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ORIGINAL ARTICLE

A new arithmetical function and its mean value properties

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Abstract

In this article, we use elementary methods and the estimate for character sums to study the mean value properties of a new arithmetical function, and obtain a sharp asymptotic formula for it.

Keywords Arithmetical function · Asymptotic formula · Mean value · Primitive roots · The estimate for character sums

Mathematics Subject Classification 11A07 · 11A25

1 Introduction

Let p and q be two fixed odd primes, $\mathbb{A}(p)$ denotes the set of all primitive roots modulo p. As for the definition of the primitive roots and its properties, refer to references [3, 8, 11, 21] and [22], which will not be stated here. For any positive integer n, we define a new arithmetical function $d(n; p, q)$ as follows:

$$d(n;p,q) = \sum_{\substack{ab=n \\ a\in\mathbb{A}(p),\ b\in\mathbb{A}(q)}} 1.$$

If $p = q$, then we write $d(n;p,p) = d(n;p)$ for convenience. For example, $d(4;3,5) = 1$, since $4 = 2 \times 2$, $2 \in \mathbb{A}(3)$ and $2 \in \mathbb{A}(5)$; $d(6;3,5) = 1$, since

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$6 = 2 \times 3$, $2 \in \mathbb{A}(3)$ and $3 \in \mathbb{A}(5)$. If n is any prime, then we have $d(n; 3, 5) = 0$. As we can see from these examples, the function $d(n;\, p,\, q)$ has an irregular distribution of values, and its value is zero on many integers n. Since this function is new, we know very little about its arithmetical properties. From the form, $d(n;\, p,\, q)$ is a bit like the divisor function $d(n)$, but there are fundamental differences between them. Because if n is any prime, then we have $d(n; p, q) = 0$ and $d(n) = 2$. And, of course, there are similarities between $d(n;\, p,\, q)$ and $d(n)$. For example, their mean values, which have almost similar asymptotic properties. The main purpose of this article is to illustrate this point. Why do we want to study the mean value properties of $d(n;\, p,\, q)$? The problem begins with a recent paper by the second author Teerapat Srichan. In fact, Teerapat Srichan [14] used the analytic methods to study the estimate problem of one kind of character sums:

$$S_{\chi_1,\chi_2}(x) = \sum_{ab \le x} \chi_1(a)\chi_2(b),$$

and obtained a sharp upper bound estimate for it (see Lemma 2 below), where χ_i is a primitive character modulo p_i, $i = 1,\ 2$, p_1 and p_2 are two positive integers with $\sqrt{x} \ge \max\{p_1, p_2\}$.

The main purpose of this article is to give a specific application of [14]. Based on this idea, we studied the mean value properties of $d(n;\, p,\, q)$, and proved the following main result:

Theorem 1 *Let p and q be two fixed odd primes. Then for any positive number $\sqrt{x} \ge \max\{p, q\}$, we have the asymptotic formula*

$$\begin{aligned}\sum_{n \le x} d(n; p, q) &= \frac{\phi(p-1)\phi(q-1)}{pq} \cdot x \cdot \left(\ln x + 2\gamma - 1 + \frac{\ln p}{p-1} + \frac{\ln q}{q-1}\right) \\ &+ \frac{\phi(p-1)\phi(q-1)}{q(p-1)} \cdot x \cdot \sum_{\substack{h \mid p-1 \\ h > 1}} \frac{\mu(h)}{\phi(h)} \sideset{}{'}\sum_{\chi^h = \chi_p^0} L(1, \chi) \\ &+ \frac{\phi(p-1)\phi(q-1)}{p(q-1)} \cdot x \cdot \sum_{\substack{k \mid q-1 \\ k > 1}} \frac{\mu(k)}{\phi(k)} \sideset{}{'}\sum_{\chi^k = \chi_q^0} L(1, \chi) \\ &+ O\left(x^{\frac{1}{3}} p^{\frac{5}{9}} q^{\frac{7}{9}} 2^{\omega(p-1)+\omega(q-1)} \ln p\right) + O\left(x^{\frac{1}{3}} q^{\frac{5}{9}} p^{\frac{7}{9}} 2^{\omega(p-1)+\omega(q-1)} \ln q\right),\end{aligned}$$

where $\phi(n)$ and γ denote the Euler function and Euler constant, respectively, $\mu(n)$ is the Möbius function, $\omega(n)$ denotes the number of all distinct prime divisors of n, $L(s, \chi_p)$ denotes the Dirichlet L-function corresponding to character χ_p modulo p, $\sideset{}{'}\sum_{\chi^h = \chi_p^0}$ denotes the summation over all h-order characters modulo p, and χ_p^0 denotes the principal character modulo p.

If $p = q$, then from this theorem we may immediately deduce the following:

Corollary 1 *Let p be fixed odd prime. Then for any $x > p^2$, we have the asymptotic formula*:

$$\sum_{n\leq x} d(n;p) = \frac{\phi^2(p-1)}{p^2}\cdot x\cdot\left(\ln x + 2\gamma - 1 + \frac{2\ln p}{p-1}\right)$$
$$+ 2\cdot\frac{\phi^2(p-1)}{p(p-1)}\cdot x\cdot \sum_{\substack{h|p-1\\ h>1}} \frac{\mu(h)}{\phi(h)} \sum_{\chi^h=\chi_p^0}{}' L(1,\chi) + O\left(x^{\frac{1}{3}}p^{\frac{4}{3}}4^{\omega(p-1)}\ln p\right).$$

Some notes: It is clear that if $\sqrt{x} > \max\{p,q\}$, then our asymptotic formula is nontrivial. In addition, if one can give a valid calculation formula for the mean value of the Dirichlet L-functions

$$\sum_{\chi^h=\chi_p^0}{}' L(1,\chi) \text{ and } \sum_{\chi^k=\chi_q^0}{}' L(1,\chi),$$

that will make our theorem more concise and beautiful.

