_\lambda \longmapsto f_{/Sp(\mathbf{T}_\lambda)}$$

is a well-defined $*$-morphism. Furthermore, by Urysohn's extension theorem ([9], p. 43), this last map is surjective. Moreover, it is an isometry. It follows that:

$$\mathcal{C}(Sp(\mathbf{T}))_\lambda = \mathcal{C}(Sp(\mathbf{T}_\lambda)), \quad for\ every\ \lambda \in \Lambda,$$

up to a topological isomorphism. So

$$\mathcal{C}(Sp(\mathbf{T})) \hookrightarrow \varprojlim \mathcal{C}(Sp(\mathbf{T}))_\lambda = \varprojlim \mathcal{C}(Sp(\mathbf{T}_\lambda)).$$

Now, by Theorem 2.1, there exists a unique unitary $*$-morphism $\Phi_{\mathbf{T}_\lambda}$ of $\mathcal{C}(Sp(\mathbf{T}_\lambda))$ into $\mathcal{L}(\mathcal{H}_\lambda)$ such that:

$$\Phi_{\mathbf{T}_\lambda}(\mathbf{z}_i) = T_i, \quad \text{for every } \ i \in I,$$

where $\mathbf{z}_i$ denotes the function $z \longmapsto z_i$ on $Sp\left(\mathbf{T}_\lambda\right)$. Moreover, one has

$$\rho_{\lambda\mu} \circ \Phi_{\mathbf{T}_\mu} = \Phi_{\mathbf{T}_\lambda} \circ i_{\lambda\mu},$$

where $i_{\lambda\mu}$ is the natural embedding of $\mathcal{C}\left(Sp\left(\mathbf{T}_\mu\right)\right)$ in $\mathcal{C}\left(Sp\left(\mathbf{T}_\lambda\right)\right)$. It follows that the map:

$$\Phi_{\mathbf{T}} = \underleftarrow{\lim}\Phi_{\mathbf{T}_\lambda} : \quad \underleftarrow{\lim}\mathcal{C}\left(Sp\left(\mathbf{T}\right)\right)_\lambda \longrightarrow B = \underleftarrow{\lim}B_\lambda, \quad \lambda \in \Lambda$$

is a unique unitary $*$-morphism such that, for every $\lambda \in \Lambda$,

$$\Phi_{\mathbf{T}}\left(f\right)|_{\mathcal{H}_\lambda} = \Phi_{\mathbf{T}_\lambda}\left(f|_{Sp\left(\mathbf{T}_\lambda\right)}\right), \quad \text{for every } \ f \in \mathcal{C}\left(Sp\left(\mathbf{T}\right)\right). \qquad (**).$$

Moreover, its restriction to the C^*-convex subalgebra $\mathcal{C}\left(Sp\left(\mathbf{T}\right)\right)$ of the locally C^*-algebra $\underleftarrow{\lim}\mathcal{C}\left(Sp\left(\mathbf{T}\right)\right)_\lambda$ is uniquely determined by:

$$\Phi_{\mathbf{T}_\lambda}\left(\mathbf{1}\right) = \mathbf{I}_{\mathcal{H}} \text{ and } \Phi_{\mathbf{T}_\lambda}\left(\mathbf{z}_i\right) = T_i, \quad \text{for every } \ i \in I.$$

Applying now the Stone–Weierstass theorem, we conclude that the subalgebra of $\mathcal{C}\left(Sp\left(\mathbf{T}\right)\right)$ generated by $\mathbf{1}$, $\mathbf{z}_i$ and $\overline{\mathbf{z}}_i$,for $i \in I$, is dense in $\mathcal{C}\left(Sp\left(\mathbf{T}\right)\right)$. So, we get the following:

