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ORIGINAL ARTICLE

Fixed point theorems for (χ,F)-Dass–Gupta contraction mappings in b-metric spaces with applications to integral equations

Ouanassa Zahi[1,2] · Hichem Ramoul[3]

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Abstract

In this paper, we introduce new types of (χ, F)-contraction mappings by involving rational expressions and establish two new fixed point theorems for this class of mappings in the setting of b-metric spaces. Furthermore, our results allow us to deduce, extend and improve some previous works in the existing literature. Along with these, some illustrative examples are also constructed in the support of our obtained fixed point theorems. As applications of our results, we investigate sufficient criteria for the existence and uniqueness of solution for certain types of nonlinear integral equations.

Keywords b-metric space · F-contraction · Fixed point · Integral equations

Mathematics Subject Classification 47H09 · 47H10 · 45B05 · 45D05

Ouanassa Zahi and Hichem Ramoul have contributed equally to this work.

✉ Hichem Ramoul
ramoul.h@gmail.com

Ouanassa Zahi
zahiw15@gmail.com

1 Department of Mathematics and Informatics, Abbes Laghrour University-Khenchela, P.O. Box 1252, Khenchela 40004, Algeria

2 Department of Mathematics, University of Batna 2, 53, Route de Constantine-Fésdis, Batna 05078, Algeria

3 ICOSI Laboratory, Department of Mathematics and Informatics, Abbes Laghrour University-Khenchela, P.O. Box 1252, Khenchela 40004, Algeria

1 Introduction

The concept of b-metric space was introduced by Bakhtin [4] (see also Czerwik [11]) as a generalization of the usual metric by providing an axiom which is weaker than the triangle inequality. Later on, b-metric theory has undergone a great development and therefore a lot of fixed point results in the setting of b-metric spaces have been investigated by many authors (see, e.g., [6–9, 13, 15, 16, 20, 24, 26, 28, 33, 34, 39].

One of the noteworthy generalizations of the celebrate *Banach contraction principle* [5] is due to Wardowski [42]. He introduced a new contraction based on an auxiliary function F fulfilling certain conditions, called *F-contraction* and proved a new fixed point theorem. Subsequently, several interesting modifications and extensions dealing with the original result of Wardowski have been elaborated in various ways by many mathematicians. For an exhaustive review concerning F-contractions, the reader may consult [23] and references therein.

Contractive conditions involving rational forms were firstly initiated by Dass and Gupta [14], Jaggi [22] and Khan [25] (revised by Fisher [18]). Over the last few years, many interesting variants for rational contractions have been provided in various frameworks, see, for example, [2, 19, 30–32] and references therein.

In this paper, we initiate two new types of contractions, called in this order (χ, F) -Dass–Gupta-contraction of type (A) and (χ, F) -Dass–Gupta-contraction of type (B). These contraction mappings involving both (χ, F)-contraction of Wardowski [43] and the rational expression appeared in [14] are therefore used to establish two new fixed point theorems in the setting of b-metric spaces. Through the first one, we did a lot of improvements in Theorem 3.2 of Lukács and Kajántó [29] and we have generalized Theorem 4.3 in [31] in the context of b-metric spaces. In addition, we have obtained the Dass–Gupta fixed point theorem in a complete b-metric space which led to improve greatly Corollary 3.7 in [34]. Our second fixed point theorem is based on (χ, F) -Dass–Gupta-contraction of type (B) and it is proved with less conditions imposed on the function F. Actually, this latter theorem can allow us to derive many fixed point results of Dass–Gupta type mappings. Besides, examples are provided to justify the validity of the presented results. As applications, we utilize our obtained results to study the existence of the unique solution for nonlinear Fredholm and Volterra integral equations. At the end of this paper, we pose two questions should be of interest to readers as well as ourselves in the near future.

2 Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}$ denote the set of all positive integers, the set of all natural numbers and the set of all real numbers, respectively.

In this section, we collect some known theoretic results and prerequisites which will be needed in the sequel. In the rest of the paper, unless otherwise stated, X stands for a nonempty set and the Picard sequence of a mapping $T : X \to X$ based on an arbitrary $x_0 \in X$ is defined by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$, where T^n means the n-fold composition of T with itself.

For the sake of completeness, we should first recall the Dass–Gupta fixed point theorem in connection with our main results.

Theorem 1 ([14]: *Dass–Gupta fixed point in metric spaces*) *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose that there exist* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta < 1$ *such that*

$$d(Tx, Ty) \leq \alpha(x, y) + \beta \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)}$$

for all $x, y \in X$. Then T has a unique fixed point $x^* \in X$ and for any $x \in X$ the sequence $\{T^n x\}$ converges to this fixed point.

2.1 Background and some recent fixed point results on *b*-metric spaces

We first recall the definition of a b-metric space, as follows.

Definition 1 (See [4] and [12]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $\sigma : X \times X \to [0, \infty)$ is said to be a b-metric if, for all $x, y, z \in X$, the following conditions hold: (b_1) $\sigma(x, y) = 0$ if and only if $x = y$;(b_2) $\sigma(x, y) = \sigma(y, x)$;$(b_3)$ $\sigma(x, z) \leq s[\sigma(x, y) + \sigma(y, z)]$.The pair (X, σ) is called a b-metric space with coefficient $s \geq 1$.

Definition 1 allows us to remark that every metric space is a b-metric space with coefficient $s = 1$, but the converse does not hold (see [1, 4, 16]). In other words, the classical metric spaces are properly included in the class of b-metric spaces.

In what follows, we recall the following interesting examples.

Example 1 (See [39]) Let (X, d) be a metric space and let the mapping $\sigma_d : X \times X \to [0, \infty)$ be defined by

$$\sigma_d(x, y) = (d(x, y))^p \quad \text{for all } x, y \in X,$$

where $p > 1$ is a fixed real number. Then (X, σ_d) is a b-metric space with coefficient $s = 2^{p-1}$.

Example 2 Let $0 < p < 1$ and let

$$L^p([0, 1]) = \left\{ u : [0, 1] \to \mathbb{R} : \int_0^1 |u(t)|^p \mathrm{d}t < \infty \right\},$$

together with the functional $\sigma : L^p([0, 1]) \times L^p([0, 1]) \to [0, \infty)$ given by

$$\sigma(u, v) = \left(\int_0^1 |u(t) - v(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} \text{ for all } u, v \in L^p([0, 1]),$$

is a b-metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

Example 3 The space $l_p(\mathbb{R})$ with $0<p<1$, where

$$l_p(\mathbb{R}) = \left\{ (x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with function $\sigma : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to [0,\infty)$ defined by

$$\sigma(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \text{ for all } x = (x_n), y = (y_n) \in l_p(\mathbb{R}),$$

is a b-metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

Remark 1 It is worth noting that the two last examples are also given in [6] and many others (see for instance [8, 9]), but the coefficient of b-metric mentioned therein is $s = 2^{\frac{1}{p}}$. After an elementary calculation, we have realized that the suitable coefficient is $s = 2^{\frac{1}{p}-1}$ (see also [28 Example 1.1]). For the sake of readability, we give the basic inequalities have allowed us to obtain the desired coefficient.

First, we use

$$(a+b)^p < a^p + b^p$$

for any $a, b > 0$ and $0<p<1$.

Next, we apply the following inequality (obtained by the convexity of the function $t \mapsto t^r$, $t \in \mathbb{R}$ and $r > 1$)

$$(a+b)^r < 2^{r-1}(a^r + b^r)$$

for any $a, b > 0$ with $r = \dfrac{1}{p}$ and $0<p<1$.

Hence, we can prove that inequality (b_3) for the two aforementioned examples is satisfied with coefficient $s = 2^{\frac{1}{p}-1}$.

Now, we sum up the notions of convergence, Cauchy sequence and completeness in the setting of b-metric spaces.

Definition 2 (See [7–9]) Let (X,σ) be a b-metric space with coefficient $s \geq 1$. Then a sequence $\{x_n\}$ in X is called: (a) convergent if and only if there exists $x \in X$ such that $\lim_{n\to\infty} \sigma(x_n, x) = 0$ and in this case we write $\lim_{n\to\infty} x_n = x$; (b) Cauchy if and only if $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$.

Definition 3 (See [7–9]) The b-metric space (X,σ) is said complete if every Cauchy sequence in X converges in X.

What follows lemma will be required for the proof of one of our results.

Lemma 1 ([40, *Lemma* 11]) *Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in X. Assume that*

$$\{d(x_n, x_{n+1})\} \in \bigcup \{O(n^{-\gamma}) : \gamma > 1 + \log_2 s\}.$$

Then $\{x_n\}$ is Cauchy.

It is well known from [3, Example 3.10] that a b-metric fails to be continuous in general. The following example illustrates also this fact.

Example 4 (See [16 Example 2.13]) Let $X = [0, \infty)$. Let $\sigma : X \times X \to [0, \infty)$ be a mapping defined by

$$\sigma(x, y) = \begin{cases} d(x, y), & xy \neq 0, \\ 4d(x, y), & xy = 0, \end{cases}$$

where $d(x, y) = |x - y|$. Then the following hold:

(1) (X, σ) is a complete b-metric space with coefficient $s = 4$;
(2) σ is not a metric on X;
(3) σ is not continuous in each variable.

