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ORIGINAL ARTICLE

Coefficient functionals for alpha-convex functions associated with the exponential function

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Abstract
Let $\mathcal{A}$ be the class of all normalized analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. We give the sharp bound for the modulus of the functional $a_2 a_3 - a_4$, and the second Hankel determinant $H_{2,2}(f) = a_2 a_4 - a_3^2$ when $f \in \mathcal{M}_\alpha(\exp) \subset \mathcal{A}$, the class of α-convex functions $(0 \leq \alpha \leq 1)$, associated with the exponential function.

Keywords Univalent function · α-convex function · Starlike function · Exponential function · Hankel determinant · Zalcman functional

Mathematics Subject Classification 30C45 · 30C50 · 30C55

1 Introduction

Let $\mathcal{H}$ denote the class of all analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the subclass of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Denote by $\mathcal{S} \subset \mathcal{A}$ the subclass of univalent functions.

For $\alpha \in [0, 1]$, denote by $\mathcal{M}_\alpha \subset \mathcal{A}$, the so-called α-convex functions f satisfying

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad z \in \mathbb{D}.$$

The class $\mathcal{M}_\alpha$ was introduced by Mocanu [16] (see also [8, Vol. I, pp. 142–147]), who showed that $\mathcal{M}_\alpha \subset \mathcal{S}$.

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We note that when $\alpha = 0$ the class $\mathcal{M}_0$ reduces to the class of starlike functions denoted by $\mathcal{S}^*$, introduced by Alexander [1] ([17], see also [8, Vol. I, Chapter 8]), and when $\alpha = 1$ the class $\mathcal{M}_1$ reduces to the class of convex functions denoted by $\mathcal{S}^c$ defined by Study [24] (see also [8, Vol. I, Chapter 8]). In [15] it was shown that $\mathcal{M}_\alpha \subset \mathcal{M}_0$ for every $\alpha \in [0, 1]$, and so all functions in $\mathcal{M}_\alpha$ are starlike, which was observed by Sakaguchi [23] before the advent of the α-convexity concept (cf. [8, Vol. I. pp. 142-143]). Also in [15] Mocanu and Reade showed that $\mathcal{M}_{\alpha_1} \subset \mathcal{M}_{\alpha_2}$ for every $0 \le \alpha_2 \le \alpha_1 \le 1$, and Mocanu [16], showed that functions in $\mathcal{M}_\alpha$ have some interesting geometrical properties.

Thus the class $\mathcal{M}_\alpha$ creates a "continuous passage" on $\alpha \in [0, 1]$ from the family of starlike functions $\mathcal{S}^* = \mathcal{M}_0$ to the family of convex functions $\mathcal{M}_1 = \mathcal{S}^c$.

The class $\mathcal{M}_\alpha$ plays an important role in geometric function theory and has been studied by many authors (e.g., [20, 19, Chapter 7] for further references).

We say that a function $f \in \mathcal{H}$ is subordinate to a function $g \in \mathcal{H}$, if there exists a function $\omega \in \mathcal{H}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$ (called a Schwarz function), such that $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. We write $f \prec g$. If g is univalent and $f(0) = g(0)$, then $f \prec g$ is equivalent to $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Suppose that the function φ is analytic and univalent in $\mathbb{D}$ and is starlike with respect to the point $\varphi(0) = 1$ with $\varphi'(0) > 0$, and is symmetric about the real axis, then Ma and Minda [13] generalized the classes of starlike and convex functions as follows:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z),\ z \in \mathbb{D} \right\}$$

and

$$\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z),\ z \in \mathbb{D} \right\}.$$

Clearly, $\varphi(z) = \exp(z), z \in \mathbb{D}$, is a valid choice of the super-ordinate, which appears to have been first considered by Mendiratta et al. [14], and recently several authors have considered problems in the resulting classes of starlike and convex functions (see e.g. [25, 26], and the references therein).

Also Breaz et al. [2] have recently defined the following subclass of $\mathcal{M}_\alpha$.

Definition 1.1 A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_\alpha(\exp)$, $\alpha \in [0, 1]$, if f satisfies the following condition:

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \exp(z), \quad z \in \mathbb{D}. \tag{2}$$

In this paper we consider problems in the class $\mathcal{M}_\alpha(\exp)$, $\alpha \in [0, 1]$, of α-convex functions associated with the exponential function, noting that $\mathcal{S}^*(\exp) := \mathcal{M}_0(\exp)$ and $\mathcal{C}(\exp) := \mathcal{M}_1(\exp)$.