Theorem 2.2 (Simultaneous continuous functional calculus). *Let $\mathcal{H}$ be a locally Hilbert space. Let $\mathbf{T} = (T_i)_{i\in I}$ be a commutative family of normal locally bounded operators on $\mathcal{H}$ and $\mathcal{C}\left(Sp\left(\mathbf{T}\right)\right)$ the locally C^*-algebra of continuous complex-valued functions on $Sp\left(\mathbf{T}\right)$. Then, there is a unique unitary $*$-morphism $\Phi_{\mathbf{T}}$ of $\mathcal{C}\left(Sp\left(\mathbf{T}\right)\right)$ into $\mathcal{L}(\mathcal{H})$ such that:*

$$\Phi_{\boldsymbol{T}}\left(z_i\right) = T_i, \quad \textit{for every } i \in I,$$

where $\mathbf{z}_i$ denotes the function $z \longmapsto z_i$ on $Sp\left(\mathbf{T}\right)$. Moreover, this $*$-morphism and its image $\Phi_{\mathbf{T}}\left(\mathcal{C}\left(Sp\left(\mathbf{T}\right)\right)\right)$ is the full subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $(T_i)_{i\in I}$, that is the sub-locally C^*-algebra of $\mathcal{L}(\mathcal{H})$ generated by $Id_{\mathcal{H}}$ and $(T_i)_{i\in I}$.

Remark 2.3 Let $\mathcal{H}$ be a locally Hilbert space. Let $\mathbf{T} = (T_i)_{i\in I}$ be a commutative family of normal locally bounded operators on $\mathcal{H}$. Then, for every $\lambda \in \Lambda$, one has:

$$p_\lambda(\Phi_{\mathbf{T}}\left(f\right)) = \left\|f\right\|_\lambda, \quad \text{for every } \ f \in \mathcal{C}\left(Sp\left(\mathbf{T}\right)\right).$$

Indeed, since $\Phi_{\mathbf{T}}\left(f\right)_\lambda = \Phi_{\mathbf{T}}\left(f\right)|_{\mathcal{H}_\lambda}$ and, by $(**)$,

$$\Phi_{\mathbf{T}}\left(f\right)|_{\mathcal{H}_\lambda} = \Phi_{\mathbf{T}_\lambda}\left(f|_{Sp\left(\mathbf{T}_\lambda\right)}\right),$$

one has

$$p_\lambda(\Phi_{\mathbf{T}}\left(f\right)) = \left\|\Phi_{\mathbf{T}}\left(f\right)|_{\mathcal{H}_\lambda}\right\|_\lambda = \left\|f|_{Sp\left(\mathbf{T}_\lambda\right)}\right\|_\lambda$$

and

$$\left\| f|_{Sp(\mathbf{T}_\lambda)} \right\|_\lambda = \sup_{t \in Sp(\mathbf{T}_\lambda)} |f(t)| .$$

So the result follows from $(*)$.

3 Simultaneous Baire functional calculus for bounded-locally operators

Let $\mathcal{H} = \underrightarrow{\lim} \mathcal{H}_\lambda$, $\lambda \in \Lambda$, be a locally Hilbert space, and let $\mathbf{T} = (\mathbf{T}_i)_{i \in I}$ be a commutative family of normal locally bounded operators on $\mathcal{H}$. Let

$$\Phi_{\mathbf{T}} : \mathcal{C}[Sp(\mathbf{T})] \longrightarrow \mathcal{L}(\mathcal{H})$$

be the unique continuous unitary $*$-morphism which defines simultaneous continuous functional calculus for $\mathbf{T}$ (Theorem 2.2). For every $x, y \in \mathcal{H}$, the map $\varphi_{x,y} : \mathcal{C}(Sp(\mathbf{T})) \longrightarrow \mathbb{C}$ defined by:

$$\varphi_{x,y}(f) = \langle \Phi_{\mathbf{T}}(f)(x), y \rangle, \quad \text{for every} \quad f \in \mathcal{C}(Sp(\mathbf{T}))$$

is a continuous linear form on $\mathcal{C}(Sp(\mathbf{T}))$. By Theorem 6.19, p.131 of [8]**,** it identifies with a complex Radon measure denoted by $\mu_{x,y}$. Now, if f is a Baire function, then it is Baire-measurable and bounded and so $\mu_{x,y}$-integrable. This allows us to extend $\Phi_{\mathbf{T}}$ to a $*$-homomorphism $\Psi_T : Ba(Sp(\mathbf{T})) \longrightarrow \mathcal{L}(\mathcal{H})$ called simultaneous Baire functional calculus for $\mathbf{T}$ given by the following result:

Theorem 3.1 *Let $\mathbf{T} = (\mathbf{T}_i)_{i \in I}$ be a commutative family of normal locally bounded operators on a locally Hilbert space $\mathcal{H}$, and let $\Phi_{\boldsymbol{T}} : \mathcal{C}[Sp(\boldsymbol{T})] \longrightarrow \mathcal{L}(\mathcal{H})$ be the simultaneous continuous functional calculus for $\mathbf{T}$. Then $\Phi_{\mathbf{T}}$ extends to a $*$-homomorphism*

$$\Psi_{\boldsymbol{T}} : Ba(Sp(\boldsymbol{T})) \longrightarrow \mathcal{L}(\mathcal{H})$$

*such that $\Psi_{\mathbf{T}}(Ba(Sp(\mathbf{T}))) \subset B^{cc}$, where B^{cc} is the bi-commutant of B, where B***,** *as considered above. Moreover, $\Psi_{\mathbf{T}}$satisfies the following properties:*

(1) $\Psi_{\mathbf{T}}$ *is determined by the equality:*

$$\langle \Psi_{\boldsymbol{T}}(f)(x), y \rangle = \int_{Sp(\boldsymbol{T})} f \, d\mu_{x,y}, \quad \textit{for every } f \in Ba(Sp(\boldsymbol{T})), \; x, y \in \mathcal{H}.$$

(2) *It is unique provided that it fulfills the following additional condition: For every sequence $(f_n)_n \in Ba(Sp(\mathbf{T}))$, such that $|f_n| \leq 1$ and $f_n \longrightarrow 0$, one has*

$\Psi_{\mathbf{T}}(f_n)x \longrightarrow 0$, for every $x \in \mathcal{H}$, in the sense that, for every $\gamma \in \Lambda$ such that $x \in \mathcal{H}_\gamma$ and $\Psi_{\mathbf{T}}(f_n) \in \mathcal{H}_\gamma$, for every $n \in \mathbb{N}$, we have

$$\|\Psi_{\mathbf{T}}(f_n)x\|_\gamma \longrightarrow 0.$$

(3) *$Sp_{\mathcal{L}(\mathcal{H})}(\Psi_{\mathbf{T}}(f)) \subset \overline{f(Sp(\mathbf{T}))}$, for every $f \in Ba(Sp(\mathbf{T}))$.*
As with simultaneous continuous functional calculus, we will also write $f(\mathbf{T})$ as a notation for $\Psi_{\mathbf{T}}(f)$ when $f \in Ba(Sp(\mathbf{T}))$.

Proof of Theorem 3.1 Observe first that $\Psi_{\mathbf{T}}$ satisfies the condition given in **2).** Indeed, by ([2], c), p. 36), $\mu_{x,x}$ is a positive measure, for every $x \in \mathcal{H}_\gamma$ and

$$\|\Psi_{\mathbf{T}}(f_n)x\|_\gamma = \int_{Sp(\mathbf{T}_\gamma)} f_n^2 d\mu_{x,x}.$$