2 Several lemmas

To complete the proof of our main result, we need following several simple lemmas. Of course, some of the lemmas can be found in the references, and the proofs of several other lemmas require some elementary and analytic number theory knowledge. In particular, the contents of primitive roots and Dirichlet characters modulo p are required. All these can be found in references [3, 11] and [21]. We also provide several relevant references [1, 2, 4, 6, 7, 10, 13] and [15-20] for interested readers. First we have the following:

Lemma 1 *Let p be an odd prime, then for any integer a coprime to p (i.e., $(a,p)=1$), we have the identity*

$$\frac{\phi(p-1)}{p-1}\sum_{k|p-1}\frac{\mu(k)}{\phi(k)}\sum_{r=1}^{k}{}' e\left(\frac{r\cdot \mathrm{ind}(a)}{k}\right) = \begin{cases} 1 & \text{if } a \text{ is a primitive root } \bmod p; \\ 0 & \text{if } a \text{ is not a primitive root } \bmod p, \end{cases}$$

where $e(y) = e^{2\pi i y}$, $\sum_{r=1}^{k}{}'$ denotes the summation over all integers $1\leq r\leq k$ such that r is coprime to k, $\mu(n)$ is the Möbius function, and $\mathrm{ind}(a)$ denotes the index of a relative to some fixed primitive root $g \bmod p$.

Proof See Proposition 2.2 in reference [11] . □

Lemma 2 *For $x > 1$ we have*

$$\sum_{n\le x}\frac{1}{n}=\ln x+\gamma-\psi(x)x^{-1}+O(x^{-2}).$$

Proof See Equation 14.41 in reference [9]. □

Lemma 3 *Let l and k be positive integers. For $x>1$ we have*

$$\sum_{\substack{n\le x\\ n\equiv l \bmod k}}\frac{1}{n}=\frac{\ln x}{k}+\frac{\gamma(l,k)}{k}-\psi\Big(\frac{x-l}{k}\Big)x^{-1}+O(kx^{-2}),$$

where

$$\gamma(l,k)=\lim_{x\to\infty}\left(\sum_{\substack{n\le x\\ n\equiv l \bmod k}}\frac{1}{n}-\frac{\ln x}{k}\right).$$

Proof It follows form using Euler's summation formula. □

Lemma 4 *For any non-principal character χ modulo k. Define for $0<a\le 1$*

$$\gamma(a)=\lim_{N\to\infty}\Big(\sum_{n=0}^{\infty}\frac{1}{n+a}-\ln(N+a)\Big).$$

Then

$$L(1,\chi)=\frac{1}{k}\sum_{n=1}^{k-1}\chi(n)\gamma(n/k).$$

Proof See Lemma 5 in reference [5]. □

Lemma 5 *I. Let $\omega\in\mathbb{R}$, $\omega>0$, $q\in\mathbb{N}$, $q\ge 2$. If f(n) is an arithmetic function, then*

$$\sum_{n\le\omega,\,(n,q)=1}f(n)=\sum_{d|q}\mu(d)\sum_{m\le\omega/d}f(md).$$

Proof See Lemma 13 in reference [12]. □

II. For real $x>1$ and any non-principal character χ modulo q, we have

$$\sum_{k \le x} \chi(k) = \sum_{j \le q} \chi(j) \left\lfloor \frac{x}{q} - \frac{j}{q} + 1 \right\rfloor, \tag{1}$$

and for any arithmetical function f,

$$\sum_{k \le x} \chi(k) \cdot f(k) = \sum_{j \le q} \chi(j) \sum_{\substack{k \le x \\ k \equiv j \bmod q}} f(k). \tag{2}$$

Proof The equations (1) and (2) follow from the periodicity of the primitive character modulo q. Namely,

$$\begin{aligned} \sum_{a \le z} \chi(a) &= \sum_{j \le q} \sum_{\substack{a \le z \\ a \equiv j \bmod q}} \chi(a) = \sum_{j \le q} \sum_{\substack{a \le z \\ a \equiv j \bmod q}} \chi(j) \\ &= \sum_{j \le q} \chi(j) \sum_{\substack{a \le z \\ a \equiv j \bmod q}} 1 = \sum_{j \le q} \chi(j) \left\lfloor \frac{z}{q} - \frac{j}{q} + 1 \right\rfloor. \end{aligned}$$

□

III. Let x, η, α, ω be real numbers with $x \ge 1, \alpha > 0, \eta \ge 1$, let j and q be positive integers with $1 \le j \le q$, let (k, ℓ) be an exponent pair with $k > 0$ and let

$$R(x, \eta, \alpha; q, j; \omega) = \sum_{\substack{n \le \eta \\ n \equiv j \bmod q}} \psi\left(\frac{x}{n^\alpha} + \omega\right),$$

where ω is independent of n. Then

$$R(x, \eta, \alpha; q, j; \omega) = O(1) + O(x^{-1/2} \eta^{1+\alpha/2} q^{-1}) + \begin{cases} O\left(x^{\frac{k}{k+1}} \eta^{\frac{\ell - \alpha k}{k+1}} q^{\frac{-\ell}{k+1}}\right) & \text{for } \ell > \alpha k, \\ O\left(x^{\frac{k}{k+1}} \ln \eta \, q^{\frac{-\alpha k}{k+1}}\right) & \text{for } \ell = \alpha k, \\ O\left((x q^{-\alpha})^{\frac{k}{1+(1+\alpha)k-\ell}}\right) & \text{for } \ell < \alpha k, \end{cases}$$

where the constants in the O-symbols depend on only α.

