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Recovering differential operators with two retarded arguments

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Abstract
This paper deals with non-self-adjoint second-order Differential Operators with two constant delays τ_i, $i = 1, 2$ which are less than half the length of the interval. We consider the case when $\frac{2\pi}{5} \leq \tau_i \leq \frac{\pi}{2}$ and potentials q_k are functions from $L_2[\tau_k, \pi]$, $k = 1, 2$. We study the inverse spectral problem of recovering operators from their spectral characteristics. Four boundary value problems are considered and we prove that delays and potentials are uniquely determined from their spectra.

Keywords Differential operators with delays · Inverse spectral problems · Fourier coefficients

Mathematics Subject Classification 34B09 · 34B24 · 34L10

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1 Introduction

The main results in the inverse spectral problems for classical Sturm–Liouville operators can be found in the monographs [1, 2]. Some of the main methods in the inverse problem theory for classical Sturm–Liouville operators turned out to be unsuitable for operators with delays. In this paper, we use the method of Fourier coefficients. This method is based on the determination of direct relations between Fourier coefficients of the potential or functions containing the potential, and Fourier coefficients of some known function. Some of the results of the inverse spectral problem for Sturm–Liouville operators with a delay can be found in [3–10]. Studying of the spectral problems for Differential Operators with two or more constant delays is of recent origin and some of the results can be found in [11–16]. One of the interesting features of the case with two (and more) delays is the requirement of specifying the spectra for two (and accordingly more) different differential equation. Negative answer to the question whether one can find an appropriate inverse problem statement involving only one equation can be found in [17]. The paper [18] is devoted to the studying of direct problems for operators with N constant delays. In what follows, we always take $i = 0, 1$ and $k = 1, 2$.

We consider the boundary value problems $D_{i,k}$

$$-y''(x) + q_1(x)y(x-\tau_1) + (-1)^i q_2(x)y(x-\tau_2) = \lambda y(x), \quad x \in [0, \pi], \tag{1.1}$$

$$y'(0) - hy(0) = 0, \tag{1.2}$$

$$y'(\pi) + H_k y(\pi) = 0, \tag{1.3}$$

where $\frac{2\pi}{5} \le \tau_2 \le \tau_1 < \frac{\pi}{2}$, $h, H_k \in \mathbb{R}$, $H_1 \neq H_2$, and λ is a spectral parameter. We assume that q_1, q_2 are complex valued potential functions from $L_2[0, \pi]$ such that $q_1(x) = 0$ as $x \in [0, \tau_1)$ and $q_2(x) = 0$ as $x \in [0, \tau_2)$.

We study the inverse spectral problem of recovering operators from the spectra of $D_{i,k}$ and generalize the results from the paper [3] which deals with operators with one constant delay from the interval $\left[\frac{2\pi}{5}, \frac{\pi}{2}\right]$ to the operators with two constant delays from the same interval.

Let $(\lambda_{n,i,k})_{n=0}^{\infty}$ be the eigenvalues of the boundary value problems $D_{i,k}$. The inverse problem is formulated as follows.

Inverse problem 1: Given $(\lambda_{n,i,k})_{n=0}^{\infty}$ find delays τ_k, parameters h, H_k, and potential functions q_k.

The organization of the paper is the following. In Sect. 2, we study the spectral properties of the boundary value problems $D_{i,k}$. In Sect. 3, we prove that delays and parameters are uniquely determined by the spectra. Then we prove that the potentials are uniquely determined by the system of two Volterra linear integral equations.

2 Spectral properties

It can be easily shown that the differential Eq. (1.1) under the initial condition (1.2) along with the normalizing condition $y(0) = 1$ and conditions $q_k(x) = 0$ as $x \in [0, \tau_k)$ is equivalent to the integral equation

$$\begin{aligned} y_i(x,z) = &\cos xz + \frac{h}{z}\sin zx + \frac{1}{z}\int_{\tau_1}^{x} q_1(t)\sin z(x-t)y(t-\tau_1,z)\,\mathrm{d}t \\ &+ \frac{(-1)^i}{z}\int_{\tau_2}^{x} q_2(t)\sin z(x-t)y(t-\tau_2,z)\,\mathrm{d}t. \end{aligned} \tag{2.1}$$

Here and in the sequel, we take $\lambda = z^2$. By the method of steps, it can be easily verified that the solution of the integral equation (2.1) on the interval $(2\tau_1, \pi]$ is

$$\begin{aligned} y_i(x,z) = &\cos zx + \frac{h}{z}\sin zx + \frac{1}{z}\left(b_{sc}^{(1)}(x,z) + (-1)^i b_{sc}^{(2)}(x,z)\right) \\ &+ \frac{h}{z^2}\left(b_{s^2}^{(1)}(x,z) + (-1)^i b_{s^2}^{(2)}(x,z)\right) \\ &+ \frac{1}{z^2}\left(b_{s^2c}^{(1)}(x,z) + b_{s^2c}^{(2)}(x,z) + (-1)^i b_{s^2c}^{(1,2)}(x,z) + (-1)^i b_{s^2c}^{(2,1)}(x,z)\right) \\ &+ \frac{h}{z^3}\left(b_{s^3}^{(1)}(x,z) + b_{s^3}^{(2)}(x,z) + (-1)^i b_{s^3}^{(1,2)}(x,z) + (-1)^i b_{s^3}^{(2,1)}(x,z)\right) \end{aligned} \tag{2.2}$$

where the integral terms in this equation are given by

$$\begin{aligned} b_{sc}^{(k)}(x,z) &= \int_{\tau_k}^{x} q_k(t)\sin z(x-t)\cos z(t-\tau_k)\,\mathrm{d}t, \\ b_{s^2}^{(k)}(x,z) &= \int_{\tau_k}^{x} q_k(t)\sin z(x-t)\sin z(t-\tau_k)\,\mathrm{d}t, \\ b_{s^2c}^{(k)}(x,z) &= \int_{2\tau_k}^{x} q_k(t)\sin z(x-t) b_{sc}^{(2)}(t-\tau_k,z)\,\mathrm{d}t, \\ b_{s^2c}^{(k,l)}(x,z) &= \int_{\tau_1+\tau_2}^{x} q_k(t)\sin z(x-t) b_{sc}^{(l)}(t-\tau_k,z)\,\mathrm{d}t, \quad (l = 1,2,\ k \neq l), \\ b_{s^3}^{(k)}(x,z) &= \int_{2\tau_k}^{x} q_k(t)\sin z(x-t) b_{s^2}^{(2)}(t-\tau_k,z)\,\mathrm{d}t, \\ b_{s^3}^{(k,l)}(x,z) &= \int_{\tau_1+\tau_2}^{x} q_k(t)\sin z(x-t) b_{s^2}^{(l)}(t-\tau_k,z)\,\mathrm{d}t, \quad (l = 1,2,\ k \neq l). \end{aligned}$$