Now by Lebesgue's dominated convergence theorem,

$$\int_{Sp(\mathbf{T}_\gamma)} f_n^2 d\mu_{x,x} \longrightarrow 0.$$

So $\|\Psi_{\mathbf{T}}(f_n)x\|_\gamma \longrightarrow 0$. Let us show that

$$\Psi_{\mathbf{T}} : Ba(Sp(\mathbf{T})) \longrightarrow \mathcal{L}(\mathcal{H}) : \quad \Psi_{\mathbf{T}}(f) = f(\mathbf{T})$$

is a $*$-homomorphism. To do this, consider $\mathcal{M}$ the class of functions $f \in B(Sp(\mathbf{T}))$ such that:

$$(fg)(\mathbf{T}) = f(\mathbf{T})g(\mathbf{T}), \quad \text{for every } g \in \mathcal{C}(Sp(\mathbf{T})).$$

Obviously, $\mathcal{C}(Sp(\mathbf{T})) \subset \mathcal{M}$. Moreover, using the condition given in **2),** it is easy to prove that $\mathcal{M}$ is a bounded-pointwise-class of $Sp(\mathbf{T})$. This implies that $Ba(Sp(\mathbf{T})) \subset \mathcal{M}$. Thus, for every $f \in Ba(Sp(\mathbf{T}))$ and $g \in \mathcal{C}(Sp(\mathbf{T}))$, one has:

$$(fg)(\mathbf{T}) = f(\mathbf{T})\, g(\mathbf{T}).$$

Now, replacing the functions $g \in \mathcal{C}(Sp(\mathbf{T}))$ by the functions $g \in Ba(Sp(\mathbf{T}))$, the previous reasoning shows that

$$(fg)(\mathbf{T}) = f(\mathbf{T})\, g(\mathbf{T}), \text{ for every } f, g \in Ba(Sp(\mathbf{T})).$$

Analogously, one has

$$f(\mathbf{T})^* = \overline{f}(\mathbf{T}), \quad \text{for every } f \in Ba(Sp(\mathbf{T})).$$

2) Let us show that $\Psi_{\mathbf{T}}$ is unique subject to satisfying the extra condition given in **2).** Let $\Psi'_{\mathbf{T}}$ be another extension of $\Phi_{\mathbf{T}}$ satisfying the previous extra condition. Then, the class $\mathcal{M}'$ of functions $f \in B(Sp(\mathbf{T}))$ such that $f(\mathbf{T}) = \Psi'_{\mathbf{T}}(f)$ contains $\mathcal{C}(Sp(\mathbf{T}))$ and it is obviously bounded-pointwise-class of $Sp(\mathbf{T})$. Whence $\Psi_{\mathbf{T}} = \Psi'_{\mathbf{T}}$.

3) Let $f \in Ba(Sp(\mathbf{T}))$. Then,

$$Sp_{\mathcal{L}(\mathcal{H})}(f(\mathbf{T})) = Sp_{\mathcal{L}(\mathcal{H})}(\Psi_{\mathbf{T}} f) \subset Sp_{Ba(Sp(\mathbf{T}))}(f).$$

On the other hand, $Ba(Sp(\mathbf{T}))$ is a full subalgebra of $B(Sp(\mathbf{T}))$. Thus, by ([2], Proposition 1.1.16, p. 8), one has:

$$Sp_{Ba(Sp(\mathbf{T}))}(f) = Sp_{B(Sp(\mathbf{T}))}(f).$$

Whence

$$Sp_{\mathcal{L}(\mathcal{H})}(f(\mathbf{T})) \subset \overline{f(Sp(\mathbf{T}))}$$

since $Sp_{B(Sp(\mathbf{T}))}(f) = \overline{f(Sp(\mathbf{T}))}$ by ([1], Corollary 7.3, p. 287). □

Now let $f \in Ba(Sp(\mathbf{T}))$. Then, by **3)** of Theorem 3.1, one has $g \circ f \in Ba(Sp(\mathbf{T}))$, for every $g \in Ba\left(\overline{f(Sp(\mathbf{T}))}\right)$. Using the extra condition given in **2)** instead of the monotone-convergence property of Theorem 7.16 of [1], p. 295, the proof of Corollary 7.18 of [1] applies, mutatis mutandis, to this setting as well and we have the following result:

Proposition 3.2 *Let T be a normal bounded-locally operator on a locally Hilbert space $\mathcal{H}$. Then, for every $g \in Ba\left(\overline{f(Sp(\mathbf{T}))}\right)$ and $f \in Ba(Sp(\mathbf{T}))$, one has:*

$$(g \circ f)(T) = g(f(T)).$$

4 Some applications of simultaneous Baire functional calculus

In this section, we give some applications of the simultaneous continuous functional calculus as explored in the preceding section. Its applications concern the existence of an orthonormal basis on a locally Hilbert space $\mathcal{H}$, consisting of eigenvectors of commuting normal locally bounded operators acting in $\mathcal{H}$. The applications also concern the polar decomposition and the existence of a proper hyper-invariant subspace as they are given in the Hilbert case (see [1]) and this without any loss. Note that the proof of the last two applications goes along the lines of [1] with the necessary modification.

4.1 Existence of an orthonormal basis on a locally Hilbert space

In this section, we go along with our first application, as mentioned above:

Theorem 4.1 *Let $\mathcal{H} = \underrightarrow{\lim} \mathcal{H}_\lambda$, $\lambda \in \Lambda$, be a locally Hilbert space whose associated Hilbert spaces $\mathcal{H}_\lambda$, $\lambda \in \Lambda$, have finite dimensions and $\mathbf{T} = (T_i)_{i \in I}$ is a commutative* *family of normal locally bounded operators on $\mathcal{H}$. For every $i \in I$, let $T_{i,\lambda} \in \mathcal{L}(\mathcal{H}_\lambda)$, $\lambda \in \Lambda$, such that $T_i = \underrightarrow{\lim} T_{i,\lambda}$. Then, there exists an orthonormal basis of $\mathcal{H}$, whose elements are eigenvectors of the operators $T_{i,\lambda}$.*

Proof Let $\mathbf{T}_\lambda = (T_{i,\lambda})_{i \in I}$ and B_λ be the full subalgebra of $\mathcal{L}(\mathcal{H}_\lambda)$ generated by $\mathbf{T}_\lambda$. Then

$$Sp(\mathbf{T}_\lambda) = \left\{ \left(\chi \left(T_{i,\lambda} \right) \right)_{i \in I} : \chi \in Sp\left(B_\lambda \right) \right\}.$$

Since $\mathcal{H}_\lambda$ is finite dimensional, B_λ is also of finite dimension. So $Sp(\mathbf{T}_\lambda)$ is finite, say

$$Sp(\mathbf{T}_\lambda) = \left\{ \alpha_1^{(\lambda)}, ..., \alpha_{p_\lambda}^{(\lambda)} \right\},$$

with $\alpha_k^{(\lambda)} = \left(\alpha_{i,k}^{(\lambda)} \right)_{i \in I}$, for every $1 \leq k \leq p_\lambda$. As $1_{\left\{\alpha_k^{(\lambda)}\right\}} \in \mathcal{C}\left(Sp\left(\mathbf{T}_\lambda \right) \right)$ and since $1_{Sp(\mathbf{T}_\lambda)} = \sum\limits_{k=1}^{p_\lambda} 1_{\left\{\alpha_k^{(\lambda)}\right\}}$, the simultaneous continuous calculus gives

$$Id_\lambda = \Phi_{\mathbf{T}_\lambda}\left(1_{Sp(\mathbf{T}_\lambda)} \right) = \sum_{k=1}^{p_\lambda} P_k^{(\lambda)},$$

where $P_k^{(\lambda)} = \Phi_{\mathbf{T}_\lambda}\left(1_{\left\{\alpha_k^{(\lambda)}\right\}} \right)$ is a hermitian projector of $\mathcal{H}_\lambda$, for $k = 1, ..., p_\lambda$. Moreover, one has

