Bol. Soc. Mat. Mex. (2021) 27:53
https://doi.org/10.1007/s40590-021-00353-z

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

Some families of differential equations associated with the 2-iterated 2D Appell and related polynomials

Subuhi Khan[1] · Shahid Ahmad Wani[1]

Received: 21 November 2020 / Accepted: 17 April 2021 / Published online: 2 June 2021

Abstract
In the present paper, the differential, integro-differential and partial differential equations satisfied by the 2-iterated 2D Appell polynomials are obtained via the factorization method. Results for certain 2D hybrid families related to the Appell polynomials are derived. Further, corresponding results for certain mixed type and 2-iterated members of these families are obtained.

Keywords 2-Iterated 2D Appell polynomials · Factorization method · Differential equation

Mathematics Subject Classification 45J05 · 65Q30

1 Introduction

The theory of Appell and Sheffer polynomials [1, 2] arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. Many authors expressed and studied these polynomials via different approaches. Blasiak et al. [3] construct explicit representations of the Heisenberg–Weyl algebra and established a link between the monomiality principle and the umbral calculus. Dattoli and Zhukovsky [4] and Dattoli et al. [5] obtained the series expansion and connection coefficients for particular expressions of Appell polynomials, respectively. He and Ricci [6], derived the differential equation of the Appell polynomials via the factorization method. Multidimensional extensions of the Bernoulli and Appell polynomials are defined and corresponding equations are

✉ Shahid Ahmad Wani
shahidwani177@gmail.com

Subuhi Khan
subuhi2006@gmail.com

1 Department of Mathematics, Aligarh Muslim University, Aligarh, India

obtained in [7]. By using this method, differential, integro-differential and partial differential equations satisfied by the extended 2D Bernoulli and Euler polynomials are obtained in [8]. Moreover, corresponding results for the Hermite-based Appell polynomials are established in [9]. This methodology is additionally reached out to determine the integro-differential equations for the hybrid, 2D extraordinary and mixed type polynomials identified with the Appell family, see for instance [10–14]. Further, the factorization method was extended via k-times shift operators and a set of finite order differential equations for the Appell polynomials are obtained in [15].

Recently, Subuhi and Raza [16] introduced and studied the 2-iterated Appell polynomials defined by the generating function:

$$A_1(t)A_2(t)e^{xt} = \sum_{n=0}^{\infty} A_n^{[2]}(x)\frac{t^n}{n!} \tag{1}$$

and obtained their multiplicative and derivative operators, differential equations and operational rules. For certain special cases of $A_1(t)$ and $A_2(t)$, the 2-iterated Bernoulli, 2-iterated Euler and Bernoulli–Euler (or Euler–Bernoulli) polynomials are defined as [16]:

$$\left(\frac{t}{e^t - 1}\right)^2 e^{xt} = \sum_{n=0}^{\infty} B_n^{[2]}(x)\frac{t^n}{n!}, \tag{2}$$

$$\left(\frac{2}{e^t + 1}\right)^2 e^{xt} = \sum_{n=0}^{\infty} E_n^{[2]}(x)\frac{t^n}{n!} \tag{3}$$

and

$$\left(\frac{2t}{e^{2t} - 1}\right) e^{xt} = \sum_{n=0}^{\infty} {}_B E_n(x)\frac{t^n}{n!}. \tag{4}$$

In the present paper, we take use of the 2-iterated 2D Appell polynomials, which are constructed as the discrete Appell convolution of the Gould–Hopper based Appell polynomials, i.e., the multiplicative operator of Gould–Hopper based Appell polynomials is inserted in the generating function of Appell polynomials, which on further simplification gives the desired generating function. Thus, 2-iterated 2D Appell polynomials are defined by means of the following generating function:

$$A_1(t)A_2(t)e^{xt+yt^j} = \sum_{n=0}^{\infty} A_n^{[2],j}(x,y)\frac{t^n}{n!}, \tag{5}$$

where

$$A_1(t) = \sum_{k=0}^{\infty} A_k \frac{t^k}{k!} \text{ and } A_2(t) = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{6}$$

These polynomials satisfy the following relations:

$$\frac{\partial}{\partial x} A_n^{[2],j}(x,y) = n A_{n-1}^{[2],j}(x,y), \; \frac{\partial}{\partial y} A_n^{[2],j}(x,y) = \frac{n!}{(n-j)!} A_{n-j}^{[2],j}(x,y). \tag{7}$$

We derive the differential, integro-differential and partial differential equations satisfied by these polynomials. It is worth to consider certain special cases of generating function (5)

In the case $A_1(t) = 1$ and $A_2(t) = A(t)$, we obtain the bivariate Appell polynomials [7] and in the case $y = 0$, the 2-iterated 2D Appell polynomials reduce to the 2-iterated Appell polynomials [16]. Taking $A_1(t) = A(t)$ (of the Appell polynomials) and corresponding to suitable choices of $A_2(t)$ (of the Bernoulli, Euler and Hermite polynomials), the generating functions for the 2D Bernoulli–Appell, 2D Euler–Appell and 2D Hermite–Appell polynomials are obtained as:

$$A(t) \frac{t}{e^t - 1} e^{xt + yt^j} = \sum_{n=0}^{\infty} {}_B A_n^{(j)}(x,y) \frac{t^n}{n!}, \tag{8}$$

$$A(t) \frac{2}{e^t + 1} e^{xt + yt^j} = \sum_{n=0}^{\infty} {}_E A_n^{(j)}(x,y) \frac{t^n}{n!} \tag{9}$$

and

$$A(t) e^{xt - \frac{t^2}{2} + yt^j} = \sum_{n=0}^{\infty} {}_H A_n^{(j)}(x,y) \frac{t^n}{n!}, \tag{10}$$

respectively.

