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Ideals in topological ternary semigroups

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Abstract
In this paper, we discuss the notions of minimal ideals, maximal ideals and principal ideals on a topological ternary semigroup. We express the kernel of a topological ternary semigroup in different ways and provide characterizations of kernel in topological ternary semigroups. We establish equivalences between minimal left ideal, minimal ideal and maximal ternary subgroup on a paratopological ternary group.

Keywords Ternary semigroup · Ternary group · Ternary subgroup · Topological ternary semigroup · Paratopological ternary group · Ideal · Kernel

Mathematics Subject Classification 20M99 · 22A99

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1 Introduction

Generalized theory of algebra, namely n-ary algebra was studied by Kasner (1904) [10], Dörnte (1928) [4], Post (1940) [14] and many others. Ternary algebraic system which is known as triplexes was first introduced by Lehmer [11] in 1932. Los [12] proved that every ternary semigroup can be embedded in a semigroup.

After many pioneering works in ternary semigroups, in 1965 Sioson [19] studied ideal theory in ternary semigroups. He introduced the concept of prime ideals, semiprime ideals, quasi-ideals to characterize regular ternary semigroups in terms of these ideals. In 1980, Dudek and Grozdzinska [5] studied ideals in regular n-semigroups. Dixit and Dewan [3] studied properties of ideals in ternary semigroups.

In paper [6] Dutta et al. studied properties of regular ternary semigroups, completely regular ternary semigroups, intra-regular ternary semigroups and their characterizations in terms of different ideals. Sabir and Bano [16] introduced the notion of prime ideals, semiprime ideals and strongly prime bi-ideals in ternary semigroups. Choosuwn and Chinram in [2] gave some characterizations of minimal and maximal quasi ideals in ternary semigroups. Iampan in paper [7] introduced a concept of ideal extension in ternary semigroups. He also considered the connection between an ideal extension and semilattice congruence in ternary semigroups. Kar and Maity in paper [9] studied different types of ideals in ternary semigroups and studied some properties of these ideals. In recent times much works are going on quasi ideals and fuzzy ideals in ternary semigroups. Very recently Petchkaew and Chinram [13] studied the minimality and maximality of n-ideals in n-ary semigroups. Srinivasan Rao et al. [15] discussed the properties of maximal ideal on compact connected topological ternary semigroups.

In this paper, we discuss ideals on a topological ternary semigroup. After giving some preliminary results, we study minimal ideals, maximal ideals and principal ideals of topological ternary semigroups. We express the kernel of topological ternary semigroups in terms of union of minimal left ideals, union of minimal right ideals and union of maximal ternary subgroups. Finally, we establish equivalences between minimal left ideal, minimal ideal and maximal ternary subgroup on a paratopological ternary group.

2 Preliminaries

Definition 2.1 [19] A non-empty set S together with a ternary operation, called *ternary multiplication* and denoted by juxtaposition, is said to be a *ternary semigroup* if $(abc)de = a(bcd)e = ab(cde)$, for all $a, b, c, d, e \in S$.

It is said to be an *abelian ternary semigroup* if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ for every permutation σ of $\{1, 2, 3\}$ and $x_1, x_2, x_3 \in S$.

Definition 2.2 Let S be a ternary semigroup. If there exists an element $0 \in S$ such that $0xy = x0y = xy0 = 0$, for all $x, y \in S$ then '0' is called the *zero element* or simply the *zero* of the ternary semigroup S. In this case we say that S is a ternary semigroup with zero.

Definition 2.3 [19] An element a of a ternary semigroup S is called *idempotent* if $a^3 = a$. A ternary semigroup S is called an *idempotent ternary semigroup* if each element of S is idempotent.

The set of all idempotents of a ternary semigroup S is denoted by $E(S)$.

Definition 2.4 [8] A ternary semigroup S admits an identity (or unital element) if there exists an element $e \in S$ such that $eex = exe = xee = x$, for all $x \in S$. In this case 'e' is called an *identity element* (*or unital element*) of the ternary semigroup S.

Definition 2.5 [8] A ternary semigroup S admits a bi-unital element if there exists an element $e \in S$ such that $eex = xee = x$, for all $x \in S$. In this case 'e' is called a *bi-unital* element of the ternary semigroup S.

Definition 2.6 [8] A non-empty subset T of a ternary semigroup S is called a *ternary subsemigroup* if $t_1t_2t_3 \in T$, for all $t_1, t_2, t_3 \in T$.

Definition 2.7 [19] A ternary semigroup S is said to be

(i) *left cancellative* (LC) if $abx = aby \Longrightarrow x = y$, for all $a, b, x, y \in S$;
(ii) *right cancellative* (RC) if $xab = yab \Longrightarrow x = y$, for all $a, b, x, y \in S$;
(iii) *laterally cancellative* (LLC) if $axb = ayb \Longrightarrow x = y$, for all $a, b, x, y \in S$;
(iv) *cancellative* if S is left, right and laterally cancellative.

Definition 2.8 [8] A ternary semigroup S is called a *ternary group* if for $a, b, c \in S$ the equations $abx = c, axb = c$ and $xab = c$ have solutions in S.

From above definition, we can easily have the following characterization.

Result 2.9 [17] A non-empty subset A of a ternary semigroup S will be a ternary subgroup of S iff $xyA = xAy = Axy = A$, for all $x, y \in A$.

Definition 2.10 [8] An element a of a ternary semigroup S is said to be *invertible in* S if there exists an element $b \in S$ such that $abx = bax = xab = xba = x$, for all $x \in S$. Then b is called the *inverse* of a (it is unique, if exists).

Theorem 2.11 *A ternary subsemigroup T of a ternary semigroup S is a ternary subgroup of S iff every element of T has an inverse in T.*

This theorem is an easy consequence of equivalent conditions of Theorem 3.9 of [6].