$$T_{i,\lambda} = \mathbf{z}_i\left(\mathbf{T}_\lambda \right) = \sum_k \alpha_{i,k}^{(\lambda)} P_k^{(\lambda)},$$

where $\mathbf{z}_i$ denotes the function $z \longmapsto z_i$ on $Sp\left(\mathbf{T}_\lambda \right)$. It follows that the restriction of $T_{i,\lambda}$ to $\mathcal{H}_k^{(\lambda)} = P_k^{(\lambda)}\left(\mathcal{H}_\lambda \right)$ is a homothety with ratio $\alpha_{i,k}^{(\lambda)}$. Indeed, let $x \in \mathcal{H}_k^{(\lambda)}$, there exists $y \in \mathcal{H}_\lambda$ such that $x = P_k^{(\lambda)}(y)$. One has

$$\begin{aligned} T_{i,\lambda}(x) &= \sum_{k'} \alpha_{i,k'}^{(\lambda)} P_{k'}^{(\lambda)}(x) = \sum_{k'} \alpha_{i,k'}^{(\lambda)} P_{k'}^{(\lambda)}(P_k^{(\lambda)}(y)) \\ &= \sum_{k'} \alpha_{i,k'}^{(\lambda)} \left(P_{k'}^{(\lambda)}(P_k^{(\lambda)} \right)(y). \end{aligned}$$

But for $k' \neq k$, $P_{k'}^{(\lambda)}(P_k^{(\lambda)} = 0$, because

$$\begin{aligned} P_{k'}^{(\lambda)}(P_k^{(\lambda)} &= \Phi_{T_\lambda}\left(1_{\{\alpha_{k'}^{(\lambda)}\}} \right) \Phi_{T_\lambda}\left(1_{\{\alpha_k^{(\lambda)}\}} \right) = \Phi_{T_\lambda}\left(1_{\{\alpha_{k'}^{(\lambda)}\}} 1_{\{\alpha_k^{(\lambda)}\}} \right) \\ &= \Phi_{T_\lambda}(1_{\{\alpha_{k'}^{(\lambda)}\} \cap \{\alpha_k^{(\lambda)}\}}) = \Phi_{T_\lambda}(1_\varnothing) = \Phi_{T_\lambda}(0) = 0. \end{aligned}$$

Thus

$$T_{i,\lambda}(x) = \alpha_{i,k}^{(\lambda)} P_k^{(\lambda)}(P_k^{(\lambda)}(y) = \alpha_{i,k}^{(\lambda)} P_k^{(\lambda)}(y) = \alpha_{i,k}^{(\lambda)} x.$$

For $k = 1, ..., p_\lambda$, let $B_k^{(\lambda)}$ be an orthonormal basis of $\mathcal{H}_k^{(\lambda)}$. Now let $x \in B_k^{(\lambda)}$ and $y \in B_l^{(\lambda)}$ $(k \neq l)$, one has:

$$\begin{aligned}\langle x, y\rangle &= \left\langle P_k^{(\lambda)}(a), P_l^{(\lambda)}(b)\right\rangle \\ &= \left\langle a, P_k^{(\lambda)} P_l^{(\lambda)}(b)\right\rangle = 0 \quad \text{for } P_k^{(\lambda)} P_l^{(\lambda)} = 0.\end{aligned}$$

It follows that $B^{(\lambda)} = B_1^{(\lambda)} \cup B_2^{(\lambda)} \cup ... \cup B_{p_\lambda}^{(\lambda)}$ is an orthonormal basis of $\mathcal{H}_\lambda$, whose elements are eigenvectors of the operators $T_{i,\lambda}$. Finally, since $B^{(\lambda)} \subset B^{(\mu)}$ if $\lambda \leq \mu$,, one has

$$B = \underset{\longrightarrow}{\lim} B^{(\lambda)} = \bigcup_{\lambda \in \Lambda} B^{(\lambda)}$$