Again for $A_1(t) = A_2(t) = \frac{t}{e^t - 1}$, $A_1(t) = A_2(t) = \frac{2}{e^t + 1}$ and $A_1(t) = A_2(t) = e^{\frac{-t^2}{2}}$, we obtain the generating functions of the 2-iterated 2D Bernoulli, 2-iterated 2D Euler and 2-iterated 2D Hermite polynomials as:

$$\left(\frac{t}{e^t - 1} \right)^2 e^{xt + yt^j} = \sum_{n=0}^{\infty} B_n^{[2],j}(x,y) \frac{t^n}{n!}, \tag{11}$$

$$\left(\frac{2}{e^t + 1} \right)^2 e^{xt + yt^j} = \sum_{n=0}^{\infty} E_n^{[2],j}(x,y) \frac{t^n}{n!} \tag{12}$$

and

$$e^{xt - t^2 + yt^j} = \sum_{n=0}^{\infty} H_n^{[2],j}(x,y) \frac{t^n}{n!}, \tag{13}$$

respectively.

Further, for suitable combinations of $A_1(t)$ and $A_2(t)$, we obtain the generating functions of the 2D Bernoulli–Euler (or Euler–Bernoulli), 2D Hermite–Bernoulli (or Bernoulli–Hermite) and 2D Hermite–Euler (or Euler–Hermite) polynomials as:

$$\left(\frac{2t}{e^{2t}-1}\right)e^{xt+yt^j} = \sum_{n=0}^{\infty} {}_E B_n^{(j)}(x)\frac{t^n}{n!}, \tag{14}$$

$$\left(\frac{t}{e^{t}-1}\right)e^{xt-\frac{t^2}{2}+yt^j} = \sum_{n=0}^{\infty} {}_H B_n^{(j)}(x,y)\frac{t^n}{n!} \tag{15}$$

and

$$\left(\frac{2}{e^{t}+1}\right)e^{xt-\frac{t^2}{2}+yt^j} = \sum_{n=0}^{\infty} {}_H E_n^{(j)}(x,y)\frac{t^n}{n!}, \tag{16}$$

respectively.

Note that the Gould–Hopper polynomials $H_n^j(x,y)$ are of great importance, since these polynomials are the solutions of the generalized heat equation

$$\frac{\partial}{\partial y}H_n^j(x,y) = \frac{\partial^j}{\partial x^j}H_n^j(x,y), \tag{17}$$

$$H_n^j(x,0) = x^n. \tag{18}$$

In the following Lemma, we obtain an explicit form of the Gould–Hopper polynomials in terms of the 2-iterated 2D Appell polynomials:

Lemma 1.1 *For the Gould–Hopper polynomials, the following expansion formula holds true*:

$$H_n^j(x,y) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}A_{n-k}^{[2],j}(x,y)a_{k-m}b_m, \tag{19}$$

where

$$A_1^{-1}(t) = \sum_{k=0}^{\infty} a_k\frac{t^k}{k!},\ A_2^{-1}(t) = \sum_{m=0}^{\infty} b_m\frac{t^m}{m!}. \tag{20}$$

Proof Dividing both sides of generating function (5) by $A_1(t)A_2(t)$ and applying the Cauchy product rule, we get the desired result. □

Moreover, we have the following addition theorem for the 2-iterated 2D Appell polynomials:

Theorem 1.2 *For the* 2*-iterated* 2*D Appell polynomials, the following addition theorem holds true*:

$$A_n^{[2],j}(x,y) = \sum_{k=0}^{n}\binom{n}{k}(-z)^k A_{n-k}^{[2],j}(x+z,y). \tag{21}$$

Proof Rewriting generating function (5) as:

$$A_1(t)A_2(t)e^{(x+z)t+yt^j}e^{-zt} = \sum_{n=0}^{\infty} A_n^{[2],j}(x,y)\frac{t^n}{n!}, \tag{22}$$

which on making use of generating function (5) and expanding the exponential function in the l.h.s gives

$$\sum_{n=0}^{\infty} A_n^{[2],j}(x+z,y)\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{(-zt)^k}{k!} = \sum_{n=0}^{\infty} A_n^{[2],j}(x,y)\frac{t^n}{n!}. \tag{23}$$

Application of the Cauchy product gives the desired result. □

The article is organized as follows. In Sect. 2, the recurrence relations and shift operators for the 2-iterated 2D Appell polynomials are derived. Further, the differential, integro-differential and partial differential equations for this family are established. In Sect. 3, the recurrence relations, shift operators, differential, integro-differential and partial differential equations for the 2D Bernoulli–Appell, 2D Euler–Appell and 2D Hermite–Appell polynomials are established. In Sect. 4, results for certain hybrid members of these families are obtained. Finally, in Appendix the corresponding results for certain 2-iterated members are considered.

2 Main results

First, we derive the recurrence relation and obtain the expressions for the shift operators by proving the following result:

Theorem 2.1 *The* 2*-iterated* 2*D Appell polynomials* $A_n^{[2],j}(x,y)$ *satisfy the following recurrence relation*:

$$\begin{aligned}(x+\alpha_0+\beta_0)A_n^{[2],j}(x,y) + \sum_{k=1}^{n}\binom{n}{k}(\alpha_k+\beta_k)A_{n-k}^{[2],j}(x,y)\\ + jy\frac{n!}{(n-j+1)!}A_{n+1-j}^{[2],j}(x,y) = A_{n+1}^{[2],j}(x,y),\end{aligned} \tag{24}$$

where the coefficients $\{\alpha_k\}_{k\in N_0}$ *and* $\{\beta_k\}_{k\in N_0}$ *are given by*:

$$\frac{A_1'(t)}{A_1(t)} = \sum_{k=0}^{\infty}\alpha_k\frac{t^k}{k!} \text{ and } \frac{A_2'(t)}{A_2(t)} = \sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!} \tag{25}$$

and the shift operators are given by the following expressions:

$${}_x\mathcal{L}_n^- := \frac{1}{n}D_x, \tag{26}$$

$$ {}_x\mathcal{L}_n^+ := x + \alpha_0 + \beta_0 + \sum_{k=1}^{n} \frac{(\alpha_k + \beta_k)}{k!} D_x^k + jyD_x^{j-1}, \tag{27} $$

$$ {}_y\mathcal{L}_n^- := \frac{1}{n} D_x^{1-j} D_y, \tag{28} $$

$$ {}_y\mathcal{L}_n^+ := (x + \alpha_0 + \beta_0) + \sum_{k=1}^{n} \frac{(\alpha_k + \beta_k)}{k!} D_x^{(1-j)k} D_y^k + jyD_x^{-(j-1)^2} D_y^{j-1}, \tag{29} $$

where

$$ D_x := \frac{\partial}{\partial x};\ D_y := \frac{\partial}{\partial y}. $$

Proof Differentiating generating function (5) with respect to t and equating the coefficients of same powers of t, recurrence relation (24) is obtained.