Definition 2.12 [19] A non-empty subset A of a ternary semigroup S is called a *left* (*right*, *lateral*, *two sided*) *ideal* of S if $SSA \subseteq A$ (respectively $ASS \subseteq A, SAS \subseteq A$ and $SSA \cup ASS \cup SSASS \subseteq A$). If A is a left ideal, right ideal and lateral ideal, then A is called an *ideal* of S.

Definition 2.13 A *minimal left* (*right*) *ideal* of a ternary semigroup S is a left (right) ideal containing no other left (right) ideal of S. We denote by $\mathscr{L}(S)$ and $\mathscr{R}(S)$ respectively the collections of all minimal left ideals and minimal right ideals of S.

Definition 2.14 A *minimal ideal* of ternary semigroup S is an ideal that contains no other proper ideal of S.

3 Ideals on a topological ternary semigroup

In this section we discuss ideals in topological ternary semigroups. For this we need the following definition:

Definition 3.1 [15] A ternary semigroup S is said to be a *topological ternary semigroup* if there exists a **Hausdorff** topology on S such that the ternary multiplication $\left.\begin{array}{l} S\times S\times S\longrightarrow S \\ (x,y,z)\longmapsto xyz \end{array}\right\}$ is continuous, $S\times S\times S$ being equipped with the product topology.

Now we are in a position to discuss ideals in topological ternary semigroups. First we discuss a few elementary results related to ternary semigroups that will be required later on.

Definition 3.2 [19] Left ideal, right ideal, lateral ideal, two sided ideal and ideal generated by a non-empty subset A of a ternary semigroup S is defined as follows:

$L(A) = A\cup SSA, R(A) = A\cup ASS, M(A) = A\cup SAS\cup SSASS$, $T(A) = A\cup SSA\cup ASS\cup SSASS, J(A) = A\cup SSA\cup ASS\cup SAS\cup SSASS$.

In particular if A is a singleton set say $\{a\}$ then we call it as ideal generated by an element a. It is called *principal ideal* generated by the element a and is denoted by $J(a)$.

Theorem 3.3 *Let S be a ternary semigroup, L be a left ideal, R be a right ideal, M be a lateral ideal, I and J be two ideals, G be a ternary subgroup of S and A be a non-empty subset of S, then*:

(a) SSA is a left ideal, ASS is a right ideal and $SAS\cup SSASS$ is a lateral ideal as well as an ideal of S;
(b) $RML\subseteq R\cap M\cap L$ and hence $R\cap M\cap L\neq\emptyset$;
(c) $I\cap J$ is an ideal of S;
(d) if $G\cap I\neq\emptyset$, then $G\subseteq I$;
(e) if J is a minimal ideal then $J = J(x)$ for all $x\in J$;
(f) if I and J are two minimal ideals then $I = J$;
(g) if I is a ternary subgroup of S, then I is a minimal ideal of S.

Proof

(a) Follows from Definition 2.12.
(b) We have that $RML\subseteq RSS\subseteq R$. Similarly $RML\subseteq SMS\subseteq M$ and $RML\subseteq L$. Therefore $RML\subseteq R\cap M\cap L$. Hence $R\cap M\cap L\neq\emptyset$.
(c) $I\cap J$ is non-empty because $ISJ\subseteq I\cap J$. Now $S(I\cap J)S\subseteq SIS\subseteq I$ and $S(I\cap J)S\subseteq SJS\subseteq J$, since I and J are two ideals of S and so lateral ideals too. These two imply that $S(I\cap J)S\subseteq I\cap J$. This implies that $I\cap J$ is a lateral

ideal of S. Similarly we can show that $I \cap J$ is also a left ideal and a right ideal of S. Hence $I \cap J$ is an ideal of S.

(d) Let $a \in G \cap I$. Since G is a ternary group, a^{-1} exists in G [by Theorem 2.11]. Let $x \in G$. Then we have that $x = xa^{-1}a \in SSI \subseteq I$ (as I is an ideal of S). Since $x \in G$ is any arbitrary element, it follows that $G \subseteq I$.

(e) From Definition 3.2 we have that for $x \in J$, $J(x) = \{x\} \cup xSS \cup SSx \cup SxS \cup SSxSS$ which is an ideal of S contained in J. Now minimality of J implies that $J = J(x)$.

(f) We have shown in (c) of this Theorem that $I \cap J$ is an ideal whenever I and J are two ideals of S. Now $I \cap J \neq \emptyset$ as $ISJ \subseteq I \cap J$ and $ISJ \neq \emptyset$ if I and J both are non-void. But $I \cap J \subseteq I$ and $I \cap J \subseteq J$ imply that $I \cap J = I = J$, since I and J are two minimal ideals of S. Hence the result.

(g) Given that I is an ideal of S such that it is a ternary subgroup of S. Then for any ideal J of S with $I \cap J \neq \emptyset$, we have that $I \subseteq J$ by (d) of this Theorem. Therefore I is a minimal ideal of S.

□

Next we discuss some property related to topology for ideals.

Theorem 3.4 *Closure of an ideal in a topological ternary semigroup S is also an ideal of S.*

Proof Let I be an ideal of a topological ternary semigroup S. Then we have that $SSI \subseteq I$, $SIS \subseteq I$ and $ISS \subseteq I$. Now applying continuity of ternary multiplication we have that $SS\bar{I} \subseteq \overline{SSI} \subseteq \bar{I}$. Similarly we can show that $\bar{I}SS \subseteq \bar{I}$ and $S\bar{I}S \subseteq \bar{I}$. Therefore $\bar{I}$ is an ideal of S.

Note 3.5 It is to be noted that same is also true for left ideals, right ideals, lateral ideals and two sided ideals of a topological ternary semigroup.

Now we discuss some results related to compact topological ternary semigroups.