The following lemma plays a crucial role to overcome the absence of the continuity of a b-metric.

Lemma 2 (*See* [33, *Lemma* 1.7]) *Let* (X, σ) *be a b-metric space with coefficient* $s \geq 1$ *and let* $\{x_n\}$ *be a sequence in X such that*

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence in (X, σ), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following items hold:

$$\varepsilon \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon;$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\varepsilon;$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\varepsilon;$$
$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\varepsilon.$$

In what follows, we recall two famous fixed point results in the framework of b-metric spaces.

Theorem 2 (*See* [17, *Theorem* 2.1]: *Banach fixed point in b-metric spaces*). *Let* (X, σ) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $T : X \to X$ *be a mapping such that for all* $x, y \in X$ *and some* $k \in [0, 1)$,

$$\sigma(Tx,Ty)\leq k\sigma(x,y).$$

Then T has a unique fixed point $x^{*}\in X$ and the sequence $\{T^{n}x\}$ converges to this fixed point for all $x\in X$.

In the paper [34], Samet extended the Dass–Gupta fixed point theorem (see Theorem 1) to b-metric spaces as follows:

Theorem 3 ([34, *Corollary* 3.7]) *Let* (X,σ) *be a complete b-metric space with coefficient* $s\geq 1$ *and* $T:X\to X$ *be a mapping. Assume that there exist* $\alpha,\beta\in[0,1)$ *with* $\alpha s+\beta<1$ *such that*

$$\sigma(Tx,Ty)\leq\alpha\sigma(x,y)+\beta\frac{\sigma(y,Ty)(1+\sigma(x,Tx))}{1+\sigma(x,y)}$$

for all $x,y\in X$. Then T has a unique fixed point $x\in X$ and the sequence $\{T^{n}x\}$ converges to this fixed point.

2.2 *F*-contractions and some related fixed point results

In 2012, Wardowski [42] defined the so-called F-contraction as follows:

Definition 4 Let (X,d) be a metric space. A mapping $T:X\to X$ is called an F-contraction if there exist $F\in\mathcal{F}$ and $\tau>0$ such that for all $x,y\in X$,

$$d(Tx,Ty)>0\Rightarrow\tau+F(d(Tx,Ty))\leq F(d(x,y)), \tag{1}$$

where $\mathcal{F}$ is the family of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the following conditions:(F_1) F is strictly increasing;(F_2) For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:

$$\lim_{n\to\infty}\alpha_n=0 \; ifandonlyif \; \lim_{n\to\infty}F(\alpha_n)=-\infty;$$

(F_3) There exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.

Example 5 (See [42]) Let $\alpha\in(0,\infty)$. The following functions $F_1(\alpha)=\ln\alpha$, $F_2(\alpha)=\ln\alpha+\alpha$, $F_3(\alpha)=\dfrac{-1}{\sqrt{\alpha}}$ and $F_4(\alpha)=\ln(\alpha^2+\alpha)$ belong to the family $\mathcal{F}$.

Remark 2 Taking $F(\alpha)=\ln\alpha$ in (1), one can get a Banach contraction (see [42, example 2.1]).

Wardowski established the following result.

Theorem 4 ([42, *Theorem* 2.1]) *Let* (X,d) *be a complete metric space and let* $T:X\to X$ *be an F-contraction. Then T has a unique fixed point* x^{*} *and for every* $x_0\in X$ *the sequence* $\{T^{n}x_0\}$ *converges to* x^{*}.

In 2018, Wardowski [43] fine-tuned the class of contractions $\mathcal{F}$ by introducing the concept of (χ,F)-contraction on a metric space. The author substitute a function χ for the positive constant τ and relaxed some assumptions on the mapping F.

Definition 5 (See [43]) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a (χ, F)-contraction if there exist two functions $F : (0, \infty) \to \mathbb{R}$ and $\chi : (0, \infty) \to (0, \infty)$ satisfying the following conditions:

1. F satisfies (F_1);
2. (F_2'): $\lim_{t\to 0^+} F(t) = -\infty$;
3. (H_0): $\liminf_{t\to\varepsilon^+} \chi(t) > 0$ for all $\varepsilon \geq 0$;
4. $\chi(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y))$ for all $x, y \in X$ with $Tx \neq Ty$.

Additionally, Wardowski [43] proved the following theorem.

Theorem 5 ([43, *Theorem* 2.1]) *On a complete metric space* (X, d), *every* (χ, F)-*contraction mapping has a unique fixed point.*

Very recently, Vujaković et al. [41] improved Theorem 5 by using only the first condition (F_1).

Remark 3 Notice that condition (F_2') is weaker than condition (F_2). However, the next lemma allows us to drop definitively condition (F_2') (see also Lemma 2.4 in [29]).

Lemma 3 ([35, *Lemma* 3.2]) *Let* $F : (0, \infty) \to \mathbb{R}$ *be a nondecreasing function and* $\{t_n\}$ *a sequence of positive real numbers such that* $\lim_{n\to\infty} F(t_n) = -\infty$. *Then* $\lim_{n\to\infty} t_n = 0$.

Example 6 Let $F_1, F_2 : (0, \infty) \to \mathbb{R}$ be given by : $F_1(t) = \ln(t + 1)$ and $F_2(t) = -\dfrac{1}{t+1}$ for all $t \in (0, \infty)$. It is clear that F_1 and F_2 belong to the class $\mathcal{F}$ but do not satisfy condition (F_2').

Recently, Lukács and Kajántó [29] introduced the following class of functions to work in b-metric spaces.

We denote by $\mathbb{F}^*$ the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying both conditions (F_1) and (F_3).

Definition 6 (See [29, Definition 2.7]) Let $s \geq 1$ and $\tau > 0$. We say that $F \in \mathbb{F}^*$ belongs to $\mathcal{F}_{s,\tau}$ if it is also satisfies

$(\mathrm{F}_{s,\tau})$ if $\inf F = -\infty$ and $x, y, z \in (0, \infty)$ are such that $\tau + F(sx) \leq F(y)$ and $\tau + F(sy) \leq F(z)$ then

$$\tau + F\left(s^2 x\right) \leq F(sy).$$

Remark 4 In [29, Proposition 2.8], Lukács and Kajántó claimed that if F is a nondecreasing function, then $(\mathrm{F}_{s,\tau})$ is equivalent to the following one (denoted by (F_4) in [10, Definition 3.1]):

(F_4) Let $s \geq 1$. If $\{\alpha_n\} \subset (0, \infty)$ is a sequence such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$ for all $n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^n \alpha_n) \leq F(s^{n-1}\alpha_{n-1})$ for all $n \in \mathbb{N}$.

In addition, the authors in [29] proved the following result in the setting of b-metric spaces.

Theorem 6 ([29, *Theorem* 3.2]) *Let* (X, σ) *be a complete* b*-metric space with coefficient* $s \geq 1$ *and let* $T : X \to X$ *be a mapping. If there exist* $\tau > 0$ *and* $F \in \mathcal{F}_{s,\tau}$ *such that for all* $x, y \in X$ *the inequality* $\sigma(Tx, Ty) > 0$ *implies*

$$(F) \qquad \tau + F(s\sigma(Tx, Ty)) \leq F(\sigma(x, y)),$$

then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

3 Main results

Henceforth, we denote by $\mathcal{L}$ the family of all functions $\chi : (0, \infty) \to (0, \infty)$ which satisfy the following condition:

$$\liminf_{t \to \eta^+} \chi(t) > 0 \quad \text{for all } \eta > 0. \qquad (H)$$

Example 7 (See [16, Example 3.3] and [37, Example 2.2])

(a) Let $\chi > 0$ be a fixed real number and $\chi_1(t) = \chi$ for all $t \in (0, \infty)$. Then $\chi_1 \in \mathcal{L}$.

(b) Let $\chi_2(t) = \delta t$ for all $t \in (0, \infty)$, where $\delta > 0$. Then $\chi_2 \in \mathcal{L}$.

(c) Let $\chi_3(t) = e^t$ for all $t \in (0, \infty)$. Then $\chi_3 \in \mathcal{L}$.

Remark 5 It is easy to see that condition (H) is slightly weaker than condition (H_0) given in Definition 5. For instance, we can observe that χ^2 does not satisfy condition (H_0) since $\liminf_{t \to 0^+} \chi_2(t) = 0$.

Definition 7 Let (X, σ) be a b-metric space with coefficient $s \geq 1$. The mapping $T : X \to X$ is said to be a (χ, F) -Dass–Gupta-contraction of type (A) if there exist two functions $F : (0, \infty) \to \mathbb{R}$ and $\chi : (0, \infty) \to (0, \infty)$ such that for all $x, y \in X$ with $\sigma(Tx, Ty) > 0$, the following condition is satisfied:

$$\chi(\sigma(x, y)) + F(\sigma(Tx, Ty)) \leq F(m(x, y)), \tag{2}$$

where

$$m(x, y) = \max\left\{\sigma(x, y), \frac{\sigma(y, Ty)(1 + \sigma(x, Tx))}{1 + \sigma(x, y)}\right\}.$$

Remark 6 Clearly, if T is a (χ, F)-Dass–Gupta-contraction of type (A) with F is a nondecreasing function, then we get

$$\sigma(Tx, Ty) < m(x, y) \tag{3}$$

for all $x, y \in X$ with $Tx \neq Ty$.