We also note that in [2], Breaz et al. gave non-sharp bounds for various coefficient functionals in $\mathcal{M}_\alpha$.

In recent years, there has been a great deal of attention given to finding bounds for the modulus of the second Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^3$, when f belongs to various subclasses of $\mathcal{A}$ (cf. [4] and [9] with further references).

In this paper, we find the sharp bound for $|H_{2,2}(f)|$ when $f \in \mathcal{M}_\alpha(\exp)$, $\alpha \in [0,1]$, together with the sharp bound of the functional

$$|J_{2,3}(f)| := |a_2a_3 - a_4|,$$

when $f \in \mathcal{M}_\alpha(\exp)$, $\alpha \in [0,1]$.

Note that $|J_{2,3}(f)|$ is a specific case of the generalized Zalcman functional $|a_na_m - a_{n+m+1}|$ investigated by Ma [12] for $f \in \mathcal{S}$ (cf. [21] for further references), and that sharp bounds for $|J_{2,3}(f)|$ for some specific general cases such as $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ have been found in [5].

2 Preliminary lemmas

Denote by $\mathcal{P}$, the class of analytic functions p in $\mathbb{D}$ with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_nz^n, \quad z \in \mathbb{D}. \tag{3}$$

Clearly if ω is a Schwarz function, then there exists $p \in \mathcal{P}$ such that

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1}, \quad z \in \mathbb{D}, \tag{4}$$

and vice versa, if $p \in \mathcal{P}$, then there exists a Schwarz function $\omega \in \mathcal{H}$ such that

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)}, \quad z \in \mathbb{D}.$$

In the proofs of our results, we will use the following lemma given in [6]. It contains the well known formulas (5) for c_1 [3] and (6) for c_2 (e.g., [18, p. 166]). The formula (7) for c_3 in the case when $\zeta_1 \in [0,1]$ is due to Libera and Złotkiewicz [10] and [11]. Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 2.1 *If $p \in \mathcal{P}$ and is given by* (3), *then*

$$c_1 = 2\zeta_1, \tag{5}$$

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \tag{6}$$

and

$$c_3 = 2\zeta_1^3 + 2(1 - |\zeta_1|^2)(2\zeta_1 - \overline{\zeta_1}\zeta_2)\zeta_2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3 \tag{7}$$

for some $\zeta_1, \zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (5), *namely,*

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}.$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (6) and (7), namely,

$$p(z) = \frac{1 + (\overline{\zeta}_1\zeta_2 + \zeta_1)z + \zeta_2 z^2}{1 + (\overline{\zeta}_1\zeta_2 - \zeta_1)z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{8}$$

We will also use the following lemma.

Lemma 2.2 [7] *For real numbers A, B, C, let*

$$Y(A,B,C) := \max\left\{|A + Bz + Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}}\right\}.$$

If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 - |C|)}, & |B| < 2(1 - |C|). \end{cases}$$

If $AC < 0$, then

$$Y(A,B,C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1 - |C|)}, & -4AC(C^{-2} - 1) \leq B^2 \wedge |B| < 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 + |C|)}, & B^2 < \min\left\{4(1 + |C|)^2, -4AC(C^{-2} - 1)\right\}, \\ R(A,B,C), & \text{otherwise}, \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A| + |B| - |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|C| + |A|)\sqrt{1 - \dfrac{B^2}{4AC}}, & \text{otherwise}. \end{cases}$$

3 The Zalcman functional

We first consider the Zalcman functional $|a_2 a_3 - a_4|$, noting that a non-sharp inequality was found in [2].

Theorem 3.1 *Let $\alpha \in [0,1]$. If $f \in \mathcal{M}_\alpha(\exp)$ and is given by (1), then*

$$
|a_2a_3 - a_4| \leq \begin{cases} \dfrac{2(\alpha+2)(4\alpha+1)J(\alpha)}{9(\alpha+1)(2\alpha+1)(3\alpha+1)(26\alpha^3+92\alpha^2+49\alpha+7)}, & \alpha \in [0,\alpha'], \\ \dfrac{1}{3(3\alpha+1)}, & \alpha \in (\alpha',1], \end{cases} \tag{9}
$$

where $J(\alpha) := \sqrt{2(26\alpha^3+92\alpha^2+49\alpha+7)(4\alpha+1)(\alpha+2)(\alpha+1)}$ and $\alpha' \approx 0.814445$ is the unique root in [0,1] of the equation

$$
424\alpha^6 + 1728\alpha^5 + 1014\alpha^4 - 1134\alpha^3 - 735\alpha^2 - 108\alpha - 1 = 0.
$$

Both inequalities are sharp.

