Bol. Soc. Mat. Mex. (2021) 27:78
https://doi.org/10.1007/s40590-021-00387-3

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

The ϕ-Brunn–Minkowski inequalities for general convex bodies

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Received: 12 April 2021 / Accepted: 5 September 2021 / Published online: 7 October 2021

Abstract
In this paper, we first give a new proof of the log-Minkowski inequality of general planar convex bodies and then extend the L_p-Brunn–Minkowski inequality and L_p-Minkowski inequality of o-symmetric planar convex bodies for $p \in (0,1)$ to ϕ-Brunn–Minkowski inequality and ϕ-Minkowski inequality of general planar convex bodies. As an application, a family of ϕ-measures of asymmetry for planar convex bodies is introduced.

Keywords Brunn–Minkowski inequality · Minkowski inequality · Mixed volume · Measure of asymmetry

Mathematics Subject Classification 52A20 · 52A40

1 Introduction

The classical Brunn–Minkowski inequality for convex bodies (compact convex sets with nonempty interiors) states that for convex bodies K, L in Euclidean n-space, $\mathbb{R}^n$, the volume of the bodies and of their Minkowski sum $K + L = \{x + y : x \in \text{ and } y \in L\}$, are related by

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}, \tag{1}$$

with equality if and only if K and L are homothetic.

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The Brunn–Minkowski inequality has an equivalent formulation as for all real $\lambda \in [0,1]$,

$$V((1-\lambda)K+\lambda L) \geq V(K)^{1-\lambda}V(L)^{\lambda}, \tag{2}$$

and for $\lambda \in (0,1)$, there is equality if and only if K and L are translates.

The excellent survey article of Gardner [3] gives a comprehensive account of various aspects and consequences of the Brunn–Minkowski inequality.

In the 1960s, Firey [2] introduced for $p \geq 1$ the so-called Minkowski–Firey L_p sum of convex bodies that contain the origin in their interiors, and established the L_p-Brunn–Minkowski inequality, which states as follows:

$$V\big((1-\lambda)\cdot K +_p \lambda \cdot L\big)^{\frac{p}{n}} \geq (1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}, \tag{3}$$

with equality for $\lambda \in (0,1)$ if and only if K and L are dilates.

In the mid-1990s, it was shown in Refs. [12, 13] that a study of the volume of L_p-Minkowski addition leads to an L_p-Brunn–Minkowski theory. This theory has expanded rapidly.

If K and L are convex bodies that contain the origin in their interiors and $0 \leq \lambda \leq 1$ then the Minkowski–Firey L_p-combination ($p > 0$), $(1-\lambda)\cdot K +_p \lambda \cdot L$, is defined by

$$(1-\lambda)\cdot K +_p \lambda \cdot L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq ((1-\lambda)h_K(u)^p + \lambda h_L(u)^p)^{1/p}\}. \tag{4}$$

It has been noticed that the L_p-Minkowski addition makes sense for all $p > 0$. The case $p = 0$ is known as the log-Minkowski addition, $(1-\lambda)\cdot K +_0 \lambda \cdot L$, of convex bodies K and L that contain the origin in their interior, defined by

$$(1-\lambda)\cdot K +_0 \lambda \cdot L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda}h_L(u)^{\lambda}\}. \tag{5}$$

In Ref. [1], Böröczky, Lutwak, Yang and Zhang conjectured the log-Brunn–Minkowski inequality: If K and L are o-symmetric convex bodies in $\mathbb{R}^n$, then for all $\lambda \in [0,1]$,

$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{(1-\lambda)}V(L)^{\lambda}. \tag{6}$$

The log-Brunn–Minkowski inequality is stronger than the L_p-Brunn–Minkowski inequality for $p > 0$. It was shown in Ref. [1] that the log-Brunn–Minkowski inequality is equivalent to the following log-Minkowski mixed volume inequality: If K and L are o-symmetric convex bodies in $\mathbb{R}^n$, then

$$\int_{S^{n-1}} \log \frac{h_L}{h_K} d\bar{V}_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)}. \tag{7}$$

Here $\bar{V}_K$ denotes the cone-volume probability measure of K.

The planar case of the log-Brunn–Minkowski inequality and the equivalent log-Brunn–Minkowski inequality were proved in Ref. [1].

Theorem 1.1 ([1]) *If K and L are o-symmetric convex bodies in* $\mathbb{R}^2$, *then for all real* $\lambda \in [0, 1]$,

$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{(1-\lambda)} V(L)^{\lambda}, \tag{8}$$

with equality for $\lambda \in (0, 1)$ if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Theorem 1.2 ([1]) *If K and L are o-symmetric convex bodies in* $\mathbb{R}^2$, *then,*

$$\int_{S^1} \log \frac{h_L}{h_K} d\bar{V}_K \geq \frac{1}{2} \log \frac{V(L)}{V(K)}, \tag{9}$$

with equality if and only if K and L are dilates or K and L are parallelograms with parallel sides.