Our first fixed point result is the following.

Theorem 7 *Let* (X, σ) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $T : X \to X$ *be a* (χ, F) *-Dass–Gupta-contraction of type* (A)*. Assume that*:

(H_1) *F is nondecreasing;*

(H_2) $\chi \in \mathcal{L}$*;*

(H_3) *there exists* $k \in \left(0, \dfrac{1}{1+\log_2 s}\right)$ *such that* $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0.$

Then T has a unique fixed point x^* *and for every* $x_0 \in X$ *the sequence* $\{T^n x_0\}$ *converges to* x^**.*

Proof Let $x_0 \in X$ be an arbitrary point and $\{x_n\}$ be the Picard sequence based on x_0. If there exists $n_0 \in \mathbb{N}_0$, such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is the fixed point of T. If $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$, we get

$$\sigma_n := \sigma(x_n, x_{n+1}) = \sigma(Tx_{n-1}, Tx_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Then we can apply the contractive inequality (2) with $x = x_{n-1}$ and $y = x_n$. Hence, we obtain

$$\begin{aligned} &\chi(\sigma(x_{n-1}, x_n)) + F(\sigma(x_n, x_{n+1})) \\ \leq F&\left(\max\left\{\sigma(x_{n-1}, x_n), \frac{\sigma(x_n, x_{n+1})(1+\sigma(x_{n-1}, x_n))}{1+\sigma(x_{n-1}, x_n)}\right\}\right) \\ &= F(\max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}) \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$.

Equivalently, (4) takes the form

$$\chi(\sigma_{n-1}) + F(\sigma_n) \leq F(\max\{\sigma_{n-1}, \sigma_n\}). \tag{5}$$

If there exists $m \in \mathbb{N}$ such that

$$\max\{\sigma_{m-1,}\sigma_m\} = \sigma_m.$$

Then from (5), we deduce that

$$F(\sigma_m) < \chi(\sigma_{m-1}) + F(\sigma_m) \leq F(\sigma_m),$$

a contradiction. Hence, for all $n \in \mathbb{N}$,

$$\max\{\sigma_{n-1}, \sigma_n\} = \sigma_{n-1}. \tag{6}$$

In view of (5), (6) and the monotonicity of F, we get

$$\sigma_n < \sigma_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

The last inequality implies that $\{\sigma_n\}$ is a strictly decreasing sequence of positive numbers. Therefore, there exists $\sigma \geq 0$ such that

$$\lim_{n\to\infty} \sigma_n = \sigma^+. \tag{7}$$

First, we prove that $\sigma = 0$. We argue by contradiction, i.e., we suppose that $\sigma > 0$.

On the other hand, via (5) and (6), we obtain the following chain of inequalities

$$\begin{aligned} F(\sigma_n) &\le F(\sigma_{n-1}) - \chi(\sigma_{n-1}) \\ &\le F(\sigma_{n-2}) - \chi(\sigma_{n-2}) - \chi(\sigma_{n-1}) \\ &\le \cdots \\ &\le F(\sigma_0) - \sum_{i=0}^{n-1} \chi(\sigma_i). \end{aligned} \tag{8}$$

By virtue of assumption (H_2) and (7), there exist $n_1 \in \mathbb{N}$ and $\mu > 0$ such that

$$\chi(\sigma_n) \ge \mu \quad \text{for all} \quad n \ge n_1.$$

Consequently, inequality (8) can be written in the following form

$$\begin{aligned} F(\sigma_n) &\le F(\sigma_0) - \sum_{i=0}^{n_1-1} \chi(\sigma_i) - \sum_{i=n_1}^{n-1} \chi(\sigma_i) \\ &\le F(\sigma_0) - \sum_{i=n_1}^{n-1} \chi(\sigma_i) \\ &\le F(\sigma_0) - \sum_{i=n_1}^{n-1} \mu \\ &= F(\sigma_0) - (n - n_1)\mu \end{aligned} \tag{9}$$

for all $n \ge n_1$.

Using (7), (9) and the monotonicity of F, we obtain

$$F(\sigma) \le F(\sigma_n) \le F(\sigma_0) - (n - n_1)\mu \quad \text{for all } n \ge n_1. \tag{10}$$

Letting $n \to \infty$ in (10), one gets

$$F(\sigma) \le -\infty,$$

a contradiction. Thus $\sigma = 0$, that is,

$$\lim_{n\to\infty} \sigma_n = 0^+. \tag{11}$$

Next, by (H_3) and (11), there exists $k \in \left(0, \frac{1}{1+\log_2 s}\right)$ such that

$$\lim_{n\to\infty} \sigma_n^k F(\sigma_n) = 0. \tag{12}$$

By (9), we obtain

$$0 \le \sigma_n^k (n - n_1)\mu \le \sigma_n^k F(\sigma_0) - \sigma_n^k F(\sigma_n) \quad \text{for all } n \ge n_1. \tag{13}$$

Owing to (11), (12) and (13), we get

$$\lim_{n\to\infty} \sigma_n^k (n - n_1)\mu = 0,$$

which implies

$$\lim_{n\to\infty} n\sigma_n^k = 0.$$

Therefore, there exists $n_2 \in \mathbb{N}$ such that

$$\sigma_n \le n^{-\frac{1}{k}} \quad \text{for all } n \ge n_2.$$

Hence,

$$\{\sigma_n\} = \{\sigma(x_n, x_{n+1})\} \in O(n^{-\frac{1}{k}}). \tag{14}$$

Since $\frac{1}{k} > 1 + \log_2 s$ in (14), Lemma 1 allows us to deduce that $\{x_n\}$ is a Cauchy sequence. By the completeness of (X, σ), $\{x_n\}$ converges to some x^* in X, that is,

$$\lim_{n\to\infty} \sigma(x_n, x^*) = 0. \tag{15}$$

Now, we will prove that x^* is a fixed point of T, that is, $Tx^* = x^*$. Arguing by contradiction, i.e., $\sigma(x^*, Tx^*) > 0$. Then, in view of (15), there exists $n_3 \in \mathbb{N}$ such that

$$\sigma(x_n, x^*) \le \frac{\sigma(x^*, Tx^*)}{2s}, \quad \forall n \ge n_3. \tag{16}$$

On the other hand, using (b_3), we get

$$\sigma(x^*, Tx^*) \le s\sigma(x^*, Tx_n) + s\sigma(Tx_n, Tx^*). \tag{17}$$

Utilizing (16), the inequality (17) gives

$$\begin{aligned} \sigma(Tx_n, Tx^*) &\ge \frac{1}{s}(\sigma(x^*, Tx^*) - s\sigma(x^*, Tx_n)) \\ &= \frac{1}{s}\sigma(x^*, Tx^*) - \sigma(x^*, x_{n+1}) \\ &\ge \frac{\sigma(x^*, Tx^*)}{2s} > 0, \end{aligned} \tag{18}$$

for all $n \ge n_3$.

By (18), the contractive inequality (3) can be applied with $x = x^*$ and $y = x_n$. Hence, (17) turns into

$$\begin{aligned}\sigma(x^*, Tx^*) &\leq s\sigma(x^*, Tx_n) + s\sigma(Tx_n, Tx^*).\\ &= s\sigma(x^*, Tx_n) + s\sigma(Tx^*, Tx_n).\\ &< s\sigma(x^*, Tx_n) + sm(x^*, x_n)\\ &= s\sigma(x^*, x_{n+1}) + s\max\left\{\sigma(x^*, x_n), \sigma_n\frac{1+\sigma(x^*, Tx^*)}{1+\sigma(x^*, x_n)}\right\}\end{aligned} \tag{19}$$

for all $n \geq n_3$.

Tending with $n \to \infty$ in (19) and using together (11) and (15), we get

$$\sigma(x^*, Tx^*) \leq 0,$$

a contradiction. Consequently, x^* is a fixed point of T, that is, $Tx^* = x^*$.

Finally, we prove the uniqueness of the fixed point of T. Assume that x^* and y^* are two different fixed points of T, i.e., $Tx^* = x^* \neq y^* = Ty^*$. Thus

$$\sigma(Tx^*, Ty^*) = \sigma(x^*, y^*) > 0. \tag{20}$$

From (20), the contractive inequality (2) (with $x = x^*$ and $y = y^*$) yields

$$\chi(\sigma(x^*, y^*)) + F(\sigma(x^*, y^*)) \leq F(\sigma(x^*, y^*)).$$

It is a contradiction since $\chi(\sigma(x^*, y^*)) > 0$. Thus, we conclude that $x^* = y^*$, which completes the proof. □

Remark 7 If $s = 1$, then condition (H_3) coincides with condition (F_3).

In the case of metric spaces, i.e., for $s = 1$, Theorem 7 reduces to the following result.