Proof Fix $\alpha \in [0,1]$ and let $f \in \mathcal{M}_\alpha(\exp)$ be of the form (1). Then by (2), we can write

$$
(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \exp(\omega(z)), \quad z \in \mathbb{D}, \tag{10}
$$

where ω is a Schwarz function. Thus there exists $p \in \mathcal{P}$ given by (3) such that (4) is satisfied, and so (10) can be written as

$$
(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \exp\left(\frac{p(z)-1}{p(z)+1}\right), \quad z \in \mathbb{D}. \tag{11}
$$

Substituting (1) and (3) into (11) and equating the coefficients gives

$$
\begin{aligned}
a_2 &= \frac{c_1}{2(1+\alpha)}, \quad a_3 = \frac{c_2}{4(1+2\alpha)} + \frac{c_1^2(1+4\alpha-\alpha^2)}{16(1+2\alpha)(1+\alpha)^2}, \\
a_4 &= \frac{c_3}{6(1+3\alpha)} - \frac{c_1c_2(4\alpha^2-9\alpha-1)}{24(1+3\alpha)(1+2\alpha)(1+\alpha)} \\
&\quad + \frac{c_1^3(4\alpha^4-31\alpha^3+21\alpha^2-17\alpha-1)}{288(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.
\end{aligned} \tag{12}
$$

Since both the class $\mathcal{M}_\alpha(\exp)$ and the functional $\mathcal{M}_\alpha(\exp) \ni f \mapsto |a_2a_3 - a_4|$ are rotationally invariant, without loss of generality we may assume that $c_1 \in [0,2]$, i.e., by (5) that $\zeta_1 \in [0,1]$. Using Lemma 2.1 in (12) we then obtain

$$
|a_2a_3 - a_4| = \frac{1}{144(3\alpha+1)(2\alpha+1)(\alpha+1)^2}|\Psi|, \tag{13}
$$

where

$$\begin{aligned}\Psi &:= (2\alpha^3 - 4\alpha^2 - 35\alpha - 5)c_1^3 - 12(\alpha+1)(2\alpha^2+1)c_1c_2 \\ &\quad + 24(\alpha+1)^2(2\alpha+1)c_3 \\ &= 8\big[(2\alpha^3 + 14\alpha^2 - 17\alpha - 5)\zeta_1^3 + 6(\alpha+1)(2\alpha^2+6\alpha+1)(1-\zeta_1^2)\zeta_1\zeta_2 \\ &\quad - 6(\alpha+1)^2(2\alpha+1)(1-\zeta_1^2)\zeta_1\zeta_2^2 \\ &\quad + 6(\alpha+1)^2(2\alpha+1)(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\big]\end{aligned} \tag{14}$$

for some $\zeta_1, \zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

(A) Suppose first that $\zeta_1 = 1$. Note now that

$$2\alpha^3 + 14\alpha^2 - 17\alpha - 5 < 0, \quad \alpha \in [0,1], \tag{15}$$

and that from (13) and (14) we have

$$|a_2a_3 - a_4| = \frac{-2\alpha^3 - 14\alpha^2 + 17\alpha + 5}{18(3\alpha+1)(2\alpha+1)(\alpha+1)^2} =: a. \tag{16}$$

(B) Suppose next that $\zeta_1 \in [0,1)$. Using the fact that $|\zeta_3| \le 1$, we obtain from (14) that

$$|\Psi| \le 48(1-\zeta_1^2)(2\alpha+1)(\alpha+1)^2\Phi(A,B,C),$$

where

$$\Phi(A,B,C) := \left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2,$$

with

$$A := \frac{(2\alpha^3 + 14\alpha^2 - 17\alpha - 5)\zeta_1^3}{6(2\alpha+1)(\alpha+1)^2(1-\zeta_1^2)}, \quad B := \frac{(2\alpha^2+6\alpha+1)\zeta_1}{(2\alpha+1)(\alpha+1)}, \quad C := -\zeta_1.$$

Hence and from (15) it follows that $AC > 0$.

