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ORIGINAL ARTICLE

Semi Riemannian hypersurfaces with a canonical principal direction

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Abstract

We study semi-Riemannian hypersurfaces with a canonical principal direction (CPD) with respect to a nondegenerate closed conformal vector field on a semi-Riemannian ambient manifold. We give a characterization of such hypersurfaces. In the case when such hypersurface is a surface with constant mean curvature in a semi-Riemannian space form, we prove that it has an intrinsic Killing vector field. A special case of hypersurfaces with a CPD are those with constant angle with respect to a parallel vector field in the semi-Riemannian ambient. We prove that a surface with zero mean curvature and constant angle, in a Loretzian ambient of arbitrary dimension, is necessarily flat. When the surface is timelike and the ambient has non positive curvature then the surface is totally geodesic. When the surface is spacelike and the ambient has non negative curvature then the surface is totally geodesic. In general when the ambient is of dimension three then the surface is always totally geodesic.

Mathematics Subject Classification Primary: 53C42

Keywords Canonical principal direction · Closed conformal vector field · Constant mean curvature · semi-Riemannian hypersurface · Constant angle surface

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1 Introduction

It is well known that a hypersurface in a semi-Riemannian ambient does not necessarily have a principal direction, which is an eigenvector of its shape operator. Of course spacelike hypersurfaces have a diagonalizable shape operator and so a basis of principal directions. But for example a timelike or a hypersurface whose metric have index greater than one could not have one. Let us recall that in Minkowski three dimensional space there are non trivial flat surfaces with zero mean curvature with a unique principal direction which is lightlike: For example the last surface given in the classification Theorem 5.1 of [3]. So it is interesting to investigate those hypersurfaces with a principal direction, in particular those hypersurfaces with a special one.

We say that a nondegenerate hypersurface isometrically immersed in a semi-Riemannian ambient manifold has a canonical principal direction if the tangent part of a vector field in the ambient is a principal direction. In this manuscript, we ask this vector field in the ambient to be closed conformal, in particular it could be a parallel vector field. The first works in the literature about surfaces with a canonical principal direction with respect to a parallel vector field in the ambient manifold were [5] and [6]. The Riemannian ambient manifolds in these works are $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, respectively. Since then, many other works have appeared: [4, 8, 9]. In [8], where the authors investigated CPD hypersurfaces with respect to a closed conformal vector field in a Riemannian ambient manifold. In [14], the authors extended some results of [8] into a Lorentz ambient. We want to remark the recent work [2], where the authors consider the concept of canonical vector field with respect to a radial (which is closed conformal) vector field in semi-Euclidean space.

We should remark that helix or constant angle hypersurfaces (see Definition 3.1) with respect to a parallel or closed conformal vector field in some semi-Riemannian ambient manifolds are examples of CPD hypersurfaces as we can check in [10] and [7]. This latter class of surfaces have been also studied with respect to a Killing vector field in instead of a parallel or a closed conformal one, see for example [12] and [11].

We now describe the three main results in this manuscript. In Theorem 2.14, we give a characterization of CPD semi-Riemannian hypersurfaces with respect to a closed conformal vector field Z in the ambient. In particular, to have a CPD is equivalent to: The integral curves of the normalized tangent part $T := Z^T/|Z^T|$ of Z are geodesics of the hypersurface. In Theorem 2.20, we prove that in a CPD nondegenerate surface with constant mean curvature the vector field fT is closed conformal and fW is Killing, where W is unitary and orthogonal to T and f is an explicit function on M. This can be interpreted as the surface having some symmetry. Finally, in Theorem 3.7, we consider nondegenerate surfaces in a Lorentzian ambient of arbitrary dimension, with zero mean curvature and constant angle with respect to a parallel vector field. We deduce that. they are flat. We also have that such surfaces are totally geodesic under any of the following conditions:

- The ambient has dimension three.
- The surface is timelike and the ambient has non positive curvature.

- The surface is spacelike and the ambient has non negative curvature.

2 Hypersurfaces with a canonical principal direction

Notation 2.1 *In this manuscript we assume the next facts and use the following notation:*

- *All the manifolds, functions, vector fields are of class* C^∞.
- *N is a semi-Riemannian manifold. Standard references are* [13] *and also* [1].
- *Z is a vector field on N.*
- *M is a nondegenerate immersed submanifold of N, i. e. the induced metric on M is nondegenerate.*
- *We denote by D the Levi–Civita connection of N and by* ∇ *the Levi–Civita connection of M.*
- ξ *is an unitary vector field orthogonal to M.*
- A_ξ *is the shape operator of M in N with respect to the normal vector field* ξ.
- α *denotes the second fundamental form of M in N.*
- $\nabla^\perp$ *is the normal connection on the normal bundle of M in N.*
- Z^T *is the tangent part of Z on M and* $Z^\perp$ *its orthogonal part. We will assume that* $Z^\top$ *and* $Z^\perp$ *are nowhere zero on M and are nondegenerate in the sense that they are either spacelike or timelike vector fields.*

Definition 2.2 Let N be a semi-Riemannian manifold. Let Z be a vector field on N. We say that Z is a closed conformal vector field, if there exists a function φ on N such that for every $Y \in \mathfrak{X}(N)$ we have the relation

$$D_Y Z = \varphi Y.$$

Notation 2.3 *In the next results of this section we will assume that:*

- *Z will be a nondegenerate closed conformal vector field on the semi-Riemannian manifold N. So, Z will be either timelike or spacelike.*
- *M will be a hypersurface of N.*

Definition 2.4 Let us consider a nondegenerate hypersurface $M \subset N$. We say that a vector field X on M is a principal direction of M, if $A_\xi(X) = \kappa X$ for some function κ on M.