is the desired orthonormal basis of $\mathcal{H}$. □

4.2 Polar decomposition

Let $a \in Sp(T)$ and let $U = Sp(T)\setminus\{a\}$. Then $\chi_{\{a\}}$, the characteristic function of $\{a\}$, is a Baire function on $Sp(T)$. Moreover, $\chi_U \in Ba\,(Sp(T))$ since χ_U is the pointwise limit of a sequence of continuous functions on $Sp(T)$. It follows that if $f \in B(Sp(T)) \cap \mathcal{C}(Sp(T)\setminus\{a\})$, then $f = f(a)\chi_{\{a\}} + \chi_U f$ is a Baire function on $Sp(T)$. Now, as in the classic case, define bounded functions r and u on $Sp(T)$ by:

$$r(\lambda) = |\lambda|\,, u(\lambda) = \frac{\lambda}{|\lambda|},\ (\lambda \neq 0),\ u(0) = 1.$$

Then $r \in \mathcal{C}(Sp(T))$, $u \in Ba(Sp(T))$ and $ru = z$, where z denotes the function $\lambda \longmapsto \lambda$ on $Sp\,(T)$. Let

$$R = \Phi_T(u) = r(T) \text{ and } U = \Psi_T(u) = u(T).$$

It follows that R is a positive operator on $\mathcal{H}$, U is a unitary operator on $\mathcal{H}$ and $T = RU$. Furthermore R, U, and T are pairwise commuting. So, one has the following result:

Theorem 4.2 *(Polar decomposition) Let $\mathcal{H}$ be a locally Hilbert space, and let T be a normal bounded-locally operator on $\mathcal{H}$. Then there exists a positive bounded-locally operator R on $\mathcal{H}$ and a unitary bounded-locally operator U on $\mathcal{H}$ such that R, U, and T are pairwise commuting and $T = RU$.*

4.3 Invariant subspaces

Recall that a closed subspace F of a locally Hilbert space $\mathcal{H}$ is a hyper-invariant subspace for $T \in \mathcal{L}(\mathcal{H})$ if $S(F) \subset F$, for every $S \in \mathcal{L}(\mathcal{H})$ with $ST = TS$. Using Baire functional calculus, we show that for normal bounded operator on locally Hilbert spaces there are proper-invariant subspaces. Note that if $\dim \mathcal{H} = 1$, the subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$. Thus, the invariant subspace problem arises if $\dim \mathcal{H} \geq 2$ **.**

Theorem 4.3 *Let* $\mathcal{H} = \underrightarrow{\lim} \mathcal{H}_\lambda$, $\lambda \in \Lambda$, *be a locally Hilbert space with* $\dim \mathcal{H} \geq 2$, *and* T *be a normal bounded-locally operator on* $\mathcal{H}$ *with* $T \notin \mathbb{C} I_{\mathcal{H}}$. *Then, there is a proper hyper-invariant subspace for* T.

For the proof we will need the following classical result. Notice that the case of a Hilbert space is given in Corollary 6. 28 of [1], p. 273.

Lemma 4.4 *Let* $\mathcal{H} = \underrightarrow{\lim} \mathcal{H}_\lambda$, $\lambda \in \Lambda$, *be a locally Hilbert and* $T \in \mathcal{L}(\mathcal{H})$ *be a normal bounded-locally operator. Suppose that* γ *is an isolated point of* $Sp(T)$. *Then,* γ *is eigenvalue of* T.

