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ORIGINAL ARTICLE

Radial operators on polyanalytic weighted Bergman spaces

Roberto Moisés Barrera-Castelán[1] · Egor A. Maximenko[1] · Gerardo Ramos-Vazquez[2]

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Abstract

Let μ_α be the Lebesgue plane measure on the unit disk with the radial weight $\frac{\alpha+1}{\pi}(1-|z|^2)^\alpha$. Denote by $\mathcal{A}_n^2$ the space of the n-analytic functions on the unit disk $\mathbb{D}$, square-integrable with respect to μ_α. Extending results of Ramazanov (1999, 2002), we explain that disk polynomials (studied by Koornwinder in 1975 and Wünsche in 2005) form an orthonormal basis of $\mathcal{A}_n^2$. Using this basis, we provide the Fourier decomposition of $\mathcal{A}_n^2$ into the orthogonal sum of the subspaces associated with different frequencies. This leads to the decomposition of the von Neumann algebra of radial operators, acting in $\mathcal{A}_n^2$, into the direct sum of some matrix algebras. In other words, all radial operators are represented as matrix sequences. In particular, we represent in this form the Toeplitz operators with bounded radial symbols, acting in $\mathcal{A}_n^2$. Moreover, using ideas by Engliš (1996), we show that the set of the Toeplitz operators with bounded generating symbols is not weakly dense in $\mathcal{B}(\mathcal{A}_n^2)$.

Keywords Radial operator · Polyanalytic function · Weighted Bergman space · Mean value property · Reproducing kernel · Von Neumann algebra · Jacobi polynomial · Disk polynomial

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✉ Egor A. Maximenko
egormaximenko@gmail.com

Roberto Moisés Barrera-Castelán
rmoisesbarrera@gmail.com

Gerardo Ramos-Vazquez
ger.ramosv@gmail.com

[1] Instituto Politécnico Nacional, Escuela Superior de Física y Matemáticas, Ciudad de México, Mexico

[2] Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Ciudad de México, Mexico

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1 Introduction

1.1 Background

Polyanalytic functions naturally arise in some physical models (plane elasticity theory, Landau levels) and in some methods of signal processing, see [1–3, 15, 17, 18]. Koshelev [23] computed the reproducing kernel of the n-analytic Bergman space $\mathcal{A}_n^2(\mathbb{D})$ on the unit disk. Balk in the book [4] explained fundamental properties of polyanalytic functions. Dzhuraev [9] related polyanalytic projections with singular integral operators. Vasilevski [40, 41] studied polyanalytic Bergman spaces on the upper half-plane and polyanalytic Fock spaces using the Fourier transform. Ramazanov [29, 30] constructed an orthonormal basis in $\mathcal{A}_n^2(\mathbb{D})$ and studied various properties of $\mathcal{A}_n^2(\mathbb{D})$. In fact, the elements of this basis are well-known disk polynomials studied by Koornwinder [21], Wünsche [43], and other authors. Pessoa [27] related Koshelev's formula with Jacobi polynomials and gave a very clear proof of this formula. He also obtained similar results for some other one-dimensional domains. Hachadi and Youssfi [14] developed a general scheme for computing the reproducing kernels of the spaces of polyanalytic functions on radial plane domains (disks or the whole plane) with radial measures.

There are general investigations about bounded linear operators in reproducing kernel Hilbert spaces (RKHS), especially about Toeplitz operators in Bergman or Fock spaces [5, 44, 45], but the complete description of the spectral properties is found only for some special classes of operators, in particular, for Toeplitz operators with generating symbols invariant under some group actions, see Vasilevski [42], Grudsky, Quiroga-Barranco, and Vasilevski [11], Dawson, Ólafsson, and Quiroga-Barranco [8]. The simplest class of this type consists of Toeplitz operators with bounded radial generating symbols. Various properties of these operators (boundedness, compactness, and eigenvalues) have been studied by many authors, see [13, 22, 28, 46]. The C*-algebra generated by such operators, acting in the Bergman space, was explicitly described in [6, 12, 16, 37]. Loaiza and Lozano [24] obtained similar results for radial Toeplitz operators in harmonic Bergman spaces. Maximenko and Tellería-Romero [26] studied radial operators in the polyanalytic Fock space.