Remark 2.5 In general, when the hypersurface M is timelike the principal directions do not necessarily exist. On other hand, when M is spacelike its shape operator A_ξ is diagonalizable.

Definition 2.6 We say that a nondegenerate hypersurface L of N is umbilical if and only if the shape operator A^L of L satisfies:

$$A^L_v(u) = fv,$$

for some function f on L and for every $u \in \mathfrak{X}(L)$. Here v is a unitary vector field orthogonal to L.

A direct computation proves the next result.

Lemma 2.7 *Let Z be a nondegenerate closed conformal vector field on N.*

(1) *If $Y \in \mathfrak{X}(N)$ is orthogonal to Z, then $Y \cdot |Z| = 0$.*
(2) *The integral curves of Z are geodesics in N.*
(3) *If Z is nowhere zero along N then the orthogonal distribution to Z is integrable with umbilical leaves.*

Example 2.8 Of course, parallel vector fields are closed conformal vector fields with $\varphi \equiv 0$. But there are closed conformal vector fields which are non parallel.

Let $I \subset \mathbb{R}$ be an open interval. In the semi-Riemannian warped product $I \times_\rho N$ there is a natural closed conformal vector field: the vector field $\rho\partial_t$. Here $\rho : I \longrightarrow \mathbb{R}$ is a C^∞ positive function. ∂_t is the lift into $I \times N$ of the canonical unitary vector field on the interval I. We apply the properties of the Levi–Civita connection D of the above semi-Riemannian warped product. In particular, we apply Proposition 35, p. 206 of [13]. We have that

$$D_{\partial_t}\partial_t = 0, \quad D_V\partial_t = \frac{\partial_t \cdot \rho}{\rho} V,$$

for every vector field V on $I \times_\rho N$ tangent to the leaves $\{t\} \times N$. Therefore,

$$D_{\partial_t}(\rho\partial_t) = (\partial_t \cdot \rho)\partial_t, \quad D_V(\rho\partial_t) = (V \cdot \rho)\partial_t + \rho D_V\partial_t = (\partial_t \cdot \rho)V.$$

This proves that for every vector field Y on $I \times_\rho N$, $D_Y(\rho\partial_t) = (\partial_t \cdot \rho)Y$, i.e. $\rho\partial_t$ is a closed conformal vector field.

Lemma 2.9 *Let M be a nondegenerate hypersurface of a semi-Riemannian manifold N. Then we can decompose Z as*

$$Z = \lambda \cdot T + \mu \cdot \xi$$

where T, ξ, λ y μ are defined by

- $T := \dfrac{Z^T}{|Z^T|}$
- $\xi := \dfrac{Z^\perp}{|Z^\perp|}$
- $\lambda := |Z^T|$
- $\mu := |Z^\perp|$.

Proof We can decompose Z in its tangent and normal parts to obtain

$$
\begin{aligned}
Z &= Z^T + Z^\perp \\
&= |\langle Z^T, Z^T\rangle|^{1/2} \cdot T + |\langle Z^\perp, Z^\perp\rangle|^{1/2} \cdot \xi \\
&= \lambda \cdot T + \mu \cdot \xi.
\end{aligned}
$$

□

Proposition 2.10 *The following equations are valid for the immersed hypersurface $M \subset N$: Let $X \in \mathfrak{X}(M)$, then*

(a) $\varphi \cdot X = X(\lambda) \cdot T + \lambda \cdot \nabla_X T - \mu \cdot A_\xi X$
(b) $0 = X(\mu) \cdot \xi + \lambda \cdot \alpha(X, T)$

where

- $\varphi \in C^\infty(N)$ *such that* $D_X Z = \varphi X$
- $\lambda = |Z| \cdot |\lambda|^{1/2} = |Z^\top|$ *and* $\mu = |Z| \cdot |\mu|^{1/2} = |Z^\perp|$.

Proof Let $X \in \mathfrak{X}(M)$. Let us compute $D_X Z$ using the decomposition of Z in Lemma 2.9

$$
\begin{aligned}
D_X Z &= D_X[\lambda \cdot T + \mu \cdot \xi] \\
&= (X \cdot \lambda)T + \lambda D_X T + (X \cdot \mu)\xi + \mu D_X \xi \\
&= (X \cdot \lambda)T + \lambda(\nabla_X T + \alpha(X, T)) + (X \cdot \mu)\xi - \mu A_\xi X.
\end{aligned}
$$

On other hand, since Z is a closed and conformal vector field, $D_X Z = \varphi \cdot X$. Thus

$$
\varphi \cdot X = (X \cdot \lambda)T + (X \cdot \mu)\xi + \lambda(\nabla_X T + \alpha(X, T)) - \mu A_\xi X.
$$

Now, we can take the tangent and normal parts of the above equation to get

$$
\begin{aligned}
\varphi \cdot X &= (X \cdot \lambda)T + \lambda \nabla_X T - \mu A_\xi X \\
0 &= (X \cdot \mu)\xi + \lambda \alpha(X, T).
\end{aligned}
$$

□

Example 2.11 We will see that if M is an umbilical hypersurface of N, then Z^T is a closed conformal vector field on M.