Corollary 1 *Let (X, d) be a complete metric space and let T be a self-mapping on X. Assume that there exist a nondecreasing function $F : (0, \infty) \to \mathbb{R}$ satisfying condition (F_3) and $\chi \in \mathcal{L}$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, the following condition holds*

$$\chi(d(x, y)) + F(d(Tx, Ty)) \leq F(m_d(x, y)),$$

where

$$m_d(x, y) = \max\left\{d(x, y), \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)}\right\}.$$

Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Corollary 2 *Let (X, σ) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping. Assume that there exist two functions $F : (0, \infty) \to \mathbb{R}$ and $\chi : (0, \infty) \to (0, \infty)$ such that for all $x, y \in X$ with $\sigma(Tx, Ty) > 0$, the following condition holds*

$$\chi(\sigma(x,y)) + F(\sigma(Tx,Ty)) \leq F(\sigma(x,y)). \tag{21}$$

Furthermore, assume that (H_1), (H_2) and (H_3) are satisfied. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Proof The desired result follows immediately through the fact that $\sigma(x,y) \leq m(x,y)$ and the monotonicity of F. □

Remark 8 Corollary 2 improves Theorem 6 on several sides. Indeed, the contractive condition (F) from Theorem 6 implies the contractive condition (21) since $s \geq 1$ and F is nondecreasing. Moreover, condition $(\mathrm{F}_{s,\tau})$ from Theorem 6 is omitted and the constant τ is replaced by a function χ in Corollary 2. Besides these, the strictness of the monotonicity of F is not necessary. It is worth mentioning that Corollary 2 does not completely improve Theorem 6 since condition (F_3) is weaker than condition (H_3). However, we will show in Example 9 that Corollary 2 is more convenient in use than Theorem 6.

Remark 9 As in Theorem 6, Corollary 2 has been also proved without condition (F_2'). In addition, and contrary to what was done in the work [29], we did not use Lemma 3 to drop the aforementioned condition.

If, in Theorem 7, we take $F(t) = \ln(t)$ and $\chi(t) = \chi$ for some $\chi > 0$, one can recover the following result in the setting of b-metric spaces.

Corollary 3 (*See* [31, *Theorem* 4.3 *with Remark* 4.4-(1)]) Let (X, σ) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying

$$\sigma(Tx,Ty) \leq \lambda m(x,y) \tag{22}$$

for all $x, y \in X$, where $\lambda \in [0,1)$. Then T has a unique fixed point $x^* \in X$ and for any $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Remark 10

(1) The constant λ appeared in Corollary 3 is given by $\lambda = e^{-\chi} \in [0,1)$.
(2) We note also that Theorem 2 is easily deduced from Corollary 3.

Now, we deduce the Dass–Gupta fixed point theorem in the setting of b-metric spaces.

Corollary 4 (*Dass–Gupta fixed point in b-metric spaces*). *Let (X, σ) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping. Suppose that there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ such that for all $x, y \in X$,*

$$\sigma(Tx,Ty) \leq M(x,y),$$

where

$$M(x,y) = \alpha\sigma(x,y) + \beta\frac{\sigma(y,Ty)(1+\sigma(x,Tx))}{1+\sigma(x,y)}. \tag{23}$$

Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Proof It is easy to see that $M(x,y) \leq \lambda m(x,y)$, where $\lambda = \alpha + \beta$ with $\alpha, \beta \in [0,1)$ and $\alpha + \beta < 1$. Hence, the conclusion follows immediately from Corollary 3. □

Remark 11

(i) Corollary 4 is an improvement of Theorem 3 due to Samet [34] since the condition $\alpha s + \beta < 1$ is relaxed to the following one $\alpha + \beta < 1$.
(ii) If $s = 1$, Corollary 4 recovers Dass and Gupta's fixed point theorem (see Theorem 1).

Corollary 5 *Let (X,σ) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping. Assume that there exist two functions $F : (0,\infty) \to \mathbb{R}$ and $\chi : (0,\infty) \to (0,\infty)$ such that for all $x, y \in X$ with $\sigma(Tx,Ty) > 0$, the following condition holds*

$$\chi(\sigma(x,y)) + F(\sigma(Tx,Ty)) \leq F(M(x,y)),$$

where $M(x,y)$ is given by (23) with $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. Furthermore, we suppose that F is nondecreasing. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Proof Since F is nondecreasing, we obtain

$$\sigma(Tx,Ty) \leq M(x,y)$$

for all $x, y \in X$ with $Tx \neq Ty$. It is clear that the above inequality also holds when $Tx = Ty$. Hence, the desired result follows immediately from Corollary 4. □

Remark 12 It is worth mentioning that the conclusion of Corollary 5 can not be deduced if $\alpha + \beta = 1$.

Example 8 Let

$$X = \left\{ x_n = \frac{n(n+1)}{2},\ n \in \mathbb{N} \right\}.$$

Let us consider the functional $\sigma : X \times X \to [0,\infty)$ given by $\sigma(x,y) = (x-y)^2$. According to Example 1, it is easy to see that (X,σ) is a complete b -metric space with coefficient $s = 2$. Let us define the mapping $T : X \to X$ as follows

$$Tx = \begin{cases} x_{n-1}, & \text{if } x = x_n, n \geq 2, \\ x_1, & \text{if } x = x_1. \end{cases}$$

First, we are going to show that T does not satisfy the contractive condition (22) in Corollary 3. Indeed, this fact is checked through the following:

$$\lim_{n\to\infty}\frac{\sigma(Tx_n,Tx_1)}{m(x_n,x_1)}=\lim_{n\to\infty}\frac{(x_{n-1}-x_1)^2}{(x_n-x_1)^2}=\lim_{n\to\infty}\frac{(n^2-n-2)^2}{(n^2+n-2)^2}=1.$$

Hence, we cannot use Corollary 3.

Next, we will show that T is a (χ,F)-Dass–Gupta-contraction of type (A). Now, we observe that for every $n,k\in\mathbb{N}$,

$$\sigma(Tx_{n+k},Tx_n)>0\Leftrightarrow(k\geq 2\wedge n=1)\vee(k\in\mathbb{N}\wedge n>1).$$

Case 1. For every $k\geq 2$ and $n=1$, we have

$$\begin{aligned}\sigma(Tx_{k+1},Tx_1)-m(x_{k+1},x_1)&\leq\sigma(x_k,x_1)-\sigma(x_{k+1},x_1)\\&=(x_k-x_1)^2-(x_{k+1}-x_1)^2\\&=\left(\frac{k^2+k-2}{2}\right)^2-\left(\frac{k^2+3k}{2}\right)^2\\&=-(k+1)\left(k^2+2k-1\right)\\&\leq-21\leq-\frac{1}{\sigma(x_{k+1},x_1)+1}.\end{aligned}$$

Case 2. For every $k\in\mathbb{N}$ and $n>1$, the following holds

$$\begin{aligned}\sigma(Tx_{n+k},Tx_n)-m(x_{n+k},x_n)&\leq\sigma(x_{n+k-1},x_{n-1})-\sigma(x_{n+k},x_n)\\&=(x_{n+k-1}-x_{n-1})^2-(x_{n+k}-x_n)^2\\&=\frac{k^2}{4}\left((2n+k-1)^2-(2n+k+1)^2\right)\\&=-k^2(2n+k)\\&\leq-5\leq-\frac{1}{\sigma(x_{n+k},x_n)+1}.\end{aligned}$$

From the above cases, T is a (χ,F) -Dass–Gupta-contraction of type (A) with $\chi(t)=\dfrac{1}{1+t}$ and $F(t)=t$ for all $t>0$. In addition, all the conditions of Theorem 7 are satisfied. Hence, T has a unique fixed point $x^*=x_1=1$. Also, we observe that the function F does not satisfy condition (F_2').

Remark 13 Example 8 shows that Theorem 7 is a real generalization of Corollary 3.

Example 9 Let

$$X=\left\{x_n=\frac{1}{2^{\frac{n-1}{2}}\sqrt{n}},n\in\mathbb{N}\right\}\cup\{0\}.$$

Let $\sigma:X\times X\to[0,\infty)$ be the mapping defined by $\sigma(x,y)=(x-y)^2$. As in

Example 8, we can see that (X,σ) is a complete b -metric space with coefficient $s=2$. Let us consider the mapping $T:X\to X$ defined by

$$Tx=\begin{cases} x_{n+1}, & \text{if } x=x_n,\ n\in\mathbb{N},\\ 0, & \text{if } x=0.\end{cases}$$

First, we will show that Theorem 6 is not applicable.