Hutník, Loaiza, Ramírez-Mora, Ramírez-Ortega, Sánchez-Nungaray, and other authors [19, 20, 25, 31, 32, 35] studied vertical and angular Toeplitz operators in polyanalytic and true-polyanalytic Bergman spaces. In particular, vertical Toeplitz operators in the n-analytic Bergman space over the upper half-plane are represented in [31] as $n \times n$ matrices whose entries are continuous functions on $(0, +\infty)$, with some additional properties at 0 and $+\infty$.

Rozenblum and Vasilevski [33] investigated Toeplitz operators with distributional symbols and showed that Toeplitz operators in true-polyanalytic spaces

Bergman or Fock spaces are equivalent to some Toeplitz operators with distributional symbols in the analytic Bergman or Fock spaces.

1.2 Objects of study

Denote by μ the Lebesgue plane measure and its restriction to the unit disk $\mathbb{D}$, and by μ_α the weighted Lebesgue plane measure

$$\mathrm{d}\mu_\alpha(z) := \frac{\alpha+1}{\pi}\,(1-|z|^2)^\alpha\,\mathrm{d}\mu(z).$$

This measure is normalized: $\mu_\alpha(\mathbb{D}) = 1$. We use notation $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for the inner product and the norm in $L^2(\mathbb{D},\mu_\alpha)$.

Let $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ be the space of n-analytic functions square-integrable with respect to μ_α. We denote by $\mathcal{A}_{(n)}^2(\mathbb{D},\mu_\alpha)$ the orthogonal complement of the subspace $\mathcal{A}_{n-1}^2(\mathbb{D},\mu_\alpha)$ in $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$.

For every τ in the unit circle $\mathbb{T} := \{z \in \mathbb{C}:\ |z| = 1\}$, let $\rho_n^{(\alpha)}(\tau)$ be the rotation operator acting in $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ by the rule

$$(\rho_n^{(\alpha)}(\tau)f)(z) := f(\tau^{-1}z).$$

In Proposition 16, we verify that $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ is invariant under such rotations, i.e., $\rho_n^{(\alpha)}(\tau)$ is well defined. The family $\rho_n^{(\alpha)}$ is a unitary representation of the group $\mathbb{T}$ in the Hilbert space $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$. We denote by $\mathcal{R}_n^{(\alpha)}$ its commutant, i.e., the von Neumann algebra that consists of all bounded linear operators acting in $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ that commute with $\rho_n^{(\alpha)}(\tau)$ for every τ in $\mathbb{T}$. In other words, the elements of $\mathcal{R}_n^{(\alpha)}$ are the operators intertwining the representation $\rho_n^{(\alpha)}$. The elements of $\mathcal{R}_n^{(\alpha)}$ are called *radial* operators in $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$.

In a similar manner, we denote by $\rho_{(n)}^{(\alpha)}(\tau)$ the rotation operators acting in $\mathcal{A}_{(n)}^2(\mathbb{D},\mu_\alpha)$ and by $\mathcal{R}_{(n)}^{(\alpha)}$ the von Neumann algebra of radial operators in $\mathcal{A}_{(n)}^2(\mathbb{D},\mu_\alpha)$. We also consider the rotation operators $\rho^{(\alpha)}(\tau)$ in $L^2(\mathbb{D},\mu_\alpha)$ and the corresponding algebra $\mathcal{R}^{(\alpha)}$ of radial operators.

1.3 Structure and results of this paper

- In Sect. 2, we list some necessary facts about Jacobi polynomials. They play a crucial role in Sects. 3 and 4.
- In Sect. 3, we recall various equivalent formulas for the disk polynomials that can be obtained by orthogonalizing the monomials in z and $\bar{z}$. Using this orthonormal basis $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{N}_0}$, we decompose $L^2(\mathbb{D},\mu_\alpha)$ into the orthogonal sum of subspaces $\mathcal{W}_\xi^{(\alpha)}$ indexed by different frequencies ξ in $\mathbb{Z}$.