(B1) Consider first the condition $|B| \ge 2(1-|C|)$, i.e.,

$$\frac{(2\alpha^2+6\alpha+1)\zeta_1}{(2\alpha+1)(\alpha+1)} \ge 2(1-\zeta_1),$$

which is equivalent to

$$\frac{3\zeta_1(2\alpha^2+4\alpha+1) - 2(2\alpha+1)(\alpha+1)}{(2\alpha+1)(\alpha+1)} \ge 0$$

and is true when $\zeta_1 \ge \zeta'$, where

$$\zeta' := \frac{2(2\alpha+1)(\alpha+1)}{3(2\alpha^2+4\alpha+1)}.$$

Note that the inequality $\zeta' < 1$ is equivalent to $-2\alpha^2 - 6\alpha - 1 < 0$ which is true for

$\alpha \in [0, 1]$.

Assume now that $\zeta_1 \in [\zeta', 1)$. Then applying Lemma 2.2 we have

$$|\Psi| \leq 48(1-\zeta_1^2)(2\alpha+1)(\alpha+1)^2(|A|+|B|+|C|).$$

Hence, and by (13),

$$|a_2a_3 - a_4| = \frac{1}{144(3\alpha+1)(2\alpha+1)(\alpha+1)^2}|\Psi| \leq \gamma(\zeta_1), \tag{17}$$

where

$$\mathbb{R}\ni t \mapsto \gamma(t) := -\frac{t}{18(3\alpha+1)(2\alpha+1)(\alpha+1)^2} \left[(26\alpha^3+92\alpha^2+49\alpha+7)t^2 - 6(4\alpha+1)(\alpha+2)(\alpha+1)\right].$$

Since $\gamma'(t) = 0$ is equivalent to

$$(26\alpha^3+92\alpha^2+49\alpha+7)t^2 - 2(4\alpha+1)(\alpha+2)(\alpha+1) = 0,$$

it follows that γ has the unique positive critical point

$$t' := \frac{\sqrt{2(26\alpha^3+92\alpha^2+49\alpha+7)(4\alpha^3+13\alpha^2+11\alpha+2)}}{26\alpha^3+92\alpha^2+49\alpha+7}, \tag{18}$$

where the function γ has a local maximum with

$$\gamma(t') = \frac{2(\alpha+2)(4\alpha+1)\sqrt{2(26\alpha^3+92\alpha^2+49\alpha+7)(4\alpha+1)(\alpha+2)(\alpha+1)}}{9(\alpha+1)(2\alpha+1)(3\alpha+1)(26\alpha^3+92\alpha^2+49\alpha+7)}.$$

Note that $t' < 1$ for all $\alpha \in [0, 1]$, which is equivalent to

$$3(6\alpha^3+22\alpha^2+9\alpha+1)(26\alpha^3+92\alpha^2+49\alpha+7) > 0, \quad \alpha \in [0, 1].$$

Moreover $t' \geq \zeta'$ if, and only if,

$$-64\alpha^6 - 252\alpha^5 - 36\alpha^4 + 384\alpha^3 + 258\alpha^2 + 57\alpha + 4 \geq 0$$

which is true for all $\alpha \in [0, 1]$. Consequently, $\gamma(t) \leq \gamma(t')$ for $t \in [\zeta', 1)$, and in particular for $t := \zeta_1$, so we obtain $\gamma(\zeta_1) \leq \gamma(t')$. Hence, and by (17) we have

$$|a_2a_3 - a_4| \leq \gamma(t'). \tag{19}$$

(B2) Suppose now that $\zeta_1 \in [0, \zeta')$, then applying Lemma 2.2 we have

$$|\Psi| \leq 48(1-\zeta_1^2)(2\alpha+1)(\alpha+1)^2\left(1+|A|+\frac{B^2}{4(1-|C|)}\right),$$

and so by (13) we obtain

$$|a_2a_3 - a_4| = \frac{1}{144(3\alpha+1)(2\alpha+1)(\alpha+1)^2}|\Psi| \le \varrho(\zeta_1), \tag{20}$$

where

$$\mathbb{R} \ni t \mapsto \varrho(t) := \frac{1}{36(3\alpha+1)(2\alpha+1)^2(\alpha+1)^2}\left[(4\alpha^4+12\alpha^3+160\alpha^2+90\alpha+13)t^3 \right.$$
$$\left. -9(2\alpha^2+1)(2\alpha^2+4\alpha+1)t^2 + 12(\alpha+1)^2(2\alpha+1)^2\right].$$