On the contrary, let us suppose that all the conditions of Theorem 6 are fulfilled. Therefore, there exist a nondecreasing $F:(0,\infty)\to\mathbb{R}$ and $\tau>0$ such that for $x=x_n, n\in\mathbb{N}$ and $y=0$, we have

$$\tau+F(2\sigma(Tx_n,T0))\le F(\sigma(x_n,0)),$$

which leads to

$$\tau+F\left(2x_{n+1}^2\right)\le F\left(x_n^2\right) \quad \text{for all } n\in\mathbb{N}. \tag{24}$$

Let us put $\alpha_n=x_{n+1}^2$. By Remark 4, (24) implies that

$$\tau+F(2^n\alpha_n)\le F\left(2^{n-1}\alpha_{n-1}\right) \quad \text{for all } n\in\mathbb{N}.$$

If we set $r_n=2^n\alpha_n$, one can write

$$\sum_{n=1}^{p}\tau\le\sum_{n=1}^{p}[F(r_{n-1})-F(r_n)], \quad p\in\mathbb{N},$$

which implies

$$\tau\le\frac{1}{p}\left(F(r_0)-F\left(r_p\right)\right), \quad p\in\mathbb{N},$$

or, equivalently

$$\tau\le\frac{1}{p}\left(F(1)-F\left(\frac{1}{p+1}\right)\right). \tag{25}$$

By condition (F_3), there exists $k\in(0,1)$ such that

$$\lim_{p\to\infty}\frac{1}{(p+1)^k}F\left(\frac{1}{p+1}\right)=0.$$

Consequently,

$$\lim_{p\to\infty}\frac{1}{p}F\left(\frac{1}{p+1}\right)=\lim_{p\to\infty}\frac{(p+1)^k}{p}\frac{1}{(p+1)^k}F\left(\frac{1}{p+1}\right)=0.$$

Letting $p\to\infty$ in (25), we obtain $\tau\le 0$, a contradiction. Thus, the contractive condition (F) from Theorem 6 is not satisfied.

Next, we are going to show that T is a (χ,F)-Dass–Gupta-contraction of type (A).

Let us consider the following cases:

Case 1. If $x = x_n, n \in \mathbb{N}$ and $y = 0$, we have

$$\begin{aligned}\sqrt{\sigma(Tx_n, T0)} - \sqrt{\sigma(x_n, 0)} &= x_{n+1} - x_n \\ &= \sqrt{\frac{n}{2(n+1)}} x_n - x_n \\ &\leq \left(\frac{1}{\sqrt{2}} - 1\right) x_n \\ &= \left(\frac{1}{\sqrt{2}} - 1\right) \sqrt{\sigma(x_n, 0)}.\end{aligned}$$

Case 2. If $x = x_{n+k}$ and $y = x_n$ for every $n, k \in \mathbb{N}$. Bearing in mind that $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence, the following holds:

$$\begin{aligned}\sqrt{\sigma(Tx_{n+k}, Tx_n)} - \sqrt{\sigma(x_{n+k}, x_n)} &= |x_{n+k+1} - x_{n+1}| - |x_{n+k} - x_n| \\ &= x_{n+1} - x_n - x_{n+k+1} + x_{n+k} \\ &= \left(\sqrt{\frac{n}{2(n+1)}} - 1\right) x_n - \left(\sqrt{\frac{n+k}{2(n+k+1)}} - 1\right) x_{n+k} \\ &\leq \left(\sqrt{\frac{n}{2(n+1)}} - 1\right) (x_n - x_{n+k}) \\ &\leq \left(\frac{1}{\sqrt{2}} - 1\right) |x_{n+k} - x_n| \\ &= \left(\frac{1}{\sqrt{2}} - 1\right) \sqrt{\sigma(x_{n+k}, x_n)}.\end{aligned}$$

In view of the above cases, it is easy to see that T is a (χ, F)-Dass–Gupta-contraction of type (A) with $\chi(t) = \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{t}$ and $F(t) = \sqrt{t}$ for all $t > 0$. Moreover, all the conditions of Corollary 2 are satisfied. Hence, T has a unique fixed point $x^* = 0$. Notice that the function F does not satisfy condition (F_2').

Motivated by Remark 12, we will attempt to study Corollary 5 in the case when $\alpha + \beta = 1$.

Definition 8 Let (X, σ) be a b-metric space with coefficient $s \geq 1$. A mapping $T : X \to X$ is called to be a (χ, F)-Dass–Gupta-contraction of type (B) if there exist a nondecreasing function $F : (0, \infty) \to \mathbb{R}$ and $\chi \in \mathcal{L}$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, the following condition is satisfied:

$$\chi(\sigma(x, y)) + F(\sigma(Tx, Ty)) \leq F\big(\mathcal{R}_{\alpha,\beta}(x, y)\big), \tag{26}$$

where

$$\mathcal{R}_{\alpha,\beta}(x,y)=\alpha\sigma(x,y)+\beta\frac{\sigma(y,Ty)(1+\sigma(x,Tx))}{1+\sigma(x,y)} \tag{27}$$

and α,β are nonnegative real numbers.

Remark 14 Obviously, we derive from Definition 8 that every T which is a (χ,F)-Dass–Gupta-contraction of type (B) satisfies the following condition

$$\sigma(Tx,Ty)<\mathcal{R}_{\alpha,\beta}(x,y) \tag{28}$$

for all $x,y\in X$ with $Tx\neq Ty$.

Let $s\geq 1$ be a given real number. For convenience, we set

$$\mathcal{B}_{\alpha,\beta}=\left\{(\alpha,\beta)\in\mathbb{R}^2:0<\alpha<\frac{1}{s^2},\ 0<\beta<1,\ \alpha+\beta=1\right\}.$$

Now, we are ready to state and prove our second fixed point result of Dass–Gupta type mappings. The proof used herein is essentially inspired by the technique developed in [16, Theorem 3.26].

Theorem 8 *Let (X,σ) be a complete b-metric space with coefficient $s\geq 1$ and $T:X\to X$ be a (χ,F)-Dass–Gupta-contraction of type (B) with $(\alpha,\beta)\in\mathcal{B}_{\alpha,\beta}$. Then T has a unique fixed point x^* and for every $x_0\in X$ the sequence $\{T^n x_0\}$ converges to x^*.*

Proof By similar reasoning as in the proof of Theorem 7, one can assume without loss of generality that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}_0$. Consequently,

$$\sigma_n=\sigma(x_n,x_{n+1})=\sigma(Tx_{n-1},Tx_n)>0\quad\text{for all } n\in\mathbb{N}.$$

Hence, by applying the contractive inequality (26) with $x=x_{n-1}$ and $y=x_n$, we get

$$\chi(\sigma_{n-1})+F(\sigma_n)\leq F(\alpha\sigma_{n-1}+\beta\sigma_n)\quad\text{for all } n\in\mathbb{N}. \tag{29}$$

Using the monotonicity of F and the fact that $\chi(t)>0,\forall t>0$, we get

$$\sigma_n<\alpha\sigma_{n-1}+\beta\sigma_n\quad\text{for all } n\in\mathbb{N}. \tag{30}$$

By assumptions of the theorem, (30) becomes

$$\sigma_n<\sigma_{n-1}\quad\text{for all } n\in\mathbb{N}. \tag{31}$$

Thus, $\{\sigma_n\}$ is a strictly decreasing sequence of positive numbers and thereby there exists $\sigma\geq 0$ such that

$$\lim_{n\to\infty}\sigma_n=\sigma^+.$$

Now, we show that $\sigma=0$ (for convenience, we use a different method to the one established in Theorem 7). Arguing by contradiction, we assume that $\sigma>0$. Since F is nondecreasing, the right limit of F exists, that is,

$$\lim_{t\to r^+} F(t) = F(r+0) = F(r^+) \quad \text{for all } r \in (0,\infty). \tag{32}$$

On the other hand, by substituting (31) into (29) through the monotonicity of F and $\alpha+\beta=1$, we obtain

$$\chi(\sigma_{n-1}) + F(\sigma_n) \leq F(\sigma_{n-1}) \quad \text{for all } n \in \mathbb{N}. \tag{33}$$

Keeping in mind (32) and letting $n\to\infty$ in (33), one gets

$$\begin{aligned}\liminf_{t\to\sigma^+} \chi(t) &\leq \liminf_{n\to\infty} \chi(\sigma_{n-1})\\ &\leq \lim_{n\to\infty} (F(\sigma_{n-1}) - F(\sigma_n))\\ &= F(\sigma^+) - F(\sigma^+)\\ &= 0,\end{aligned}$$

which contradicts (). Hence,

$$\lim_{n\to\infty} \sigma_n = 0^+. \tag{34}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, i.e., $\{x_n\}$ is not a Cauchy sequence. From (34) and the first statement of Lemma 2, there exist $\varepsilon > 0$ and two sequences $\{m(k)\}, \{n(k)\}$ of positive integers such that

$$\varepsilon \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon.$$

Consequently, there exists $k_1 \in \mathbb{N}$ such that $\left\{\sigma\left(x_{m(k)}, x_{n(k)}\right)\right\}$ is bounded for all $k \geq k_1$ and hence it has a convergent subsequence. Therefore, there exist a real number μ and a subsequence $\{k(j)\}_{j\geq k_1}$ of $\{k\}_{k\geq k_1}$ such that

$$\lim_{j\to\infty} \sigma\left(x_{m(k(j))}, x_{n(k(j))}\right) = \mu \tag{35}$$

with

$$0 < \varepsilon \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \mu \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon. \tag{36}$$

Taking into account the condition that $\alpha < \frac{1}{s^2}$ with the fact that $\mu > 0$, the following equation

$$\beta s^2 t^2 + 2s^2 t - \mu\left(1 - \alpha s^2\right) = 0, \quad t \in \mathbb{R}, \tag{37}$$

admits a positive root, denoted by q_s.