- In Sect. 4, we give an elementary proof of the weighted mean value property of polyanalytic functions and show the boundedness of the evaluation functionals for the spaces of polyanalytic functions over general domains in $\mathbb{C}$. In the unweighted case, this mean value property was proven by Koshelev [23] and Pessoa [27]. Hachadi and Youssfi [14] generalized this property to the weighted case and used it to compute the reproducing kernel of $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$.
- In Sect. 5, extending results by Ramazanov [29, 30] to the weighted case, we verify that the family $(b_{p,q}^{(\alpha)})_{p\geq 0, 0\leq q<n}$ is an orthonormal basis of $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$. Using this fact, we decompose $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$ into subspaces $\mathcal{W}_\xi^{(\alpha)} \cap \mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$.
- In Sect. 6, we prove that the set of all Toeplitz operators with bounded generating symbols is not weakly dense in $\mathcal{B}(\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha))$. This simple result was surprising for us.
- In Sect. 7, we decompose the von Neumann algebras $\mathcal{R}^{(\alpha)}$, $\mathcal{R}_n^{(\alpha)}$, and $\mathcal{R}_{(n)}^{(\alpha)}$, into direct sums of factors. In particular, Theorems 2 and 3 imply that the algebra $\mathcal{R}_n^{(\alpha)}$ is noncommutative for $n\geq 2$, whereas $\mathcal{R}_{(n)}^{(\alpha)}$ is commutative for every n in $\mathbb{N}$.
- In Sect. 8, we find explicit representations of the radial Toeplitz operators acting in the spaces $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$ and $\mathcal{A}_{(n)}^2(\mathbb{D}, \mu_\alpha)$. The results of Sects. 7 and 8 are similar to [26]. The main difference is that the orthonormal bases are given by other formulas.

Most of the facts in Sects. 2–5 are known. We recall them in a logical and almost self-contained form, emphasizing some aspects relevant for us.

We hope that this paper can serve as a basis for further investigations about polyanalytic or polyharmonic Bergman spaces and operators acting on these spaces. For example, an interesting task is to describe the C*-algebra generated by radial Toeplitz operators acting in $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$.

2 Necessary facts about Jacobi polynomials

In this section, we recall some necessary facts about Jacobi polynomials. Most of them are explained in [38, Chapter 4]. For every α and β in $\mathbb{R}$, the Jacobi polynomials can be defined by Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) := \frac{(-1)^n}{2^n\, n!}\,(1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big((1-x)^{n+\alpha}(1+x)^{n+\beta}\Big). \tag{1}$$

This definition and the general Leibniz rule imply an expansion into powers of $x-1$ and $x+1$

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}. \tag{2}$$

Formula (2) yields a symmetry relation, the values at the points 1 and -1, and a formula for the derivative

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \tag{3}$$

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \qquad P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n}, \tag{4}$$

$$\left(P_n^{(\alpha,\beta)}\right)'(x) = \frac{\alpha+\beta+n+1}{2}\, P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{5}$$

With the above properties, it is easy to compute the derivatives of $P_n^{(\alpha,\beta)}$ at the point 1. Now, Taylor's formula yields another explicit expansion for $P_n^{(\alpha,\beta)}$

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{\alpha+\beta+n+k}{k} \binom{\alpha+n}{n-k} \left(\frac{x-1}{2}\right)^k. \tag{6}$$

For $\alpha > -1$ and $\beta > -1$, we equip $(-1,1)$ with the weight $(1-x)^\alpha(1+x)^\beta$, then denote by $\langle\cdot,\cdot\rangle_{(-1,1),\alpha,\beta}$ the corresponding inner product

$$\langle f, g\rangle_{(-1,1),\alpha,\beta} := \int_{-1}^{1} f(x)\overline{g(x)}\,(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x.$$

Then, $L^2((-1,1),(1-x)^\alpha(1+x)^\beta)$ is a Hilbert space, and the set $\mathcal{P}$ of the univariate polynomials is a dense subset of this space. Using (1) and integrating by parts, for every f in $\mathcal{P}$, we get

$$\langle f, P_n^{(\alpha,\beta)}\rangle_{(-1,1),\alpha,\beta} = \frac{1}{2n}\langle f', P_{n-1}^{(\alpha+1,\beta+1)}\rangle_{(-1,1),\alpha+1,\beta+1}. \tag{7}$$

Applying (7) and induction, it is easy to prove that the sequence $(P_n^{(\alpha,\beta)})_{n=0}^{\infty}$ is an orthogonal basis of $L^2((-1,1),(1-x)^\alpha(1+x)^\beta)$, that is, for every polynomial h of degree less than n

$$\int_{-1}^{1} h(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\mathrm{d}x = 0. \tag{8}$$

Furthermore

$$\langle P_m^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}\rangle_{(-1,1),\alpha,\beta} = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\,n!}\delta_{m,n}. \tag{9}$$

Formulas (1) and (8), and induction allow us to compute the following integral for $\beta > 0$:

$$\int_{-1}^{1} P_n^{(\alpha,\beta+1)}(x)\,(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x = 2^{\alpha+\beta+1}(-1)^n\ \mathrm{B}\ (\alpha+n+1,\beta+1), \tag{10}$$

where B is the well-known Beta function.