Again, differentiating both sides of generating Eq. (5) with respect to x and equating the coefficients of same powers of t, we get

$$ \frac{\partial}{\partial x} A_n^{[2],j}(x,y) = n\,A_{n-1}^{[2],j}(x,y), \tag{30} $$

which yields expression (26) of the lowering operator ${}_x\mathcal{L}_n^-$ and we can write

$$ {}_x\mathcal{L}_n^- \left\{ A_n^{[2],j}(x,y) \right\} = A_{n-1}^{[2],j}(x,y). \tag{31} $$

Using the operator ${}_x\mathcal{L}_n^-$, we have

$$ A_{n-k}^{[2],j}(x,y) = \left[{}_x\mathcal{L}_{k+1}^- \, {}_x\mathcal{L}_{k+2}^- \cdots {}_x\mathcal{L}_n^- \right] \left\{ A_n^{[2],j}(x,y) \right\}, $$

that is

$$ A_{n-k}^{[2],j}(x,y) = \frac{(n-k)!}{n!} D_x^k A_n^{[2],j}(x,y). \tag{32} $$

Similarly, we have the following expression:

$$ A_{n+1-j}^{[2],j}(x,y) = \left[{}_x\mathcal{L}_{n+2-j}^- \, {}_x\mathcal{L}_{n+3-j}^- \cdots {}_x\mathcal{L}_n^- \right] \left\{ A_n^{[2],j}(x,y) \right\}, $$

that is

$$ A_{n+1-j}^{[2],j}(x,y) = \frac{(n-j+1)!}{n!} D_x^{j-1} A_n^{[2],j}(x,y). \tag{33} $$

Inserting expressions (32) and (33) in recurrence relation (24), it follows that:

$$\left[x+\alpha_0+\beta_0+\sum_{k=1}^{n}\frac{(\alpha_k+\beta_k)}{k!}D_x^k+jyD_x^{j-1}\right]\left\{A_n^{[2],j}(x,y)\right\}=A_{n+1}^{[2],j}(x,y), \tag{34}$$

which yields expression (27) of the raising operator ${}_x\mathcal{L}_n^+$ and we can write

$${}_x\mathcal{L}_n^+\left\{A_n^{[2],j}(x,y)\right\}=A_{n+1}^{[2],j}(x,y). \tag{35}$$

Further, differentiation of generating function (5) with respect to y gives

$$\frac{n!}{(n-j)!}A_{n-j}^{[2],j}(x,y)=\frac{\partial}{\partial y}A_n^{[2],j}(x,y). \tag{36}$$

Making use of Eq. (30) in above equation, it follows that

$$D_y A_n^{[2],j}(x,y)=n\,D_x^{j-1}A_{n-1}^{[2],j}(x,y), \tag{37}$$

which yields expression (28) of the lowering operator ${}_y\mathcal{L}_n^-$ and we can write

$${}_y\mathcal{L}_n^-\left\{A_n^{[2],j}(x,y)\right\}=A_{n-1}^{[2],j}(x,y). \tag{38}$$

Using the lowering operator $\mathcal{L}_n^-$, we have

$$A_{n-k}^{[2],j}(x,y)=\frac{(n-k)!}{n!}D_x^{(1-j)k}D_y^k A_n^{[2],j}(x,y) \tag{39}$$

and

$$A_{n+1-j}^{[2],j}(x,y)=\frac{(n+1-j)!}{n!}D_x^{-(j-1)^2}D_y^{j-1}A_n^{[2],j}(x,y). \tag{40}$$

Inserting expressions (39) and (40) in recurrence relation (24), it follows that:

$$\begin{aligned}{}_y\mathcal{L}_n^+ :=&\left[x+\alpha_0+\beta_0+\sum_{k=1}^{n}\frac{\alpha_k+\beta_k}{k!}D_x^{(1-j)(k)}D_y^k+jyD_x^{-(j-1)^2}D_y^{j-1}\right]\left\{A_n^{[2],j}(x,y)\right\}\\ =&A_{n+1}^{[2],j}(x,y),\end{aligned} \tag{41}$$

which yields expression (29) of the raising operator ${}_y\mathcal{L}_n^+$ and we can write

$${}_y\mathcal{L}_n^+\left\{A_n^{[2],j}(x,y)\right\}=A_{n+1}^{[2],j}(x,y). \tag{42}$$

□

Remark 2.1 Applying the factorization identity

$${}_x\mathcal{L}_{n+1}^+\,{}_x\mathcal{L}_n^-\left(A_n^{[2],j}(x,y)\right)=A_n^{[2],j}(x,y)$$

and in view of the expressions (26) and (27), we deduce the following consequence of Theorem 2.1.

Corollary 2.1 *The 2-iterated 2D Appell polynomials $A_n^{[2],j}(x,y)$ satisfy the following differential equation*:

$$\left((x+\alpha_0+\beta_0)D_x+\sum_{k=1}^{n}\frac{(\alpha_k+\beta_k)}{(k)!}D_x^{k+1}+jyD_x^j-n\right)A_n^{[2],j}(x,y)=0. \tag{43}$$

Remark 2.2 Making use of factorization identity

$${}_y\mathcal{L}_{n+1}^{+}\,{}_y\mathcal{L}_n^{-}\{A_n^{[2],j}(x,y)\}=A_n^{[2],j}(x,y)$$

and in view of the expressions (28) and (29), we deduce the following consequence of Theorem 2.1.