Proof See Lemma 17 in reference [12]. □

Lemma 6 *Let p and q be two fixed primes, χ_p^0 be the principal character modulo p and χ_q^0 be the principal character modulo q. Then for any real number $x > \max\{p, q\}$, we have the asymptotic formula*

$$\sum_{ab \le x} \chi_p^0(a)\chi_q^0(b) = \frac{(p-1)(q-1)}{pq} \cdot x \cdot \left(\ln x + 2\gamma - 1 + \frac{\ln p}{p-1} + \frac{\ln q}{q-1} \right) + O(x^{1/3}q^{7/9}p^{5/9} + x^{1/3}p^{7/9}q^{5/9}).$$

Proof For any $x > 1$,

$$\begin{aligned} &\sum_{ab \le x} \chi_p^0(a)\chi_q^0(b) \\ &= \sum_{a \le \sqrt{x}} \chi_p^0(a) \sum_{b \le \frac{x}{a}} \chi_q^0(b) + \sum_{b \le \sqrt{x}} \chi_q^0(b) \sum_{a \le \frac{x}{b}} \chi_p^0(a) - \sum_{a \le \sqrt{x}} \chi_p^0(a) \sum_{b \le \sqrt{x}} \chi_q^0(b) \\ &= \sum_{\substack{a \le \sqrt{x} \\ (a,p)=1}} \sum_{\substack{b \le \frac{x}{a} \\ (b,q)=1}} 1 + \sum_{\substack{b \le \sqrt{x} \\ (b,q)=1}} \sum_{\substack{a \le \frac{x}{b} \\ (a,p)=1}} 1 - \sum_{\substack{a \le \sqrt{x} \\ (a,p)=1}} 1 \sum_{\substack{b \le \sqrt{x} \\ (b,q)=1}} 1. \end{aligned}$$

In view of I in Lemma 5, we have

$$\begin{aligned} &\sum_{ab \le x} \chi_p^0(a)\chi_q^0(b) \\ &= \sum_{\substack{a \le \sqrt{x} \\ (a,p)=1}} \sum_{d|q} \mu(d) \left\lfloor \frac{x}{ad} \right\rfloor + \sum_{\substack{b \le \sqrt{x} \\ (b,q)=1}} \sum_{t|p} \mu(t) \left\lfloor \frac{x}{bt} \right\rfloor \\ &\quad - \Big(\sum_{t|p} \mu(t) \left\lfloor \frac{\sqrt{x}}{t} \right\rfloor \Big) \Big(\sum_{d|q} \mu(d) \left\lfloor \frac{\sqrt{x}}{d} \right\rfloor \Big) \\ &= \sum_{d|q} \sum_{t|p} \mu(d)\mu(t) \sum_{a \le \sqrt{x}/t} \left\lfloor \frac{x}{atd} \right\rfloor + \sum_{d|q} \sum_{t|p} \mu(d)\mu(t) \sum_{b \le \sqrt{x}/d} \left\lfloor \frac{x}{bdt} \right\rfloor \\ &\quad - \Big(\sum_{t|p} \mu(t) \left\lfloor \frac{\sqrt{x}}{t} \right\rfloor \Big) \Big(\sum_{d|q} \mu(d) \left\lfloor \frac{\sqrt{x}}{d} \right\rfloor \Big). \end{aligned}$$

In view of $\lfloor z \rfloor = z - \psi(z) - \frac{1}{2}$, we have

$$\begin{aligned}
&\sum_{ab\le x}\chi_p^0(a)\chi_q^0(b)\\
&=\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{a\le\sqrt{x}/t}\left(\frac{x}{atd}-\psi\left(\frac{x}{atd}\right)-\frac{1}{2}\right)\\
&\quad+\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{b\le\sqrt{x}/d}\left(\frac{x}{bdt}-\psi\left(\frac{x}{bdt}\right)-\frac{1}{2}\right)\\
&\quad-\left(\sum_{t|p}\mu(t)\left(\frac{\sqrt{x}}{t}-\psi\left(\frac{\sqrt{x}}{t}\right)-\frac{1}{2}\right)\right)\left(\sum_{d|q}\mu(d)\left(\frac{\sqrt{x}}{d}-\psi\left(\frac{\sqrt{x}}{d}\right)-\frac{1}{2}\right)\right).
\end{aligned}$$