2.1 Jacobi polynomials for the unit interval

The function $t \mapsto 2t-1$ is a bijection from (0, 1) onto $(-1,1)$. Denote by $Q_n^{(\alpha,\beta)}$ the "shifted Jacobi polynomial" obtained from $P_n^{(\alpha,\beta)}$ by composing it with this change of variables

$$Q_n^{(\alpha,\beta)}(t) := P_n^{(\alpha,\beta)}(2t-1).$$

The properties of $Q_n^{(\alpha,\beta)}$ follow easily from the properties of $P_n^{(\alpha,\beta)}$. In particular, here are analogs of (1) and (6)

$$Q_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{n!}\,(1-t)^{-\alpha}t^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big((1-t)^{n+\alpha}t^{n+\beta}\Big), \tag{11}$$

$$Q_n^{(\alpha,\beta)}(t) = \sum_{k=0}^{n}\binom{\alpha+\beta+n+k}{k}\binom{\beta+n}{n-k}(-1)^{n-k}t^k. \tag{12}$$

The sequence $(Q_n^{(\alpha,\beta)})_{n=0}^{\infty}$ is orthogonal on (0, 1) with respect to the weight $(1-t)^{\alpha}t^{\beta}$, and

$$\int_0^1 Q_m^{(\alpha,\beta)}(t)Q_n^{(\alpha,\beta)}(t)(1-t)^{\alpha}t^{\beta}\mathrm{d}t = \frac{\delta_{m,n}\,\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\,n!}. \tag{13}$$

Also, here are analogs of (8) and (10)

$$\int_0^1 h(t)Q_n^{(\alpha,\beta)}(t)(1-t)^{\alpha}t^{\beta}\mathrm{d}t = 0 \qquad (^{\circ}(h) < n), \tag{14}$$

$$\int_0^1 Q_n^{(\alpha,\beta+1)}(t)(1-t)^{\alpha}t^{\beta}\mathrm{d}t = (-1)^n\ \mathrm{B}\ (\alpha+n+1,\beta+1). \tag{15}$$

Substituting in (11) t by tu and applying the chain rule, we get

$$\frac{\partial^n}{\partial t^n}\Big((1-tu)^{n+\alpha}t^{n+\beta}\Big) = n!\,(1-tu)^{\alpha}\,t^{\beta}\,Q_n^{(\alpha,\beta)}(tu). \tag{16}$$

Inspired by (13), we define the function $\mathcal{J}_n^{(\alpha,\beta)}$ on (0, 1) as

$$\mathcal{J}_n^{(\alpha,\beta)}(t) := c_n^{(\alpha,\beta)}(1-t)^{\alpha/2}t^{\beta/2}Q_n^{(\alpha,\beta)}(t), \tag{17}$$

where

$$c_n^{(\alpha,\beta)} := \sqrt{\frac{(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,n!}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}. \tag{18}$$

Then

$$\int_0^1 \mathcal{J}_m^{(\alpha,\beta)}(t)\mathcal{J}_n^{(\alpha,\beta)}(t)\,\mathrm{d}t = \delta_{m,n}. \tag{19}$$

2.2 Reproducing property for the polynomials on the unit interval

Given n in $\mathbb{N}_0$ and $\alpha,\beta > -1$, we denote by $R_n^{(\alpha,\beta)}$ the polynomial

$$R_n^{(\alpha,\beta)}(t) := \frac{(-1)^n\ \mathrm{B}\ (\alpha+1,\beta+1)}{\mathrm{B}\ (\alpha+n+1,\beta+1)}\ Q_n^{(\alpha,\beta+1)}(t). \tag{20}$$

Proposition 1 *Let $n \in \mathbb{N}_0$ and $\alpha,\beta > -1$. Then, for every polynomial h with $^\circ(h) \leq n$*

$$\frac{1}{\mathrm{B}\ (\alpha+1,\beta+1)}\int_0^1 h(t)R_n^{(\alpha,\beta)}(t)\,(1-t)^\alpha t^\beta\,\mathrm{d}t = h(0). \tag{21}$$