By Proposition 2.10, we have that for every $X \in \mathfrak{X}(M)$

$$
\lambda \cdot \nabla_X T = -X(\lambda) \cdot T + \mu \cdot A_\xi X + \varphi \cdot X.
$$

Equivalently,

$$
\begin{aligned}
\nabla_X(\lambda T) &= \mu A_\xi X + \varphi X \\
\nabla_X(|Z^\top|\, T) &= |Z^\perp| A_\xi X + \varphi X \\
\nabla_X Z^\top &= |Z^\perp| A_\xi X + \varphi X.
\end{aligned}
$$

Since M is an umbilical hypersurface, there is a function κ on M such that

$A_\xi X = \kappa X$, for every $X \in \mathfrak{X}(M)$.

We deduce that,

$$\nabla_X Z^\top = |Z^\perp|\kappa X + \varphi X = (|Z^\perp|\kappa + \varphi)X.$$

This proves that $Z^\top$ is closed conformal on M.

Let us observe that a semi-Euclidean space have parallel vector fields. This imply that, the semi-Riemannian space forms have closed conformal vector fields because they are umbilical hypersurfaces in a semi-Euclidean ambient manifold.

Corollary 2.12 *Let $W \in \mathfrak{X}(M)$ be a vector field orthogonal to T. Let us define $u_\xi := \langle \xi, \xi \rangle = \pm 1$. Then the following equations are valid*:

(a) $A_\xi W = \dfrac{1}{\mu} \cdot [W(\lambda) \cdot T + \lambda \cdot \nabla_W T - \varphi \cdot W]$

(b) $A_\xi T = \dfrac{T(\lambda) - \varphi}{\mu} \cdot T + \dfrac{\lambda}{\mu} \cdot \nabla_T T$

(c) $\alpha(W, T) = -\dfrac{W(\mu)}{\lambda} \cdot \xi$

(d) $\alpha(T, T) = -\dfrac{T(\mu)}{\lambda} \cdot \xi$

(e) $\langle \alpha(W, T), \xi \rangle = -\dfrac{W(\mu)}{\lambda} \cdot u_\xi$

(f) $\langle \alpha(T, T), \xi \rangle = -\dfrac{T(\mu)}{\lambda} \cdot u_\xi$

(g) $W(\mu) = -\dfrac{\lambda^2}{\mu} \cdot u_\xi \cdot \langle \nabla_T T, W \rangle$.

Proof

(a) In Proposition 2.10 (a), we can take $X = W$ and solve for A_ξ:

$$A_\xi W = \frac{1}{\mu} \cdot [W(\lambda) \cdot T + \lambda \cdot \nabla_W T - \varphi \cdot W].$$

(b) Similarly as in (a) above, now we can take $X = T$:

$$A_\xi T = \frac{T(\lambda) - \varphi}{\mu} \cdot T + \frac{\lambda}{\mu} \cdot \nabla_T T.$$

(c) We apply again Proposition 2.10 (b), taking $X = W$ and solving for $\alpha(W, T)$:

$$\alpha(W, T) = -\frac{W(\mu)}{\lambda} \cdot \xi.$$

(d) Analogously as in (c) above, now we take $X = T$:

$$\alpha(T, T) = -\frac{T(\mu)}{\lambda} \cdot \xi.$$

(e) We apply (c) to get

$$\langle \alpha(W,T),\xi\rangle = -\frac{W(\mu)}{\lambda}\cdot\langle \xi,\xi\rangle = -\frac{W(\mu)}{\lambda}\cdot u_\xi.$$

(f) Now, we apply (d) above:

$$\langle \alpha(T,T),\xi\rangle = -\frac{T(\mu)}{\lambda}\cdot\langle \xi,\xi\rangle = -\frac{T(\mu)}{\lambda}\cdot u_\xi.$$

(g) We can use that W y T are orthogonal and (b) above.

$$\begin{aligned}\langle \alpha(W,T),\xi\rangle &= \langle W, A_\xi T\rangle \\ &= \frac{1}{\mu}\cdot[T(\lambda)-\varphi]\cdot\langle W,T\rangle + \frac{\lambda}{\mu}\cdot\langle W,\nabla_T T\rangle \\ &= \frac{\lambda}{\mu}\cdot\langle W,\nabla_T T\rangle.\end{aligned}$$

On other side, we have that

$$\langle \alpha(W,T),\xi\rangle = -\frac{W(\mu)}{\lambda}, \quad \text{equivalently} \quad -\frac{W(\mu)}{\lambda} = \frac{\lambda}{\mu}\cdot\langle W,\nabla_T T\rangle$$

and therefore

$$W(\mu) = -\frac{\lambda^2}{\mu}\cdot u_\xi\cdot\langle \nabla_T T, W\rangle.$$

□

Definition 2.13 We say that a nondegenerate hypersurface $M\subset N$ has a canonical principal direction with respect to Z, if the tangent part $Z^\top$ of Z along M is nowhere degenerate and it is a principal direction of M.