It is easily seen from definition (4) *that for fixed convex bodies* K, L *and fixed* $\lambda \in [0, 1]$, *the* L_p*-Minkowski–Firey combination* $(1-\lambda)\cdot K +_p \lambda \cdot L$ *is increasing with respect to set inclusion, as p increases, i.e., if* $0 \leq p \leq q$,

$$(1-\lambda)\cdot K +_p \lambda \cdot L \subset (1-\lambda)\cdot K +_q \lambda \cdot L. \tag{10}$$

From (9), the L_p-Brunn–Minkowski inequality and the L_p-Minkowski inequality were proved in Ref. [1] for $p \in (0, 1)$.

Theorem 1.3 ([1]) *Suppose* $0 < p < 1$. *If K and L are o-symmetric convex bodies in* $\mathbb{R}^2$, *then for all real* $\lambda \in [0, 1]$,

$$V\big((1-\lambda)\cdot K +_p \lambda \cdot L\big) \geq V(K)^{(1-\lambda)} V(L)^{\lambda}, \tag{11}$$

with equality for $\lambda \in (0, 1)$ if and only if $K = L$.

Theorem 1.4 ([1]) *Suppose* $0 < p < 1$. *If K and L are o-symmetric convex bodies in* $\mathbb{R}^2$, *then for all* $\lambda \in [0, 1]$,

$$\left(\int_{S^1} \left(\frac{h_L}{h_K}\right)^p d\bar{V}_K\right)^{\frac{1}{p}} \geq \left(\frac{V(L)}{V(K)}\right)^{\frac{1}{2}}, \tag{12}$$

with equality for $\lambda \in (0, 1)$ if and only if K and L are dilates.

In Ref. [18], *Ma gave an alternative proof of Theorem* 1.2. *Some results of the log-Brunn–Minkowski inequality for* $n \geq 3$, *see Refs.* [19, 21, 25].

There is a counterexample, showing that, if K is an o-centered cube and L is a distinct translate of K, then (6) *does not hold for general non-o-symmetric convex bodies. By introducing the notion of "dilation position", Xi and Leng* [23] *proved the log-Brunn–Minkowski inequality and the equivalent log-Minkowski mixed volume inequality for general planar convex bodies.*

Theorem 1.5 ([23]) *If K and L are convex bodies in $\mathbb{R}^2$ with $o \in K \cap L$, and K, L are in dilation position, then for all real $\lambda \in [0,1]$,*

$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{(1-\lambda)} V(L)^{\lambda}, \tag{13}$$

with equality for $\lambda \in (0,1)$ if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Theorem 1.6 ([24]) *If K and L are convex bodies in $\mathbb{R}^2$ with $o \in K \cap L$, and K, L are in dilation position, then*

$$\int_{S^1} \log \frac{h_L}{h_K} d\bar{V}_K \geq \frac{1}{2} \log \frac{V(L)}{V(K)}, \tag{14}$$

with equality if and only if K and L are dilates or K and L are parallelograms with parallel sides.

The Orlicz–Brunn–Minkowski theory originated with the work of Lutwak et al. [15, 16]. *By introducing the Orlicz–Minkowski addition, Gardner, Hug and Weil* [4], *and Xi et al.* [24] *proved the Orlicz–Brunn–Minkowski inequality and Orlicz–Minkowski inequality. It is a natural extension of the L_p-Brunn–Minkowski theory for $p \geq 1$. For dual Orlicz–Brunn–Minkowski theory see* [5, 26].

Let Φ be the set of strictly increasing functions $\phi : (0,\infty) \to I \subset \mathbb{R}$ which are continuously differentiable on $(0,\infty)$ with positive derivative, and satisfy that $\lim_{t\to\infty} \phi(t) = \infty$ *and that $\log \circ \phi^{-1}$ is concave. Observe that whenever $\phi \in \Phi$ is convex, the composite function $\log \circ \phi^{-1}$ is concave. The collection of convex functions from Φ shall be denoted by $\mathcal{C}$.*

Let $\lambda \in [0,1]$ and $\phi \in \Phi$. For $u \in S^{n-1}$, we define a function $h_\lambda(u)$ as

$$h_\lambda(u) = \inf\{\tau > 0 : (1-\lambda)\phi\left(\frac{h_K(u)}{\tau}\right) + \lambda\phi\left(\frac{h_L(u)}{\tau}\right) \leq \phi(1)\}. \tag{15}$$

By the strict monotonicity of ϕ, we have

$$\phi(1) = (1-\lambda)\phi\left(\frac{h_K(u)}{h_\lambda(u)}\right) + \lambda\phi\left(\frac{h_L(u)}{h_\lambda(u)}\right). \tag{16}$$