Since $q_s > 0$, (34) implies that there exist $j_1 \geq k_1, j_2 \geq k_1$ such that

$$\begin{aligned} \sigma_{m(k(j))} &= \sigma\left(x_{m(k(j))}, x_{m(k(j))+1}\right) \leq q_s \quad \text{for all } j \geq j_1, \\ \sigma_{n(k(j))} &= \sigma\left(x_{n(k(j))}, x_{n(k(j))+1}\right) \leq q_s \quad \text{for all } j \geq j_2. \end{aligned} \tag{38}$$

Using (35) and the fact that $q_s > 0$, it follows that there exists $j_3 \geq k_1$ such that

$$\sigma\left(x_{m(k(j))}, x_{n(k(j))}\right) \leq \mu + q_s \quad \text{for all } j \geq j_3. \tag{39}$$

On the other hand, using (b_3), we obtain

$$\begin{aligned} &\sigma\left(x_{m(k(j))}, x_{n(k(j))}\right) \\ &\leq s\sigma\left(x_{m(k(j))}, x_{m(k(j))+1}\right) + s\sigma\left(x_{m(k(j))+1}, x_{n(k(j))}\right) \\ &\leq s\sigma\left(x_{m(k(j))}, x_{m(k(j))+1}\right) + s^2\sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) \\ &\quad + s^2\sigma\left(x_{n(k(j))}, x_{n(k(j))+1}\right) \\ &= s\sigma_{m(k(j))} + s^2\sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) + s^2\sigma_{n(k(j))} \end{aligned}$$

for all $j \geq k_1$.

This implies

$$\begin{aligned} &\sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) \\ &\geq \frac{1}{s^2}\left(\sigma\left(x_{m(k(j))}, x_{n(k(j))}\right) - s\sigma_{m(k(j))} - s^2\sigma_{n(k(j))}\right) \end{aligned} \tag{40}$$

for all $j \geq k_1$.

Taking limit inferior as $j \to \infty$ in (40) with (34) and (35), we get

$$\liminf_{j \to \infty} \sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) \geq \frac{\mu}{s^2}. \tag{41}$$

Then, by virtue of (41) and $q_s > 0$, there exists $j_4 \geq k_1$ such that

$$\sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) > \frac{\mu}{s^2} - q_s \quad \text{for all } j \geq j_4. \tag{42}$$

Having in mind that q_s is a positive root of the equation given by (37), we have

$$\frac{\mu}{s^2} - q_s \geq \frac{\mu}{s^2}\left(1 - \alpha s^2\right) - q_s = \beta q_s^2 + q_s > 0. \tag{43}$$

Accordingly, inequalities (42) and (43) allow us to obtain

$$\sigma\left(Tx_{m(k(j))}, Tx_{n(k(j))}\right) > 0 \quad \text{for all } j \geq j_4. \tag{44}$$

Putting $u_j = \sigma\left(x_{m(k(j))}, x_{n(k(j))}\right)$ and $v_j = \sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right)$, it follows through (44), that the contractive inequality (26) with $x = x_{m(k(j))}$ and $y = x_{n(k(j))}$ takes the form

$$\chi(u_j) + F(v_j) \le F\left(\alpha u_j + \beta \sigma_{n(k(j))}\left(\frac{1+\sigma_{m(k(j))}}{1+u_j}\right)\right) \quad \text{for all } j \ge j_4. \tag{45}$$

We set $N = \max\{j_1, j_2, j_3, j_4\}$. Then, using (38), (39), (42), (43) and (45) with $1 + u_j > 1$, $\alpha + \beta = 1$ and the monotonicity of F, we get

$$\begin{aligned}\chi(u_j) + F\left(\frac{\mu}{s^2} - q_s\right) &\le F(\alpha(\mu + q_s) + \beta q_s(1 + q_s)) \\ &= F(\alpha\mu + q_s + \beta q_s^2) \\ &= F\left(\alpha\mu + \frac{\mu}{s^2}(1 - \alpha s^2) - q_s\right) \\ &= F\left(\frac{\mu}{s^2} - q_s\right)\end{aligned}$$

for all $j \ge N$.

Consequently, the last inequality implies that $\chi(u_j) \le 0$, for all $j \ge N$, which is a contradiction. Then, $\{x_n\}$ is a Cauchy sequence. Since (X, σ) is a complete b-metric space, there exists x^* in X such that

$$\lim_{n\to\infty} \sigma(x_n, x^*) = 0. \tag{46}$$

Following the same steps as those used in the proof of Theorem 7, we obtain that there exists $n_4 \in \mathbb{N}$ such that for all $n \ge n_4$,

$$\sigma(Tx_n, Tx^*) > 0. \tag{47}$$

Through (47), the contractive inequality (28) can be applied with $x = x^*$ and $y = x_n$. Thus we have

$$\begin{aligned}\sigma(x^*, Tx^*) &\le s\sigma(x^*, Tx_n) + s\sigma(Tx_n, Tx^*) \\ &= s\sigma(x^*, Tx_n) + s\sigma(Tx^*, Tx_n) \\ &< s\sigma(x^*, Tx_n) + s\mathcal{R}_{\alpha,\beta}(x^*, x_n) \\ &= s\sigma(x^*, x_{n+1}) + s\alpha\sigma(x^*, x_n) + s\beta\sigma_n \frac{1+\sigma(x^*, Tx^*)}{1+\sigma(x^*, x_n)}\end{aligned} \tag{48}$$

for all $n \ge n_4$.

Next, passing to the limit as $n \to \infty$ in (48) and using together (15) and (34), we obtain

$$\sigma(x^*, Tx^*) \le 0,$$

which is a contradiction. Therefore, x^* is a fixed point of T, i.e., $Tx^* = x^*$.

To prove the uniqueness, we assume that x^* and y^* are two distinct fixed points of T, i.e., $Tx^* = x^* \ne y^* = Ty^*$. Then

$$\sigma(Tx^*, Ty^*) = \sigma(x^*, y^*) > 0. \tag{49}$$

Using (49) and the fact that F is nondecreasing, the contractive inequality (26) (with $x = x^*$ and $y = y^*$) yields

$$\begin{aligned}\chi(\sigma(x^*, y^*)) + F(\sigma(x^*, y^*)) &\leq F(\alpha\sigma(x^*, y^*)) \\ &\leq F(\sigma(x^*, y^*)),\end{aligned}$$

which is a contradiction since $\chi(\sigma(x^*, y^*)) > 0$. Thus, the fixed point of T is unique and the proof is finished. □

Remark 15 Theorem 8 is proved without conditions (F_2'), (H_3) and the strictness of the monotonicity of F.

As a corollary of Theorem 8, taking $\chi(t) = \ln(t + \alpha + 1)$ (with $0 < \alpha < \frac{1}{s^2}$) and $F(t) = \ln(t)$, we obtain the following result.

Corollary 6 *Let (X, σ) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping such that*

$$\sigma(Tx, Ty) \leq \frac{\mathcal{R}_{\alpha,\beta}(x, y)}{\sigma(x, y) + \alpha + 1},$$

where $(\alpha, \beta) \in \mathcal{B}_{\alpha,\beta}$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^*.

Remark 16 One can list further consequences of Theorem 8 by varying the mappings F and χ suitably such as in the above corollary.

Example 10 Let $X = [\frac{1}{3}, 5]$ and the mapping $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \begin{cases} \max\{x, y\}, & x \neq y, \\ 0, & x = y. \end{cases}$$

for all $x, y \in X$. Then (X, d) is a complete metric space (see [21]).

Let $\sigma : X \times X \to [0, \infty)$ be the mapping defined as follows

$$\sigma(x, y) = (d(x, y))^2 \quad \text{for } x, y \in X.$$

In view of Example 1, (X, σ) is a complete b-metric space with coefficient $s = 2$.

Let $T : X \to X$ be a mapping given by

$$Tx = \begin{cases} \frac{1}{3}, & x \in [\frac{1}{3}, 3) \cup (3, 5], \\ \frac{2}{3}, & x = 3. \end{cases}$$

First, we observe that

$$\sigma(Tx,Ty)=\frac{4}{9}>0\Leftrightarrow\left[\left(x=3\wedge y\in[\frac{1}{3},3)\cup(3,5]\right)\vee\left(y=3\wedge x\in[\frac{1}{3},3)\cup(3,5]\right)\right].$$

Let $x,y\in X$ and denote

$$\mathcal{R}_{\frac{1}{8},\frac{7}{8}}(x,y)=\frac{1}{8}\sigma(x,y)+\frac{7}{8}\frac{\sigma(y,Ty)(1+\sigma(x,Tx))}{1+\sigma(x,y)}.$$

Next, in both cases $\left[\left(x=3\wedge y\in[\frac{1}{3},3)\cup(3,5]\right)\vee\left(y=3\wedge x\in[\frac{1}{3},3)\cup(3,5]\right)\right]$, we easily obtain

$$\mathcal{R}_{\frac{1}{8},\frac{7}{8}}(x,y)\geq\frac{1}{8}\sigma(x,y)=\frac{1}{8}(\max\{x,y\})^2\geq\frac{9}{8}.$$

On the other hand, we have

$$\begin{aligned}\frac{\sigma(x,y)}{26}+\frac{(-1)^q-1}{2(\sigma(Tx,Ty))^2}+\frac{((-1)^q+1)\sigma(Tx,Ty)}{2}&\leq\frac{\sigma(x,y)}{26}+\sigma(Tx,Ty)\\&\leq\frac{25}{26}+\frac{4}{9}\leq2\\&\leq\mathcal{R}_{\frac{1}{8},\frac{7}{8}}(x,y)+\frac{1}{\mathcal{R}_{\frac{1}{8},\frac{7}{8}}(x,y)}\end{aligned}\tag{50}$$

for all $x,y\in X$ with $\sigma(Tx,Ty)>0$ and $q\in\mathbb{N}_0$.