Corollary 2.2 *The 2-iterated 2D Appell polynomials $A_n^{[2],j}(x,y)$ satisfy the following integro-differential equation*:

$$\begin{aligned}&\left((x+\alpha_0+\beta_0)D_y+\sum_{k=1}^{n}\frac{(\alpha_k+\beta_k)}{k!}D_x^{(1-j)k}D_y^{k+1}+jD_x^{-(j-1)^2}D_y^{j-1}\right.\\&\left.+jyD_x^{-(j-1)^2}D_y^j-(n+1)D_x^{j-1}\right)A_n^{[2],j}(x,y)=0.\end{aligned} \tag{44}$$

Remark 2.3 Differentiation of Eq. (44), $(j-1)(n-1)$ times w.r.t. x yields the following partial differential equation satisfied by the 2-iterated 2D Appell polynomials $A_n^{[2],j}(x,y)$:

$$\begin{aligned}&\Big((x+\alpha_0+\beta_0)D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y\\&+\sum_{k=1}^{n}\frac{(\alpha_k+\beta_k)}{k!}D_x^{(1-j)(1-n+k)}D_y^{k+1}\\&+jD_x^{(j-1)(n-j)}D_y^{j-1}+jyD_x^{(j-1)(n-j)}D_y^j-(n+1)D_x^{(j-1)n}\Big)A_n^{[2],j}(x,y)=0,\quad n\geq j.\end{aligned} \tag{45}$$

In the next section, the recurrence relation, shift operators, differential, integro-differential and partial differential equations satisfied by certain hybrid families related to the Appell polynomials are obtained.

3 Results for certain hybrid 2D families

Choosing $A_1(t)=A(t)$ and $A_2(t)=\frac{t}{e^t-1}$ in Eq. (5), we obtain generating function (8) of the 2D Bernoulli–Appell polynomials ${}_BA_n^{(j)}(x,y)$.

Following the same lines of proof as in Theorem 2.1, we obtain recurrence relation, shift operators, differential, integro-differential and partial differential equations satisfied by the 2D Bernoulli–Appell polynomials ${}_BA_n^{(j)}(x,y)$.

Theorem 3.1 *The* 2D *Bernoulli–Appell polynomials* ${}_BA_n^{(j)}(x,y)$ *satisfy the following recurrence relation:*

$$\begin{aligned}{}_BA_{n+1}^{(j)}(x,y) &= (x+\alpha_0-\frac{1}{2}){}_BA_n^{(j)}(x,y)+\sum_{k=1}^{n}\binom{n}{k}\alpha_k{}_BA_{n-k}^{(j)}(x,y)\\ &\quad -\sum_{k=1}^{n}\binom{n}{k}\frac{B_{k+1}(1)}{k+1}{}_BA_{n-k}^{(j)}(x,y)+jy\frac{n!}{(n-j+1)!}{}_BA_{n+1-j}^{(j)}(x,y),\end{aligned} \tag{46}$$

where the coefficients $\{\alpha_k\}_{k\in N_0}$ *are given by Eq.* (25), B_k *are the Bernoulli numbers and the shift operators are given by the following expressions*:

$${}_x\mathcal{L}_n^- := \frac{1}{n}D_x, \tag{47}$$

$${}_x\mathcal{L}_n^+ := x+\alpha_0-\frac{1}{2}+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k-\sum_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_x^k+jyD_x^{j-1}, \tag{48}$$

$${}_y\mathcal{L}_n^- := \frac{1}{n}D_x^{1-j}D_y, \tag{49}$$

$${}_y\mathcal{L}_n^+ := x+\alpha_0-\frac{1}{2}+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(1-j)k}D_y^k+jyD_x^{-(j-1)^2}D_y^{j-1}. \tag{50}$$

The 2D *Bernoulli–Appell polynomials* ${}_BA_n^{(j)}(x,y)$ *satisfy the following differential, integro-differential and partial differential equations*:

$$\begin{aligned}&\left(\left(x+\alpha_0-\frac{1}{2}\right)D_x+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{k+1}\right.\\ &\quad\left.-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{k+1}+jyD_x^j-n\right){}_BA_n^j(x,y)=0,\end{aligned} \tag{51}$$

$$\begin{aligned}&\left(\left(x+\alpha_0-\frac{1}{2}\right)D_y+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{(1-j)k}D_y^{k+1}\right.\\ &\quad-\sum_{k=0}^{n-1}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(1-j)k}D_y^{k+1}+jD_x^{-(j-1)^2}D_y^{j-1}\\ &\quad\left.+jyD_x^{-(j-1)^2}D_y^j-(n+1)D_x^{j-1}\right){}_BA_n^{(j)}(x,y)=0\end{aligned} \tag{52}$$

and

$$\begin{aligned}&\left(\left(x+\alpha_0-\frac{1}{2}\right)D_x^{(j-1)(n-1)}D_y+(n-1)(j-1)D_x^{(j-1)(n-1)-1}D_y\right.\\&+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{(1-j)(1-n+k)}D_y^{k+1}\\&-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(1-j)(1-n+k)}D_y^{k+1}+jD_x^{(j-1)(n-j)}D_y^{j-1}+jyD_x^{(j-1)(n-j)}D_y^{j}\\&\left.-(n+1)D_x^{(j-1)n}\right)\,{}_BA_n^{(j)}(x,y)=0,\ n\geq j,\end{aligned}\tag{53}$$

respectively.

Taking $A_1(t)=A(t)$ and $A_2(t)=\frac{2}{e^t+1}$ in generating function (5), we obtain generating function (9) of the 2D Euler–Appell polynomials ${}_EA_n^{(j)}(x,y)$.

Following the same lines of proof as in Theorem 2.1, we obtain recurrence relation, shift operators, differential, integro-differential and partial differential equations satisfied by the 2D Euler–Appell polynomials.