The ϕ-combination $(1-\lambda)\cdot K +_\phi \lambda \cdot L$ of $K, L \in \mathcal{K}_o^n$ is defined in Ref. [17] *by*

$$(1-\lambda)\cdot K +_\phi \lambda \cdot L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h_\lambda(u)\}. \tag{17}$$

Note that if $\phi(t) = t^p$ with $p > 0$, then the ϕ-combination reduces to the L_p-Minkowski combination. Further, if $\phi(t) = \alpha \log(t)(\alpha > 0)$, then we retrieve the log-Minkowski combination. In Ref. [17], *Lv proved the ϕ-Minkowski inequality and ϕ-Brunn–Minkowski inequality for general functions ϕ for o-symmetric planar convex bodies K, L. If $\phi(t) = t^p, p \in (0,1)$, then the ϕ-Minkowski inequality reduces to the L_p-Minkowski inequality* (12) *and L_p-Brunn–Minkowski inequality* (11).

In this paper, we first present a new proof Theorem 1.6, and extend Theorems 1.3 and 1.4 from $p \in (0,1)$ and o-symmetric convex bodies K, Lto general case ϕ and general convex bodies K, L. More precisely, we have the following main results.

Theorem 1.7 *Let $\phi \in \Phi$ with $\phi \neq \alpha \log(\alpha > 0)$, and K and L are planar convex bodies containing the origin o in their interiors, and $o \in K \cap L$. If K and L are at a dilation position, then*

$$\int_{S^1} \phi\left(\frac{h_L}{h_K}\right) d\bar{V}_K \geq \phi\left(\frac{V(L)^{\frac{1}{2}}}{V(K)^{\frac{1}{2}}}\right), \tag{18}$$

with equality if and only if K and L are dilates.

Theorem 1.8 *Let $\phi \in \Phi$, $\phi \neq \alpha \log(\alpha > 0)$ be concave on $(0, \infty)$, and K and L are planar convex bodies containing the origin o in their interiors, and $o \in K \cap L$. If K and L are at a dilation position, then for all real $\lambda \in [0,1]$,*

$$V\big((1-\lambda) \cdot K +_\phi \lambda \cdot L\big) \geq V(K)^{(1-\lambda)} V(L)^{\lambda}, \tag{19}$$

with equality for $\lambda \in (0,1)$ if and only if $K = L$.

2 Preliminaries

Let $\mathcal{K}^n$ be the class of convex bodies (compact convex sets with nonempty interiors) in $\mathbb{R}^n$, and let $\mathcal{K}_o^n$ be those sets in $\mathcal{K}^n$ containing the origin in their interiors.

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$, of compact convex subset K of $\mathbb{R}^n$ is defined by $h_K(x) = \{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$, and uniquely determines the convex set.

A boundary point $x \in \partial K$ of the convex body K is said to have $u \in S^{n-1}$ as one of its outer unit normals provided $x \cdot u = h_K(u)$. A boundary point is said to be singular if it has more than one unit normal vector. It is well known that the set of singular boundary points of a convex body has $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ equal to 0.

Let $K \in \mathcal{K}^n$ and $\nu_K : \partial K \to S^{n-1}$ the generalized Gauss map. For each Borel set $\omega \subset S^{n-1}$, the inverse spherical image $\nu_K^{-1}(\omega)$ of ω is the set of all boundary points of K which have an outer unit normal belonging to the set ω. The surface area measure S_K of $K \in \mathcal{K}^n$ is defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)), \tag{20}$$

for each Borel set $\omega \subset S^{n-1}$, i.e., $S_K(\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all points on ∂K that have a unit normal that lies in ω.

The Hausdorff distance $d_H(K, L)$ of compact convex sets K, L is defined by $d_H(K, L) = \|h_K - h_L\|_\infty$. A sequence of convex bodies, K_i, is said to converge to a body K, i.e., $\lim_{i\to\infty} K_i = K$ if $d_H(K_i, K) \to 0$. If K is a convex body and K_i is a sequence of convex bodies then

$$\lim_{i\to\infty} K_i = K \Rightarrow \lim_{i\to\infty} S_{K_i} = S_K, \text{ weakly.} \tag{21}$$