Our Theorem 2.14 is an extension into a semi-Riemannian context of part of Theorem 5 in [8] given in a Riemannian context. A similar result is Theorem 2.3 in [14] for a spacelike hypersurface in a Lorentz ambient.

Theorem 2.14 *Let M be a nondegenerate hypersurface of N. Then the following are equivalent*:

(1) *M has a canonical principal direction with respect to Z, i.e. T is a principal direction.*
(2) *$|Z^\perp|$ is constant along directions tangent to M and orthogonal to T.*
(3) *The integral curves of T are geodesics of M.*
(4) *In Z is nowhere zero on M, the function $\langle Z/|Z|,\xi\rangle$ is constant in the directions tangent to M and ortogonal to T.*

Proof Let W be a vector field in $\mathfrak{X}(M)$ orthogonal to T. We have the following implications:

- (1)⇒(2)
  Since T is a principal direction, we have that $A_\xi T = \theta \cdot T$. By Corollary 2.12 (b), we deduce that

$$\theta \cdot T = A_\xi T = \frac{T(\lambda) - \varphi}{\mu} \cdot T + \frac{\lambda}{\mu} \cdot \nabla_T T.$$

  Solving for $\nabla_T T$ we get

$$\nabla_T T = \left[\frac{\mu \cdot \theta - T(\lambda) - \varphi}{\lambda}\right] \cdot T.$$

  Now, we substitute the latter equation in Corollary 2.12 (g):

$$\begin{aligned} W(\mu) &= -\frac{\lambda^2}{\mu} \cdot u_\xi \cdot \langle \nabla_T T, W \rangle \\ &= -\left[\frac{\mu \cdot \theta - T(\lambda) - \varphi}{\lambda}\right] \cdot \frac{\lambda^2}{\mu} \cdot u_\xi \cdot \langle T, W \rangle = 0. \end{aligned}$$

  Therefore

$$\mu = |Z^\perp|$$

  is constant along the direction W.
- (2)⇒(3)
  Since μ is constant, by Corollary 2.12 (g) we deduce that $\langle \nabla_T T, W \rangle = 0$. Moreover, since T is unitary, we obtain that $\langle \nabla_T T, T \rangle = (1/2) \cdot T \langle T, T \rangle = 0$. This proves that $\nabla_T T = 0$.
- (3)⇒(1)
  We have that $\nabla_T T = 0$, so by Corollary 2.12 (b) we get that

$$A_\xi T = \frac{T(\lambda) - \varphi}{\mu} \cdot T + \frac{\lambda}{\mu} \cdot \nabla_T T = \frac{T(\lambda) - \varphi}{\mu} \cdot T.$$

  As a consequence, T is a principal direction.
- (2)⇒(4)
  Let us recall that $Z^\perp = \langle \xi, \xi \rangle \langle Z, \xi \rangle \xi$. Thus,

$$\langle Z^\perp, Z^\perp \rangle = \langle \xi, \xi \rangle \langle Z, \xi \rangle^2.$$

  So, $|Z^\perp| = |\langle Z, \xi \rangle|$. We deduce that, $|Z^\perp|/|Z| = |\langle Z/|Z|, \xi \rangle|$. This says that $\langle Z/|Z|, \xi \rangle = \epsilon |Z^\perp|/|Z|$, where $\epsilon = \pm 1$. Let $X \in \mathfrak{X}(M)$ and orthogonal to T. Therefore, X is also orthogonal to Z. By our assumption (2), $X \cdot |Z^\perp| = 0$. By Lemma 2.7, $X \cdot |Z| = 0$. This implies that

$$X \cdot \langle Z/|Z|, \xi \rangle = \epsilon (X \cdot |Z^\perp|/|Z|) = 0.$$

- (4)⇒(2)
  In a similar fashion, we just have to apply the relation $|Z^{\perp}| = \epsilon|Z|\langle Z/|Z|, \xi\rangle$ and Lemma 2.7.

□

Definition 2.15 Let $F : N \longrightarrow \mathbb{R}$ be a function. We say that F is a transnormal function if there is a function $b : I \longrightarrow \mathbb{R}$ defined on some interval $I \subset \mathbb{R}$ such that

$$|\nabla F| = b \circ F.$$

Lemma 2.16 *Let $F : N \longrightarrow \mathbb{R}$ be a transnormal function. Let ∇F be a nowhere zero vector field which is either spacelike or timelike. Then $X \cdot \langle \nabla F, \nabla F\rangle = 0$ for every $X \in \mathfrak{X}(N)$ orthogonal to ∇F.*

Proof By our hypothesis, the level hypersurfaces of F are nondegenerate hypersurfaces of N orthogonal to ∇F. Let $X \in \mathfrak{X}(N)$ be orthogonal to ∇F, in consequence X is tangent to the level hypersurfaces of F. Now, let us observe that $|\langle \nabla F, \nabla F\rangle| = (b \circ F)^2$ and that it is nowhere zero. So, $\langle \nabla F, \nabla F\rangle = \epsilon(b \circ F)^2$, where $\epsilon = \pm 1$. Therefore,

$$X \cdot \langle \nabla F, \nabla F\rangle = 2\epsilon(b \circ F)[X \cdot (b \circ F)] = 0,$$

because the function $b \circ F$ is constant along the level hypersurfaces of F. □

Example 2.17 We use a transnormal function to construct a hypersurface with a canonical principal direction.