Proof The difference $h(t)-h(0)$ divides by t. Denote by $q(t)$ the corresponding quotient. Therefore, q is a polynomial of degree $^\circ(q) \leq n-1$ and $h(t) = h(0) + tq(t)$. Then

$$\begin{aligned}
&\frac{1}{\mathrm{B}\ (\alpha+1,\beta+1)}\int_0^1 h(t)R_n^{(\alpha,\beta)}(t)\,(1-t)^\alpha t^\beta\,\mathrm{d}t \\
&\qquad = \frac{(-1)^n h(0)}{\mathrm{B}\ (\alpha+n+1,\beta+1)}\int_0^1 Q_n^{(\alpha,\beta+1)}(t)\,(1-t)^\alpha t^\beta\,\mathrm{d}t \\
&\qquad\quad + \frac{(-1)^n}{\mathrm{B}\ (\alpha+n+1,\beta+1)}\int_0^1 q(t)Q_n^{(\alpha,\beta+1)}(t)\,(1-t)^\alpha t^{\beta+1}\,\mathrm{d}t.
\end{aligned}$$

By (15) and (14), the first summand is $h(0)$ and the second is 0. □

As a particular case of (21), for $\beta = 0$ and $m \leq n$:

$$\frac{1}{\alpha+1}\int_0^1 t^m R_n^{(\alpha,0)}(t)(1-t)^\alpha \mathrm{d}t = \delta_{m,0}. \tag{22}$$

Formula (22) was proven in [14] in other way.

3 Orthonormal basis and Fourier decomposition of $L^2(\mathbb{D},\mu_\alpha)$

For each $p,q \in \mathbb{N}_0$, denote by $m_{p,q}$ the monomial function

$$m_{p,q}(z) := z^p \overline{z}^q.$$

The inner product of two monomial functions is

$$\langle m_{p,q}, m_{j,k} \rangle = (\alpha+1)\ \mathrm{B}\ (p+j+1, \alpha+1)\,\delta_{p-q,j-k}. \tag{23}$$

In particular, this means that the family $(m_{p,q})_{p,q\in\mathbb{N}_0}$ is not orthogonal.

In this section, we recall various equivalent formulas for an orthonormal basis in $L^2(\mathbb{D}, \mu_\alpha)$, that can be obtained by orthonormalizing $(m_{p,q})_{p,q\in\mathbb{N}_0}$, and whose elements are known as *Jacobi polynomials in* z *and* $\overline{z}$, see Koornwinder [21], or *disk polynomials*, see Wünsche [43], among others. These polynomials, in the unweighted case, were also rediscovered in [23, 29], and [27], in the context of polyanalytic functions. We work with a normalized version of the disk polynomials and define them by

$$b_{p,q}^{(\alpha)}(z) := (-1)^{p+q}\,\widetilde{c}_{p,q}^{(\alpha)}\,(1-z\overline{z})^{-\alpha}\,\frac{\partial^q}{\partial z^q}\,\frac{\partial^p}{\partial \overline{z}^p}\left((1-z\,\overline{z})^{p+q+\alpha}\right), \tag{24}$$

where

$$\widetilde{c}_{p,q}^{(\alpha)} = \sqrt{\frac{(\alpha+p+q+1)\Gamma(\alpha+p+1)\Gamma(\alpha+q+1)}{(\alpha+1)p!\,q!\,\Gamma(\alpha+p+q+1)^2}}. \tag{25}$$

Since $\dfrac{\partial}{\partial z}(1-z\overline{z}) = -\overline{z}$ and $\dfrac{\partial}{\partial \overline{z}}(1-z\overline{z}) = -z$, the expression in (24) can be rewritten in other equivalent forms

$$b_{p,q}^{(\alpha)}(z) = (-1)^q \sqrt{\frac{(\alpha+p+q+1)\Gamma(\alpha+p+1)}{(\alpha+1)p!\,q!\,\Gamma(\alpha+q+1)}}\,\frac{\frac{\partial^q}{\partial z^q}\left(z^p(1-z\,\overline{z})^{\alpha+q}\right)}{(1-z\overline{z})^\alpha}, \tag{26}$$

$$b_{p,q}^{(\alpha)}(z) = (-1)^p \sqrt{\frac{(\alpha+p+q+1)\Gamma(\alpha+q+1)}{(\alpha+1)p!\,q!\,\Gamma(\alpha+p+1)}}\,\frac{\frac{\partial^p}{\partial \overline{z}^p}\left(\overline{z}^q(1-z\,\overline{z})^{\alpha+p}\right)}{(1-z\overline{z})^\alpha}. \tag{27}$$