Theorem 3.2 *The 2D Euler–Appell polynomials ${}_EA_n^{(j)}(x,y)$ satisfy the following recurrence relation:*

$$\begin{aligned}{}_EA_{n+1}^{(j)}(x,y)=&\left(x+\alpha_0-\frac{1}{2}\right){}_EA_n^{(j)}(x,y)+\sum_{k=1}^{n}\binom{n}{k}\alpha_k\,{}_EA_{n-k}^{(j)}(x,y)\\&+\frac{1}{2}\sum_{k=1}^{n}\varepsilon_k\,{}_EA_{n-k}^{(j)}(x,y)\\&+jy\,\frac{n!}{(n-j+1)!}\,{}_EA_{n+1-j}^{j}(x,y),\end{aligned}\tag{54}$$

where the coefficients $\{\alpha_k\}_{k\in N_0}$ are given by Eq. (25), ε_k are the Euler numbers and the shift operators are given by the following expressions:

$${}_x\mathcal{L}_n^{-}:=\frac{1}{n}D_x,\tag{55}$$

$${}_x\mathcal{L}_n^{+}:=\left(x+\alpha_0-\frac{1}{2}\right)+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k+\frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^k+jyD_x^{j-1},\tag{56}$$

$${}_y\mathcal{L}_n^{-}:=\frac{1}{n}D_x^{1-j}D_y,\tag{57}$$

$${}_y\mathcal{L}_n^{+}:=\left(x+\alpha_0-\frac{1}{2}\right)+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{(1-j)k}D_y^k+\frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{(1-j)k}D_y^k+jyD_x^{-(j-1)^2}D_y^{j-1}.\tag{58}$$

The 2D Euler–Appell polynomials ${}_EA_n^{(j)}(x,y)$ satisfy the following differential, integro-differential and partial differential equations:

$$\left(\left(x+\alpha_0-\frac{1}{2}\right)D_x+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{k+1}+\frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{k+1}+jyD_x^j-n\right){}_EA_n^j(x,y)=0, \tag{59}$$

$$\begin{aligned}&\left(\left(x+\alpha_0-\frac{1}{2}\right)D_y+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{(1-j)k}D_y^{k+1}\right.\\&+\frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{(1-j)k}D_y^{k+1}+jD_x^{-(j-1)^2}D_y^{j-1}\\&\left.+jyD_x^{-(j-1)^2}D_y^j-(n+1)D_x^{j-1}\right){}_EA_n^j(x,y)=0\end{aligned} \tag{60}$$

and

$$\begin{aligned}&\left(\left(x+\alpha_0-\frac{1}{2}\right)D_x^{(j-1)(n-1)}D_y+(n-1)(j-1)D_x^{(j-1)(n-1)-1}D_y\right.\\&+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{(1-j)(1-n+k)}D_y^{k+1}\\&+\frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{(1-j)(1-n+k)}D_y^{k+1}+jD_x^{(j-1)(n-j)}D_y^{j-1}+jyD_x^{(j-1)(n-j)}D_y^j\\&\left.-(n+1)D_x^{(j-1)n}\right){}_EA_n^j(x,y)=0,\ n\geq j,\end{aligned} \tag{61}$$

respectively.

Further, taking $A_1(t)=A(t)$ and $A_2(t)=e^{-\frac{t^2}{2}}$ in Eq. (5), we obtain generating function (10) of the 2D Hermite–Appell polynomials ${}_HA_n^{(j)}(x,y)$.

Following the same lines of proof as in Theorem 2.1, we deduce the recurrence relation, shift operators, differential, integro-differential and partial differential equations satisfied by 2D Hermite–Appell polynomials.

Theorem 3.3 *The 2D Hermite–Appell polynomials ${}_HA_n^{(j)}(x,y)$ satisfy the following recurrence relation:*

$$\begin{aligned}{}_HA_{n+1}^{(j)}(x,y)=&(x+\alpha_0){}_HA_n^{(j)}(x,y)+\sum_{k=1}^{n}\binom{n}{k}\alpha_k\,{}_HA_{n-k}^{(j)}(x,y)\\&+jy\frac{n!}{(n-j+1)!}{}_HA_{n+1-j}^{(j)}(x,y)-n\,{}_HA_{n-1}^{(j)}(x,y),\end{aligned} \tag{62}$$

where the coefficients $\{\alpha_k\}_{k\in N_0}$ are given by Eq. (25) *and the shift operators are given by the following expressions*:

$${}_x\mathcal{L}_n^-:=\frac{1}{n}D_x, \tag{63}$$

$$ {}_x\mathcal{L}_n^+ := x + \alpha_0 - D_x^{n-1} + jyD_x^{j-1} + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{n-k}, \tag{64} $$

$$ {}_y\mathcal{L}_n^- := \frac{1}{n} D_x^{1-j} D_y, \tag{65} $$

$$ {}_y\mathcal{L}_n^+ := x + \alpha_0 - D_x^{(1-j)(n-1)} D_y^{n-1} + jyD_x^{-(j-1)^2} D_y^{j-1} + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{(1-j)(n-k)} D_y^{n-k}. \tag{66} $$

The 2D *Hermite–Appell polynomials satisfy the following differential, integro-differential and partial differential equations*:

$$ \left((x+\alpha_0)D_x - D_x^n + jyD_x^j + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{n-k+1} - n \right) {}_HA_n^{(j)}(x,y) = 0, \tag{67} $$

$$ \begin{aligned} &\Bigg((x+\alpha_0)D_y - D_x^{(1-j)(n-1)} D_y^n + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{(1-j)(n-k)} D_y^{n-k+1} + jD_x^{-(j-1)^2} D_y^{j-1} \\ &\quad + jyD_x^{-(j-1)^2} D_y^j - (n+1)D_x^{j-1} \Bigg) {}_HA_n^{(j)}(x,y) = 0 \end{aligned} \tag{68} $$

and

$$ \begin{aligned} &\Big((x+\alpha_0)D_x^{(j-1)(n-1)} D_y + (j-1)(n-1)D_x^{(j-1)(n-1)-1} D_y \\ &\quad + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{(j-1)(k-1)} D_y^{n-k+1} \\ &\quad + jD_x^{(j-1)(n-j)} D_y^{j-1} + jyD_x^{(j-1)(n-j)} D_y^j \\ &\quad - (n+1)D_x^{(j-1)n} - D_y^n \Big) {}_HA_n^{(j)}(x,y) = 0, \end{aligned} \tag{69} $$

respectively.