Since $\varrho'(t) = 0$ is equivalent to

$$t\left[t(4\alpha^4+12\alpha^3+160\alpha^2+90\alpha+13) - 6(2\alpha^2+1)(2\alpha^2+4\alpha+1)\right] = 0,$$

it follows that ϱ has the unique positive critical point

$$t'' := \frac{6(2\alpha^2+1)(2\alpha^2+4\alpha+1)}{4\alpha^4+12\alpha^3+160\alpha^2+90\alpha+13},$$

which is a local minimum point. Observe now that $t'' < \zeta'$ if, and only if,

$$64\alpha^6 + 252\alpha^5 + 36\alpha^4 - 384\alpha^3 - 258\alpha^2 - 57\alpha - 4 < 0$$

which holds for all $\alpha \in [0,1]$. Therefore,

$$\varrho(t) \le \max\{\varrho(0), \varrho(\zeta')\}, \quad 0 < t < \zeta',$$

and in particular when $t = \zeta_1$ we have $\varrho(\zeta_1) \le \max\{\varrho(0), \varrho(\zeta')\}$, and hence by (20) we obtain

$$|a_2a_3 - a_4| \le \max\{\varrho(0), \varrho(\zeta')\}. \tag{21}$$

It is easy to check that $\gamma(\zeta') = \varrho(\zeta')$, so the function

$$[0,1] \ni t \mapsto \psi(t) := \begin{cases} \varrho(t), & t \in [0,\zeta'], \\ \gamma(t), & t \in [\zeta',1], \end{cases}$$

is continuous, has a local minimum at $t = t''$ and a local maximum at $t = t'$. Since $t'' < t'$ and $\psi(1) = \gamma(1) = a$, where a is defined by (16), it follows from (16), (19) and (21) that

$$|a_2a_3 - a_4| \le \max\{\psi(t) : t \in [0,1]\} = \max\{\varrho(0), \gamma(t')\}.$$

A simple calculation shows that

$$\begin{aligned}
&\gamma(t') - \varrho(0)\\
&= \frac{2(\alpha+2)(4\alpha+1)\sqrt{2(26\alpha^3+92\alpha^2+49\alpha+7)(4\alpha+1)(\alpha+2)(\alpha+1)}}{9(\alpha+1)(2\alpha+1)(3\alpha+1)(26\alpha^3+92\alpha^2+49\alpha+7)} - \frac{1}{3(3\alpha+1)}\\
&= \frac{\mu(\alpha)}{9(\alpha+1)(2\alpha+1)(3\alpha+1)(26\alpha^3+92\alpha^2+49\alpha+7)} \geq 0
\end{aligned}$$

if, and only if,

$$\begin{aligned}
\mu(\alpha) :=& -3(\alpha+1)(2\alpha+1)(26\alpha^3+92\alpha^2+49\alpha+7)\\
&+2(\alpha+2)(4\alpha+1)\sqrt{2(26\alpha^3+92\alpha^2+49\alpha+7)(4\alpha+1)(\alpha+2)(\alpha+1)} \geq 0,
\end{aligned}$$

or equivalently, if, and only if,

$$\begin{aligned}
&2(\alpha+2)(4\alpha+1)\sqrt{2(26\alpha^3+92\alpha^2+49\alpha+7)(4\alpha+1)(\alpha+2)(\alpha+1)}\\
&\geq 3(\alpha+1)(2\alpha+1)(26\alpha^3+92\alpha^2+49\alpha+7).
\end{aligned}$$

Squaring both sides of the above inequality gives

$$\begin{aligned}
(\alpha+1)(26\alpha^3+92\alpha^2+49\alpha+7)(424\alpha^6+1728\alpha^5+1014\alpha^4-1134\alpha^3\\
-735\alpha^2-108\alpha-1) \leq 0
\end{aligned}$$

which is true for $\alpha \in [0, \alpha']$, where $\alpha' \approx 0.814445$ is the unique root in [0, 1] of the equation

$$424\alpha^6+1728\alpha^5+1014\alpha^4-1134\alpha^3-735\alpha^2-108\alpha-1=0.$$

(C) It remains to show that both inequalities in Theorem 3.1 are sharp. If $\alpha \in (\alpha', 1]$, then the function f given by (10) with $\omega(z) := z^3$, $z \in \mathbb{D}$, for which $a_2 = 0, a_3 = 0$ and $a_4 = 1/3(1+3\alpha)$ is extremal for the second inequality in (9).