Theorem 3.6 *If S is a compact (connected) topological ternary semigroup (with identity) and A is a compact (connected) subset of S, then J(A) is compact (connected).*

Proof We have that from Definition 3.2, $J(A) = A \cup SSA \cup ASS \cup SAS \cup SSASS$. Since S and A are compact and ternary multiplication is continuous, therefore each of the following subsets SSA, ASS, SAS, and $SSASS$ are compact and hence $J(A)$ is compact. On the other hand, if S is connected and has an identity e (say), and A is a connected subset of S, then for all $a \in A$, we have that $a = aee = eeaee \in SSASS$. Therefore $A \subseteq SSASS$. Now let $p \in SAS$. Then $p = xay$, where $x, y \in S$ and $a \in A$. But $p = epe = e(xay)e \in SSASS \Rightarrow SAS \subseteq SSASS$. Similarly we can show that $ASS \subseteq SSASS$, $SSA \subseteq SSASS$. Therefore $J(A) = SSASS$. Hence that $J(A)$ is connected follows from the fact that S and A both are connected and ternary multiplication is continuous.

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Corollary 3.7 *Let S be a compact (connected) topological ternary semigroup (with identity) and $a \in S$. Then the principal ideal $J(a)$ generated by a is compact (connected).*

Corollary 3.8 *Let S be a compact topological ternary semigroup. Then principal left ideals, principal right ideals, principal lateral ideals, principal two sided ideals and principal ideals are closed. If e is any idempotent element of S, then eSe is a closed ternary subsemigroup of S with e as bi-unital element.*

Proof First part follows from Corollary 3.7 and the fact that compact subset of a Hausdorff space is closed. Now we see that $eSeeSeeSe \subseteq eSSSSSSSe \subseteq eSSSe \subseteq eSe$, as S is a ternary semigroup. Therefore eSe is a ternary subsemigroup of S. Now for $p \in eSe$, we have that $eep = eeese$ for some $s \in S$ with $p = ese$. Therefore $eep = ese = p$. Similarly we can show that $pee = p$ for all $p \in eSe$. Therefore for all $p \in eSe$ we have that $eep = pee = p$. Hence e is a bi-unital element of eSe. Closedness follows from the continuity of ternary multiplication and compactness of Hausdorff space S.

If S is a topological ternary semigroup and $A \subseteq S$, then we define $J_0(A)$ as the union of all ideals of S contained in A, provided those ideals exist. We see that if A contains an ideal of S, then $J_0(A)$ is an ideal of S. Otherwise $J_0(A) = \emptyset$. Similarly we can define $L_0(A), R_0(A), M_0(A)$ and $T_0(A)$ respectively as the union of all left ideals, right ideals, lateral ideals and two sided ideals of S contained in A. If $L_0(A)$ (respectively $R_0(A), M_0(A), T_0(A)$) is non-empty, then $L_0(A)$ (respectively $R_0(A), M_0(A), T_0(A)$) is the largest left (respectively right, lateral, two sided) ideal of S contained in A. Next we discuss some properties of $J_0(A)$.

Theorem 3.9 *Let S be a topological ternary semigroup and $A \subseteq S$ with $J_0(A) \neq \emptyset$. Then*

(a) A is closed in S implies $J_0(A)$ is closed in S;
(b) S is compact and A is open in S implies $J_0(A)$ is open in S.

Proof

(a) Let S be a topological ternary semigroup. From Theorem 3.4, we have that closure of an ideal of S is again an ideal of S. Therefore $\overline{J_0(A)}$ is an ideal of S. Since $J_0(A) \subseteq A$, we have that $\overline{J_0(A)} \subseteq \overline{A} = A$, since A is closed in S. But maximality of $J_0(A)$ implies that $\overline{J_0(A)} = J_0(A)$. This implies that $J_0(A)$ is closed in S.
(b) If $J_0(A) \neq \emptyset$, fix $x \in J_0(A)$. Then $J(x) = \{x\} \cup xSS \cup SSx \cup SxS \cup SSxSS \subseteq A$. So there exists an open set W in S containing x such that $J(W) \subseteq A$, by repeated application of continuity of ternary multiplication and compactness of S and $\{x\}$. But $x \in W \subseteq J(W) \subseteq J_0(A)$ and hence $J_0(A)$ is open in S.

□

Corollary 3.10 *Let S be a compact topological ternary semigroup, I be a proper ideal of S and $x \in S\backslash I$. Then $J_0(S\backslash\{x\})$ is an open proper ideal of S.*

Proof Since $S\backslash\{x\}$ contains an ideal I, $J_0(S\backslash\{x\})$ is an ideal of S. Now S being Hausdorff, $\{x\}$ is closed. So $S\backslash\{x\}$ is open. Therefore from Theorem 3.9(b), $J_0(S\backslash\{x\})$ is an open proper ideal of S. □

Definition 3.11 An ideal I of a ternary semigroup S is a *maximal proper ideal* of S if I is a proper ideal of S and is not contained in other proper ideal of S.

Theorem 3.12 *Let S be a compact topological ternary semigroup. Then each proper ideal of S is contained in a maximal proper ideal of S and each maximal proper ideal is open.*

Proof Proof is exactly same as that is done in Theorem 1.33 of [1].

Same result holds for left, right, lateral and two sided ideals also.

Corollary 3.13 *If S is a compact connected topological ternary semigroup and J is a maximal proper ideal of S, then J is dense in S.*

Proof By Theorem , J is open in S. Now $\bar{J}$ is an ideal of S, by Theorem 3.4. Then maximality of J implies that either $J = \bar{J}$ or $\bar{J} = S$. If $J = \bar{J}$, then J is a clopen proper ideal of S, contradicting the fact that S is connected. Therefore $\bar{J} = S$. In other words, J is dense in S.