Proof Since $\mathcal{L}(\mathcal{H}) = \underleftarrow{\lim} \mathcal{L}(\mathcal{H}_\lambda)$, $\lambda \in \Lambda$ is a locally C^*-algebra, it admits a holomorphic functional calculus. Using this last calculus, one has as in the classical case ([1], Corollary 4.97), that there is a non-zero hermitian projection $P \in \mathcal{L}(\mathcal{H})$ such that $PT = TP$ and

$$Sp_{\mathcal{L}(P(\mathcal{H}))}\left(T_{|P(\mathcal{H})}\right) = \{\gamma\}.$$

It follows that $PT^* = T^*P$ and TP is normal. Thus $(T - \gamma I_{\mathcal{H}})P$ is normal and $\rho_{\mathcal{L}(\mathcal{H})}((T - \gamma I_{\mathcal{H}})P) = 0$. Now since

$$\rho_{\mathcal{L}(\mathcal{H})}((T - \gamma I_{\mathcal{H}})P) = \sup_\lambda \left(T_\lambda - \gamma I_{\mathcal{H}_\lambda}\right)P_\lambda),$$

it follows that $\left(T_\lambda - \gamma I_{\mathcal{H}_\lambda}\right)P_\lambda = 0$, for every $\lambda \in \Lambda$. Thus $TP = \gamma P$, and so γ is an eigenvalue of T. □

Proof of Theorem 4.3 Suppose first that $Sp(T)$ has an isolated point. Then, by Lemma 4.4, T has an eigenvalue γ, and the corresponding eigenspace $E(\gamma)$ is a proper (given that $T \notin \mathbb{C} I_{\mathcal{H}}$) hyper-invariant subspace for T. Suppose now that $Sp(T)$ has no isolated points. Let U be an open subset of $\mathbb{C}$ such that $U \cap Sp(T)$ and $Sp(T) \backslash \overline{U}$ are non-empty. Then $P = \Psi_T(\chi_U)$ is a self-adjoint bounded-locally operator such that $P^2 = P$. Let $f \in \mathcal{C}_{\mathbb{R}}(Sp(T))$ a non-zero function such that $0 \leq f \leq 1$ on $Sp(T)$ and $f(x) = 0$ for all $x \in Sp(T) \backslash U$. Then $0 \leq f \leq \chi_U$. This implies that: $\Psi_T(\chi_U) \geq \Phi_T(f) > 0$ since Φ_T is isometric and order-preserving. It follows that $P \neq 0$ and $P \neq I_{\mathcal{H}}$. Whence $F = P(\mathcal{H})$ is a closed subspace of $\mathcal{H}$, with $F \neq \{0\}$ and $F \neq \mathcal{H}$. Moreover, $P \in B^{cc}$. Suppose that $S \in \mathcal{L}(\mathcal{H})$ with $ST = TS$. Then, by Fuglede, Putnam and Rosenblum's theorem ([7], Theorem 12.16, p. 315), $ST^* = T^*S$, and so $S \in B^{cc}$**.** Thus $SP = PS$ and so $S(F) \subset F$. This shows that F is a proper hyper-invariant subspace for T. □

Declarations

Conflict of interest On behalf of all authors, there is no conflict of interest.

References

1. Allan, G.R.: Introduction to Banach spaces and algebras. Oxford Graduate Texts in Mathematics, 20. Oxford University Press, Oxford (2011)
2. Buchwalter, H., Tarral, D.: Théorie spectrale. Publ. Dép. Math. (Lyon) (N.S.) (1982)
3. Dixmier, J.: C^*-algebras. North-Holland, Amsterdam (1981)
4. Fragoulopoulou, M.: *Topological Algebras with Involution*. North-Holland Mathematics Studies, 200. Elsevier Science B.V., Amsterdam (2005)
5. Inoue, A.: Locally C^**-algebra.* Mem. Fac. Sci. Kyushu Univ. Ser. A 25 , 197–235 (1971)
6. Mazighi, M., El Kinani, A.: Some applications of simultaneous continuous functional calculus. Rend. Circ. Mat. Palermo II, Ser (2022)
7. Rudin, W.: Functional Analysis. McGraw-Hill, New York (1973)
8. Rudin, W.: Real and complex analysis, 3rd edn. McGraw-Hill Book Co., New York (1987)
9. Gillman, L., Jerison, M.: Rings of Continuous Functions. Van Nostrand, Princeton (1960)