In the next section, we consider certain 2D hybrid special polynomials and derive the recurrence relation, shift operators and differential equations for these mixed-type polynomials.

4 Examples

Example 4.1 Taking $A(t) = \frac{t}{e^t - 1}$ in the generating function (10), we obtain the following generating function of the 2D Hermite–Bernoulli polynomials ${}_HB_n^{(j)}(x,y)$:

$$ \frac{t}{e^t - 1} \exp\left(xt - \frac{t^2}{2} + yt^j \right) = {}_HB_n^{(j)}(x,y) \frac{t^n}{n!}. \tag{70} $$

Proceeding on the same lines as in the previous section, certain results for the 2D

Hermite–Bernoulli polynomials can be obtained and these results are mentioned in the following Table 1.

Example 4.2 Taking $A(t) = \frac{2}{e^t+1}$ in generating function (10), we obtain the following generating function of the 2D Hermite–Euler polynomials ${}_HE_n^{(j)}(x, y)$:

$$\frac{2}{e^t + 1}\exp\left(xt - \frac{t^2}{2} + yt^j\right) = {}_HE_n^{(j)}(x, y)\frac{t^n}{n!}. \tag{71}$$

Proceeding on the same lines as in the previous section, the recurrence relation, shift operators, differential, integro-differential and partial differential equations of the 2D Hermite–Euler polynomials can be obtained. We mention these results in the following Table 2.

Note Taking $y = 0$ in the generating functions (70) and (71) of the 2D Hermite–Bernoulli and 2D Hermite–Euler polynomials, we can obtain the corresponding results for the Hermite–Bernoulli polynomials ${}_HB_n^{(j)}(x)$ and Hermite–Euler polynomials ${}_HE_n^{(j)}(x)$ respectively.

Table 1 Results of the 2D Hermite–Bernoulli polynomials ${}_HB_n^{(j)}(x, y)$

Recurrence relation	${}_HB_{n+1}^{(j)}(x, y) = \left(x - \frac{1}{2}\right){}_HB_n^{(j)}(x, y) - \sum_{k=1}^{n}\binom{n}{k}\frac{B_{k+1}(1)}{k+1}\,{}_HB_{n-k}^{(j)}(x, y)$ $+jy\frac{n!}{(n-j+1)!}\,{}_HB_{n+1-j}^{(j)}(x, y) - n\,{}_HB_{n-1}^{(j)}(x, y)$.
Shift operators	${}_x\mathcal{L}_n^- := \frac{1}{n}D_x$, ${}_x\mathcal{L}_n^+ := \left(x - \frac{1}{2}\right) - D_x^{n-1} + jyD_x^{j-1} - \sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{n-k}$, ${}_y\mathcal{L}_n^- := \frac{1}{n}D_x^{1-j}D_y$, ${}_y\mathcal{L}_n^+ := \left(x - \frac{1}{2}\right) - D_x^{(1-j)(n-1)}D_y^{n-1} + jyD_x^{(1-j)^2}D_y^{j-1} - \sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(1-j)(n-k)}D_y^{n-k}$.
Differential equation	$\left(\left(x - \frac{1}{2}\right)D_x - D_x^n + jyD_x^j - \sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{n-k+1} - n\right){}_HB_n^{(j)}(x, y) = 0.$
Integro-differential equation	$\Big(\left(x - \frac{1}{2}\right)D_y - D_x^{(1-j)(n-1)}D_y^n - \sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(1-j)(n-k)}D_y^{n-k+1}$ $+jD_x^{-(j-1)^2}D_y^{j-1} + jyD_x^{-(j-1)^2}D_y^j - (n+1)D_x^{j-1}\Big){}_HB_n^{(j)}(x, y) = 0.$
Partial differential equation	$\Big(\left(x - \frac{1}{2}\right)D_x^{(j-1)(n-1)}D_y + (j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y + jD_x^{(j-1)(n-j)}D_y^{j-1}$ $+jyD_x^{(j-1)(n-j)}D_y^j - \sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(j-1)(k-1)}D_y^{n-k+1} - D_y^n - (n+1)D_x^{(j-1)n}\Big){}_HB_n^{(j)}(x, y) = 0.$

Table 2 Results of the 2D Hermite–Bernoulli polynomials ${}_HE_n^{(j)}(x,y)$

Recurrence relation	${}_HE_{n+1}^{(j)}(x,y) = \left(x-\frac{1}{2}\right){}_HE_n^{(j)}(x,y) + \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\varepsilon_k {}_HE_{n-k}^{(j)}(x,y)$ $+jy\frac{n!}{(n-j+1)!}{}_HE_{n+1-j}^{(j)}(x,y) - n\,{}_HE_{n-1}^{(j)}(x,y).$
Shift operators	${}_x\mathcal{L}_n^- := \frac{1}{n}D_x,$ ${}_x\mathcal{L}_n^+ := \left(x-\frac{1}{2}\right) - D_x^{n-1} + jyD_x^{j-1} + \frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{n-k},$ ${}_y\mathcal{L}_n^- := \frac{1}{n}D_x^{1-j}D_y,$ ${}_y\mathcal{L}_n^+ := \left(x-\frac{1}{2}\right) - D_x^{(1-j)(n-1)}D_y^{n-1} + jyD_x^{(1-j)^2}D_y^{j-1} + \frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{(1-j)(n-k)}D_y^{n-k}.$
Differential equation	$\left(\left(x-\frac{1}{2}\right)D_x - D_x^n + jyD_x^j + \frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{n-k+1} - n\right){}_HE_n^{(j)}(x,y) = 0.$
Integro-differential equation	$\Big(\left(x-\frac{1}{2}\right)D_y - D_x^{(1-j)(n-1)}D_y^n + \frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{(1-j)(n-k)}D_y^{n-k+1}$ $+jD_x^{-(j-1)^2}D_y^{j-1} + jyD_x^{-(j-1)^2}D_y^j - (n+1)D_x^{j-1}\Big){}_HE_n^{(j)}(x,y) = 0.$
Partial differential equation	$\Big(\left(x-\frac{1}{2}\right)D_x^{(j-1)(n-1)}D_y + (j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y + \frac{1}{2}\sum_{k=1}^{n}\frac{\varepsilon_k}{k!}D_x^{(j-1)(k-1)}D_y^{n-k+1}$ $+jD_x^{(j-1)(n-j)}D_y^{j-1} + jyD_x^{(j-1)(n-j)}D_y^j - D_y^n - (n+1)D_x^{(j-1)n}\Big){}_HE_n^{(j)}(x,y) = 0.$