For the first inequality let $\alpha \in [0, \alpha']$, and set $\tau := t'$, where t' is defined by (18). Since $\tau < 1$, the function p given by (8) with $\zeta_1 = \tau$ and $\zeta_2 = -1$, i.e., the function

$$p(z) := \frac{1-z^2}{1-2\tau z+z^2} = 1+2\tau z+(4\tau^2-2)z^2+\cdots, \quad z \in \mathbb{D},$$

belongs to $\mathcal{P}$. Thus the function f given by (11), with p as above and

$$\begin{aligned}
a_2 &= \frac{\tau}{1+\alpha}, \quad a_3 = \frac{\tau^2(3\alpha^2+12\alpha+5)-2(1+\alpha)^2}{4(1+2\alpha)(1+\alpha)^2},\\
a_4 &= \frac{\tau((52\alpha^4+317\alpha^3+633\alpha^2+355\alpha+59)\tau^2-6(8\alpha^2+27\alpha+7)(1+\alpha)^2)}{36(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.
\end{aligned}$$

belongs to $\mathcal{M}_\alpha(\exp)$ and is extremal for the first inequality in (9), which completes the proof of the Theorem 3.1. □

For $\alpha = 0$, we deduce the following ([25, Corollary 2]).

Corollary 3.1 *If* $f \in \mathcal{M}_0(\exp)$ *and is given by* (1), *then*

$$|a_2a_3 - a_4| \leq \frac{8\sqrt{7}}{63}.$$

The inequality is sharp.

For $\alpha = 1$, we deduce the following [25, Corollary 5].

Corollary 3.2 *If* $f \in \mathcal{M}_1(\exp)$ *and is given by* (1), *then*

$$|a_2a_3 - a_4| \leq \frac{1}{12}.$$

The inequality is sharp.

4 The Hankel determinant $H_{2,2}(f)$

In this section, we find the sharp bound for the modulus of the second Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$ when $f \in \mathcal{M}_\alpha(\exp)$.

Theorem 4.1 *Let* $\alpha \in [0, 1]$. *If* $f \in \mathcal{M}_\alpha(\exp)$ *and is given by* (1), *then*

$$|H_{2,2}(f)| = |a_2a_4 - a_3^2| \leq \begin{cases} \dfrac{1}{4(2\alpha+1)^2}, & \alpha \in \left[0, (\sqrt{6}-1)/5\right], \\ \dfrac{34\alpha^3 + 82\alpha^2 + 27\alpha + 3}{(3\alpha+1)(173\alpha^4 + 546\alpha^3 + 440\alpha^2 + 126\alpha + 11)}, & \alpha \in \left((\sqrt{6}-1)/5, 1\right]. \end{cases} \tag{22}$$

Both inequalities are sharp.

Proof Fix $\alpha \in [0, 1]$ and let $f \in \mathcal{M}_\alpha(\exp)$ be of the form (1). Since both the class $\mathcal{M}_\alpha(\exp)$ and the functional $\mathcal{M}_\alpha(\exp) \ni f \mapsto H_{2,2}(f)$ are rotationally invariant, without loss of generality we may assume that $c_1 \in [0, 2]$, i.e., by (5) that $\zeta_1 \in [0, 1]$. From (12) applying Lemma 2.1 we obtain

$$|a_2a_4 - a_3^2| = \frac{1}{2304(3\alpha+1)(2\alpha+1)^2(\alpha+1)^3}|\Psi|, \tag{23}$$