The cone-volume measure V_K of $K \in \mathcal{K}^n$ is a Borel measure on the unit sphere S^{n-1} defined for a Borel set $\omega \subset S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n}\int_{x\in \nu_K^{-1}(\omega)} x\cdot \nu_K(x)\mathrm{d}\mathcal{H}^{n-1}(x), \tag{22}$$

and thus

$$\mathrm{d}V_K = \frac{1}{n} h_K dS_K. \tag{23}$$

Since,

$$V(K) = \frac{1}{n}\int_{u\in S^{n-1}} h_K(u)\mathrm{d}S_K(u), \tag{24}$$

we can define the cone-volume probability measure $\bar{V}_K$ of K by

$$\bar{V}_K = \frac{1}{V(K)} V_K. \tag{25}$$

Suppose $K, L \in \mathcal{K}_o^n$. For $p \neq 0$, the L_p-mixed volume $V_p(K,L)$ can be defined as

$$V_p(K,L) = \int_{u\in S^{n-1}} \left(\frac{h_L}{h_K}\right)^p \mathrm{d}V_K. \tag{26}$$

The normalized L_p-mixed volume $\bar{V}_p(K,L)$ was first defined in Ref. [14],

$$\bar{V}_p(K,L) = \left(\int_{u\in S^{n-1}} \left(\frac{h_L}{h_K}\right)^p \mathrm{d}\bar{V}_K\right)^{\frac{1}{p}}. \tag{27}$$

For $p = \infty$, we define

$$\bar{V}_\infty(K,L) = \max\{h_L/h_K : u \in \mathrm{supp}S_K\}, \tag{28}$$

and we have

$$\lim_{p\to\infty} \bar{V}_p(K,L) = \bar{V}_\infty(K,L). \tag{29}$$

Letting $p \to 0$ gives

$$\bar{V}_0(K,L) = \exp\left(\int_{u\in S^{n-1}} \log\frac{h_L}{h_K}\mathrm{d}\bar{V}_K\right), \tag{30}$$

which is the normalized log-mixed volume of K and L. From Jesen's inequality we know that $p \mapsto \bar{V}_p(K,L)$ is strictly monotone increasing, unless h_L/h_K is constant on $\mathrm{supp}S_K$.

Suppose $K, L \in \mathcal{K}^n$. The inradius $r(K, L)$ and $R(K, L)$ of K with respect to L are defined by

$$r(K, L) = \sup\{t > 0 : x + tL \subset K \text{ and } x \in \mathbb{R}^n\},$$

$$R(K, L) = \inf\{t > 0 : x + tL \supset K \text{ and } x \in \mathbb{R}^n\}.$$

From the definition, it follows that $r(K, L) = 1/R(L, K)$.

If K, L happen to be o-symmetric convex bodies, then clearly

$$r(K, L) = \min_{u \in S^{n-1}} \frac{h_K(u)}{h_L(u)} \text{ and } R(K, L) = \max_{u \in S^{n-1}} \frac{h_K(u)}{h_L(u)}. \tag{31}$$

Let $K, L \in \mathcal{K}^n$. K and L are said to be at a *dilation position*, if $o \in K \cap L$, and

$$r(K, L)L \subset K \subset R(K, L)L. \tag{32}$$

The definition and some properties of dilation position were first given by Xi and Leng [23]. It is easy to prove that if K, L are o-symmetric convex bodies, then K and L are at a dilation position.

In general, we refer the reader to [20] for standard notation concerning convex bodies.

3 A new proof of Theorem 1.6

In Ref. [18], Ma gave a proof of Theorem 1.1. In the following, we demonstrate an alternate proof of Theorem 1.5 by employing Ma's approach [18]. The following lemma is needed in our proof.

Lemma 3.1 ([23]) *Let $K, L \in \mathcal{K}^2$ with $o \in K \cap L$. If K and L are at a dilation position, then*

$$\int_{S^1} \frac{h_K}{h_L} \mathrm{d}\bar{V}_K \leq \frac{V(L, K)}{V(L)}, \tag{33}$$

with equality if and only if K and L are dilates, or K and L are parallelograms with parallel sides.

We repeat the statement of Theorem 1.6, and present our approach.

Theorem 3.2 ([23]) *If K and L are convex bodies in $\mathbb{R}^2$ with $o \in K \cap L$, and K, L are at a dilation position, then*

$$\int_{S^1} \log \frac{h_L}{h_K} \mathrm{d}\bar{V}_K \geq \frac{1}{2} \log \frac{V(L)}{V(K)}, \tag{34}$$

with equality if and only if K and L are dilates or K and L are parallelograms with parallel sides.

Proof Set

$$F(t) = \int_{S^1} \log\left(\frac{h_{L+tK}}{h_K}\right) d\bar{V}_K - \frac{1}{2}\log\left(\frac{V(L+tK)}{V(K)}\right), \quad t \in [0,\infty). \tag{35}$$

Since $h_{L+tK} = h_L + th_K$ and $V(L+tK) = V(L) + 2V(L,K)t + V(K)t^2$, we have

$$\begin{aligned} F'(t) &= \int_{S^1} \frac{h_K}{h_L + th_K} d\bar{V}_K - \frac{(V(L,K) + V(K)t)}{V(L) + 2V(L,K)t + V(K)t^2} \\ &= \int_{S^1} \frac{h_K}{h_{L+tK}} d\bar{V}_K - \frac{V(L+tK,K)}{V(L+tK)}. \end{aligned}$$

By Lemma 5.2 of Ref. [23], we have K and $L+tK$ are at a dilation position. Therefore, we get $F'(t) \leq 0$ from Lemma3.1, which implies that $F(t)$ is decreasing on $[0,\infty)$.