Denote

$$\Delta_{i,k} = F_{i,k}(z) = y_i'(\pi,z) + H_k y(\pi,z)$$

From (2.2), we obtain

$$\begin{aligned}
F_{i,k}(z) = {} & \left(-z + \frac{hH_k}{z}\right)\sin \pi z + (h + H_k)\cos \pi z + b_{c^2}^{(1)}(z) + (-1)^i b_{c^2}^{(2)}(z) \\
& + \frac{h}{z}\left(b_{cs}^{(1)}(z) + (-1)^i b_{cs}^{(2)}(z)\right) + \frac{H_k}{z}\left(b_{sc}^{(1)}(z) + (-1)^i b_{sc}^{(2)}(z)\right) \\
& + \frac{H_k h}{z^2}\left(b_{s^2}^{(1)}(z) + (-1)^i b_{s^2}^{(2)}(z)\right) \\
& + \frac{1}{z}\left(b_{csc}^{(1)}(z) + b_{csc}^{(2)}(z) + (-1)^i b_{csc}^{(1,2)}(z) + (-1)^i b_{csc}^{(2,1)}(z)\right) \\
& + \frac{h}{z^2}\left(b_{cs^2}^{(1)}(z) + b_{cs^2}^{(2)}(z) + (-1)^i b_{cs^2}^{(1,2)}(z) + (-1)^i b_{cs^2}^{(2,1)}(z)\right) \\
& + \frac{H_k}{z^2}\left(b_{s^2c}^{(1)}(z) + b_{s^2c}^{(2)}(z) + (-1)^i b_{s^2c}^{(1,2)}(z) + (-1)^i b_{s^2c}^{(2,1)}(z)\right) \\
& + \frac{Hh}{z^3}\left(b_{s^3}^{(1)}(z) + b_{s^3}^{(2)}(z) + (-1)^i b_{s^3}^{(1,2)}(z) + (-1)^{(i)} b_{s^3}^{(2,1)}(z)\right)
\end{aligned}$$

where

$$\begin{aligned}
b_{cs}^{(k)}(z) &= \int_{\tau_k}^{\pi} q_k(t)\cos z(\pi - t)\sin z(t-\tau_k)\,\mathrm{d}t, \\
b_{c^2}^{(k)}(z) &= \int_{\tau_k}^{\pi} q_k(t)\cos z(\pi - t)\cos z(t-\tau_k)\,\mathrm{d}t, \\
b_{csc}^{(k)}(z) &= \int_{2\tau_k}^{\pi} q_k(t)\cos z(\pi - t) b_{sc}^{(k)}(t-\tau_k, z)\,\mathrm{d}t, \\
b_{csc}^{(k,l)}(z) &= \int_{\tau_1+\tau_2}^{x} q_k(t)\cos z(\pi - t) b_{sc}^{(l)}(t-\tau_k, z)\,\mathrm{d}t, \quad (k \neq l), \\
b_{cs^2}^{(k)}(z) &= \int_{2\tau_2}^{\pi} q_k(t)\cos z(\pi - t) b_{s^2}^{(k)}(t-\tau_k, z)\,\mathrm{d}t, \\
b_{cs^2}^{(k,l)}(z) &= \int_{\tau_1+\tau_2}^{x} q_k(t)\cos z(x - t) b_{s^2}^{(l)}(t-\tau_k, z)\,\mathrm{d}t, \quad (k \neq l).
\end{aligned}$$

Here, for the sake of simplifying the above-given equations for $F_{i,k}(z)$, we write (z) as the argument of the functions instead of (π, z) .

To simplify further consideration, we define the so-called transitional function $\tilde{q}_i$ as follows:

$$\tilde{q}_i(t) = \begin{cases} q_1\left(t+\frac{\tau_1}{2}\right) + (-1)^i q_2\left(t+\frac{\tau_2}{2}\right), & t \in \left[\frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right], \\ (-1)^i q_2\left(t+\frac{\tau_2}{2}\right), & t \in \left[\frac{\tau_2}{2}, \tau_2\right) \cup \left(\pi - \frac{\tau_1}{2}, \pi - \frac{\tau_2}{2}\right], \\ 0, & t \in \left[0, \frac{\tau_2}{2}\right) \cup \left(\pi - \frac{\tau_2}{2}, \pi\right]. \end{cases}$$

Let us also define the function

$$K_i(t) = K^{(1)}(t) + K^{(2)}(t) + (-1)^i K^{(1,2)}(t) + (-1)^i K^{(2,1)}(t),$$

where

$$\begin{aligned} K^{(k)}(t) &= q_k(t+\tau_k)\int_{\tau_k}^{t} q_k(s)\,\mathrm{d}s - q_k(t)\int_{t+\tau_k}^{\pi} q_k(s)\,\mathrm{d}s \\ &\quad - \int_{t+\tau_k}^{\pi} q_k(s-t)q_k(s)\,\mathrm{d}s, \quad t \in [\tau_k, \pi-\tau_k], \\ K^{(k)}(t) &= 0, \quad t \in [0,\tau_k) \cup (\pi-\tau_k, \pi], \\ K^{(k,l)}(t) &= q_k\left(t+\frac{\tau_1+\tau_2}{2}\right)\int_{\tau_l}^{t} q_l(s)\,\mathrm{d}s - q_l\left(t-\frac{\tau_k}{2}+\frac{\tau_l}{2}\right)\int_{t+\frac{\tau_1+\tau_2}{2}}^{\pi} q_k(s)\,\mathrm{d}s \\ &\quad - \int_{t+\frac{\tau_1+\tau_2}{2}}^{\pi} q_l\left(s-t-\frac{\tau_k}{2}+\frac{\tau_l}{2}\right)q_k(s)\,\mathrm{d}s, \quad t \in \left[\frac{\tau_1+\tau_2}{2}, \pi - \frac{\tau_1+\tau_2}{2}\right], \end{aligned}$$

and

$$K^{(k,l)}(t) = 0, \quad t \in \left[0, \frac{\tau_1+\tau_2}{2}\right) \cup \left(\pi - \frac{\tau_1+\tau_2}{2}, \pi\right].$$

Moreover, if we introduce the notations

$$\begin{aligned} J_1^{(k)} &= \int_{\tau_k}^{\pi} q_k(t)\,\mathrm{d}t, \quad J_2^{(k)} = \int_{2\tau_k}^{\pi} q_k(t)\left(\int_{\tau_k}^{t-\tau_k} q_k(s)\,\mathrm{d}s\right)\mathrm{d}t, \\ J_2^{(k,l)} &= \int_{\tau_1+\tau_2}^{\pi} q_k(t)\left(\int_{\tau_l}^{t-\tau_k} q_l(s)\,\mathrm{d}s\right)\mathrm{d}t \end{aligned}$$

and the functions

$$\begin{aligned} \tilde{a}_{i,c}(z) &= \int_0^{\pi} \tilde{q}_i(t)\cos z(\pi-2t)\,\mathrm{d}t, \quad \tilde{a}_{i,s}(z) = \int_0^{\pi} \tilde{q}_i(t)\sin z(\pi-2t)\,\mathrm{d}t, \\ k_{i,s}(z) &= \int_0^{\pi} K_i(t)\sin z(\pi-2t)\,\mathrm{d}t, \quad k_c(z) = \int_0^{\pi} K_i(t)\cos z(\pi-2t)\,\mathrm{d}t, \\ u_{i,s}(z) &= \int_0^{\pi} U_i(t)\sin z(\pi-2t)\,\mathrm{d}t, \quad u_c(z) = \int_0^{\pi} U_i(t)\cos z(\pi-2t)\,\mathrm{d}t \end{aligned}$$

where the functions U_i in the last two equations are defined as

$$U(t) = U^{(1)}(t) + U^{(2)}(t) + (-1)^i U^{(1,2)}(t) + (-1)^i U^{(2,1)}(t)$$

we can easily show that the following relations hold:

$$\begin{cases} \int_{\tau_k}^{\pi-\tau_k} K^{(k)}(t)\,\mathrm{d}t = -J_2^{(k)}, & \int_{\tau_k}^{\pi-\tau_k} U^{(2)}(t)\,\mathrm{d}t = J_2^{(2)}, \\ \int_{\frac{\tau_1+\tau_2}{2}}^{\pi-\frac{\tau_1+\tau_2}{2}} K^{(k,l)}(t)\,\mathrm{d}t = -J_2^{(k,l)}, & \int_{\frac{\tau_1+\tau_2}{2}}^{\pi-\frac{\tau_1+\tau_2}{2}} U^{(k,l)}(t)\,\mathrm{d}t = J_2^{(k,l)}. \end{cases} \tag{2.3}$$

Here we note that the functions $U^{(k,l)}(t)$ differ from functions $K^{(k,l)}(t)$ only with the sign in front of the third integral in (2.3).