The last inequality in (50) holds through the following inequality:

$$h+\frac{1}{h}\geq2\qquad\text{for all } h>0.$$

Keeping in mind that $\sigma(Tx,Ty)=\frac{4}{9}<1$ and $\mathcal{R}_{\frac{1}{8},\frac{7}{8}}(x,y)\geq\frac{9}{8}>1$, one can consider $\chi:(0,\infty)\to(0,\infty)$ given by $\chi(t)=\frac{t}{26}$ and $F:(0,\infty)\to\mathbb{R}$ defined as follows

$$F(t)=\begin{cases}\frac{(-1)^q-1}{2t^2}+\frac{((-1)^q+1)t}{2}, & \text{if } 0<t\leq1,\quad q\in\mathbb{N}_0,\\ t+\frac{1}{t}, & \text{if } t>1.\end{cases}$$

Hence, T is a (χ,F) -Dass–Gupta-contraction of type (B) and all the conditions of Theorem 8 are satisfied for $\alpha=\frac{1}{8}$ and $\beta=\frac{7}{8}$. Therefore, T has a fixed point x^* (which is $\frac{1}{3}$). Notice that F does not satisfy condition (F_2') when q is even and does not satisfy condition (H_3) when q is odd.

4 Applications

4.1 An application to a nonlinear Fredholm integral equation

In this subsection, we apply Corollary 2 to guarantee the existence and uniqueness of a solution for a kind of nonlinear Fredholm integral equation.

Let $W = \mathcal{C}([0,1];[0,\infty))$ be the space of all nonnegative continuous functions defined on $[0,1]$ and the mapping $d : W \times W \to [0,\infty)$ given by

$$d_\infty(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)| \quad \text{for all } x,y \in W.$$

For some $p > 1$, we define

$$\sigma_\infty(x,y) = (d_\infty(x,y))^p = \sup_{t\in[0,1]} |x(t) - y(t)|^p \quad \text{for all } x,y \in W. \tag{51}$$

By Example 1, (W, σ_∞) is a complete b-metric space with coefficient $s = 2^{p-1}$.

In this application we deal with the following nonlinear Fredholm integral equation:

$$u(t) = \int_0^1 G(t,r)f(r,u(r))\mathrm{d}r, \quad u \in W,\ t \in [0,1], \tag{52}$$

where $f : [0,1] \times [0,\infty) \to \mathbb{R}$ and

$$G(t,r) = \begin{cases} t(1-r), & 0 \le t \le r \le 1, \\ r(1-t), & 0 \le r \le t \le 1. \end{cases}$$

Theorem 9 *Suppose the following hypothesis holds:*

(*A*) *For all* $r \in [0,1]$ *and for all* $z,w \in [0,\infty)$,

$$|f(r,z) - f(r,w)| \le \mu e^r |z - w|,$$

where

$$\mu := \frac{4}{\left(\frac{1}{p} + \frac{1}{p-1}\right)^{\frac{1}{p}} e}. \tag{53}$$

Then the integral equation (52) has a unique solution in W.

Proof Let $T : (W, \sigma_\infty) \to (W, \sigma_\infty)$ be the mapping defined as follows

$$(Tu)(t) = \int_0^1 G(t,r)f(r,u(r))\mathrm{d}r, \quad u \in W,\ t \in [0,1].$$

First, we easily observe that T is well defined. We set

$$q := \frac{p}{p-1} > 1, \quad \text{that is,} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Next, assume that $u, v \in W$ with $Tu \neq Tv$ and $t \in [0,1]$. Using Hölder inequality and assumption (A), one gets

$$\begin{aligned} |(Tu)(t) - (Tv)(t)|^p &\leq \left(\int_0^1 G(t,r)^q \mathrm{d}r \right)^{\frac{p}{q}} \int_0^1 |f(r,u(r)) - f(r,v(r))|^p \mathrm{d}r \\ &\leq \mu^p \left(\sup_{t\in[0,1]} \int_0^1 G(t,r)^q \mathrm{d}r \right)^{\frac{p}{q}} \int_0^1 e^{pr} |u(r) - v(r)|^p \mathrm{d}r \\ &\leq \mu^p \sigma_\infty(u,v) \left(\sup_{t\in[0,1]} \int_0^1 G(t,r)^q \mathrm{d}r \right)^{\frac{p}{q}} \int_0^1 e^{pr} \mathrm{d}r \\ &= \frac{\mu^p \sigma_\infty(u,v)}{p} (e^p - 1) \left(\sup_{t\in[0,1]} \int_0^1 G(t,r)^q \mathrm{d}r \right)^{\frac{p}{q}}. \end{aligned} \tag{54}$$

It is easy to obtatin that

$$\int_0^1 G(t,r)^q \mathrm{d}r = \frac{t^q(1-t)^q}{q+1}$$

and so

$$\sup_{t\in[0,1]} \int_0^1 G(t,r)^q \mathrm{d}r = \frac{1}{q+1} \frac{1}{2^{2q}} = \frac{1}{\frac{p}{p-1}+1} \frac{1}{2^{2\frac{p}{p-1}}}.$$

Hence, (54) turns into

$$
\begin{aligned}
|(Tu)(t)-(Tv)(t)|^p &\le \frac{\mu^p\sigma_\infty(u,v)}{p}(e^p-1)\left[\frac{1}{\frac{p}{p-1}+1}\frac{1}{2^{\frac{2p}{p-1}}}\right]^{p-1}\\
&= \frac{\mu^p\sigma_\infty(u,v)}{p}(e^p-1)\frac{1}{\left(\frac{p}{p-1}+1\right)^{p-1}}\frac{1}{2^{2p}}\\
&\le \frac{1}{2^{2p}}\frac{\mu^p\sigma_\infty(u,v)}{p}(e^p-1)\frac{1}{\left(\frac{p}{p-1}+1\right)^{p-1}}\\
&\le \frac{\sigma_\infty(u,v)(e^p-1)}{e^p\left(\frac{p}{p-1}+1\right)}\frac{1}{\left(\frac{p}{p-1}+1\right)^{p-1}}\\
&\le \sigma_\infty(u,v)(1-e^{-p})\\
&= \sigma_\infty(u,v)-\sigma_\infty(u,v)e^{-p}.
\end{aligned}
$$

Taking the supremum with respect to $t \in [0,1]$, we get

$$\sigma_\infty(Tu,Tv) \le \sigma_\infty(u,v)-\sigma_\infty(u,v)e^{-p},$$

or, equivalently,

$$\sigma_\infty(u,v)e^{-p}+\sigma_\infty(Tu,Tv) \le \sigma_\infty(u,v).$$

Hence, all the assumptions of Corollary 2 are fulfilled for $F(t)=t$ and $\chi(t)=e^{-p}t$ for all $t>0$. Therefore, T has a unique fixed point u^* in $\mathcal{C}([0,1];[0,\infty))$. □

Example 11 Let $r \in [0,1]$, $u \in W$ and μ given by (53). It is easy to see that the function f given by

$$f(r,u)=\frac{\mu e^r u}{1+u},$$

satisfies assumption (A).

4.2 An application to a nonlinear Volterra integral equation

In this subsection, we apply Corollary 6 to prove the existence and uniqueness of a solution for the following nonlinear Volterra integral equation:

$$x(t)=h(t)+\int_0^t G(t,r)f(r,x(r))\mathrm{d}r, \quad t \in I, \tag{55}$$

where $I=[0,\lambda]$ with $\lambda>0$, $f: I\times\mathbb{R}\to\mathbb{R}$, $G: I\times I\to\mathbb{R}$ and $h: I\to\mathbb{R}$ are mappings.

Let $X = \mathcal{C}(I;\mathbb{R})$ be the set of all continuous functions $x : I \to \mathbb{R}$. For any $x \in \mathcal{C}(I;\mathbb{R})$ and fixed arbitrary $\tau > 0$, we define the following norm

$$\|x\|_\tau = \sup_{t\in I} e^{-\tau t}|x(t)|.$$

As it is mentioned in the paper [36], the space $(X, \|.\|_\tau)$ is a Banach space. Hence, X endowed with the metric d_τ associated to the aforementioned norm, given by

$$d_\tau(x,y) = \sup_{t\in I} e^{-\tau t}|x(t) - y(t)| \quad \text{for all } x,y \in X,$$

is a complete metric space.