By mean value theorem for integrals, there exists $u_0 \in S^1$ such that

$$\int_{S^1} \log\left(\frac{h_{L+tK}}{h_K}\right) d\bar{V}_K = \log\left(\frac{h_{L+tK}(u_0)}{h_K(u_0)}\right). \tag{36}$$

Let $t \to \infty$, then

$$\begin{aligned} F(t) &= \log\left(\frac{h_{L+tK}(u_0)}{h_K(u_0)}\right) - \frac{1}{2}\log\left(\frac{V(L+tK)}{V(K)}\right) \\ &= \log\left(\frac{h_L(u_0) + th_K(u_0)}{h_K(u_0)} \cdot \frac{V(K)^{\frac{1}{2}}}{V(L+tK)^{\frac{1}{2}}}\right) \\ &= \log\left(\frac{h_L(u_0) + th_K(u_0)}{h_K(u_0)} \cdot \frac{V(K)^{\frac{1}{2}}}{(V(L) + 2tV(L,K) + t^2V(K))^{\frac{1}{2}}}\right) \\ &\to 0. \end{aligned}$$

Therefore, $F(t) \geq 0$ for $t \in [0,\infty)$. In particular, $F(0) \geq 0$, which implies

$$\int_{S^1} \log\frac{h_L}{h_K} d\bar{V}_K \geq \frac{1}{2}\log\frac{V(L)}{V(K)}.$$

If the equality holds in (34), then $F(0) = 0$, which implies $F(t) \equiv 0$ for $t \in [0,\infty)$. Therefore, $F'(t) \equiv 0$ for all $t \in [0,\infty)$. By Lemma 3.1, we have K and $L+tK$ are dilates, or K and $L+tK$ are parallelograms with parallel sides. So, K and L are dilates, or K and L are parallelograms with parallel sides. Conversely, if K and L are dilates, or K and L are parallelograms with parallel sides, the equality of (34) holds. □

Remark 3.3 In Ref. [23], Xi and Leng proved that Theorems 1.5 and 1.6 are equivalent.

4 Proofs of Theorems 1.7 and 1.8

Suppose $K, L \in \mathcal{K}_o^n$. For $\phi \in \Phi$, the ϕ-mixed volume $V_\phi(K, L)$ was defined in Ref. [17] by

$$V_\phi(K, L) = \int_{S^{n-1}} \phi\left(\frac{h_L}{h_K}\right) \mathrm{d}V_K. \tag{37}$$

The normalized ϕ-mixed volume $\bar{V}_\phi(K, L)$ of $K, L \in \mathcal{K}_o^n$ was defined in Ref. [17] by

$$\bar{V}_\phi(K, L) = \phi^{-1}\left(\int_{S^{n-1}} \phi\left(\frac{h_L}{h_K}\right) d\bar{V}_K\right). \tag{38}$$

In particular, if $\phi(t) = t^p$ with $p > 0$, the normalized ϕ-mixed volume $\bar{V}_\phi(K, L)$ reduces to the normalized L_p-mixed volume $\bar{V}_p(K, L)$.

We repeat the statements of Theorems 1.7 and 1.8.

Theorem 4.1 *Suppose that $\phi \in \Phi$ with $\phi \neq \alpha \log(\alpha > 0)$, and $K, L \in \mathcal{K}_o^2$ with $o \in K \cap L$. If K and L are at a dilation position, then*

$$\int_{S^1} \phi\left(\frac{h_L}{h_K}\right) \mathrm{d}\bar{V}_K \geq \phi\left(\frac{V(L)^{\frac{1}{2}}}{V(K)^{\frac{1}{2}}}\right), \tag{39}$$

with equality if and only if K and L are dilates.