Using the aforementioned notations and relations given with (2.3), we can rewrite the characteristic functions $F_{i,k}(z)$ as follows

$$\begin{aligned} F_{i,k}(z) =& \left(-z + \frac{hH_k}{z}\right)\sin \pi z + (h + H_k)\cos \pi z \\ &+ \frac{1}{2}\left(\tilde{a}_{i,c}(z) + J_{i,c}(z)\right) + \frac{h}{2z}\left(-\tilde{a}_{i,s}(z) + J_{i,s}(z)\right) \\ &+ \frac{H_k}{2z}\left(\tilde{a}_{i,s}(z) + J_{i,s}(z)\right) + \frac{hH_k}{2z^2}\left(\tilde{a}_{i,c}(z) - J_{i,c}(z)\right) \\ &+ \frac{1}{4z}\left(J_{2,i,s}(z) - u_{i,s}(z)\right) - \frac{h}{4z^2}\left(J_{2,i,c}(z) + k_{i,c}(z)\right) \\ &- \frac{H_k}{4z^2}\left(J_{2,i,c}(z) - u_{i,c}(z)\right) - \frac{hH_k}{4z^3}\left(J_{2,i,s}(z) + k_{i,s}(z)\right) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} J_{i,c}(z) &= J_1^{(1)}\cos z(\pi - \tau_1) + (-1)^i J_1^{(2)}\cos z(\pi - \tau_2), \\ J_{i,s}(z) &= J_1^{(1)}\sin z(\pi - \tau_1) + (-1)^i J_1^{(2)}\sin z(\pi - \tau_2), \\ J_{2,i,c}(z) =& J_2^{(1)}\cos z(\pi - 2\tau_1) + J_2^{(2)}\cos z(\pi - 2\tau_2) \\ &+ (-1)^i\left(J_2^{(1,2)} + J_2^{(2,1)}\right)\cos z(\pi - \tau_1 - \tau_2), \\ J_{2,i,s}(z) =& J_2^{(1)}\sin z(\pi - 2\tau_1) + J_2^{(2)}\sin z(\pi - 2\tau_2) \\ &+ (-1)^i\left(J_2^{(1,2)} + J_2^{(2,1)}\right)\sin z(\pi - \tau_1 - \tau_2). \end{aligned}$$

The functions $F_{i,k}(z)$ have one singular point $z = 0$. It can be easily proved that $z = 0$ is an apparent singularity. Also, the functions $F_{i,k}(z)$ are entire in λ of order 1/2. Indeed, it is well known that the functions $\sin z\pi$ and $\cos z\pi$ are entire of order 1/2. Since any entire function has the form of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and we can determine its order as $\limsup_{n\to\infty} \frac{n \ln n}{-\ln |a_n|}$, we conclude that every given function in (2.4) has the same order as the functions $\sin z\pi$ and $\cos z\pi$. Further, using (2.4) and a well-known method (see [1]), we obtain the asymptotic formulas for the eigenvalues $(\lambda_{n,i,k})_{n=0}^{\infty}$ of the boundary value problems $D_{i,k}$ as

$$\lambda_{n,i,k} = n^2 + \frac{2(h + H_k)}{\pi} + \frac{J_1^{(1)}}{\pi}\cos n\tau_1 + \frac{(-1)^i J_1^{(2)}}{\pi}\cos n\tau_2 + o(1), \ n \to \infty. \quad (2.5)$$

Now, by Hadamard's factorization theorem, from the spectra of $D_{i,k}$, we can construct the characteristic functions $F_{i,k}(z)$. The next lemma holds.

Lemma 1 *The specification of the spectrum $(\lambda_{n,i,k})_{n=0}^{\infty}$ of the boundary value problems $D_{i,k}$ uniquely determines the characteristic functions $F_{i,k}(z)$ by the formulas*

$$F_{i,k}(z) = \pi\left(\lambda_{0,i,k} - z^2\right)\prod_{n=1}^{\infty}\frac{\lambda_{n,i,k} - z^2}{n^2}. \quad (2.6)$$

3 Main results

In this section, we prove that the delays and the parameters are uniquely determined by the spectra.

Lemma 2 *The delays τ_k, the integrals $J_1^{(k)}$, and the sums $h + H_k$ are uniquely determined by the eigenvalues $(\lambda_{n,i,k})_{n=0}^{\infty}$.*

Proof Let us consider the sequences $\rho_{n,k} = \frac{1}{2}\left(\lambda_{n,0,k} + \lambda_{n,1,k}\right)$ and $\sigma_n = \frac{1}{2}\left(\lambda_{n,0,1} - \lambda_{n,1,1}\right)$. From (2.5), we have

$$\rho_{n,k} = n^2 + \frac{2}{\pi}(h + H_k) + \frac{J_1^{(1)}}{\pi}\cos n\tau_1 + o(1)$$

and

$$\sigma_{n,i} = \frac{J_1^{(2)}}{\pi}\cos n\tau_2 + o(1)$$

as $n \to \infty$. Obviously, the delays τ_1, τ_2 and integrals $J_1^{(1)}$, $J_1^{(2)}$ can be determined from sequences $\left(\rho_{n,k}\right)_{n=0}^{\infty}$ and $\left(\sigma_n\right)_{n=0}^{\infty}$ in the same way as for the operators with one delay (see [9]). Lemma is proved.

Lemma 3 *Parameters h and H_k are uniquely determined by the eigenvalues $(\lambda_{n,0,k})_{n=0}^{\infty}$.*

Proof Functions $J_{1,c}^{0}(z)$ and $J_{1,s}^{0}(z)$ are known by virtue of Lemma 2. Since the characteristic functions are uniquely determined by the spectra, by writing $\lambda = \left(\frac{4m+1}{2}\right)^2$ in (2.6), we can define the functions

$$F^*_{0,k}(m) = F_{0,k}\big(\frac{4m+1}{2}\big) + \frac{4m+1}{2} - \frac{1}{2}J_{0,c}\big(\frac{4m+1}{2}\big) - \frac{H_k+h}{4m+1}J_{0,s}\big(\frac{4m+1}{2}\big).$$

Then, using the form of the characteristic functions $F_{0,k}$ given in (2.4), we get

$$h = \frac{1}{2}\lim_{m\to\infty}\frac{4m+1}{H_2-H_1}\big(F^*_{0,2}(m) - F*_{0,1}(m)\big).$$

Thus, we determine H_k from $h+H_k$ and prove the lemma. □

To recover the potential functions from the spectra, we should transform the characteristic functions (2.4). For this purpose, we use the method of integration by parts in (2.4) once in the integrals denoted by $\tilde{a}_{i,s}(z)$, $\tilde{a}_{i,c}(z)$, $u_s(z)$, and $u_c(z)$, and twice in the integrals denoted by $k_c(z)$ and $k_s(z)$. This is how the following function appears

$$K^*_i(t) = K^{(1)*}(t) + K^{(2)*}(t) + (-1)^i K^{(1,2)*}(t) + (-1)^{(i)} K^{(2,1)*}(t),$$

where

$$K^{(k)*}(t) = \begin{cases} \int_{\tau_k}^t K^{(k)}(u)\,\mathrm{d}u, & t\in[\tau_k, \pi-\tau_k], \\ 0, & t\in[0,\tau_k)\cup(\pi-\tau_k,\pi], \end{cases}$$

and

$$K^{(k,l)*}(t) = \begin{cases} \int_{\frac{\tau_1+\tau_2}{2}}^t K^{(k,l)}(u)\,\mathrm{d}u, & t\in\big[\frac{\tau_1+\tau_2}{2}, \pi-\frac{\tau_1+\tau_2}{2}\big], \\ 0, & t\in\big[0,\frac{\tau_1+\tau_2}{2}\big)\cup\big(\pi-\frac{\tau_1+\tau_2}{2},\pi\big]. \end{cases}$$