where

$$\begin{aligned}\Psi &:= c_1^4(5\alpha^4-30\alpha^3-232\alpha^2-162\alpha-13)-144c_2^2(1+3\alpha)(1+\alpha)^3\\ &\quad -24c_1^2c_2(7\alpha^2-2\alpha+1)(1+\alpha)^2+192c_1c_3(1+\alpha)^2(1+2\alpha)^2\\ &= 16\big((5\alpha^4+42\alpha^3-88\alpha^2-90\alpha-13)\zeta_1^4\\ &\quad +12(7\alpha^2+10\alpha+1)(\alpha+1)^2(1-\zeta_1^2)\zeta_1^2\zeta_2\\ &\quad -12(\alpha+1)^2(1-\zeta_1^2)((7\alpha^2+4\alpha+1)\zeta_1^2+3(1+\alpha)(1+3\alpha))\zeta_2^2\\ &\quad +48(\alpha+1)^2(2\alpha+1)^2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\zeta_3\big)\end{aligned} \tag{24}$$

for some $\zeta_1, \zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

(A) Suppose first that $\zeta_1 = 1$. Since

$$\frac{-5\alpha^4-42\alpha^3+88\alpha^2+90\alpha+13}{144(3\alpha+1)(2\alpha+1)^2(\alpha+1)^3} > 0, \quad \alpha \in [0,1], \tag{25}$$

from (23) and (24) we have

$$|a_2a_4 - a_3^2| = \frac{-5\alpha^4-42\alpha^3+88\alpha^2+90\alpha+13}{144(3\alpha+1)(2\alpha+1)^2(\alpha+1)^3}.$$

(B) Now suppose that $\zeta_1 \in [0,1)$. Noting from (24) that $|\zeta_3| \le 1$, we obtain

$$|\Psi| \le 768\zeta_1(1-\zeta_1^2)(2\alpha+1)^2(\alpha+1)^2\Phi(A,B,C),$$

where

$$\Phi(A,B,C) := \left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2,$$

with

$$A := \frac{(5\alpha^4+42\alpha^3-88\alpha^2-90\alpha-13)\zeta_1^3}{48(2\alpha+1)^2(\alpha+1)^2(1-\zeta_1^2)}, \quad B := \frac{(7\alpha^2+10\alpha+1)\zeta_1}{4(2\alpha+1)^2},$$
$$C := -\frac{(7\alpha^2+4\alpha+1)\zeta_1^2+9\alpha^2+12\alpha+3}{4(2\alpha+1)^2\zeta_1}.$$

A simple calculation using (25) shows that $AC > 0$.

(B1) Thus, we first consider the condition $|B| \ge 2(1-|C|)$, i.e.,

$$\frac{(7\alpha^2+10\alpha+1)\zeta_1}{4(2\alpha+1)^2} \ge 2\left(1 - \frac{(7\alpha^2+4\alpha+1)\zeta_1^2+9\alpha^2+12\alpha+3}{4(2\alpha+1)^2\zeta_1}\right),$$

which can be equivalently written as

$$\frac{3(7\alpha^2+6\alpha+1)\zeta_1^2-8(2\alpha+1)^2\zeta_1+6(3\alpha+1)(\alpha+1)}{4(2\alpha+1)^2\zeta_1}>0,$$

which is true for all $\alpha \in [0,1]$ and $\zeta_1 \in [0,1)$. Thus, applying Lemma 2.2 we have

$$|\Psi| \leq 768\zeta_1(1-\zeta_1^2)(2\alpha+1)^2(\alpha+1)^2(|A|+|B|+|C|).$$

Hence and by (23)

$$|a_2a_4-a_3^3| = \frac{1}{2304(3\alpha+1)(2\alpha+1)^2(\alpha+1)^3}|\Psi| \leq \gamma(\zeta_1),$$

where

$$\begin{aligned}\mathbb{R}\ni t\mapsto\gamma(t) :=& \frac{1}{144(3\alpha+1)(2\alpha+1)^2(\alpha+1)^3}\\ &\times\left[-(173\alpha^4+546\alpha^3+440\alpha^2+126\alpha+11)t^4\right.\\ &\left.+12(5\alpha^2+2\alpha-1)(\alpha+1)^2t^2+36(3\alpha+1)(\alpha+1)^3\right].\end{aligned}$$