In view of $\sum_{d|q}\mu(d)=0$, when $q>1$, we have

$$\begin{aligned}
&\sum_{ab\le x}\chi_p^0(a)\chi_q^0(b)\\
&=\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{a\le\sqrt{x}/t}\left(\frac{x}{atd}-\psi\left(\frac{x}{atd}\right)\right)+\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\\
&\quad\sum_{b\le\sqrt{x}/d}\left(\frac{x}{bdt}-\psi\left(\frac{x}{bdt}\right)\right)\\
&\quad-\left(\sum_{t|p}\mu(t)\left(\frac{\sqrt{x}}{t}-\psi\left(\frac{\sqrt{x}}{t}\right)\right)\right)\left(\sum_{d|q}\mu(d)\left(\frac{\sqrt{x}}{d}-\psi\left(\frac{\sqrt{x}}{d}\right)\right)\right)\\
&=x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}\sum_{a\le\sqrt{x}/t}\frac{1}{a}-\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{a\le\sqrt{x}/t}\psi\left(\frac{x}{atd}\right)\\
&\quad+x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}\sum_{b\le\sqrt{x}/d}\frac{1}{b}-\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{b\le\sqrt{x}/d}\psi\left(\frac{x}{btd}\right)\\
&\quad-x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}+\sqrt{x}\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\mu(t)\psi\left(\frac{\sqrt{x}}{t}\right)\\
&\quad+\sqrt{x}\sum_{d|q}\sum_{t|p}\frac{\mu(t)}{t}\mu(d)\psi\left(\frac{\sqrt{x}}{d}\right)+O((pq)^{\epsilon}).
\end{aligned}$$

In view of Lemma 2, we have

$$
\begin{aligned}
&\sum_{ab\le x}\chi_p^0(a)\chi_q^0(b)\\
&= x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}\left(\ln\sqrt{x}/t+\gamma-\psi(\sqrt{x}/t)(\sqrt{x}/t)^{-1}+O((\sqrt{x}/t)^{-2})\right)\\
&\quad-\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{a\le\sqrt{x}/t}\psi\left(\frac{x}{atd}\right)\\
&\quad+x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}\left(\ln\sqrt{x}/d+\gamma-\psi(\sqrt{x}/d)(\sqrt{x}/d)^{-1}+O((\sqrt{x}/d)^{-2})\right)\\
&\quad-\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{b\le\sqrt{x}/d}\psi\left(\frac{x}{btd}\right)\\
&\quad-x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}+\sqrt{x}\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\mu(t)\psi\left(\frac{\sqrt{x}}{t}\right)\\
&\quad+\sqrt{x}\sum_{d|q}\sum_{t|p}\frac{\mu(t)}{t}\mu(d)\psi\left(\frac{\sqrt{x}}{d}\right)+O((pq)^{\epsilon})\\
&= x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}\left(\ln\sqrt{x}/t+\gamma\right)-\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{a\le\sqrt{x}/t}\psi\left(\frac{x}{atd}\right)\\
&\quad+x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}\left(\ln\sqrt{x}/d+\gamma\right)-\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{b\le\sqrt{x}/d}\psi\left(\frac{x}{btd}\right)\\
&\quad-x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}+O((pq)^{\epsilon}+p\ln q+q\ln p).
\end{aligned}
$$

Now we use III in Lemma 2 with the exponent pair (2/7, 4/7) to bound the two sums involving function $\psi(.)$. Thus, we have

$$
\begin{aligned}
&\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{a\le\sqrt{x}/t}\psi\left(\frac{x}{atd}\right)\\
&\quad=\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)R\left(\frac{x}{td},\frac{x^{1/2}}{d},1,1,1,0\right)\\
&\quad\ll\sum_{d|q}\sum_{t|p}\left(1+\left(\frac{x}{td}\right)^{-1/2}\left(\frac{x^{1/2}}{d}\right)^{3/2}+\left(\frac{x}{td}\right)^{2/9}\left(\frac{x^{1/2}}{d}\right)^{2/9}\right)\\
&\quad\ll\sum_{d|q}\sum_{t|p}\left(1+x^{1/4}t^{1/2}d^{-1}+x^{1/3}t^{-2/9}d^{-4/9}\right)\\
&\quad=O((pq^{\epsilon})+x^{1/4}p^{3/2}\log q+x^{1/3}p^{7/9}q^{5/9}).
\end{aligned}
$$

In the same way, we also have

$$\sum_{d|q}\sum_{t|p}\mu(d)\mu(t)\sum_{b\le\sqrt{x}/d}\psi\left(\frac{x}{btd}\right)$$
$$=O((pq^{\epsilon})+x^{1/4}q^{3/2}\ln p+x^{1/3}q^{7/9}p^{5/9}).$$

Thus, we have

$$\begin{aligned}\sum_{ab\le x}\chi_p^0(a)\chi_q^0(b)
&=x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}\left(\ln\sqrt{x}/t+\gamma\right)\\
&\quad+x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}\left(\ln\sqrt{x}/d+\gamma\right)\\
&\quad-x\sum_{d|q}\sum_{t|p}\frac{\mu(d)}{d}\frac{\mu(t)}{t}+O((pq)^{\epsilon}\\
&\quad+p\ln q+q\ln p+x^{1/3}q^{7/9}p^{5/9}+x^{1/3}p^{7/9}q^{5/9}).\end{aligned}$$

The complete proof follows from the identity $\sum_{d|q}\frac{\mu(d)\ln d}{d}=-\frac{\phi(q)}{q}\frac{\ln q}{q-1}$. □

Lemma 7 *Let p and q be two fixed primes, χ_p^0 be the principal character modulo p, χ_q be the non-principal character modulo q. Then for any real number $x>\max\{p,q\}$, we have the asymptotic formula*

$$\sum_{ab\le x}\chi_p^0(a)\chi_q(b)=\frac{p-1}{p}\cdot x\cdot L(1,\chi_q)+O\left(x^{1/3}q^{7/9}p^{5/9}+x^{1/3}p^{7/9}q^{5/9}\right),$$

where $L(1,\chi_q)$ denotes the Dirichlet L-function corresponding to χ_q.