One can show that the following relations hold

$$\int_{\tau_k}^{\pi-\tau_k}\Big(\int_{\tau_k}^t K^{(k)}(u)\,\mathrm{d}u\Big)\,\mathrm{d}t = -(\pi-2\tau_k)J_2^{(k)},$$

$$\int_{\frac{\tau_1+\tau_2}{2}}^{\pi-\frac{\tau_1+\tau_2}{2}}\Big(\int_{\pi-\frac{\tau_1+\tau_2}{2}}^t (K^{(1,2)}(u)+K^{(2,1)}(u))\,\mathrm{d}u\Big)\,\mathrm{d}t$$
$$= -(\pi-\tau_1-\tau_2)\big(J_2^{(1,2)}+J_2^{(2,1)}\big).$$

Then we obtain the characteristic functions in the form

$$
\begin{aligned}
F_{i,k}(z) =& \Big(-z + \frac{H_k h}{z} \Big) \sin \pi z + (h + H_k) \cos \pi z \\
&+ \frac{1}{2} \big(\tilde{a}_{i,c}(z) + \frac{H_k}{z} \tilde{a}_{i,s}(z) \big) - h \Big(\tilde{q}^{(1)}_{i,c}(z) + \frac{H_k}{z} \tilde{q}^{(1)}_{i,s}(z) \Big) \\
&- \frac{1}{2} \Big(u^{*}_{i,c}(z) + \frac{H_k}{z} u^{*}_{i,s}(z) \Big) + h \Big(k^{**}_{i,c}(z) + \frac{H_k}{z} k^{**}_{i,s}(z) \Big) \\
&+ \frac{J_{i,c}(z)}{2} + \frac{2h + H_k}{2z} J_{i,s}(z) \\
&+ \frac{1}{2z} \Big(1 + (\pi - 2\tau) h - \frac{H_k h}{z^2} \Big) J_{2,i,s}(z) \\
&+ \frac{H_k h (\pi - 2\tau)}{2z^2} J_{2,i,c}(z)
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\tilde{q}^{(1)}_{i,c}(z) &= \int_{\frac{\tau_2}{2}}^{\pi - \frac{\tau_2}{2}} \Big(\int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s) ds \Big) \cos z(\pi - 2t) \, \mathrm{d}t, \\
\tilde{q}^{(1)}_{i,s}(z) &= \int_{\frac{\tau_2}{2}}^{\pi - \frac{\tau_2}{2}} \Big(\int_{\frac{\tau_2}{2}}^{t} \tilde{q}_i(s) ds \Big) \sin z(\pi - 2t) \, \mathrm{d}t, \\
u^{*}_{i,c}(z) &= \int_{\tau_2}^{\pi - \tau_2} \Big(\int_{\tau_2}^{t} U_i(s) ds \Big) \cos z(\pi - 2t) \, \mathrm{d}t, \\
u^{*}_{i,s}(z) &= \int_{\tau_2}^{\pi - \tau_2} \Big(\int_{\tau_2}^{t} U_i(s) ds \Big) \sin z(\pi - 2t) \, \mathrm{d}t, \\
k^{**}_{i,c}(z) &= \int_{\tau_2}^{\pi - \tau_2} \Big(\int_{\tau_2}^{t} K^{*}_i(s) ds \Big) \cos z(\pi - 2t) \, \mathrm{d}t, \\
k^{**}_{i,s}(z) &= \int_{\tau_2}^{\pi - \tau_2} \Big(\int_{\tau_2}^{t} K^{*}_i(s) ds \Big) \sin z(\pi - 2t) \, \mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
(\pi - 2\tau) J_{2,i,s}(z) =& \sum_{k=1}^{2} (\pi - 2\tau_k) J_2^{(k)} \sin z(\pi - 2\tau_k) \\
&+ (-1)^i (\pi - \tau_1 - \tau_2) \big(J_2^{(1,2)} + J_2^{(2,1)} \big) \sin z(\pi - \tau_1 - \tau_2), \\
(\pi - 2\tau) J_{2,i,c}(z) =& - \sum_{k=1}^{2} (\pi - 2\tau_k) J_2^{(k)} \cos z(\pi - 2\tau_k) \\
&- (-1)^i (\pi - \tau_1 - \tau_2) \big(J_2^{(1,2)} + J_2^{(2,1)} \big) \cos z(\pi - \tau_1 - \tau_2).
\end{aligned}
$$

Since we have the transformed version of the characteristic functions (2.4), we are now ready to recover the potential functions from the spectra. To do this, first we need to define the functions

$$A_i(z) = \frac{2}{H_2 - H_1}\left(H_2 F_{i,1}(z) - H_1 F_{i,2}(z)\right) + 2z\sin \pi z - 2h\cos \pi z - J_{i,c}(z) - \frac{2h}{z} J_{i,s}(z)$$

and

$$B_i(z) = \frac{2z}{H_2 - H_1}\left(F_{i,2}(z) - F_{i,1}(z)\right) - 2h\sin \pi z - 2z\cos \pi z - J_{i,s}(z).$$

From (3.1), we obtain

$$A_i(z) = \tilde{a}_{i,c}(z) - 2h\tilde{q}^{(1)}_{i,c}(z) - u^*_{i,c}(z) + 2hk^{**}_{i,c}(z) + \alpha_i(z), \tag{3.2}$$

$$B_i(z) = \tilde{a}_{i,s}(z) - 2h\tilde{q}^{(1)}_{i,s}(z) - u^*_{i,s}(z) + 2hk^{**}_{i,s}(z) + \beta_i(z) \tag{3.3}$$

where

$$\alpha_i(z) = \frac{1 + (\pi - 2\tau)h}{2z} J_{2,i,s}(z) \tag{3.4}$$

and

$$\beta_i(z) = \frac{h(\pi - 2\tau)}{z} J_{2,i,c}(z) - \frac{h}{z^2} J_{2,i,s}(z). \tag{3.5}$$

The two below-given equations hold:

$$\beta_{0,i} = \lim_{z\to 0} \beta_i(z) = 0,$$

$$\alpha_{0,i} = \lim_{z\to 0} \alpha_i(z) = \sum_{k=1}^{2} \left(h(\pi - 2\tau_k)^2 + (\pi - 2\tau_k)\right) J_2^{(k)} \sin z(\pi - 2\tau_k) + (-1)^i \left(h(\pi - \tau_1 - \tau_2)^2 + (\pi - \tau_1 - \tau_2)\right)\left(J_2^{(1,2)} + J_2^{(2,1)}\right) \times \sin z(\pi - \tau_1 - \tau_2).$$

If we put $z = m$, $m \in \mathbb{N}$ into (3.2) and (3.3) and denote

$$A_{2m,i} = \frac{2m}{\pi}(-1)^m A_i(m), \quad B_{2m,i} = \frac{2}{\pi}(-1)^{m+1} B_i(m),$$
$$\alpha_{2m,i} = \frac{2}{\pi}(-1)^m \alpha_i(m), \quad \beta_{2m,i} = \frac{2}{\pi}(-1)^{m+1} \beta_i(m)$$