The results established in Sect. 2 are general and include new families of the 2D special polynomials related to the Appell polynomials. Results for certain hybrid members of these families are mentioned in this section. we consider results for the 2-iterated members of these families in the Appendix.

Appendix

I. Taking $A(t) = \frac{t}{e^t-1}$ in generating function (8) of the 2D Bernoulli–Appell polynomials, we obtain the following generating function of the 2-iterated 2D Bernoulli-polynomials $B_n^{[2],j}(x,y)$:

$$\left(\frac{t}{e^t-1}\right)^2 e^{xt+yt^j} = \sum_{n=0}^{\infty} B_{n+1}^{[2],j}(x,y)\frac{t^n}{n!}. \tag{72}$$

The results for the 2-iterated 2D Bernoulli polynomials are given in the following Table 3.

II. Taking $A(t) = \frac{2}{e^t+1}$ in generating function (9) of the 2D Euler–Appell polynomials, we obtain generating function of the 2-iterated 2D Euler polynomials $E_n^{[2],j}(x,y)$:

Table 3 Results for the 2-iterated 2D Bernoulli polynomials $B_n^{[2],j}(x,y)$

Recurrence relation	$B_{n+1}^{[2],j}(x,y)=(x-1)B_n^{[2],j}(x,y)-2\sum_{k=1}^{n}\binom{n}{k}\frac{B_{k+1}(1)}{k+1}B_{n-k}^{[2],j}(x,y)+jy\frac{n!}{(n-j+1)!}B_{n+1-j}^{[2],j}(x,y).$
Shift operators	${}_x\mathcal{L}_n^{-}:=\frac{1}{n}D_x,$ ${}_x\mathcal{L}_n^{+}:=(x-1)+jyD_x^{j-1}-2\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^k,$ ${}_y\mathcal{L}_n^{-}:=\frac{1}{n}D_x^{1-j}D_y,$ ${}_y\mathcal{L}_n^{+}:=(x-1)+jyD_x^{-(j-1)^2}D_y^{j-1}-2\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(1-j)k}D_y^k.$
Differential equation	$\left((x-1)D_x+jyD_x^j-2\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{k+1}-n\right)B_n^{[2],j}(x,y)=0.$
Integro-differential equation	$\Big((x-1)D_y-2\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(1-j)k}D_y^{k+1}+jD_x^{-(j-1)^2}D_y^{j-1}$ $+jyD_x^{-(j-1)^2}D_y^j-(n+1)D_x^{j-1}\Big)B_n^{[2],j}(x,y)=0.$
Partial differential equation	$\Big((x-1)D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y+jD_x^{(j-1)(n-j)}D_y^{j-1}$ $+jyD_x^{(j-1)(n-j)}D_y^j-2\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_x^{(1-j)(1-n+k)}D_y^{k+1}-(n+1)D_x^{(j-1)n}\Big)B_n^{[2],j}(x,y)=0.$

$$\left(\frac{2}{e^t+1}\right)^2 e^{xt+yt^j}=\sum_{n=0}^{\infty}E_n^{[2],j}(x,y)\frac{t^n}{n!} \tag{73}$$

The 2-iterated 2D Euler polynomials are given in the following Table 4.

III. Taking $A(t)=e^{\frac{-t^2}{2}}$ in generating function (10) of the 2D Hermite–Appell polynomials, we obtain the generating function of the 2-iterated 2D Hermite-polynomials $H_n^{[2],j}(x,y)$:

$$\exp(xt-t^2+yt^j)=\sum_{n=0}^{\infty}H_n^{[2],j}(x,y)\frac{t^n}{n!} \tag{74}$$

The results for the 2-iterated 2D Hermite polynomials are given in the following Table 5.

Note that, for $y=0$ the results obtained in this paper coincide with the corresponding recurrence formulas and differential equations for the 2-iterated Appell polynomials.

Table 4 Results for the 2-iterated 2D Euler polynomials $E_n^{[2],j}(x,y)$

Recurrence relation	$E_{n+1}^{[2],j}(x,y) = (x-1)E_n^{[2],j}(x,y) + \sum_{k=1}^{n} \binom{n}{k} \varepsilon_k E_{n-k}^{[2],j}(x,y) + jy\, \frac{n!}{(n-j+1)!} E_{n+1-j}^{[2],j}(x,y).$
Shift operators	${}_x\mathcal{L}_n^- := \frac{1}{n} D_x,$ ${}_x\mathcal{L}_n^+ := (x-1) + \sum_{k=1}^{n} \frac{\varepsilon_k}{k!} D_x^k + jyD_x^{j-1},$ ${}_y\mathcal{L}_n^- := \frac{1}{n} D_x^{1-j} D_y,$ ${}_y\mathcal{L}_n^+ := (x-1) + \sum_{k=1}^{n} \frac{\varepsilon_k}{k!} D_x^{(1-j)k} D_y^k + jyD_x^{-(j-1)^2} D_y^{j-1}.$
Differential equation	$\left((x-1)D_x + \sum_{k=1}^{n} \frac{\varepsilon_k}{k!} D_x^{k+1} + jyD_x^j - n\right) E_n^{[2],j}(x,y) = 0.$
Integro-differential equation	$\Big((x-1)D_y + \sum_{k=1}^{n} \frac{\varepsilon_k}{k!} D_x^{(1-j)k} D_y^{k+1} + jD_x^{-(j-1)^2} D_y^{j-1} + jyD_x^{-(j-1)^2} D_y^j$ $-(n+1)D_x^{j-1}\Big) E_n^{[2],j}(x,y) = 0$
Partial differential equation	$\Big((x-1)D_x^{(j-1)(n-1)} D_y + (n-1)(j-1)D_x^{(j-1)(n-1)-1} D_y + \sum_{k=1}^{n} \frac{\varepsilon_k}{k!} D_x^{(1-j)(1-n+k)} D_y^{k+1}$ $+jD_x^{(j-1)(n-j)} D_y^{j-1} + jyD_x^{(j-1)(n-j)} D_y^j - (n+1)D_x^{(j-1)n}\Big) B_n^{[2],j}(x,y) = 0, \quad n \geq j.$