Let $F : N \longrightarrow \mathbb{R}$ be a transnornal function such that $\langle \nabla F, \nabla F\rangle + 1$ is nowhere zero along N. We also assume that ∇F is nowhere zero and it is either spacelike or a timelike vector field.

We verify that the graph of F given by

$$M := \{p = (x, F(x)) \in N \times \mathbb{R}|\ x \in N\},$$

is a hypersurface in the semi-Riemannian manifold $N \times \mathbb{R}$ with a canonical principal direction with respect to the parallel vector field ∂_t.

We proceed as follows: Let us observe that a unitary vector field orthogonal to M is defined by $\xi := (\nabla F - \partial_t)/\sqrt{|\langle \nabla F, \nabla F\rangle + 1|}$. This is true because a basis on the tangent spaces T_pM is given by

$$\{X_1 + (X_1 \cdot F)\partial_t, \ldots, X_{dimN} + (X_{dimN} \cdot F)\partial_t\},$$

where $X_1, \ldots, X_{dimN}$ is any basis of $T \times N$. In particular, if we choose the basis $\nabla F, Y_2, \ldots, Y_{dimN}$ for T_xN, with $Y_2, \ldots, Y_{dimN}$ orthogonal to ∇F, the basis for T_pM becomes

$$\nabla F + \langle \nabla F, \nabla F\rangle \partial_t, Y_2, \ldots, Y_{dimN}.$$

Now, it is clear that ξ is orthogonal to the latter basis of T_pM because ∂_t is orthogonal to $\nabla F, Y_2, \ldots, Y_{dimN}$.

This observation also implies that $\partial_t^\top$ is linearly dependent of $\nabla F + \langle \nabla F, \nabla F\rangle \partial_t$. In particular, a basis for the orthogonal directions to $\partial_t^\top$ and tangent to M is given by $Y_2, \ldots, Y_{dimN}$.

Finally, we will apply Theorem 2.14 case (4): We need the following computation,

$$\langle \partial_t, \xi\rangle = -1/\sqrt{|\langle \nabla F, \nabla F\rangle + 1|}.$$

By Lemma 2.16, $\langle \nabla F, \nabla F\rangle$ is constant along the level hypersurfaces of F. Since, $Y_2, \ldots, Y_{dimN}$ are orthogonal to ∇F and tangent to N, they are also tangent to the level hypersurfaces of F. But these directions $Y_2, \ldots, Y_{dimN}$ are tangent to M as was explained above. So, $\langle \partial_t, \xi\rangle$ is constant along the directions tangent to M and orthogonal to ∇F. This proves that the hypersurface M of $N = M \times \mathbb{R}$ has a canonical principal direction with respect to $Z = \partial_t$.

Lemma 2.18 *Let M be a semi-Riemannian surface and let f be a function on M. Let U and V be orthonormal vector fields in $\mathfrak{X}(M)$, with $\epsilon_V := \langle V, V\rangle$. Let us assume that U is a geodesic vector field on $M : \nabla_U U = 0$. Then the next conditions are equivalent*

(1) *fU is a closed conformal vector field on M.*
(2) *f satisfies*

$$V \cdot f = 0, \quad U \cdot f = \epsilon_V f \langle \nabla_V U, V\rangle.$$

(3) *fV is a Killing vector field on M.*

Proof Since U is a geodesic vector field, $\nabla_U V = 0$. Let us observe that for all $X \in \mathfrak{X}(M)$,

$$\nabla_U(fU) = (U \cdot f)U + f\nabla_U U = (U \cdot f)U$$

and

$$\nabla_V(fU) = (V \cdot f)U + \epsilon_V f\langle \nabla_V U, V\rangle V.$$

From de above equations we deduce that fU is a closed conformal vector field if and only if $V \cdot f$ and $U \cdot f = \epsilon_V f\langle \nabla_V U, V\rangle$. This proves the equivalence between (1) and (2).

Another direct computation proves that (2) implies (3).