, we obtain

$$A_{2m,i} = \frac{2}{\pi}\tilde{a}_{2m,i} - \frac{4}{\pi} h\tilde{q}^{(1)}_{2m,i,c} - \frac{2}{\pi} u^*_{2m,i,c} + \frac{4}{\pi} hk^{**}_{2m,i,c} + \alpha_{2m,i} \tag{3.6}$$

$$B_{2m} = \frac{2}{\pi}\tilde{b}_{2m,i} - \frac{4}{\pi}h\tilde{q}^{(1)}_{2m,i,s} - \frac{2}{m}u^{*}_{2m,i,s} + \frac{4}{\pi}hk^{**}_{2m,i,s} + \beta_{2m,i} \tag{3.7}$$

where

$$\begin{aligned}
\tilde{a}_{2m,i} &= \int_0^{\pi} \tilde{q}_i(t)\cos 2mt\,\mathrm{d}t,\\
\tilde{b}_{2m,i} &= \int_0^{\pi} \tilde{q}_i(t)\sin 2mt\,\mathrm{d}t,\\
u^{*}_{2m,i,s} &= \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}} \left(\int_{\frac{\tau_2}{2}}^{t} U_i(t_2)dt_2\right)\sin 2mt\,\mathrm{d}t,\\
u^{*}_{2m,i,c} &= \int_{\frac{\tau_2}{2}}^{\pi-\frac{\tau_2}{2}} \left(\int_{\frac{\tau_2}{2}}^{t} U_i(t_2)dt_2\right)\cos 2mt\,\mathrm{d}t,\\
k^{**}_{2m,i,c} &= \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} K^{*}_i(t_2)dt_2\right)\cos 2mt\,\mathrm{d}t,\\
k^{**}_{2m,i,s} &= \int_{\tau_2}^{\pi-\tau_2} \left(\int_{\tau_2}^{t} K^{*}_i(t_2)dt_2\right)\sin 2mt\,\mathrm{d}t.
\end{aligned}$$

Further, we have

$$A_{0,i} = \frac{2}{\pi}\lim_{m\to 0} A_i(m) = \frac{2}{\pi}\tilde{a}_{0,i} - \frac{4}{\pi}h\tilde{q}^{(1)}_{0,i,c} - \frac{2}{\pi}u^{*}_{0,i,c} + \frac{4}{\pi}hk^{**}_{0,i,c} + \alpha_{0,i}. \tag{3.8}$$

Since sequences $\{\alpha_{2m,i}\}$, $\{\beta_{2m,i}\}$, $\{A_{2m,i}\}$, and $\{B_{2m,i}\}$ belong to the space l_2, by virtue of Riesz–Fischer theorem, there exist functions f_i and φ_i from $L_2[0,\pi]$ such that

$$f_i(t) = \frac{A_{0,i}}{2} + \sum_{m=1}^{\infty} A_{2m,i}\cos 2mt + B_{2m,i}\sin 2mt, \quad t\in[0,\pi]$$

and

$$\varphi_i(t) = \frac{\alpha_{0,i}}{2} + \sum_{m=1}^{\infty} \alpha_{2m,i}\cos 2mt + \beta_{2m,i}\sin 2mt, \quad t\in[0,\pi]$$

hold. Now multiplying (3.8) with $\frac{1}{2}$, (3.6) with $\cos 2mt$, (3.7) with $\sin 2mt$, and then summing up from $m=1$ to $m=\infty$, we get the system of integral equations

$$\tilde{q}_i(t) - 2h\int_{\frac{\tau_i}{2}}^{t}\tilde{q}_i(s)\,\mathrm{d}s - \int_{\tau_2}^{t} U(s)\,\mathrm{d}s + 2h\int_{\tau_2}^{t} K^{*}(s)\,\mathrm{d}s + \varphi_i(t) = f_i(t). \tag{3.9}$$

Substituting functions U and K^{*} into (3.9), we obtain

$$\begin{aligned}&\tilde{q}_i(t) - 2h\int_{\frac{\tau_2}{2}}^{t}\tilde{q}_i(s)\,\mathrm{d}s - \int_{\tau_2}^{t}U^{(2)}(s)\,\mathrm{d}s - \int_{\tau_1}^{t}U^{(1)}(s)\,\mathrm{d}s\\&-(-1)^i\int_{\frac{\tau_1+\tau_2}{2}}^{t}\big(U^{(1,2)}(s)+U^{(2,1)}(s)\big)\,\mathrm{d}s\\&+2h\int_{\tau_2}^{t}K^{(2)*}(s)\,\mathrm{d}s+2h\int_{\tau_1}^{t}K^{(1)*}(s)\,\mathrm{d}s\\&+(-1)^i\int_{\frac{\tau_1+\tau_2}{2}}^{t}\big(K^{(1,2)*}(s)+K^{(2,1)*}(s)\big)\,\mathrm{d}s\\&+\varphi_i(t)=f_i(t).\end{aligned}\tag{3.10}$$

From (3.4) and (3.5), we have

$$\varphi_i(t)=J_2^{(1)}S^{(1)}(t)+J_2^{(2)}S^{(2)}(t)+(-1)^i\big(J_2^{(1,2)}+J_2^{(2,1)}\big)S^{(1,2)}(t)$$

where

$$\begin{aligned}S^{(k)}=&\frac{\pi-2\tau_k}{\pi}\big(h(\pi-2\tau_k)+1\big)-\frac{2\big(h(\pi-2\tau_k)+1\big)}{\pi}\sum_{m=1}^{\infty}\frac{\sin 2m\tau_i}{m}\cos 2mt\\&-\frac{2h(\pi-2\tau_k)}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2m\tau_i}{m}\sin 2mt-\frac{2h}{\pi}\sum_{m=1}^{\infty}\frac{\sin 2m\tau_i}{m^2}\sin 2mt\end{aligned}$$

and

$$\begin{aligned}S^{(1,2)}(t)=&\frac{\pi-\tau_2-\tau_1}{\pi}\big(h(\pi-\tau_2-\tau_1)+1\big)\\&-\frac{2\big(h(\pi-\tau_2-\tau_1)+1\big)}{\pi}\sum_{m=1}^{\infty}\frac{\sin m(\tau_2+\tau_1)}{m}\cos 2mt\\&-\frac{2h(\pi-\tau_2-\tau_1)}{\pi}\sum_{m=1}^{\infty}\frac{\cos m(\tau_2+\tau_1)}{m}\sin 2mt\\&-\frac{2h}{\pi}\sum_{m=1}^{\infty}\frac{\sin m(\tau_2+\tau_1)}{m^2}\sin 2mt.\end{aligned}$$

Further, we have

$$\sum_{m=1}^{\infty}\frac{\sin 2ma}{m}\cos 2mt=\begin{cases}-a, & t\in(a,\pi-a),\\ \pi/2-a, & t\in(0,a)\cup(\pi-a,\pi),\\ \pi/4-a, & t=a,\ t=\pi-a,\end{cases}$$

$$\sum_{m=1}^{\infty}\frac{\cos 2ma}{m}\sin 2mt=\begin{cases}-t, & t\in(0,a),\\ \pi/2-t, & t\in(a,\pi-a),\\ \pi-t, & t\in(\pi-a,\pi),\\ \pi/4-a, & t=a,\\ -\pi/4+a, & t=\pi-a,\end{cases}$$

and

$$\sum_{m=1}^{\infty}\frac{\sin 2ma}{m^2}\sin 2mt=\begin{cases}(\pi-2a)t, & t\in(0,a),\\ a(\pi-2t), & t\in(a,\pi-a),\\ (\pi-2a)(t-\pi), & t\in(\pi-a,\pi),\\ (\pi-2a)a, & t=a,\\ -(\pi-2a)a, & t=\pi-a.\end{cases}$$

Then we get

$$S^{(k)}(t)=\begin{cases}0, & t\in(0,\tau_k)\cup(\pi-\tau_k,\pi),\\ 1+2h(t-\tau_k), & t\in(\tau_k,\pi-\tau_k),\\ 1/2, & t=\tau_k,\\ 1/2+h(\pi-2\tau_k), & t=\pi-\tau_k,\end{cases}\tag{3.11}$$

and

$$S^{(1,2)}(t)=\begin{cases}0, & t\in\left(0,\dfrac{\tau_1+\tau_2}{2}\right)\cup\left(\pi-\dfrac{\tau_1+\tau_2}{2},\pi\right),\\ 1+2h\big(t-(\tau_1+\tau_2)/2\big), & t\in\left(\dfrac{\tau_1+\tau_2}{2},\pi-\dfrac{\tau_1+\tau_2}{2}\right),\\ 1/2, & t=(\tau_1+\tau_2)/2,\\ 1/2+h(\pi-\tau_1-\tau_2), & t=\pi-(\tau_1+\tau_2)/2.\end{cases}\tag{3.12}$$