None of the above theorems do not ensure the existence of minimal left ideals, minimal right ideals, minimal two sided ideals, minimal lateral ideals and minimal ideals of a topological ternary semigroup. Following theorem ensures existence of minimal ideals on a topological ternary semigroup.

Theorem 3.14 *Let S be a compact topological ternary semigroup. Then each left ideal of S contains at least one minimal left ideal of S and each minimal left ideal of S is closed and hence compact.*

Proof Let L be any left ideal of a compact topological ternary semigroup S and $\mathscr{T}$ be the collection of all closed left ideals of S that are contained in L. First of all, for each $x \in L$, SSx is a left ideal of S, since $SS(SSx) = (SSS)Sx \subseteq SSx$ (since S is a ternary semigroup). Also $SSx \subseteq SSL \subset L$ (since L is a left ideal of S). Again, SSx is compact, since S and $\{x\}$ being compact, $S \times S \times \{x\}$ is compact and continuous image of a compact set is compact. So S being Hausdorff, SSx is closed. Therefore $\mathscr{T}$ is non-empty.

We introduce partial ordering '$\leq$' on $\mathscr{T}$ by : for all $L_1, L_2 \in \mathscr{T}$, $L_1 \leq L_2$ if $L_2 \subseteq L_1$. Consider a linearly ordered subcollection $\{\mathscr{T}_i\}$ of $\mathscr{T}$. Then $\bigcap_i \mathscr{T}_i$ is non-empty (since S is compact) and hence is a left ideal in L. Therefore $\{\mathscr{T}_i\}$ has a lower bound in $\mathscr{T}$. So by Zorn's lemma, there exists a minimal element L_0 (say) in $\mathscr{T}$.

Let L_3 be a left ideal of S contained in L_0 and let $x \in L_3$. Then SSx is a closed left ideal of S contained in L. Also we note that $SSx \subseteq SSL_3 \subseteq L_3 \subseteq L_0$. So L_0 being a minimal element in $\mathscr{T}$, we have that $SSx = L_0 = L_3$. Thus L_0 is a minimal left ideal of S. Since $L_0 \in \mathscr{T}$, it is closed in S. Then compactness of S implies L_0 is compact. This completes the proof. □

Note 3.15 It is to be noted that same results can be obtained for right ideals, lateral ideals and ideals also.

4 Kernel of a topological ternary semigroup

We begin with the following definition:

Definition 4.1 The intersection of all ideals of a ternary semigroup S, if it is non-empty, is called the *kernel* of S. We denote it by K. It is to be noted that kernel is the smallest ideal of S.

In this section we discuss about kernel—its algebraic expression, characterizations and some other results related to it. Again, if a ternary semigroup contains a zero, then K only contains a zero. The ternary semigroups having ternary subgroups and minimal ideals are frequently useful. The following theorem gives us a sufficient condition that kernel is a ternary subgroup of a ternary semigroup S.

Theorem 4.2 *Let S be a ternary semigroup such that the kernel K is a minimal ideal and $K \subseteq Z(S)$, where $Z(S) = \{x \in S : xyz = xzy = yxz = yzx = zxy = zyx, \forall\, y, z \in S\}$. Then K is a ternary subgroup of S.*

Proof Let $x, y \in K$. Then for $p, q \in S$ we have that $pq(xyK) = (pqx)yK = x(pqy)K = xy(pqK) \subseteq xyK$, since $K \subseteq Z(S)$ and K is an ideal of S. Since $p, q \in S$ are any two arbitrary elements, we have that $SS(xyK) \subseteq xyK$. Therefore, xyK is a left ideal of S. Similarly we can show that xyK is a right ideal of S. Now let $x, y \in K$ and $p, q \in S$ be any two arbitrary elements. Then $p(xyK)q = (pxy)Kq = xy(pKq) \subseteq xy(SKS) \subseteq xyK$, since $K \subseteq Z(S)$ and K is a lateral ideal. Since $p, q \in S$ are any two arbitrary elements, we have that $S(xyK)S \subseteq xyK$ and therefore, xyK is a lateral ideal of S. Hence xyK is an ideal of S. Now $xyK \subseteq SSK \subseteq K$, since K is an ideal of S. So minimality of K ensures that $xyK = K$. Similarly we can show that $xKy = Kxy = K$. Therefore for all $x, y \in K$, we have that $xyK = xKy = Kxy = K$. Hence by Result 2.9, K is a ternary subgroup of S. □

Corollary 4.3 *If S is an abelian ternary semigroup such that the kernel K is a minimal ideal, then K is a ternary subgroup of S.*

Our next result is an important one. It will be required later to express kernel in different ways. The result is available in [18]. Here we present an alternative proof.

Theorem 4.4 *Let S be a ternary semigroup with kernel K. Also let us assume that K^* is the intersection of all two sided ideals of S and $K^* \neq \emptyset$. Then $K = K^*$.*

Proof We can easily show that K^* is a two sided ideal of S. Also we know that K is an ideal of S. Now is easy to verify that K^* is a ternary subsemigroup of S. It follows from definitions of K and K^* that $K^* \subseteq K$. So assumption that $K^* \neq \emptyset$ implies that

$K \neq \emptyset$. So K is a minimal ideal of S. Now $K^* \subseteq K$ implies that $SK^*S \subseteq SKS \subseteq K$, as K is an ideal of S. We now consider the set $K^* \cup SK^*S$. Then $SS(K^* \cup SK^*S) = SSK^* \cup SSSK^*S \subseteq K^* \cup SK^*S$. Therefore $K^* \cup SK^*S$ is a left ideal of S. Similarly we can show that $K^* \cup SK^*S$ is a right ideal of S. Now $S(K^* \cup SK^*S)S = SK^*S \cup SSK^*SS \subseteq SK^*S \cup K^*$, as K^* is a two sided ideal of S. Hence $K^* \cup SK^*S$ is a lateral ideal of S. Therefore it is an ideal of S. Now K is minimal implies that $K \subseteq K^* \cup SK^*S$. Again it is easy to verify that SK^*S is a two sided ideal of S. Therefore $K^* \subseteq SK^*S$. Hence $K \subseteq SK^*S$. Already we have proved that $SK^*S \subseteq K$. Therefore we have that $K = SK^*S$. Again $K = SK^*S$ implies that $SKS = SSK^*SS \subseteq K^*$. Again SKS is a two sided ideal of S. Therefore $K^* \subseteq SKS$. So $K^* = SKS$. Now $K^* \subseteq K$ implies that $SK^*S \subseteq SKS$ which implies that $K \subseteq K^*$. Hence $K = K^*$.