Let us prove that (3) implies (2):

Since fV is Killing,

$$0 = \langle \nabla_V(fV), V \rangle = (V \cdot f)\langle V, V \rangle = (V \cdot f)\epsilon_V.$$
$$(U \cdot f)\epsilon_V = \langle \nabla_U(fV), V \rangle = -\langle \nabla_V(fV), U \rangle = -f\langle \nabla_V V, U \rangle = f\langle \nabla_V U, V \rangle.$$

This finishes the proof. □

Remark 2.19 Let us recall that the mean curvature vector field H of our nondegenerate surface M in a three dimensional semi-Riemannian manifold N is given by

$$2H = \epsilon_T \alpha(T, T) + \epsilon_W \alpha(W, W),$$

where T is as before and T, W are orthonormal vector fields on M. This implies that

$$\begin{aligned} 2\langle H, \xi \rangle &= \epsilon_T \langle \alpha(T, T), \xi \rangle + \epsilon_W \langle \alpha(W, W), \xi \rangle \\ &= \epsilon_T \langle A_\xi(T), T \rangle + \epsilon_W \langle A_\xi(W), W \rangle. \end{aligned}$$

When $\langle H, \xi \rangle$ is a constant function, we say that M has constant mean curvature (CMC).

The next Theorem 2.20, is an extension into the semi-Riemannian context of Lemma 3.4 and part of Theorem 3.6 in [4].

Theorem 2.20 *Let M be a nondegenerate surface of the three dimensional either Riemannian or Lorentzian space form N without umbilical points. Let us assume that M has a canonical principal direction with respect to Z. If M has constant mean curvature (CMC) then $\frac{1}{\sqrt{|\kappa_1 - \kappa_2|}} T$ is closed and conformal vector field and $\frac{1}{\sqrt{|\kappa_1 - \kappa_2|}} W$ is a Killing vector field on M, where κ_1 and κ_2 are the principal curvatures in the orthogonal directions T and W, respectively. We also have that $W \cdot \kappa_1 = W \cdot \kappa_2 = 0$.*

Proof The case when N is a Riemannian space form was considered in Lemma 3.4 and Theorem 3.6 of [4]. So, let us assume that N is a Lorentzian space form. And the proof in the following lines is an adaptation of Lemma 3.4 of [4].

By Theorem 2.14, T is a principal direction of M and $\nabla_T T = 0$. This latter condition implies that $\nabla_T W = 0$. Since M is nondegenerate, W is also a principal direction.

We will prove that the function $f := \frac{1}{\sqrt{|\kappa_1 - \kappa_2|}}$, $U := T$ and $V := W$ satisfy condition (1) in Lemma 2.18. We proceed as follows.

Let us apply the Codazzi equation in space forms:

$$\nabla_T(A_\xi W) - A_\xi(\nabla_T W) = \nabla_W(A_\xi T) - A_\xi(\nabla_W T).$$

So we obtain

$$\begin{gathered} \nabla_T(\kappa_2 W) = \nabla_W(\kappa_1 T) - A_\xi(\nabla_W T) \\ (T \cdot \kappa_2)W = (W \cdot \kappa_1)T + \kappa_1 \nabla_W T - \kappa_2 \nabla_W T \end{gathered}$$

Now we apply the metric against T, in first place, and against W in second place:

Against T:

$$0 = \epsilon_T (W \cdot \kappa_1),$$

where $\epsilon_T = \langle T, T \rangle$. Since M is CMC, Remark 2.19 implies that there is a constant a such that $\epsilon_T \langle A_\xi(T), T \rangle + \epsilon_W \langle A_\xi(W), W \rangle = a$. In our case we have that T and W are principal directions, so $\kappa_1 + \kappa_2 = a$. Therefore, we also have that $0 = W \cdot \kappa_2$.

Against W:

$$\epsilon_W (T \cdot \kappa_2) = (\kappa_1 - \kappa_2) \langle \nabla_W T, W \rangle,$$

where $\epsilon_W = \langle W, W \rangle$. Once again, the CMC condition on M implies that

$$-\epsilon_W T \cdot \kappa_1 = (\kappa_1 - \kappa_2) \langle \nabla_W T, W \rangle$$

and so

$$T \cdot (\kappa_1 - \kappa_2) = -2\epsilon_W (\kappa_1 - \kappa_2) \langle \nabla_W T, W \rangle \,.$$

Since M does not have umbilical points, we can write

$$|\kappa_1 - \kappa_2| = \sigma(\kappa_1 - \kappa_2),$$

where $\sigma \in \{-1, +1\}$. Finally, let us observe that

$$T \cdot \frac{1}{\sqrt{|\kappa_1 - \kappa_2|}} = T \cdot \frac{1}{\sqrt{\sigma(\kappa_1 - \kappa_2)}} = -\frac{1}{2} \frac{\sigma T \cdot (\kappa_1 - \kappa_2)}{(\sigma(\kappa_1 - \kappa_2))^{3/2}} = \epsilon_W \frac{\langle \nabla_W T, W \rangle}{\sqrt{|\kappa_1 - \kappa_2|}}.$$

This proves that, $\frac{1}{\sqrt{|\kappa_1 - \kappa_2|}} T$ is closed conformal and $\frac{1}{\sqrt{|\kappa_1 - \kappa_2|}} W$ is a Killing vector field on M. □

3 Semi-Riemannian submanifolds of constant angle

As we will see in this section, in a semi-Riemannian ambient manifold N, a constant angle hypersurface with respect to a parallel vector field has a canonical principal direction. Moreover, Proposition 3.5 of [10] says that, when the ambient is a semi-Riemannian space form, a constant angle hypersurface with respect to a closed conformal vector field has a canonical principal direction.