Now, after summing and subtracting integral equations (3.10) and then introducing substitution of variables, we get the system of integral equations

$$
\begin{aligned}
&q_1(x) - 2h\int_{\tau_1}^{x} q_1(u)\,\mathrm{d}u - \int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(2)}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u - \int_{\frac{3\tau_1}{2}}^{x} U^{(1)}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u \\
&\quad + 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} K^{(2)*}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u + 2h\int_{\frac{3\tau_1}{2}}^{x} K^{(1)*}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u \\
&\quad + J_2^{(1)}S^{(1)}\big(x-\frac{\tau_1}{2}\big) + J_2^{(2)}S^{(2)}\big(x-\frac{\tau_1}{2}\big) \\
&\quad = \frac{1}{2}\big(f_0\big(x-\frac{\tau_1}{2}\big) + f_1\big(x-\frac{\tau_1}{2}\big)\big)
\end{aligned} \tag{3.13}
$$

and

$$
\begin{aligned}
&q_2(x) - 2h\int_{\tau_2}^{x} q_2(u)\,\mathrm{d}u - \int_{\tau_2+\frac{\tau_1}{2}}^{x} \Big(U^{(1,2)}\big(u-\frac{\tau_2}{2}\big) + U^{(2,1)}\big(u-\frac{\tau_2}{2}\big)\Big)\,\mathrm{d}u \\
&\quad + 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} \Big(K^{(1,2)*}\big(u-\frac{\tau_2}{2}\big) + K^{(2,1)*}\big(u-\frac{\tau_2}{2}\big)\Big)\,\mathrm{d}u \\
&\quad + \big(J_2^{(1,2)} + J_2^{(2,1)}\big)S^{(1,2)}\big(x-\frac{\tau_2}{2}\big) \\
&\quad = \frac{1}{2}\Big(f_0\big(x-\frac{\tau_2}{2}\big) - f_1\big(x-\frac{\tau_2}{2}\big)\Big).
\end{aligned} \tag{3.14}
$$

Notice that each integral Eqs. (3.13) and (3.14) contains both potential functions q_1 and q_2. To prove uniqueness of the solutions of integral equations, we reduce integral Eqs. (3.13) and (3.14) to the linear integral equations of Volterra type. For that purpose, we will prove that all functions containing q_2 in integral equation (3.13) are known, as well as all functions containing q_1 in integral equation (3.14).

However, it turns out not to be true in general and depends on the relations between delays. In integral Eqs. (3.13) and (3.14) there are unknown integrals $J_2^{(1)}$, $J_2^{(2)}$, $J_2^{(2,1)}$, $J_2^{(1,2)}$. Although they are not known, we show that they do not exist in the integral equations on certain sub-intervals created at the beginning and at the end of $[\tau_k, \pi]$. After solving integral equations on these intervals, depending on the case we consider, we will reveal that all integrals or some of them are known. Then, after solving the integral equations on certain intervals, we prove that the remaining integrals are known.

It is well known that the point $\frac{2\pi}{5}$ is a point that separates linear and nonlinear cases of the boundary value problem. The boundary value problem becomes linear on the right side of this point and becomes non-linear on the left. Furthermore, the point $\frac{2\pi}{5}$ is a point of separation in terms of the uniqueness of the solution of the boundary value problem. On the right side of $\frac{2\pi}{5}$, the boundary value problem has a unique solution, while this is not the case on the left (see [19–21]).

This is the motivation behind the following process. First, we show that the integral equations (3.13) and (3.14) have unique solutions on certain sub-intervals of $[\tau_k, \pi]$.

Theorem 1 *Let* $q_k \in L_2[\tau_k, \pi]$, $q_k(x) = 0$ *as* $x \in [0, \tau_k)$. *If* $\frac{2}{5}\pi \le \tau_2 < \tau_1 < \frac{\pi}{2}$, *then integral equation* (3.13) *has a unique solution* q_1 *on intervals* $\left(\tau_1, \tau_2 + \frac{\tau_1}{2}\right)$ *and* $\left(\pi + \frac{\tau_1}{2} - \tau_2, \pi\right)$, *and integral equation* (3.14) *has a unique solution* q_2 *on intervals* $\left[\tau_2, \tau_2 + \frac{\tau_1}{2}\right]$ *and* $\left(\pi - \frac{\tau_1}{2}, \pi\right]$.

Proof From (3.11) and (3.12), we have

$$S^{(1)}\left(x - \frac{\tau_1}{2}\right) = \begin{cases} 0, & t \in \left(\frac{\tau_1}{2}, \frac{3\tau_1}{2}\right) \cup \left(\pi - \frac{\tau_1}{2}, \pi + \frac{\tau_1}{2}\right), \\ 1 + 2h\left(x - \frac{3\tau_1}{2}\right), & t \in \left(\frac{3\tau_1}{2}, \pi - \frac{\tau_1}{2}\right), \\ 1/2, & t = \frac{3\tau_1}{2}, \\ 1/2 + h(\pi - 2\tau_1), & t = \pi - \frac{3\tau_1}{2}, \end{cases}$$

$$S^{(2)}\left(x - \frac{\tau_1}{2}\right) = \begin{cases} 0, & t \in \left(\frac{\tau_1}{2}, \tau_2 + \frac{\tau_1}{2}\right) \cup \left(\pi - \tau_2 + \frac{\tau_1}{2}, \pi + \frac{\tau_1}{2}\right), \\ 1 + 2h\left(x - \tau_2 - \frac{\tau_1}{2}\right), & t \in \left(\tau_2 + \frac{\tau_1}{2}, \pi - \tau_2 + \frac{\tau_1}{2}\right), \\ 1/2, & t = \tau_2 + \frac{\tau_1}{2}, \\ 1/2 + h(\pi - 2\tau_2), & t = \pi - \tau_2 + \frac{\tau_1}{2}, \end{cases}$$

and

$$S^{(1,2)}\left(x - \frac{\tau_2}{2}\right) = \begin{cases} 0, & x \in \left(\frac{\tau_2}{2}, \tau_2 + \frac{\tau_1}{2}\right) \cup \left(\pi - \frac{\tau_1}{2}, \pi + \frac{\tau_2}{2}\right), \\ 1 + 2h\left(x - \tau_2 - \frac{\tau_2}{2}\right), & x \in \left(\tau_2 + \frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right), \\ \frac{1}{2}, & x = \tau_2 + \frac{\tau_1}{2}, \\ \frac{1}{2} + h(\pi - \tau_1 - \tau_2), & x = \pi - \frac{\tau_1}{2}. \end{cases}$$

Now we consider the following cases:

Case 1. If $x \in \left[\tau_2, \tau_2 + \frac{\tau_1}{2}\right]$ we have $S^{(1,2)}\left(x - \frac{\tau_2}{2}\right) = 0$ and from (3.14) we obviously get the Volterra linear integral equation

$$q_2(x) = \frac{1}{2}\left(f_0\left(x - \frac{\tau_2}{2}\right) - f_1\left(x - \frac{\tau_2}{2}\right)\right) + 2h\int_{\tau_2}^{x} q_2(u)\,\mathrm{d}u$$

which has a unique solution on $\left[\tau_2, \tau_2 + \frac{\tau_1}{2}\right]$.