In the following theorem we give expressions of minimal left ideals, minimal right ideals and express K^* in terms of these ideals.

Theorem 4.5 *Let S be a topological ternary semigroup that has a minimal left ideal and a minimal right ideal. Then the following results hold:*

(a) if A_1 and A_2 both are either in $\mathscr{L}(S)$ or in $\mathscr{R}(S)$ with $A_1 \cap A_2 \neq \emptyset$, then $A_1 = A_2$;
(b) if $L \in \mathscr{L}(S)$, then $LLx = SSx = L,\ \forall x \in L$; if $R \in \mathscr{R}(S)$ then $xRR = xSS = R,\ \forall x \in R$;
(c) S has a minimal two sided ideal K^* and $K^* = \cup\{L : L \in \mathscr{L}(S)\} = \cup\{R : R \in \mathscr{R}(S)\}$.

Proof

(a) If A_1 and A_2 are in $\mathscr{L}(S)$ with $A_1 \cap A_2 \neq \emptyset$, then we have shown in 3.3(c), that $A_1 \cap A_2$ is a left ideal of S. Also $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$. Now minimality of A_1 and A_2 implies that $A_1 = A_1 \cap A_2 = A_2$.
(b) Let us assume that $L \in \mathscr{L}(S)$ and $x \in L$. Then LLx is a left ideal of S which is contained in L. Now $LLx \subseteq SSx \subseteq SSL \subseteq L$. Also SSx is a left ideal of S. So minimality of L implies that $LLx = SSx = L,\ \forall x \in L$. Similarly we can show that $xRR = xSS = R,\ \forall x \in R$.
(c) Let $L \in \mathscr{L}(S)$ and $x \in S$. Then LLx is a left ideal of S since, $SS(LLx) = (SSL)Lx \subseteq LLx$. Also we will show that $LLx \in \mathscr{L}(S)$. If possible let L_1 be a non-empty left ideal of S properly contained in LLx. Consider the set $L \cap \{a : abx \in L_1 \text{ for some } b \in L\} = L \cap P$ (say), where $P = \{a : abx \in L_1 \text{ for some } b \in L\}$. Then P is non-empty because L_1 is non-empty. Then we claim that $L \cap P$ is a left ideal of S properly contained in L. To prove it, let us assume that $s_1, s_2 \in S$ and $a \in P$. Then there exists $b \in L$ such that $abx \in L_1$ with $x \in S$. Then $s_1 s_2 a \in SSL \subseteq L$ and $s_1 s_2 abx \in SSL_1 \subseteq L_1$. Since $s_1, s_2 \in S$ are any two arbitrary elements and $a \in P$ it follows that $SSP \subseteq P$. Hence P is a left ideal of S properly contained in L which contradicts minimality of L. Therefore $\cup\{LLx : x \in S\} = LLS$ is a union of minimal left ideals. Now it is easy to show that LLS is a two sided ideal of S. Let I be any two sided ideal of S. Then $IIL \subseteq SSL \subseteq L$. Also

$IIL \subseteq ISS \subseteq I$. Again IIL is a left ideal of S. Therefore $L = IIL \subseteq I$. Thus we see that for each $L \in \mathscr{L}(S)$, $L \subseteq I$, where I is any two sided ideal of S. Thus we have that $LLS \subseteq ISS \subseteq I$ for every two sided ideal I of S. Since I is any arbitrary two sided ideal of S, it follows that $LLS = \cup\{LLx : x \in S\}$ is the minimal two sided ideal of S which is K^* by definition. Therefore $K^* = LLS = \cup\{LLx : x \in S\}$. Thus from (a), $L_2 = LLx$ for some $x \in S$. Thus we have that $K^* = \cup\{L : L \in \mathscr{L}(S)\}$. Similarly we can show that $K^* = \cup\{R : R \in \mathscr{R}(S)\}$.

□

Remark 4.6 From Theorems 4.4 and 4.5 (c), we can replace K^* by K which gives algebraic expression of K.

Already we have mentioned in Note 3.15, that a compact topological ternary semigroup contains a minimal ideal. Here we can prove it using Remark 4.6 and Theorem 4.5, which is given below as a corollary.

Corollary 4.7 *A compact topological ternary semigroup contains a minimal ideal, and hence, the kernel is non-empty.*

Proof Let S be a compact topological ternary semigroup. Since S is a left ideal of itself, by Theorem 3.14, S contains a minimal left ideal. Similarly using Note 3.15, we can say that S also contains a minimal right ideal. Since S contains a minimal left ideal and a minimal right ideal, by Theorem 4.5 (c), S contains a minimal two sided ideal K^*. But from Theorem 4.4, we have that $K = K^*$. Since K is the smallest ideal, first part is proved. Also K is non-empty because it contains at least one minimal left ideal and at least one minimal right ideal. □

Remark 4.8 It is to be noted that compactness is an essential condition for existence of minimal ideal. If this condition is not satisfied then the theorem is no longer valid. Consider the set $2\mathbb{Z} + 1$ of all odd integers with usual addition and equipped with the discrete topology. Then it is easy to see that it is a non-compact topological ternary semigroup without minimal ideal.