Proof For any $x>1$,

$$\begin{aligned}\sum_{ab\le x}\chi_p^0(a)\chi_q(b)&=\sum_{a\le\sqrt{x}}\chi_p^0(a)\sum_{b\le\frac{x}{a}}\chi_q(b)\\
&\quad+\sum_{b\le\sqrt{x}}\chi_q(b)\sum_{a\le\frac{x}{b}}\chi_p^0(a)-\sum_{a\le\sqrt{x}}\chi_p^0(a)\sum_{b\le\sqrt{x}}\chi_q(b)\\
&=\sum_{\substack{a\le\sqrt{x}\\(a,p)=1}}\sum_{b\le\frac{x}{a}}\chi_q(b)+\sum_{b\le\sqrt{x}}\chi_q(b)\sum_{\substack{a\le\frac{x}{b}\\(a,p)=1}}1-\sum_{\substack{a\le\sqrt{x}\\(a,p)=1}}1\sum_{b\le\sqrt{x}}\chi_q(b).\end{aligned}$$

In view of I in Lemma 2, we have

$$\sum_{ab\leq x}\chi_p^0(a)\chi_q(b)=\sum_{d|p}\mu(d)\sum_{a\leq\frac{\sqrt{x}}{d}}\sum_{b\leq\frac{x}{ad}}\chi_q(b)+\sum_{d|p}\mu(d)\sum_{b\leq\sqrt{x}}\chi_q(b)\left\lfloor\frac{x}{bd}\right\rfloor$$
$$-\left(\sum_{d|p}\mu(d)\left\lfloor\frac{\sqrt{x}}{d}\right\rfloor\right)\sum_{b\leq\sqrt{x}}\chi_q(b).$$

From (1) and (2) in Lemma 2, we have

$$\sum_{ab\leq x}\chi_p^0(a)\chi_q(b)=\sum_{d|p}\mu(d)\sum_{j\leq q}\chi_q(j)\sum_{a\leq\frac{\sqrt{x}}{d}}\left\lfloor\frac{x}{aqd}-\frac{j}{q}+1\right\rfloor$$
$$+\sum_{d|p}\mu(d)\sum_{j\leq q}\chi_q(j)\sum_{\substack{b\leq\sqrt{x}\\ b\equiv j \bmod q}}\left\lfloor\frac{x}{bd}\right\rfloor$$
$$-\left(\sum_{d|p}\mu(d)\left\lfloor\frac{\sqrt{x}}{d}\right\rfloor\right)\left(\sum_{j\leq q}\chi_q(j)\left\lfloor\frac{\sqrt{x}}{q}-\frac{j}{q}+1\right\rfloor\right).$$

From $\lfloor z\rfloor=z-\psi(z)-\frac{1}{2}$ and $\psi(z+1)=\psi(z)$, we have

$$\sum_{ab\leq x}\chi_p^0(a)\chi_q(b)=\sum_{d|p}\mu(d)\sum_{j\leq q}\chi_q(j)\sum_{a\leq\frac{\sqrt{x}}{d}}\left(\frac{x}{aqd}-\frac{j}{q}+\frac{1}{2}-\psi\left(\frac{x}{aqd}-\frac{j}{q}\right)\right)$$
$$+\sum_{d|p}\mu(d)\sum_{j\leq q}\chi_q(j)\sum_{\substack{b\leq\sqrt{x}\\ b\equiv j \bmod q}}\left(\frac{x}{bd}-\frac{1}{2}-\psi\left(\frac{x}{bd}\right)\right)$$
$$-\left(\sum_{d|p}\mu(d)\left(\frac{\sqrt{x}}{d}-\frac{1}{2}-\psi\left(\frac{\sqrt{x}}{d}\right)\right)\right)$$
$$\left(\sum_{j\leq q}\chi_q(j)\left(\frac{\sqrt{x}}{q}-\frac{j}{q}+\frac{1}{2}-\psi\left(\frac{\sqrt{x}}{q}-\frac{j}{q}\right)\right)\right).$$