Since $\gamma'(t)=0$ is equivalent to

$$\left[(173\alpha^4+546\alpha^3+440\alpha^2+126\alpha+11)t^2-6(5\alpha^2+2\alpha-1)(\alpha+1)^2\right]t=0,$$

it follows that for $(\sqrt{6}-1)/5<\alpha\leq 1$ the function γ has the unique positive critical point

$$t' := \frac{(\alpha+1)\sqrt{6(173\alpha^4+546\alpha^3+440\alpha^2+126\alpha+11)(5\alpha^2+2\alpha-1)}}{173\alpha^4+546\alpha^3+440\alpha^2+126\alpha+11}, \qquad (26)$$

where the function γ has a local maximum with

$$\gamma(t') = \frac{34\alpha^3+82\alpha^2+27\alpha+3}{(3\alpha+1)(173\alpha^4+546\alpha^3+440\alpha^2+126\alpha+11)}.$$

Note that $t'<1$, since this is equivalent to

$$(143\alpha^4+474\alpha^3+392\alpha^2+126\alpha+17)(173\alpha^4+546\alpha^3+440\alpha^2+126\alpha+11)>0.$$

For $0\leq\alpha\leq(\sqrt{6}-1)/5$ we have

$$\gamma(t)\leq\max\{\gamma(0),\gamma(1)\}=\gamma(0)=\frac{1}{4(2\alpha+1)^2}, \quad t\in[0,1],$$

since

$$\gamma(0)-\gamma(1)=\frac{1}{4(2\alpha+1)^2}-\frac{-5\alpha^4-42\alpha^3+88\alpha^2+90\alpha+13}{144(3\alpha+1)(2\alpha+1)^2(\alpha+1)^3}$$
$$=\frac{113\alpha^4+402\alpha^3+344\alpha^2+126\alpha+23}{144(3\alpha+1)(2\alpha+1)^2(\alpha+1)^3}>0,\quad \alpha\in[0,1].$$

(C) It remains to show that the inequalities in Theorem 4.1 are sharp. If $\alpha\in[0,(\sqrt{6}-1)/5]$, then the function f given by (10) with $\omega(z):=z^2$, $z\in\mathbb{D}$, for which $a_2=0$, $a_3=1/(2(1+2\alpha))$ and $a_4=0$ is extremal for the first inequality in (22).

For the second inequality, let $\alpha\in((\sqrt{6}-1)/5,1]$, and set $\tau:=t'$, where t' is given by (26). Since $\tau<1$, the function p given by (8) with $\zeta_1=\tau$ and $\zeta_2=-1$, i.e., the function

$$p(z):=\frac{1-z^2}{1-2\tau z+z^2}=1+2\tau z+(4\tau^2-2)z^2+\cdots,\quad z\in\mathbb{D},$$

belongs to $\mathcal{P}$. Thus the function f given by (11) has the form (1) with

$$a_2=\frac{\tau}{1+\alpha},\quad a_3=\frac{\tau^2(3\alpha^2+12\alpha+5)-2(1+\alpha)^2}{4(1+2\alpha)(1+\alpha)^2},$$
$$a_4=\frac{\tau((52\alpha^4+317\alpha^3+633\alpha^2+355\alpha+59)\tau^2-6(8\alpha^2+27\alpha+7)(1+\alpha)^2))}{36(1+\alpha)^3(1+2\alpha)(1+3\alpha)},$$

which gives equality in (22). □

When $\alpha=0$, we deduce the following [25, Corollary 3].

Corollary 4.1 *If* $f\in\mathcal{S}^*(\exp)$, *then*

$$|H_{2,2}(f)|\le\frac{1}{4}.$$

The inequality is sharp.

When $\alpha=1$, we deduce the following ([25, Corollary 6]).

Corollary 4.2 *If* $f\in\mathcal{C}(\exp)$, *then*

$$|H_{2,2}(f)|\le\frac{73}{2592}.$$

The inequality is sharp.

Remark 4.1 We end by noting that in [22] it was recently shown that for the third Hankel determinant

$$H_{3,1}(f)=2a_2a_3a_4-a_3^3-a_4^2+a_5(a_3-a_2^2),$$

when $f\in\mathcal{S}^*(\exp)$, the sharp bound is $|H_{3,1}(f)|\le 1/9$, and when $f\in\mathcal{C}(\exp)$, the sharp bound is $|H_{3,1}(f)|\le 1/144$.

Clearly finding the sharp bound for $|H_{3,1}(f)|$ when $f \in \mathcal{M}_\alpha(\exp)$ presents a significantly difficult problem.

Declarations

Conflict of interest Not applicable.

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