Definition 3.1 Let M be a nondegenerate submanifold of N. Let us assume that N admits nowhere degenerate a parallel vector field Z. We say that M is a helix submanifold or that has constant angle with respect to Z if we have that $|Z^T| = \lambda \neq 0$ is constant, where $Z = Z^T + Z^\perp$.

Remark 3.2 Let $X \in \mathfrak{X}(M)$. Since Z is parallel, $X \cdot \langle Z, Z \rangle = 2\langle D_X Z, Z \rangle = 0$, i. e. $\langle Z, Z \rangle$ is constant. Moreover, the relation

$$\langle Z, Z \rangle = \langle Z^\top, Z^\top \rangle + \langle Z^\perp, Z^\perp \rangle$$

implies that $\mu := |Z^\perp|$ is also a constant. If we assume that $\lambda, \mu \neq 0$ and using the equality

$$T = \frac{Z^T}{|\langle Z^T, Z^T \rangle|^{1/2}} \qquad \xi = \frac{Z^\perp}{|\langle Z^\perp, Z^\perp \rangle|^{1/2}}$$

we obtain that Z admits the following decomposition:

$$Z = \lambda T + \mu \xi.$$

Finally, When M is a hypersurface, we have that $\langle Z^T, Z^T \rangle$ is constant if and only if $\langle Z/|Z|, v \rangle$ is constant, where v is any unitary vector field orthogonal to M.

Notation 3.3 *In this section, we will assume the next conventions:*

- *Z is a parallel vector field on N.*
- *We have the decomposition $Z = \lambda T + \mu \xi$.*
- *M* **has constant angle with respect to** *Z.*
- *λ and μ are non zero constants.*

Proposition 3.4 *Let W in $\mathfrak{X}(M)$. Let us denote $\rho = \lambda/\mu$. Let us observe that $\mu \neq 0$. Then*

(1) *$A_\xi T = 0$, i.e. T is a principal direction of A_ξ.*
(2) *$A_\xi W = \rho \cdot \nabla_W T$.*
(3) *$\rho \cdot \alpha(T, W) = -\nabla^\perp_W \xi$.*
(4) *$\rho \cdot \alpha(T, T) = -\nabla^\perp_T \xi$.*
(5) *$\nabla_T T = 0$, i.e. the integral curves of T are geodesics of M.*
(6) *$\langle \alpha(W, T), \xi \rangle = 0$.*

Proof Let $X \in \mathfrak{X}(M)$. Since $D_X Z = 0$, then

$$\begin{aligned} 0 &= \lambda D_X T + \mu D_X \xi \\ &= \lambda(\nabla_X T + \alpha(X, T)) + \mu(-A_\xi X + \nabla^\perp_X \xi). \end{aligned}$$

We can take the tangent and normal parts to obtain

$$\begin{cases} 0 &= \lambda \nabla_X T - \mu A_\xi X \\ 0 &= \lambda \alpha(X, T) + \mu \nabla^\perp_X \xi \end{cases}$$

The items (2) and (3) are direct consequences of the latter two equations. To get case (4), we have to take $W = T$ in (3).

Now, we can assume that $X = T$ in the first equation above to get

$$0 = \lambda \nabla_T T - \mu A_\xi T.$$

Let W in $\mathfrak{X}(M)$, we have that

$$\begin{aligned}\langle \nabla_T T, W\rangle &= (\mu/\lambda)\langle A_\xi T, W\rangle = (\mu/\lambda)\langle \xi, \alpha(T, W)\rangle \\ &= (\mu/\lambda)\langle \xi, -(\mu/\lambda)\nabla^\perp_W \xi\rangle = -(\mu/\lambda)^2\ \langle \xi, \nabla^\perp_W \xi\rangle \\ &= -(1/2)(\mu/\lambda)^2\ W\cdot\langle \xi, \xi\rangle = 0.\end{aligned}$$

Thus, $\nabla_T T = 0$, which implies that $A_\xi T = 0$. So, (1) and (5) are valid. It follows from (1), that $\langle \alpha(W, T), \xi\rangle = \langle A_\xi T, X\rangle = 0$. This proves (6). □

Definition 3.5 We say that a hypersurface of N is ruled, if the hypersurface has a non singular foliation by geodesics of N.

Corollary 3.6 *If M is a hypersurface then for every vector field W in $\mathfrak{X}(M)$ we have that*:

(a) $\alpha(W, T) = 0$
(b) $\nabla^\perp_W \xi = 0$
(c) $D_T T = 0$, *i.e. the integral curves of T are also geodesics of N. Then M is a ruled hypersurface of N.*

Proof

(a) By Proposition 3.4 (6), we have that $\langle \alpha(W, T), \xi\rangle = 0$. Since M is a hypersurface, ξ generates $T^\perp M$. Therefore $\alpha(W, T) = 0$.
(b) By (a) above, we have that $\alpha(W, T) = 0$ for every vector field W in $\mathfrak{X}(M)$. By Proposition 3.4 (3), we obtain that

$$\nabla^\perp_W \xi = -\rho\cdot\alpha(T, W) = 0$$

(c) By (a), we know that $\alpha(W, T) = 0$ for every vector field W in $\mathfrak{X}(M)$. In particular, $\alpha(T, T) = 0$. By proposition 3.4 (5), $\nabla_T T = 0$. So, by Gauss formula for $D_T T$ we have that

$$D_T T = \nabla_T T + \alpha(T, T) = 0.$$

□

The next result is a generalization of part of Theorem 3.1 and Theorem 3.2 in [15], which are in a Riemannian context.