In view of $\sum_{j\leq q}\chi_q(j)=0$ and $\sum_{d|p}\mu(d)=0$, we have

$$
\begin{aligned}
&\sum_{ab\le x}\chi_p^0(a)\chi_q(b)\\
&\quad=\sum_{d|p}\mu(d)\sum_{j\le q}\chi_q(j)\sum_{a\le\frac{\sqrt{x}}{d}}\left(-\frac{j}{q}-\psi\left(\frac{x}{aqd}-\frac{j}{q}\right)\right)\\
&\qquad+\sum_{d|p}\mu(d)\sum_{j\le q}\chi_q(j)\sum_{\substack{b\le\sqrt{x}\\ b\equiv j \bmod q}}\left(\frac{x}{bd}-\psi\left(\frac{x}{bd}\right)\right)\\
&\qquad-\left(\sum_{d|p}\mu(d)\left(\frac{\sqrt{x}}{d}-\psi\left(\frac{\sqrt{x}}{d}\right)\right)\right)\left(\sum_{j\le q}\chi_q(j)\left(-\frac{j}{q}-\psi\left(\frac{\sqrt{x}}{q}-\frac{j}{q}\right)\right)\right)\\
&\quad=-\frac{1}{q}\sum_{d|p}\mu(d)j\sum_{j\le q}\chi_q(j)\left\lfloor\frac{\sqrt{x}}{d}\right\rfloor-\sum_{d|p}\mu(d)\sum_{j\le q}\chi_q(j)\sum_{a\le\frac{\sqrt{x}}{d}}\psi\left(\frac{x}{aqd}-\frac{j}{q}\right)\\
&\qquad+x\frac{\phi(p)}{p}\sum_{j\le q}\chi_q(j)\sum_{\substack{b\le\sqrt{x}\\ b\equiv j \bmod q}}\frac{1}{b}-\sum_{d|p}\mu(d)\sum_{j\le q}\chi_q(j)\sum_{\substack{b\le\sqrt{x}\\ b\equiv j \bmod q}}\psi\left(\frac{x}{bd}\right)\\
&\qquad+\frac{1}{q}\sum_{d|p}\mu(d)\sum_{j\le q}j\chi_q(j)\left(\frac{\sqrt{x}}{d}-\psi\left(\frac{\sqrt{x}}{d}\right)\right)\\
&\qquad+\sum_{d|p}\mu(d)\sum_{j\le q}\chi_q(j)\left(\frac{\sqrt{x}}{d}-\psi\left(\frac{\sqrt{x}}{d}\right)\right)\psi\left(\frac{\sqrt{x}}{q}-\frac{j}{q}\right)+O(p^{\epsilon}q)\\
&\quad=x\frac{\phi(p)}{p}\sum_{j\le q}\chi_q(j)\sum_{\substack{b\le\sqrt{x}\\ b\equiv j \bmod q}}\frac{1}{b}-S_1-S_2\\
&\qquad+\sum_{d|p}\mu(d)\sum_{j\le q}\chi_q(j)\left(\frac{\sqrt{x}}{d}-\psi\left(\frac{\sqrt{x}}{d}\right)\right)\psi\left(\frac{\sqrt{x}}{q}-\frac{j}{q}\right)+O(p^{\epsilon}q),
\end{aligned}
$$

where

$$
S_1=\sum_{d|p}\mu(d)\sum_{j\le q}\chi_q(j)\sum_{a\le\frac{\sqrt{x}}{d}}\psi\left(\frac{x}{aqd}-\frac{j}{q}\right)
$$

and

$$S_2 = \sum_{d|p} \mu(d) \sum_{j \leq q} \chi_q(j) \sum_{\substack{b \leq \sqrt{x} \\ b \equiv j \bmod q}} \psi\Big(\frac{x}{bd}\Big).$$

In view of Lemma 3, we have

$$\begin{aligned} &\sum_{ab \leq x} \chi_p^0(a)\chi_q(b) \\ &= x\frac{\phi(p)}{p}\sum_{j \leq q} \chi_q(j)\Big(\frac{\ln x}{2q} + \frac{\gamma(j,q)}{q} - \psi\Big(\frac{\sqrt{x}-j}{q}\Big)x^{-1/2} + O(qx^{-1})\Big) - S_1 - S_2 \\ &\quad + \sum_{d|p} \mu(d) \sum_{j \leq q} \chi_q(j)\left(\frac{\sqrt{x}}{d} - \psi\left(\frac{\sqrt{x}}{d}\right)\right)\psi\left(\frac{\sqrt{x}}{q} - \frac{j}{q}\right) + O(p^\epsilon q) \\ &= x\frac{\phi(p)}{pq}\sum_{j \leq q} \chi_q(j)\gamma(j,q) - S_1 - S_2 + O(p^\epsilon q^2). \end{aligned}$$

In view of Lemma 4, we have

$$\sum_{ab \leq x} \chi_p^0(a)\chi_q(b) = x\frac{\phi(p)}{p}L(1,\chi_q) - S_1 - S_2 + O(p^\epsilon q^2). \tag{3}$$

Now, we use III in Lemma 2 with exponent pair (2/7, 4/7) to bound S_1 and S_2. We have

$$\begin{aligned} S_1 &= \sum_{d|p}\sum_{j \leq q} \mu(d)\chi_q(j)R\Big(\frac{x}{dq}, \frac{x^{1/2}}{d}, 1, 1, 0, \frac{-j}{q}\Big) \\ &\ll \sum_{d|p}\sum_{j \leq q}\left(1 + \left(\frac{x}{dq}\right)^{-1/2}\left(\frac{x^{1/2}}{d}\right)^{3/2} + \left(\frac{x}{dq}\right)^{2/9}\left(\frac{x^{1/2}}{d}\right)^{2/9}\right) \\ &\ll \sum_{d|p}\sum_{j \leq q}\left(1 + x^{1/4}q^{1/2}d^{-1} + x^{1/3}q^{-2/9}d^{-4/9}\right) \\ &= O((qp^\epsilon) + x^{1/4}q^{3/2}p^\epsilon + x^{1/3}q^{7/9}p^\epsilon). \end{aligned}$$

In the same way, we also have

$$
\begin{aligned}
S_2 &= \sum_{d|p}\sum_{j\le q}\mu(d)\chi_q(j)R\Big(\frac{x}{d},\sqrt{x},1,q,j,0\Big)\\
&\ll \sum_{d|p}\sum_{j\le q}\left(1+\Big(\frac{x}{d}\Big)^{-1/2}\Big(x^{1/2}\Big)^{3/2}q^{-1}+\Big(\frac{x}{d}\Big)^{2/9}\Big(x^{1/2}\Big)^{2/9}q^{-4/9}\right)\\
&\ll \sum_{d|p}\sum_{j\le q}\Big(1+x^{1/4}q^{-1}d^{1/2}+x^{1/3}q^{-4/9}d^{-2/9}\Big)\\
&= O((qp^{\epsilon})+x^{1/4}p^{1/2}+x^{1/3}q^{5/9}p^{\epsilon}).
\end{aligned}
$$