Next, we define

$$\sigma_\tau(x,y) = (d_\tau(x,y))^2 = \sup_{t\in I} e^{-2\tau t}(x(t) - y(t))^2 \quad \text{for all } x,y \in X. \tag{56}$$

Obviously, by Example 1, (X, σ_τ) is a complete b-metric space with coefficient $s = 2$.

Theorem 10 *Assume that the following assumptions hold*:

(A_1) h is a continuous function;

(A_2) G is a continuous function and there exist $\tau > 0$ and $K > 0$ such that

$$\sup_{t\in I} \int_0^t |G(t,r)|e^{\tau(r-t)}\mathrm{d}r \leq K; \tag{57}$$

(A_3) the function f is continuous and there exists a constant $\alpha \in (0, \frac{1}{4})$ such that for all $r \in I$ and for all $z, w \in \mathbb{R}$,

$$|f(r,z) - f(r,w)| \leq \frac{\sqrt{\alpha}|z-w|}{K\sqrt{1 + \alpha + (z-w)^2}}. \tag{58}$$

Then the integral equation (55) has a unique solution in X.

Proof Let $T : (X, \sigma_\tau) \to (X, \sigma_\tau)$ be the mapping defined as follows:

$$(Tx)(t) = h(t) + \int_0^t G(t,r)f(r,x(r))\mathrm{d}r, \quad x \in X, \ t \in I.$$

Clearly, under the hypotheses of the theorem, T is well defined (i.e., if $x \in X$ then $Tx \in X$).

Let $x, y \in X$ such that $Tx \neq Ty$. By assumptions (A_2) and (A_3), one can get

$$\begin{aligned}
|(Tx)(t)-(Ty)(t)| &\leq \int_0^t |G(t,r)||f(r,x(r))-f(r,y(r))|\mathrm{d}r \\
&\leq \frac{\sqrt{\alpha}}{K}\int_0^t |G(t,r)|\frac{|x(r)-y(r)|}{\sqrt{1+\alpha+(x(r)-y(r))^2}}\mathrm{d}r \\
&\leq \frac{\sqrt{\alpha}}{K}\int_0^t |G(t,r)|\frac{|x(r)-y(r)|e^{-\tau r}e^{\tau r}}{\sqrt{1+\alpha+(x(r)-y(r))^2 e^{-2\tau r}}}\mathrm{d}r \\
&\leq \frac{\sqrt{\alpha}}{K}\frac{d_\tau(x,y)}{\sqrt{1+\alpha+\sigma_\tau(x,y)}}\int_0^t |G(t,r)|e^{\tau r}\mathrm{d}r \\
&\leq \frac{\sqrt{\alpha}}{K}\frac{d_\tau(x,y)e^{\tau t}}{\sqrt{1+\alpha+\sigma_\tau(x,y)}}\sup_{t\in I}\int_0^t |G(t,r)|e^{\tau(r-t)}\mathrm{d}r \\
&\leq \frac{\sqrt{\alpha}d_\tau(x,y)e^{\tau t}}{\sqrt{1+\alpha+\sigma_\tau(x,y)}}.
\end{aligned}$$

This leads to

$$((Tx)(t)-(Ty)(t))^2 e^{-2\tau t} \leq \frac{\alpha\sigma_\tau(x,y)}{1+\alpha+\sigma_\tau(x,y)}.$$

Taking the supremum on $t \in I$, we deduce that

$$\sigma_\tau(Tx,Ty) \leq \frac{\alpha\sigma_\tau(x,y)}{1+\alpha+\sigma_\tau(x,y)}, \tag{59}$$

which further implies that

$$\sigma_\tau(Tx,Ty) \leq \frac{\mathcal{R}^\tau_{\alpha,\beta}(x,y)}{\sigma_\tau(x,y)+\alpha+1},$$

where

$$\mathcal{R}^\tau_{\alpha,\beta}(x,y) = \alpha\sigma_\tau(x,y) + \beta\frac{\sigma_\tau(y,Ty)(1+\sigma_\tau(x,Tx))}{1+\sigma_\tau(x,y)}$$

with the constant β to be chosen such that $\beta \in (0,1)$ and $\alpha+\beta=1$.

Consequently, all the conditions of Corollary 6 are fulfilled. Hence, the integral equation (55) has a unique solution in $X = \mathcal{C}(I;\mathbb{R})$ □

The following example illustrates the results of Theorem 10.

Example 12 In this example, we present an application of Theorem 10. More precisely, we consider an application to *Damped Spring-Mass system* in engineering problems. Let $m > 0$ be the mass of the spring. Herein, we deal with critical damped motion of a spring subjected to some given external force *f*. Then such type of system is governed by the following initial value problem (see [38]):

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} + \dfrac{k}{m}\dfrac{\mathrm{d}u}{\mathrm{d}t} = f(t, u(t)), \quad t \in I = [0, 2\pi], \\ u(0) = 0, \ u'(0) = a, \end{cases} \tag{60}$$

where $k > 0$ is the damping constant, $a \in \mathbb{R}$.

Let us choose the function f as follows

$$f(t, u) = \frac{1}{8\pi^2} \max\{\sin(t), \frac{|u|}{1 + u^2}\}, \ t \in I, \ u \in \mathcal{C}(I; \mathbb{R}).$$

The initial boundary problem (60) is equivalent to the following integral equation (see again [38]).

$$u(t) = \int_0^t G(t, r) f(r, u(r)) \mathrm{d}r, \quad t \in I \tag{61}$$

where $G : I \times I \to \mathbb{R}$ is the Green's function given by

$$G(t, r) = \begin{cases} (t - r)e^{\tau(t-r)}, & 0 \le r \le t \le 2\pi, \\ 0, & 0 \le t \le r \le 2\pi, \end{cases}$$

where $\tau > 0$ being a constant, calculated in terms of m and k.

Note that the existence of a solution $u \in \mathcal{C}^2(I; \mathbb{R})$ of problem (60) is equivalent to the existence of a fixed point $u \in \mathcal{C}(I; \mathbb{R})$ of the integral equation (61).

Now, we check that the hypotheses in Theorem 10 are satisfied. Indeed, hypothesis (A_1) is immediately satisfied with $h(t) = 0$ for all $t \in I$.

Notice that the Green's function G is continuous and nonnegative on $I \times I$. Moreover, after routine calculations, we obtain

$$\sup_{t \in I} \int_0^t |G(t, r)| e^{\tau(r-t)} \mathrm{d}r = \sup_{t \in I} \frac{t^2}{2} = 2\pi^2.$$

Consequently, hypothesis (A_2) is satisfied with $K = 2\pi^2$.

Now, we check the condition (A_3). For arbitrary $z, w \in \mathbb{R}$ and using the following basic inequality (see [27])

$$|\max\{a, b\} - \max\{c, d\}| \le \max\{|a - c|, |b - d|\}, \ a, b, c, d \in \mathbb{R},$$

we obtain

$$
\begin{aligned}
|f(r,z)-f(r,w)| &\le \frac{1}{8\pi^2}\left|\max\{\sin(r),\frac{|z|}{1+z^2}\}-\max\{\sin(r),\frac{|w|}{1+w^2}\}\right| \\
&\le \frac{1}{8\pi^2}\max\left\{0,\left|\frac{|z|}{1+z^2}-\frac{|w|}{1+w^2}\right|\right\} \\
&= \frac{1}{8\pi^2}\left|\frac{|z|}{1+z^2}-\frac{|w|}{1+w^2}\right| \\
&\le \frac{1}{8\pi^2}\frac{|z-w||1-|zw||}{(1+z^2)(1+w^2)} \\
&= \frac{1}{8\pi^2}\frac{|z-w||1-|zw||}{\sqrt{(1+z^2)(1+w^2)}\sqrt{(1+z^2)(1+w^2)}} \\
&\le \frac{1}{8\pi^2}\frac{|z-w||1-|zw||}{\sqrt{(1+zw)^2}\sqrt{1+\frac{1}{2}(z-w)^2}} \\
&\le \frac{1}{2\pi^2\sqrt{8}}\frac{|z-w|}{\sqrt{2+(z-w)^2}} \\
&\le \frac{1}{2\pi^2\sqrt{8}}\frac{|z-w|}{\sqrt{\frac{9}{8}+(z-w)^2}}.
\end{aligned}
$$

Thus, assumption (A_3) is satisfied with $\alpha=\dfrac{1}{8}$ and $K=2\pi^2$. Consequently, all the conditions of Theorem 10 are fulfilled. Hence, the integral equation (61) has a solution in $\mathcal{C}(I;\mathbb{R})$. Consequently, the problem (60) has a solution $u\in\mathcal{C}^2(I;\mathbb{R})$.

We finally ask the following questions:

Question 1. Does Theorem 7 hold if condition (H_3) is replaced with condition (F_3)? i.e., does the conclusion of Theorem 7 remains true for any $k\in[\frac{1}{1+\log_2 s},1)$?

Question 2. Does the conclusion of Theorem 8 remains true for any $\alpha\in[\frac{1}{s^2},1)$?

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