Proof From the log-concavity of ϕ^{-1}, we have

$$\int_{S^{n-1}} \log \frac{h_L}{h_K} \mathrm{d}\bar{V}_K \leq \log \circ \phi^{-1}\left(\int_{S^{n-1}} \phi\left(\frac{h_L}{h_K}\right) \mathrm{d}\bar{V}_K\right), \tag{40}$$

which is equivalent to

$$\exp\left(\int_{S^{n-1}} \log \frac{h_L}{h_K} \mathrm{d}\bar{V}_K\right) \leq \phi^{-1}\left(\int_{S^{n-1}} \phi\left(\frac{h_L}{h_K}\right) \mathrm{d}\bar{V}_K\right). \tag{41}$$

That is

$$\bar{V}_0(K, L) \leq \bar{V}_\phi(K, L), \tag{42}$$

with equality if and only if h_L/h_K is constant on suppS_K. From (14), we have

$$\bar{V}_\phi(K, L) \geq \frac{V(L)^{\frac{1}{2}}}{V(K)^{\frac{1}{2}}}, \tag{43}$$

which leads to (39). From the equality condition of (14) and (42), we have equality holds in (39) if and only if K and L are dilates. □

Theorem 4.2 *Suppose that $\phi \in \Phi$, $\phi \neq \alpha \log(\alpha > 0)$ be concave on $(0, \infty)$, and $K, L \in \mathcal{K}_o^2$ with $o \in K \cap L$. If K and L are at a dilation position, then for all real $\lambda \in [0, 1]$,*

$$V\big((1-\lambda)\cdot K +_\phi \lambda \cdot L\big) \geq V(K)^{(1-\lambda)} V(L)^{\lambda}, \tag{44}$$

with equality for $\lambda \in (0, 1)$ if and only if $K = L$.

Proof Set $Q_\lambda = (1-\lambda)\cdot K +_\phi \lambda \cdot L$. From (16) and the concavity of ϕ, we have

$$\phi(1) = (1-\lambda)\phi\left(\frac{h_K(u)}{h_\lambda(u)}\right) + \lambda\phi\left(\frac{h_L(u)}{h_\lambda(u)}\right) \leq \phi\left(\frac{(1-\lambda)h_K + \lambda h_L}{h_\lambda}\right). \tag{45}$$

By the monotone property of ϕ, we have

$$h_\lambda \leq (1-\lambda)h_K + \lambda h_L. \tag{46}$$

From (17), we have $h_\lambda = h_{Q_\lambda}$ with respect to the surface area measure S_{Q_λ}. Hence, we have

$$Q_\lambda \subset (1-\lambda)K + \lambda L. \tag{47}$$

On the other hand, from (16), we have

$$1 = \phi^{-1}\left((1-\lambda)\phi\left(\frac{h_K(u)}{h_\lambda(u)}\right) + \lambda\phi\left(\frac{h_L(u)}{h_\lambda(u)}\right)\right). \tag{48}$$

From the log-concavity of ϕ, we have

$$\begin{aligned} 0 &= (\log\circ\phi^{-1})\left((1-\lambda)\phi\left(\frac{h_K(u)}{h_\lambda(u)}\right) + \lambda\phi\left(\frac{h_L(u)}{h_\lambda(u)}\right)\right) \\ &\geq (1-\lambda)\log\frac{h_K(u)}{h_\lambda(u)} + \lambda\log\frac{h_L(u)}{h_\lambda(u)} \\ &= \log\frac{h_K^{1-\lambda}h_L^{\lambda}}{h_\lambda}, \end{aligned}$$

which implies $h_K^{1-\lambda}h_L^{\lambda} \leq h_\lambda$. Hence,

$$(1-\lambda)\cdot K +_0 \lambda \cdot L \subset Q_\lambda. \tag{49}$$

From (13), we have

$$V(Q_\lambda) \geq V((1-\lambda)\cdot K +_0 \lambda\cdot L) \geq V(K)^{1-\lambda}V(L)^{\lambda}. \tag{50}$$

If equality holds in (44), then $V((1-\lambda)\cdot K +_0 \lambda \cdot L) = V(K)^{1-\lambda}V(L)^{\lambda}$. By the equality condition of (13), we have K and L are dilates. In addition, from $V(Q_\lambda) = V((1-\lambda)\cdot K +_0 \lambda\cdot L)$, we have $(1-\lambda)\cdot K +_0 \lambda \cdot L = Q_\lambda$, which implies $K = L$. □

We can get the L_p-Minkowski inequality and L_p-Brunn–Minkowski inequality for general planar convex bodies by setting $\phi(t) = t^p$ in Theorems 4.1 and 4.2.

Corollary 4.3 *Suppose that* $0<p<1$, *and* $K, L \in \mathcal{K}_o^2$ *with* $o \in K \cap L$. *If K and L are at a dilation position, then*

$$\left(\int_{S^1}\left(\frac{h_L}{h_K}\right)^p \mathrm{d}\bar{V}_K\right)^{\frac{1}{p}} \geq \left(\frac{V(L)}{V(K)}\right)^{\frac{1}{2}}, \tag{51}$$

with equality if and only if K and L are dilates.