Case 2. Let $x \in \left(\pi - \frac{\tau_1}{2}, \pi\right]$ in (3.14). Then we both have $S^{(1,2)}\left(x - \frac{\tau_2}{2}\right) = 0$ and

$$\int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(k,l)}\left(u-\frac{\tau_2}{2}\right)\mathrm{d}t=\int_{\tau_2+\frac{\tau_1}{2}}^{x} K^{(k,l)*}\left(u-\frac{\tau_2}{2}\right)\mathrm{d}t,\quad x\in\left(\pi-\frac{\tau_1}{2},\pi\right].$$

Since,

$$\int_{\tau_2}^{x} q_2(u)\,\mathrm{d}u=\int_{\tau_2}^{\pi} q_2(u)\,\mathrm{d}u-\int_{x}^{\pi} q_2(u)\,\mathrm{d}u=J_1^{(2)}-\int_{x}^{\pi} q_2(u)\,\mathrm{d}u$$

we obtain integral equation

$$q_2(x)=g_1(x)-2h\int_{x}^{\pi} q_2(u)\,\mathrm{d}u \tag{3.15}$$

where

$$g_1(x)=\frac{1}{2}\left(f_0\left(x-\frac{\tau_2}{2}\right)-f_1\left(x-\frac{\tau_2}{2}\right)\right)+2hJ_1^{(2)}.$$

Integral equation (3.15) has a unique solution on $\left(\pi-\frac{\tau_1}{2},\pi\right]$.

Case 3. On the interval $\left(\tau_1,\tau_2+\frac{\tau_1}{2}\right)$, we have $S^{(1)}\left(x-\frac{\tau_1}{2}\right)=S^{(2)}\left(x-\frac{\tau_1}{2}\right)=0$. Then integral equation (3.13) becomes a linear integral equation of Volterra type with a kernel equal to one which has a unique solution q_1 on $\left(\tau_1,\tau_2+\frac{\tau_1}{2}\right)$.

Case 4. In the same way as in *Case 2* using $S^{(1)}\left(x-\frac{\tau_1}{2}\right)=S^{(2)}\left(x-\frac{\tau_1}{2}\right)=0$ on $\left(\pi+\frac{\tau_1}{2}-\tau_2,\pi\right]$, we obtain the unique solution q_1 of integral equation (3.13) on interval $\left(\pi+\frac{\tau_1}{2}-\tau_2,\pi\right]$. The theorem is proved.

Now, let us show that the integrals $J_2^{(1)}$, $J_2^{(2)}$, $J_2^{(2,1)}$, and $J_2^{(1,2)}$ are known. The next lemma holds.

Lemma 4 *Integrals $J_2^{(1)}$, $J_2^{(2)}$, $J_2^{(2,1)}$ and $J_2^{(1,2)}$ are determined by potentials q_2 on $\left(\tau_2,\tau_2+\frac{\tau_1}{2}\right)\cup\left(\pi-\frac{\tau_1}{2},\pi\right]$ and q_1 on $\left(\tau_1,\tau_2+\frac{\tau_1}{2}\right)\cup\left(\pi+\frac{\tau_1}{2}-\tau_2,\pi\right]$.*

Proof The arguments of $J_2^{(2)}$ belong to the intervals $\left(2\tau_2,\pi\right)\subset\left(\pi-\frac{\tau_1}{2},\pi\right)$ and $\left(\tau_2,\pi-\tau_2\right)\subset\left(\tau_2,\tau+\frac{\tau_1}{2}\right)$, so $J_2^{(2)}$ is known. In the same way, we get that arguments of $J_2^{(1)}$ belong to the intervals $(2\tau_1,\pi)\subset\left(\pi+\frac{\tau_1}{2}-\tau_2,\pi\right]$ and $(\tau_1,\pi-\tau_1)\subset\left(\tau_1,\tau+\frac{\tau_1}{2}\right)$ and integral $J_2^{(1)}$ is known, too. Argument of q_2 in the integral $J_2^{(2,1)}$ belongs to the interval $(\tau_1+\tau_2,\pi)\subset\left(\pi-\frac{\tau_1}{2},\pi\right]$. Consequently, integral $J_2^{(2,1)}$ is known. Argument of q_1 in the integral $J_2^{(1,2)}$ belongs to the interval $(\tau_1+\tau_2,\pi)\subset\left(\pi+\frac{\tau_1}{2}-\tau_2,\pi\right]$ and argument of q_1 to the interval $(\tau_2,\pi-\tau_1)\subset\left(\tau_2,\tau_2+\frac{\tau_1}{2}\right)$ and then integral $J_2^{(1,2)}$ is known. Then, in further considerations, we take that summand $J_2^{(1)}S^{(1)}\left(x-\frac{\tau_1}{2}\right)+J_2^{(2)}S^{(2)}\left(x-\frac{\tau_1}{2}\right)$ in integral equation (3.13) and summand $\left(J_2^{(1,2)}+J_2^{(2,1)}\right)S^{(1,2)}\left(x-\frac{\tau_2}{2}\right)$ in integral equation (3.14) are known.

Now we come to our main result and prove that Inverse problem 1 has a unique solution.

Theorem 2 *Let* $q_k \in L_2[\tau_k, \pi]$, $q_k(x) = 0$ *for* $x \in [0, \tau_k)$ *and* $\frac{2\pi}{5} \le \tau_2 < \tau_1 \le \frac{\pi}{2}$. *The four spectra of the boundary value problem* $D_{i,k}$ *uniquely determine delays* τ_k, *parameters* h, H_k, *and potential functions* q_k.

Proof Taking Lemma 1, Lemma 2, and Theorem 1 into account, it remains to show that integral equations (3.13) and (3.14) have unique solutions q_1 on the interval $\left(\tau_2 + \dfrac{\tau_1}{2}, \pi + \dfrac{\tau_1}{2} - \tau_2\right)$ and q_2 on the interval $\left(\tau_2 + \frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right)$, respectively.

Case 1. Let $x \in \left(\tau_2 + \frac{\tau_1}{2}, \frac{3\tau_1}{2}\right]$. From (3.13), we get the integral equation

$$\begin{aligned} &q_1(x) - 2h\int_{\tau_1}^{\tau_2+\frac{\tau_1}{2}} q_1(u)\,du - 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} q_1(u)\,du \\ &\quad - \int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(2)}\left(u - \frac{\tau_1}{2}\right)du + 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} K^{(2)*}\left(u - \frac{\tau_1}{2}\right)du \\ &\quad = \frac{1}{2}\left(f_0\left(x - \frac{\tau_1}{2}\right) + f_1\left(x - \frac{\tau_1}{2}\right)\right) - J_2^{(1)}S^{(1)}\left(x - \frac{\tau_1}{2}\right) - J_2^{(2)}S^{(2)}\left(x - \frac{\tau_1}{2}\right). \end{aligned}$$

One can easily show that arguments of the potential q_2 appearing in the function

$$\int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(2)}\left(u - \frac{\tau_1}{2}\right)du$$

belong to the intervals $[2\tau_2, \tau_1 + \tau_2] \subset \left[\pi - \frac{\tau_1}{2}, \pi\right]$, $[\tau_2, \tau_1] \subset \left[\tau_2, \tau_2 + \frac{\tau_1}{2}\right]$, $[2\tau_2, \pi] \subset \left[\pi - \frac{\pi}{2}, \pi\right]$, and $[\tau_2, \pi - \tau_2] \subset \left[\tau_2, \tau_2 + \frac{\tau_1}{2}\right]$. We have the same for the function $\int_{\tau_2+\frac{\tau_1}{2}}^{x} K^{(2)*}\left(u - \frac{\tau_1}{2}\right)du$. Then we get the integral equation

$$q_1(x) = g_2(x) + 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} q_1(u)\,du \tag{3.16}$$

where

$$\begin{aligned} g_2(x) &= \frac{1}{2}\left(f_0\left(x - \frac{\tau_1}{2}\right) + f_1\left(x - \frac{\tau_1}{2}\right)\right) + \int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(2)}\left(u - \frac{\tau_2}{2}\right)du \\ &\quad - 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} K^{(2)*}\left(u - \frac{\tau_2}{2}\right)du + 2h\int_{\tau_1}^{\tau_2+\frac{\tau_1}{2}} q_1(u)\,du \\ &\quad - J_2^{(1)}S^{(1)}\left(x - \frac{\tau_1}{2}\right) + J_2^{(2)}S^{(2)}\left(x - \frac{\tau_1}{2}\right) \end{aligned}$$

is a known function. Integral equation (3.16) has a unique solution q_1 on $\left(\tau_2 + \frac{\tau_1}{2}, \frac{3\tau_1}{2}\right]$.