Now we characterize kernel. For this reason we need the following lemma.

Lemma 4.9 *Let S be a ternary semigroup and $e \in E(S)$. Then*
(a) e is a bi-unital element of eSe;
(b) $eSe = eSS \bigcap SSe$.

Proof

(a) Let e be any idempotent element of a ternary semigroup S. Let $x \in eSe$. Then there exists $s \in S$ such that $x = ese$. Now $eex = ee(ese) = (eee)se = ese = x$. Therefore e is a left identity element for eSe. Similarly we can show that e is a right identity element for eSe. Hence e is a bi-unital element of eSe.
(b) For any element $p \in eSe$, we have that $p = es_1e$ for some $s_1 \in S \Rightarrow p \in eSS$. Similarly $p \in eSe \Rightarrow p \in SSe$. Hence $p \in eSS \bigcap SSe$. Since p is any arbitrary element of eSe, we have that $eSe \subseteq eSS \bigcap SSe$. Again let $q \in eSS \bigcap SSe$. Then

$q = es_3s_4 = s_5s_6e$ for some $s_3, s_4, s_5, s_6 \in S$. Now $q = es_3s_4 = eees_3s_4 = ees_5s_6e \in eSSSe \subseteq eSe$. Since q is any arbitrary element of $eSS \bigcap SSe$ we have that $eSS \bigcap SSe \subseteq eSe$. Therefore $eSe = eSS \bigcap SSe$.

Following theorem gives us a characterization of kernel of a topological ternary semigroup.

Theorem 4.10 *Suppose that S is a compact topological ternary semigroup with the kernel K and there exists $e \in E(S) \cap K$. Then $x \in K$ if and only if xSx is a ternary group containing x.*

Proof Let us assume that xSx is a ternary group containing x. We now consider an element $a \in K$. Let us set $y = xax$. Then $y \in xSx$. Since xSx is a ternary group, then we have that $x = xy^{-1}y = xy^{-1}xax \in SSSKS \subseteq SKS \subseteq K$, as K is an ideal of S. Therefore $x \in K$ as desired.

Conversely, assume that $x \in K$. From Remark 4.6 and Theorem 4.5, there exists $L \in \mathscr{L}(S)$ and $R \in \mathscr{R}(S)$ such that $x \in L \cap R$. Consider the set RSL. It is non-empty because, $x^3 \in RSL$. Also $(RSL)^3 \subseteq RS^7L \subseteq RSL$ by repeated application of ternary semigroup property of S and the fact that $R, L \subseteq S$. Therefore RSL is a ternary subsemigroup of S. Let $a, b \in RSL$. Then $a, b \in RSS \subseteq R$, as R is a right ideal of S. Similarly $a, b \in L$ and hence $a, b \in R \cap L$. Now $abR \subseteq RSS \subseteq R$ and abR is a right ideal of S contained in R. So minimality of R implies $abR = R$. Similarly we can show that $Lab = L$ for all $a, b \in L$. Now $ab(RSL) = (abR)SL = RSL$. Similarly $(RSL)ab = RSL$. Therefore $ab(RSL) = (RSL)ab = RSL$ for all $a, b \in RSL$. Again $a(RSL)b \subseteq RSSSL \subseteq RSL$. On the other hand $RSL = ab(RSL) = ab(RSL)ab = a(bRSLa)b \subseteq a(RSSSL)b \subseteq a(RSL)b$. Therefore we have that $a(RSL)b = RSL$. So $ab(RSL) = a(RSL)b = (RSL)ab = RSL$ for all $a, b \in RSL$. Hence by Result 2.9, RSL is a ternary group. Since $e \in K$, by Theorem 4.5, $L = SSe$ and $R = eSS$. Again by Lemma 4.9, $L \cap R = SSe \cap eSS = eSe \subseteq RSL$. Also $RSL \subseteq L \cap R$. Combining we have that $RSL = eSe = L \cap R$. Therefore $x \in RSL$. Again $xSx \subseteq RSL$ and for $a \in RSL$, $a = xx^{-1}ax^{-1}x \in xSx$ as $x^{-1}ax^{-1} \in S$. Since $a \in RSL$ is any arbitrary element, it follows that $RSL \subseteq xSx$. Combining these two contentions we have that $RSL = xSx$. Hence xSx is a ternary group containing x. □

Corollary 4.11 *Suppose that S is a compact topological ternary semigroup with the kernel K and $e \in E(S) \cap K$. Then K is a union of pairwise disjoint ternary subgroups of S.*

Proof By Theorem 4.5(c) and Remark 4.6, $x \in K \Leftrightarrow x \in L$ and $x \in R$ for some $L \in \mathscr{L}(S)$ and $R \in \mathscr{R}(S)$. Therefore $K = \bigcup\{L \cap R : L \in \mathscr{L}(S) \text{ and } R \in \mathscr{R}(S)\}$.

Let us assume that $L_1, L_2 \in \mathscr{L}(S)$ and $R_1, R_2 \in \mathscr{R}(S)$. According to the proof of Theorem 4.10, $L_1 \cap R_1$ and $L_2 \cap R_2$ are two ternary subgroups of S. If possible let $x \in (L_1 \cap R_1) \cap (L_2 \cap R_2)$. Then $x \in L_1 \cap L_2$ and $x \in R_1 \cap R_2$. But $L_1 \cap L_2$ is a left ideal of S and $L_1 \cap L_2$ is contained in L_1 and L_2. But minimality of L_1 and L_2 imply

$L_1 = L_1 \cap L_2 = L_2$. Similarly we can show that $R_1 = R_2$. Hence $L_1 \cap R_1 = L_2 \cap R_2$. Hence the result. □

Corollary 4.12 *Suppose that S is a compact topological ternary semigroup with the kernel K and $e \in K$. Then SSe and eSS are respectively minimal left ideal and minimal right ideal of S.*