We put these bounds of S_1 and S_2 in to (3), we have

$$
\sum_{ab\le x}\chi_p^0(a)\chi_q(b)=x\frac{\phi(p)}{p}L(1,\chi_q)+O(x^{1/3}q^{7/9}p^{\epsilon}+x^{1/3}q^{5/9}p^{\epsilon}).
$$

This proves Lemma 7. □

Lemma 8 *Let p_1 and p_2 be two odd primes, χ_i be a non-principal Dirichlet character modulo p_i, $i=1$, 2. Then for any real number $x\ge p_i^2$, $i=1$ and 2, we have the estimate*

$$
\sum_{ab\le x}\chi_1(a)\chi_2(b)=O\Big(x^{\frac{1}{3}}\cdot p_1^{\frac{5}{9}}\cdot p_2^{\frac{7}{9}}\cdot\ln p_1\Big).
$$

Proof See Theorem 1.1 in reference [14]. □

3 Proof of the Theorem 1

In this part, we shall complete the proof of our main result. For any fixed primes p and q, from Lemma 1 and the definition of $d(n;p,q)$ we have

$$
\begin{aligned}
\sum_{n\le x}d(n;p,q) &= \sum_{n\le x}\sum_{\substack{ab=n\\ a\in\mathbb{A}(p),b\in\mathbb{A}(q)}}1=\sum_{\substack{ab\le x\\ a\in\mathbb{A}(p),b\in\mathbb{A}(q)}}1\\
&=\frac{\phi(p-1)}{p-1}\cdot\frac{\phi(q-1)}{q-1}\cdot\sum_{ab\le x}\sum_{h|p-1}\frac{\mu(h)}{\phi(h)}\sum_{r=1}^{h}{}' \, e\Big(\frac{r\cdot \mathrm{ind}(a)}{h}\Big) \qquad (4)\\
&\quad\times\sum_{k|q-1}\frac{\mu(k)}{\phi(k)}\sum_{s=1}^{k}{}' \, e\Big(\frac{s\cdot\mathrm{ind}(b)}{k}\Big).
\end{aligned}
$$

For any integer $1\le r\le h\le p-1$ with $h\mid p-1$ and $(r,h)=1$, we write: $e\Big(\frac{r\cdot\mathrm{ind}(a)}{h}\Big)=\chi_{r,h}(a)$, and $\chi_{r,h}(a)=0$, if $p\mid a$. It is clear that $\chi_{r,h}(a)$ is a Dirichlet character modulo p, $\chi_{1,1}=\chi_p^0$ denotes the principal character modulo p. So from (4) and the above notations we have

$$\begin{aligned}\sum_{n\le x} d(n;p,q) &= \frac{\phi(p-1)\phi(q-1)}{(p-1)(q-1)}\\ &\cdot \sum_{h|p-1}\sum_{k|q-1}\frac{\mu(h)\mu(k)}{\phi(h)\phi(k)}\sum_{r=1}^{h}{}'\sum_{s=1}^{k}{}'\sum_{ab\le x}\chi_{r,h}(a)\chi_{s,k}(b)\\ &= \frac{\phi(p-1)\phi(q-1)}{(p-1)(q-1)}\cdot\sum_{\substack{h|p-1\\ h>1}}\sum_{\substack{k|q-1\\ k>1}}\frac{\mu(h)\mu(k)}{\phi(h)\phi(k)}\sum_{r=1}^{h}{}'\sum_{s=1}^{k}{}'\sum_{ab\le x}\chi_{r,h}(a)\chi_{s,k}(b)\\ &+ \frac{\phi(p-1)\phi(q-1)}{(p-1)(q-1)}\cdot\sum_{\substack{h|p-1\\ h>1}}\frac{\mu(h)}{\phi(h)}\sum_{r=1}^{h}{}'\sum_{ab\le x}\chi_{r,h}(a)\chi_q^0(b)\\ &+ \frac{\phi(p-1)\phi(q-1)}{(p-1)(q-1)}\cdot\sum_{\substack{k|p-1\\ k>1}}\frac{\mu(k)}{\phi(k)}\sum_{s=1}^{k}{}'\sum_{ab\le x}\chi_p^0(a)\chi_{s,k}(b)\\ &+ \frac{\phi(p-1)\phi(q-1)}{(p-1)(q-1)}\cdot\sum_{ab\le x}\chi_p^0(a)\chi_q^0(b)\equiv W_1+W_2+W_3+W_4.\end{aligned}\tag{5}$$