Case 2. For $x \in \left(\frac{3\tau_1}{2}, \pi - \frac{\tau_1}{2}\right]$, from (3.14) we get

$$
\begin{aligned}
& q_1(x) - 2h\int_{\tau_1}^{\frac{3\tau_1}{2}} q_1(u)\,\mathrm{d}u - 2h\int_{\frac{3\tau_1}{2}}^{x} q_1(u)\,\mathrm{d}u \\
&\quad - \int_{\tau_2+\frac{\tau_1}{2}}^{\frac{3\tau_1}{2}} U^{(2)}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u - \int_{\frac{3\tau_1}{2}}^{x} U^{(2)}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u \\
&\quad - \int_{\frac{3\tau_1}{2}}^{x} U^{(1)}\big(u-\frac{\tau_1}{2}\big) + 2h\int_{\tau_2+\frac{\tau_1}{2}}^{\frac{3\tau_1}{2}} K^{(2)*}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u \\
&\quad + 2h\int_{\frac{3\tau_1}{2}}^{x} K^{(2)*}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u + 2h\int_{\frac{3\tau_1}{2}}^{x} K^{(1)*}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u \\
&\quad = \frac{1}{2}\Big(f_0\big(x-\frac{\tau_1}{2}\big) + f_1\big(x-\frac{\tau_1}{2}\big)\Big) - J_2^{(1)}S^{(1)}\big(x-\frac{\tau_1}{2}\big) \\
&\quad - J_2^{(2)}S^{(2)}\big(x-\frac{\tau_1}{2}\big).
\end{aligned}
$$

In the same way as in *Case 1* one can show that functions

$$
\int_{\frac{3\tau_1}{2}}^{x} U^{(2)}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u \quad \text{and} \quad \int_{\frac{3\tau_1}{2}}^{x} K^{(2)*}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u
$$

are known as well as the functions

$$
\int_{\frac{3\tau_1}{2}}^{x} U^{(1)}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u \quad \text{and} \quad \int_{\frac{3\tau_1}{2}}^{x} K^{(1)*}\big(u-\frac{\tau_1}{2}\big)\,\mathrm{d}u.
$$

Then, we get the integral equation

$$
q_1(x) = g_3(x) + 2h\int_{\frac{3\tau_1}{2}}^{x} q_1(u)\,\mathrm{d}u \tag{3.17}
$$

where

$$
\begin{aligned}
g_3(x) &= \frac{1}{2}\Big(f_0\big(x-\frac{\tau_1}{2}\big) + f_1\big(x-\frac{\tau_1}{2}\big)\Big) + \int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(2)}\big(u-\frac{\tau_2}{2}\big)\,\mathrm{d}u \\
&\quad - 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} K^{(2)*}\big(u-\frac{\tau_2}{2}\big)\,\mathrm{d}u + 2h\int_{\tau_1}^{\frac{3\tau_1}{2}} q_1(u)\,\mathrm{d}u \\
&\quad - J_2^{(1)}S^{(1)}\big(x-\frac{\tau_1}{2}\big) - J_2^{(2)}S^{(2)}\big(x-\frac{\tau_1}{2}\big)
\end{aligned}
$$

is a known function. Integral equation (3.17) has a unique solution q_1 on $\big(\frac{3\tau_1}{2}, \pi - \frac{\tau_1}{2}\big]$.

Case 3. In the same way we show that integral equation (3.13) has a unique solution on $\big(\pi - \frac{\tau_1}{2}, \pi + \frac{\tau_1}{2} - \tau_2\big]$.

Case 4. Finally, we prove that the integral equation (3.14) has a unique solution q_2 on $\big(\tau_2 + \frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\big]$. Notice that the potential q_1 is known. From (3.14) we get

$$
\begin{aligned}
& q_2(x) - 2h\int_{\tau_2}^{\tau_2+\frac{\tau_1}{2}} q_1(u)\,\mathrm{d}u - 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} q_1(u)\,\mathrm{d}u \\
& \quad - \int_{\tau_1+\frac{\tau_1}{2}}^{x}\left(U^{(1,2)}\left(u-\frac{\tau_2}{2}\right) + U^{(2,1)}\left(u-\frac{\tau_2}{2}\right)\right)\mathrm{d}u \\
& \quad + 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x}\left(K^{(1,2)*}\left(u-\frac{\tau_2}{2}\right) + K^{(2,1)*}\left(u-\frac{\tau_2}{2}\right)\right)\mathrm{d}u \\
& \quad = \frac{1}{2}\left(f_0\left(x-\frac{\tau_2}{2}\right) - f_1\left(x-\frac{\tau_2}{2}\right)\right) \\
& \quad - \left(J_2^{(1,2)} + J_2^{(2,1)}\right)S^{(1,2)}\left(x-\frac{\tau_2}{2}\right).
\end{aligned}
$$

It can be easily shown that the arguments of the potential q_2 in the function

$$
\int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(2,1)}\left(u-\frac{\tau_2}{2}\right)
$$

belong to the interval $[\tau_2, \pi-\tau_1] \subset \left[\tau_2, \tau_2+\frac{\tau_1}{2}\right]$ and arguments of the potential q_2 in the function

$$
\int_{\tau_2+\frac{\tau_1}{2}}^{x} U^{(1,2)}\left(u-\frac{\tau_2}{2}\right)\mathrm{d}u
$$

belong to the interval $[\tau_1+\tau_2, \pi] \subset \left[\pi-\frac{\tau_1}{2}, \pi\right]$. So, we get the integral equation

$$
q_2(x) = g_4(x)x + 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x} q_2(u)\,\mathrm{d}u \tag{3.18}
$$

where

$$
\begin{aligned}
g_4(x) = & \frac{1}{2}\left(f_0\left(x-\frac{\tau_2}{2}\right) - f_1\left(x-\frac{\tau_2}{2}\right)\right) + 2h\int_{\tau_2}^{\tau_2+\frac{\tau_1}{2}} q_1(u)\,\mathrm{d}u \\
& + \int_{\tau_2+\frac{\tau_1}{2}}^{x}\left(U^{(1,2)}\left(u-\frac{\tau_2}{2}\right) + U^{(2,1)}\left(u-\frac{\tau_2}{2}\right)\right)\mathrm{d}u \\
& - \left(J_2^{(1,2)} + J_2^{(2,1)}\right)S^{(1,2)}\left(x-\frac{\tau_2}{2}\right) \\
& - 2h\int_{\tau_2+\frac{\tau_1}{2}}^{x}\left(K^{(1,2)*}\left(u-\frac{\tau_2}{2}\right) + K^{(2,1)*}\left(u-\frac{\tau_2}{2}\right)\right)\mathrm{d}u
\end{aligned}
$$

is a known function. Integral equation (3.18) has a unique solution on $\left(\tau_2+\frac{\tau_1}{2}, \pi-\frac{\tau_1}{2}\right]$. Thus, the theorem is proved.

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