Corollary 4.4 *Suppose that* $0<p<1$, *and* $K, L \in \mathcal{K}_o^2$ *with* $o \in K \cap L$. *If K and L are at a dilation position, then for all real* $\lambda \in [0,1]$,

$$V\big((1-\lambda)\cdot K +_\phi \lambda \cdot L\big) \geq V(K)^{(1-\lambda)} V(L)^{\lambda}, \tag{52}$$

with equality for $\lambda \in (0,1)$ if and only if $K = L$.

5 ϕ-Minkowski measure of asymmetry

In the well-known paper [6], abstracting from some extremal problems arising from geometry or other mathematical branches and from the previous work of many mathematicians, Grünbaum formulated a concept of measures of asymmetry (or symmetry) for convex bodies which, among other applications, can be used to describe how far a convex set is from a (centrally) symmetric one. Since then, the properties and applications of these known asymmetry measures are studied by many mathematicians (see [7–11, 22] and references therein).

In Ref. [7], Guo introduced a family of measures of (central) asymmetry, the so-called p-measures of asymmetry, which have the well-known Minkowski measure of asymmetry as a special case, and showed some similar properties of the p-measures to the Minkowski one. In Ref. [11], Jin, Leng and Guo extended the p-Minkowski measure of asymmetry to an Orlicz version. In addition, Jin et al. [11] showed that p $(1 \leq p \leq \infty)$-Minkowski measures of asymmetry are closely related to L_p-mixed volumes. More precisely, we can define p $(1 \leq p \leq \infty)$-Minkowski measures of asymmetry by L_p-mixed volumes. In Ref. [9], Jin introduced a measure of asymmetry $\mathrm{as}_0(K)$ for planar convex bodies K in terms of the log-mixed volume, and extended the p-Minkowski measures of asymmetry to the case $0 \leq p \leq \infty$.

For $K \in \mathcal{K}^n$, $x \in \mathrm{int}(K)$ and $1 \leq p \leq \infty$, the p-Minkowski measure of asymmetry of K is defined by

$$\mathrm{as}_p(C) = \inf_{x \in \mathrm{int(C)}} \bar{V}_p(K_x, -K_x), \tag{53}$$

where K_x denotes $K + \{-x\}$. A point $x \in \mathrm{int}(K)$ satisfying $\bar{V}_p(K_x, -K_x) = \mathrm{as}_p(K)$ is called a p-critical point of K. The set of all p-critical points is denoted by $\mathcal{C}_p(K)$. The well-known Minkowski measure of asymmetry is the special case that $p = \infty$.

Theorem 5.1 ([6, 7]) *For* $1 \leq p \leq \infty$, *if* $K \in \mathcal{K}^n$ *then,*

$$1 \le \mathrm{as}_p(K) \le n, \tag{54}$$

equality holds on the left-hand side if and only if K is symmetric, and on the right-hand side if and only if K is a simplex.

For the p-critical set $\mathcal{C}_p(K)$, we have the following theorem.

Theorem 5.2 ([6, 7]) *For $1 \le p \le \infty$, and $K \in \mathcal{K}^n$, we have the following statements:*

(1) if $p = 1$, then $\mathcal{C}_1(K) = \mathrm{int}(K)$;

(2) if $p = \infty$, then $\mathcal{C}_\infty(K)$ is a convex set with $\dim(\mathcal{C}_\infty(K)) + \mathrm{as}_\infty(K) \le n$;

(3) if $p \in (1, \infty)$, then $\mathcal{C}_p(K)$ is a singleton.

Note that if $K \in \mathcal{K}^2$, then $\mathcal{C}_\infty(K)$ is a singleton, i.e., each planar convex body has a unique critical ∞-critical point.

For fixed $K \in \mathcal{K}^n$, we denotes the unique p-critical point of K by x_p for $p \in (1, \infty)$. It is easy to see that x_p are coincide with the center of K if K is symmetric; if K is a simplex, then x_p are coincide with the centroid of K. There are some other convex bodies that have this property that all $p(1 < p < \infty)$-critical points coincide.

Example 5.3 (1) If $K := a_1a_2a_3a_4$ with $a_1(-3,0), a_2(0,-3), a_3(4,0)$ and $a_4(0,3)$, then the quadrilateral K has centroid $c(\frac{1}{4}, 0)$ and $x_p(\frac{4}{15}, 0)$ for $p \in (1, \infty]$;

(2) If $K := a_1a_2a_3a_4$ with $a_1(-5,0), a_2(0,-5), a_3(12,0)$ and $a_4(0,5)$, then the quadrilateral K has centroid $c(\frac{7}{3}, 0)$ and $x_p(\frac{84}{41}, 0)$ for $p \in (1, \infty]$.

Therefore, we state the following problem.

Problem 5.4 Suppose that $K \in \mathcal{K}^n$. Is it that $\dim(\mathrm{conv}\{x_p : p \in (1, \infty)\}) = 0$?