Now we estimate W_1, W_2, W_3 and W_4 in (5), respectively. Note that the estimate

$$\sum_{\substack{h|p-1\\ h>1}}|\mu(h)| = 2^{\omega(p-1)}-1 \quad\text{and}\quad \sum_{\substack{k|q-1\\ k>1}}|\mu(k)| = 2^{\omega(q-1)}-1,$$

from Lemma 8 we have the estimate

$$\begin{aligned}W_1 &\ll \frac{\phi(p-1)\phi(q-1)}{(p-1)(q-1)}\cdot\sum_{\substack{h|p-1\\ h>1}}|\mu(h)|\sum_{\substack{k|q-1\\ k>1}}|\mu(k)|\cdot x^{\frac{1}{3}}\cdot p^{\frac{5}{9}}\cdot q^{\frac{7}{9}}\cdot\ln p\\ &\ll x^{\frac{1}{3}}\cdot p^{\frac{5}{9}}\cdot q^{\frac{7}{9}}\cdot 2^{\omega(p-1)+\omega(q-1)}\cdot\ln p.\end{aligned}\tag{6}$$

Applying Lemma 7 we have

$$\begin{aligned}W_2 &= \frac{\phi(p-1)\phi(q-1)}{(p-1)(q-1)}\cdot\sum_{\substack{h|p-1\\ h>1}}\frac{\mu(h)}{\phi(h)}\sum_{r=1}^{h}{}'\left(\frac{q-1}{q}\cdot x\cdot L(1,\chi_{r,h})+O\left(x^{\frac{1}{3}}\cdot p^{\frac{7}{9}}\cdot q^{\epsilon}\right)\right)\\ &= \frac{\phi(p-1)\phi(q-1)}{q(p-1)}x\sum_{\substack{h|p-1\\ h>1}}\frac{\mu(h)}{\phi(h)}\sum_{r=1}^{h}{}'L(1,\chi_{r,h})\\ &+O\left(x^{\frac{1}{3}}\cdot p^{\frac{7}{9}}\cdot q^{\epsilon}2^{\omega(p-1)}\right).\end{aligned}\tag{7}$$

Similarly, we also have

$$W_3 = \frac{\phi(p-1)\phi(q-1)}{p(q-1)} x \sum_{\substack{k \mid q-1 \\ k>1}} \frac{\mu(k)}{\phi(k)} \sum_{s=1}^{k}{}' L(1, \chi_{s,k}) + O\left(x^{\frac{1}{3}} \cdot q^{\frac{7}{9}} \cdot p^{\epsilon} 2^{\omega(q-1)}\right). \tag{8}$$

From Lemma 6, we have the asymptotic formula

$$\begin{aligned} W_4 =& \frac{\phi(p-1)\phi(q-1)}{(p-1)(q-1)} \cdot \sum_{ab \le x} \chi_p^0(a)\chi_q^0(b) \\ =& \frac{\phi(p-1)\phi(q-1)}{pq} \cdot x \cdot \left(\ln x + 2\gamma - 1 + \frac{\ln p}{p-1} + \frac{\ln q}{q-1}\right) \\ &+ O\left(x^{\frac{1}{3}} \cdot p^{\frac{5}{9}} \cdot q^{\frac{7}{9}} + x^{\frac{1}{3}} \cdot q^{\frac{5}{9}} \cdot p^{\frac{7}{9}}\right). \end{aligned} \tag{9}$$

Combining (5)–(9), we have the asymptotic formula

$$\begin{aligned} \sum_{n \le x} d(n; p, q) =& \frac{\phi(p-1)\phi(q-1)}{pq} \cdot x \cdot \left(\ln x + 2\gamma - 1 + \frac{\ln p}{p-1} + \frac{\ln q}{q-1}\right) \\ &+ \frac{\phi(p-1)\phi(q-1)}{q(p-1)} \cdot x \cdot \sum_{\substack{h \mid p-1 \\ h>1}} \frac{\mu(h)}{\phi(h)} \sum_{r=1}^{h}{}' L(1, \chi_{r,h}) \\ &+ \frac{\phi(p-1)\phi(q-1)}{p(q-1)} \cdot x \cdot \sum_{\substack{k \mid q-1 \\ k>1}} \frac{\mu(k)}{\phi(k)} \sum_{s=1}^{k}{}' L(1, \chi_{s,k}) \\ &+ O\left(x^{\frac{1}{3}} p^{\frac{5}{9}} q^{\frac{7}{9}} 2^{\omega(p-1)+\omega(q-1)} \ln p\right) + O\left(x^{\frac{1}{3}} q^{\frac{5}{9}} p^{\frac{7}{9}} 2^{\omega(p-1)+\omega(q-1)} \ln q\right). \end{aligned}$$

It is clear that $\chi_{r,h}^h = \chi_p^0$ and $\chi_{r,h}^j \neq \chi_p^0$ for all $1 \le j < h$. So $\chi_{r,h}$ is a h-order character modulo p. Therefore, we have the identity

$$\sum_{\substack{h \mid p-1 \\ h>1}} \frac{\mu(h)}{\phi(h)} \sum_{r=1}^{h}{}' L(1, \chi_{r,h}) = \sum_{\substack{h \mid p-1 \\ h>1}} \frac{\mu(h)}{\phi(h)} \sum_{\chi^h = \chi_p^0}{}' L(1, \chi).$$

This completes the proof of our theorem.

4 Conclusion

In this article, we first introduced a new arithmetical function $d(n; p, q)$, then we using the analytic methods and Teerapat Srichan's work [14] to study the mean value properties of $d(n; p, q)$, and prove a strong asymptotic formula for it. Especially for any fixed primes p and q, we have the following simple form

$$\sum_{n \le x} d(n; p, q) = \frac{\phi(p-1)\phi(q-1)}{pq} \cdot x \cdot \ln x + O(x), \quad x \to +\infty.$$

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Declarations

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of this paper.

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