Proof Given that K is the kernel of S with $e \in K$. Now SSe is a left ideal of S. Let us assume that L is a left ideal of S contained in SSe. Now we consider $x \in L$. Also $e \in K$ implies that eSe is a ternary group containing e [by Theorem 4.10]. It is to be noted that e is a bi-unital element of eSe [by Lemma 4.9]. Now $exe \in eSe$. So it has an inverse, say, y in eSe. Then for $a, b \in S$, we have that $abe = aby(exe)e = abye(eexee) = abyex \in SSSSL \subseteq SSL \subseteq L$ (applying cancellation property), as L is a left ideal and $x \in L$. Now $abe \in SSe \Rightarrow SSe \subseteq L$. Hence $SSe = L$. Similarly we can show that $eSS = R$ is a minimal right ideal of S. □

In the beginning of this section it is mentioned when kernel becomes a ternary group. Here is another result that also shows when kernel K becomes a ternary group.

Theorem 4.13 *Suppose that S is a compact topological ternary semigroup with kernel K and there exists $e \in E(S) \cap K$. Then the following conditions are equivalent*:

(a) K is a ternary group;
(b) S has only one minimal left ideal and one minimal right ideal.

Proof $(a) \Rightarrow (b)$ Let us assume that K is a ternary group with $e \in K$. Then we have that from Corollary 4.12, $L = SSe$ is a minimal left ideal and $R = eSS$ is a minimal right ideal of S. Now $SSe \subseteq SSK \subseteq K$ as $e \in K$ and K is an ideal of S. Again let us assume that $p \in K$ is any element. Since K is a ternary group and $e \in K$, e is a bi-unital element of K [For any element $x \in K$, $ee^3x = eex \Rightarrow ee(eex) = eex \Rightarrow eex = x$, applying left cancellation property of K. So e is a left identity of K. Similarly we can show that e is a right identity of S. So e is a bi-unital element of K.]. Then we have that $x = xee \in SSe$. Since $x \in K$ is any arbitrary element, it follows that $K \subseteq SSe$. Therefore $K = SSe$. So SSe is the only left ideal of S. Similarly we can show that eSS is the only minimal right ideal of S with $K = eSS$. Hence the proof.

$(b) \Rightarrow (a)$ Let us assume that S has only one minimal left ideal L and one minimal right ideal R. But $K = \cup\{L : L \in \mathscr{L}(S)\} = \cup\{R : R \in \mathscr{R}(S)\}$ by Remark 4.6 and Theorem 4.5. Then we have that $K = L = R = L \cap R$ and by proof of Theorem 4.10, $L \cap R$ is a ternary group. Thus K is a ternary group. □

It is well known that idempotent elements play important role in topological ternary semigroup theory. Here we state necessary and sufficient condition that kernel of a topological ternary semigroup S contains idempotent elements of S.

Theorem 4.14 *Let S be a compact topological ternary semigroup and $e \in E(S)$. Then the following conditions are equivalent*:

(a) *$K = SeS$;*
(b) *e belongs to K;*
(c) *eSe is a ternary group.*

Proof $(a) \Leftrightarrow (b)$ Let $K = SeS$. Then it is obvious that $e = eee \in SeS = K$. Conversely, let $e \in K$. Then SeS is an ideal of S, since $SS(SeS) = (SSS)eS \subseteq SeS$, $(SeS)SS = Se(SSS) \subseteq SeS$ and $S(SeS)S = S(SeeeS)S \subseteq SSSeSSS \subseteq SeS$. Also $SeS \subseteq SKS \subseteq K$ as $e \in K$ and K is an ideal of S. But minimality of K implies that $K = SeS$.

$(b) \Leftrightarrow (c)$ Since S is a compact topological ternary semigroup and $e \in K$, according to Theorem 4.10, eSe is a ternary group containing e. On the other hand, let eSe be a ternary group. Now from Lemma 4.9, $eSe = SSe \cap eSS$. Now from last part of proof of Corollary 4.12, we have that SSe and eSS are respectively minimal left and minimal right ideals of S each containing e. So by Theorem 4.5(c) and Remark 4.6, we have that $eSe \subseteq K$. Therefore $e \in K$.

In preliminaries section we defined $\mathscr{L}(S)$ and $\mathscr{R}(S)$. In this section the kernel of a ternary semigroup is defined. In next theorem, with the help of some additional conditions (algebraic and topological), we obtain expressions of $\mathscr{L}(S)$, $\mathscr{R}(S)$ and the kernel. For this purpose we need the following definition.

Definition 4.15 [20] A topological ternary semigroup S which is also a ternary group is called a *paratopological ternary group*.

Theorem 4.16 *Let S be a compact paratopological ternary group. Also let $L \in \mathscr{L}(S)$, $R \in \mathscr{R}(S)$ and $e \in L \cap R$ be an idempotent element. Then*

(a) $L \cap R$ is a ternary subgroup of S;
(b) $\mathscr{L}(S) = \{SSe : e \in K \cap E(S)\}$, $\mathscr{R}(S) = \{eSS : e \in K \cap E(S)\}$;
(c) if $H(e)$ is a maximal ternary subgroup of S containing e, then $H(e) = eSe$ and $K = \bigcup\{H(e) : e \in K \cap E(S)\}$.