The p-Minkowski measure of asymmetry for the case $p \in [0, 1)$ is introduced in Ref. [9].

Given $K \in \mathcal{K}^2$, let $s \in \mathcal{C}_\infty(K)$ be the unique ∞-critical point of K. The log-Minkowski measure $\mathrm{as}_0(K)$ of K is defined by

$$\mathrm{as}_0(K) = \bar{V}_0(K_s, -K_s). \tag{5.3}$$

Theorem 5.5 ([9])

If $K \in \mathcal{K}^2$, then,

$$1 \le \mathrm{as}_0(K) \le 2. \tag{56}$$

Equality holds on the left-hand side if and only if K is symmetric, and equality holds on the right-hand side if and only if K is a triangle.

If we define $\mathrm{as}_0(K) = \inf_{x \in \mathrm{int(K)}} \bar{V}_0(K_x, -K_x)$, then when K is a square, $\mathrm{as}_0(C) < 1$. This result shows that $\mathrm{as}_0(K)$ is not a measure of asymmetry in the sense of Grünbaum [6].

In the following, we introduce a new measure of asymmetry in terms of the normalized ϕ-mixed volume.

Definition 5.6 Suppose that $\phi \in \Phi$ be concave on $(0, \infty)$, $K \in \mathcal{K}^2$, and $s \in \mathcal{C}_\infty(K)$ be the unique ∞-critical point of K. The ϕ-Minkowski measure $\mathrm{as}_\phi(K)$ of K is defined by

$$\mathrm{as}_\phi(K) = \bar{V}_\phi(K_s, -K_s). \tag{57}$$

For the ϕ-Minkowski measure, we have the following theorem.

Theorem 5.7 *Suppose that $\phi \in \Phi$ be concave on $(0, \infty)$. If $K \in \mathcal{K}^2$, then,*

$$1 \le \mathrm{as}_\phi(K) \le 2. \tag{58}$$

Equality holds on the left-hand side if and only if K is symmetric, and equality holds on the right-hand side if and only if K is a triangle.

Proof From (57), (42) and (56), we have

$$\begin{aligned} \mathrm{as}_\phi(K) &= \bar{V}_\phi(K_s, -K_s) \\ &\ge \bar{V}_0(K_s, -K_s) \\ &= \mathrm{as}_0(K) \\ &\ge 1. \end{aligned}$$

On the other hand, from the concavity of ϕ, we have

$$\int_{S^{n-1}} \phi\left(\frac{h_{-K_s}}{h_{K_s}}\right) \mathrm{d}\bar{V}_{K_s} \le \phi\left(\int_{S^{n-1}} \frac{h_{-K_s}}{h_{K_s}} \mathrm{d}\bar{V}_{K_s}\right). \tag{59}$$

From (27), (38), (53), (54) and (59), we have

$$\begin{aligned} \mathrm{as}_\phi(K) &= \bar{V}_\phi(K_s, -K_s) \\ &= \phi^{-1}\left(\int_{S^{n-1}} \phi\left(\frac{h_{-K_s}}{h_{K_s}}\right) \mathrm{d}\bar{V}_{K_s}\right) \\ &\le \int_{S^{n-1}} \frac{h_{-K_s}}{h_{K_s}} \mathrm{d}\bar{V}_{K_s} \\ &= \bar{V}_1(K_s, -K_s) \\ &= \mathrm{as}_1(K) \\ &\le 2. \end{aligned}$$

Hence,

$$1 \le \mathrm{as}_0(K) \le \mathrm{as}_\phi(K) \le \mathrm{as}_1(K) \le 2.$$

If K is symmetric, then we have $1 = \mathrm{as}_0(K) \leq \mathrm{as}_\phi(K) \leq \mathrm{as}_1(K) = 1$, which implies $\mathrm{as}_\phi(K) = 1$; Conversely, if $\mathrm{as}_\phi(K) = 1$, then $1 \leq \mathrm{as}_0(K) \leq \mathrm{as}_\phi(K) = 1$, which implies $\mathrm{as}_0(K) = 1$, so K is symmetric.

If K is a triangle, then we have $2 = \mathrm{as}_0(K) \leq \mathrm{as}_\phi(K) \leq \mathrm{as}_1(K) = 2$, which implies $\mathrm{as}_\phi(K) = 2$; Conversely, if $\mathrm{as}_\phi(K) = 2$, then $2 = \mathrm{as}_\phi(K) \leq \mathrm{as}_1(K) \leq 2$, which implies $\mathrm{as}_1(K) = 2$, so K is a triangle. □

Funding Project supported by Postgraduate Research and Practice Innovation Program of Jiangsu Province No. KYCX20_2745 and by National Nature Science Foundation of China Nos. 12071334 and 12071277.

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