Theorem 3.7 *Let us assume that N is a Lorentzian manifold with* $\dim(N) \geq 3$ *and that M^2 is a surface with zero mean curvature.*

(a) *Then M is flat. Moreover, if* $\dim(N) = 3$, *then M is totally geodesic.*
(b) *If M is timelike and N has non positive curvature then M is totally geodesic.*
(c) *If M is spacelike and N has non negative curvature then M is totally geodesic.*

Proof Let us take X in $\mathfrak{X}(M)$ unitary and orthogonal to T. Let us recall that, since M has zero mean curvature, we have that $\epsilon_X \alpha(X, X) + \epsilon_T \alpha(T, T) = 0$, where $\epsilon_X = \langle X, X\rangle$ and $\epsilon_T = \langle T, T\rangle$.

(a) To prove that M is flat, we have to verify that

$$\nabla_T T = \nabla_X T = \nabla_T X = \nabla_X X = 0.$$

(i) $\nabla_T T = 0$: By Proposition 3.4 (5) we have that $\nabla_T T = 0$.

(ii) $\nabla_X T = 0$: Let us observe that

$$\langle \nabla_X T, T \rangle = (1/2)\ X\langle T, T \rangle = 0.$$

Moreover, By Proposition 3.4 (2),

$$\langle \nabla_X T, X \rangle = (1/\rho)\ \langle A_\xi X, X \rangle = (1/\rho)\ \langle \alpha(X, X), \xi \rangle = -\epsilon_X \epsilon_T \langle \alpha(T, T), \xi \rangle = 0.$$

This implies that $\nabla_X T = 0$.

(iii) $\nabla_X X = 0$: Since X and T are orthogonal then

$$0 = X\langle X, T \rangle = \langle \nabla_X X, T \rangle + \langle X, \nabla_X T \rangle.$$

In consequence,

$$\langle \nabla_X X, T \rangle = -\langle X, \nabla_X T \rangle = 0.$$

Since X is unitary, $\langle \nabla_X X, X \rangle = (1/2)\ X\langle X, X \rangle = 0$.
Therefore $\nabla_X X = 0$.

(iv) $\nabla_T X = 0$: We have the following equalities

$$\langle \nabla_T X, X \rangle = (1/2)\ T\langle X, X \rangle = 0,$$
$$\langle \nabla_T X, T \rangle = T\langle X, T \rangle - \langle X, \nabla_T T \rangle = 0.$$

Thus $\nabla_T X = 0$.

Now, let us consider the case when $\dim(N) = 3$. Let us assume that $\{X, T\}$ is an orthonormal frame in $\mathfrak{X}(M)$. By hypothesis, M is a hypersurface of N. By Corollary 3.6 (a), $\alpha(X, T) = 0$ and $\alpha(T, T) = 0$. So, $\alpha(X, X) = -\epsilon_X \epsilon_T \alpha(T, T) = 0$. Thus $\alpha \equiv 0$, i.e. M is totally geodesic.

(b) Let X be in $\mathfrak{X}(M)$ which is unitary and orthogonal to T. Thus we have an orthonormal frame on M. By (a) above, we have that M is flat and therefore $R^M_{TX} T = 0$.
We now apply Gauss equation:

$$\begin{aligned} 0 &= \langle R^M_{TX} T, X \rangle \\ &= \langle R^N_{TX} T, X \rangle - \langle \alpha(T, X), \alpha(X, T) \rangle + \langle \alpha(T, T), \alpha(X, X) \rangle. \end{aligned}$$

We obtain the equality

$$\langle R^N_{TX}T, X\rangle = \langle \alpha(T,X), \alpha(X,T)\rangle - \langle \alpha(T,T), \alpha(X,X)\rangle.$$

Since M has zero mean curvature, we have the condition $\alpha(X,X) + \alpha(T,T) = 0$, equivalently $\alpha(X,X) = -\alpha(T,T)$. Thus

$$\langle R^N_{TX}T, X\rangle = \langle \alpha(T,X), \alpha(X,T)\rangle + \langle \alpha(T,T), \alpha(T,T)\rangle.$$

Let us recall that M is a timelike surface, which means that the orthogonal subspace $TM^\perp$ is spacelike. So,

$$\langle \alpha(T,X), \alpha(X,T)\rangle + \langle \alpha(T,T), \alpha(T,T)\rangle \geq 0.$$

The other hypothesis says that N has non positive curvature, i.e. $\langle R^N_{TX}T, X\rangle \leq 0$. Therefore

$$\|\alpha(T,X)\|^2 + \|\alpha(T,T)\|^2 = 0.$$

This implies that $\alpha(T,X) = 0$ and $\alpha(T,T) = -\alpha(X,X) = 0$. This finishes the proof of (a).

(c) The proof is analogous to that of (b), by just reversing the above inequalities.

□

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