Table 5 Results of the 2-iterated 2D Hermite polynomials $H_n^{[2],j}(x,y)$

Recurrence relation	$H_{n+1}^{[2],j}(x,y) = xH_n^{[2],j}(x,y) - 2\,H_{n-1}^{[2],j}(x,y) + jy\, \frac{n!}{(n-j+1)!} H_{n+1-j}^{[2],j}(x,y).$
Shift operators	${}_x\mathcal{L}_n^- := \frac{1}{n} D_x,$ ${}_x\mathcal{L}_n^+ := x - \frac{2}{n} D_x + jyD_x^{j-1},$ ${}_y\mathcal{L}_n^- := \frac{1}{n} D_x^{1-j} D_y,$ ${}_y\mathcal{L}_n^+ := x - \frac{2}{n} D_x^{(1-j)k} D_y^k + jyD_x^{-(j-1)^2} D_y^{j-1}.$
Differential equation	$\left(xD_x - \frac{2}{n} D_x^2 + jyD_x^j - n\right) H_n^{[2],j}(x,y) = 0.$
Integro-differential equation	$\left(xD_y - \frac{2}{n} D_x^{(1-j)k} D_y^{k+1} + jD_x^{-(j-1)^2} D_y^{j-1} + jyD_x^{-(j-1)^2} D_y^j - (n+1)D_x^{j-1}\right) H_n^{[2],j}(x,y) = 0.$
Partial differential equation	$\Big(xD_x^{(j-1)(n-1)} D_y + (n-1)(j-1)D_x^{(j-1)(n-1)-1} D_y - \frac{2}{n} D_x^{(1-j)(1-n+k)} D_y^{k+1}$ $+jD_x^{(j-1)(n-j)} D_y^{j-1} + jyD_x^{(j-1)(n-j)} D_y^j - (n+1)D_x^{(j-1)n}\Big) H_n^{[2],j}(x,y) = 0, \quad n \geq j.$

References

1. Appell, P.: Sur une classe de polynômes. Ann. Sci. École. Norm. Sup. **9**(2), 119–144 (1880)
2. Sheffer, I.M.: Some properties of polynomial sets of type zero. Duke Math. J. **5**, 590–622 (1939)
3. Blasiak, P., Dattoli, G., Horzela, A., Penson, K.A.: Representations of monomiality principle with Sheffer-type polynomials and boson normal ordering. Phys. Lett. A **352**, 7–12 (2006)
4. Dattoli, G., Zhukovsky, K.: Appel polynomial series expansion. Int. Math. Forum **5**, 649–662 (2010)
5. Dattoli, G., Germano, B., Martinelli, M.R., Ricci, P.E.: Sheffer and non sheffer polynomial families. Int. J. Math. Math. Sci. (2012). https://doi.org/10.1155/2012/323725
6. He, M.X., Ricci, P.E.: Differential equation of Appell polynomials via factorization method. J. Comput. Appl. Math. **139**(2), 231–237 (2002)
7. Bretti, G., Ricci, P.E.: Multidimensional extension of the Bernoulli and Appell polynomials. Taiwan. J. Math. **8**(3), 415–428 (2004)
8. Yılmaz, B., Özarslan, M.A.: Differential equations for the extended 2D Bernoulli and Euler polynomials. Adv. Differ. Equ. **107**, 1–16 (2013)
9. Srivastava, H.M., Özarslan, M.A., Yılmaz, B.: Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials. Filomat **28**(4), 695–708 (2014)
10. Wani, S.A., Khan, S., Naikoo, S.: Differential and integral equations for the Laguerre–Gould–Hopper based Appell and related polynomials, Boletín de la Sociedad Matemática Mexicana (2019). https://doi.org/10.1007/s40590-019-00239-1
11. Riyasat, M., Wani, S.A., Khan, S.: On some classes of differential equations and associated integral equations for the Laguerre–Appell polynomials. Adv. Pure Appl. Math. (2017). https://doi.org/10.1515/apam-2017-0079
12. Khan, S., Riyasat, M., Wani, S.A.: Differential and integral equations associated with some hybrid families of Legendre polynomials. Tbil. Math. J. **11**(1), 127–139 (2018)
13. Araci, S., Riyasat, M., Wani, S.A., Subuhi, K.: Differential and integral equations for the 3-variable Hermite–Frobenius–Euler and Frobenius–Genocchi polynomials. Appl. Math. Inf. Sci. **11**(5), 1–11 (2017)
14. Wani, S.A., Khan, S.: Properties and applications of the Gould–Hopper–Frobenius–Euler polynomials. Tbil. Math. J. **12**(1), 93–104 (2019)
15. Özarslan, M.A., Yılmaz, B.: A set of finite order differential equations for the Appell polynomials. J. Comput. Appl. Math. **259**, 108–116 (2014)
16. Khan, S., Raza, N.: 2-Iterated Appell polynomials and related numbers. Appl. Math. Comput. **219**, 9469–9483 (2013)