Proof (a) Let us consider $L \in \mathscr{L}(S)$ and $R \in \mathscr{R}(S)$. Then we have that $RSL \subseteq RSS \subseteq R$. Similarly $RSL \subseteq SSL \subseteq L$. Then we have that $RSL \subseteq R \cap L$. Therefore $L \cap R$ is non-empty. Now for $x, y, z \in L \cap R$ we have that $xyz \in L \cap R$. Therefore $L \cap R$ is a ternary subsemigroup of S. Again for all $x, y \in L \cap R$ we have that $xy(L \cap R) \subseteq L \cap R$. Now let $p \in L \cap R$. Then $p = xyy^{-1}x^{-1}p \in xy(L \cap R)$ because, $y^{-1}x^{-1}p \in SSL \subseteq L$ and $y^{-1}x^{-1}p = xx^{-1}y^{-1}x^{-1}p \in RSSSS \subseteq RSS \subseteq R$. Since $p \in L \cap R$ is an arbitrary element, we have that $L \cap R \subseteq xy(L \cap R)$. Therefore $xy(L \cap R) = L \cap R$. Similarly we can show that $(L \cap R)xy = L \cap R$. Again we have that $x(L \cap R)y \subseteq L \cap R$. On the other hand, let $p \in L \cap R$. Then we have that $p = xx^{-1}py^{-1}y$. Now $x^{-1}py^{-1} = xx^{-1}x^{-1}py^{-1} \in RSSSS \subseteq RSS \subseteq R$, since R is a right ideal of S. Similarly we can show that $x^{-1}py^{-1} \in L$. Therefore $p \in x(L \cap R)y$. Since $p \in L \cap R$ is an arbitrary element, we have that $L \cap R \subseteq x(L \cap R)y$. Since x, y are any two arbitrary elements of $L \cap R$, we have that $xy(L \cap R) = x(L \cap R)y = (L \cap R)xy = L \cap R$, for all $x, y \in L \cap R$. Hence $L \cap R$ is a ternary subgroup of S (by Result 2.9).

(b) From Theorem 4.5(b), if $L \in \mathscr{L}(S)$ then $L = LLx = SSx$ for all $x \in L$. Given that $e \in L \cap R$ be an idempotent element. Now by Remark 4.6, $e \in K$. Therefore

$e \in K \cap E(S)$. Then we have that $L = LLe = SSe$. Similarly we can have that $R = RRe = SSe$. Hence the result follows.

(c) We see that SSe is a left ideal of S and for $e \in L \in \mathscr{L}(S)$ we have that $SSe \subseteq SSL \subseteq L$. Then $SSe = L$. Now $L \cap R = SSe \cap eSS = eSe$. Therefore eSe is a ternary subgroup of S, since $L \cap R$ is a ternary subgroup of S [by (a)]. Then $H(e) = eH(e)e \subseteq eSe = L \cap R$. So $H(e) = L \cap R = eSe$, $e \in K \cap E(S)$ [by (b) and maximality of $H(e)$]. Now from Corollary 4.11, $K = \bigcup\{L \cap R : L \in \mathscr{L}(S), R \in \mathscr{R}(S)\}$ and each $L \cap R$ is a ternary subgroup of S. Since $H(e) = L \cap R$, so $K = \bigcup\{H(e) : e \in K \cap E(S)\}$. □

We conclude by proving the equivalence of the following algebraic conditions on a compact paratopological ternary group which shows the relationship between minimal left ideal, minimal ideal and maximal ternary subgroup with the help of previous results. It is to be noted that compactness condition is not used directly throughout the proof. It is used to ensure the existence of a minimal left ideal and the kernel in the theorem given below.

Theorem 4.17 *Let S be a compact paratopological ternary group and $e \in E(S)$. Then the following conditions are equivalent*:

(a) SSe is a minimal left ideal of S;
(b) SeS is the kernel of S;
(c) eSe is a maximal ternary subgroup of S.

Proof $(a) \Rightarrow (b)$ If SSe is a minimal left ideal of S, then we have that $SSe \subseteq K$, by Theorem 4.5 and Remark 4.6. Obviously $e \in K$. Now we note that SeS is an ideal of S, since $SS(SeS) = (SSS)eS \subseteq SeS$, as S is ternary semigroup; $(SeS)SS \subseteq Se(SSS) \subseteq SeS$ and $S(SeS)S = S(SeeeS)S \subseteq S(SSeSS)S = (SSS)e(SSS) \subseteq SeS$. Again $SeS \subseteq SKS \subseteq K$, as K is an ideal of S. Then minimality of K implies that $SeS = K$. Hence SeS is the smallest ideal of S.

$(b) \Rightarrow (c)$ Let SeS be the kernel of S. Therefore $SeS = K$. Then we have that $e \in K$. Now from proof of Theorem 4.16 (c), we have that the kernel K is the union of $H(e)$ where $H(e) = eSe$ is a maximal ternary subgroup of S with $e \in K \cap E(S)$. Hence eSe is a maximal ternary subgroup of S.

$(c) \Rightarrow (a)$ Suppose that eSe is a maximal ternary subgroup of S. Also let L be a non-empty left ideal of S properly contained in SSe. We claim that $L \cap eSS$ is non-empty. To justify this, if possible let us assume that $L \cap eSS = \emptyset$. Then there exists a proper subset M of SSe such that $L \cup M = SSe$. Now $eSe = SSe \cap eSS = (L \cup M) \cap eSS = (L \cap eSS) \cup (M \cap eSS) = \emptyset \cup (M \cap eSS) = M \cap eSS \subsetneq SSe \cap eSS = eSe$, which is a contradiction. So our claim that $L \cap eSS$ is non-empty is justified. Let $a \in L \cap eSS$. Then we have that $a \in SSe \cap eSS$. Again $SSe \cap eSS = eSe$ [by Lemma 4.9(b)]. Now there exists an element $a^{-1} \in eSe$ such that $aa^{-1}x = x$ for all $x \in eSe$, as it is a ternary subgroup of S. This implies that $x = aa^{-1}x = xa^{-1}a \in SSL \subseteq L$. In particular $e = eaa^{-1} = ea^{-1}a \in SSL \subseteq L$. But any $x \in SSe$ implies that $x = xee \in SSL \subseteq L \Rightarrow SSe \subseteq L \Rightarrow SSe = L$ which is a contradiction. Therefore SSe is a minimal left ideal of S. □

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