By (16), $b_{p,q}^{(\alpha)}$ can be expressed via the shifted Jacobi polynomials, with coefficients defined by (18)

$$b_{p,q}^{(\alpha)}(z) = \begin{cases} \dfrac{c_q^{(\alpha,p-q)}}{\sqrt{\alpha+1}}\, z^{p-q} Q_q^{(\alpha,p-q)}(|z|^2), & \text{if } p \geq q; \\ \dfrac{c_p^{(\alpha,q-p)}}{\sqrt{\alpha+1}}\, \overline{z}^{q-p} Q_p^{(\alpha,q-p)}(|z|^2), & \text{if } p < q. \end{cases} \tag{28}$$

The two cases in (28) can be joined and written in terms of (17)

$$b_{p,q}^{(\alpha)}(r\tau) = \frac{c_{\min\{p,q\}}^{(\alpha,|p-q|)}}{\sqrt{\alpha+1}}\, r^{|p-q|}\tau^{p-q} Q_{\min\{p,q\}}^{(\alpha,|p-q|)}(r^2) \qquad (r \geq 0,\ \tau \in \mathbb{T}), \tag{29}$$

$$b_{p,q}^{(\alpha)}(r\tau) = \frac{\tau^{p-q}(1-r^2)^{-\alpha/2}}{\sqrt{\alpha+1}}\, \mathcal{J}_{\min\{p,q\}}^{(\alpha,|p-q|)}(r^2). \tag{30}$$

Notice that

$$c_{\min\{p,q\}}^{(\alpha,|p-q|)} = \sqrt{\frac{(\alpha+p+q+1)(\min\{p,q\})!\,\Gamma(\alpha+\max\{p,q\}+1)}{(\max\{p,q\})!\,\Gamma(\alpha+\min\{p,q\}+1)}}.$$

The family $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{N}_0}$ has the following conjugate symmetric property:

$$\overline{b_{p,q}^{(\alpha)}(z)} = b_{q,p}^{(\alpha)}(z). \tag{31}$$

Applying (12) in the right-hand side of (28), we obtain

$$b_{p,q}^{(\alpha)}(z) = \sqrt{\frac{(\alpha+p+q+1)p!\,q!}{(\alpha+1)\,\Gamma(\alpha+p+1)\Gamma(\alpha+q+1)}} \times \sum_{k=0}^{\min\{p,q\}} (-1)^k \frac{\Gamma(\alpha+p+q+1-k)}{k!\,(p-k)!\,(q-k)!} z^{p-k}\overline{z}^{q-k}, \tag{32}$$

In particular, (32) implies that $b_{p,q}^{(\alpha)}$ is a polynomial in z and $\overline{z}$ whose leading term, corresponding to $k=0$, is a positive multiple of the monomial $m_{p,q}$.

Let $\mathcal{P}$ be the set of all polynomials functions in z and $\overline{z}$, i.e., the linear span of the monomials

$$\mathcal{P} := \text{span}\{m_{p,q} : p,q \in \mathbb{N}_0\}.$$

For every $\xi \in \mathbb{Z}$ and every $s \in \mathbb{N}$, denote by $\mathcal{W}_{\xi,s}^{(\alpha)}$ the subspace of $\mathcal{P}$ generated by $m_{p,q}$ with $p-q=\xi$ and $\min\{p,q\} < s$

$$\mathcal{W}_{\xi,s}^{(\alpha)} := \text{span}\{m_{p,q} :\ p-q=\xi,\ \min\{p,q\} < s\}. \tag{33}$$

The vector space $\mathcal{W}_{\xi,s}^{(\alpha)}$ does not depend on α, but we endow it with the inner product from $L^2(\mathbb{D},\mu_\alpha)$. Obviously, $\dim(\mathcal{W}_{\xi,s}^{(\alpha)}) = s$. Let us show that

$$\mathcal{W}_{\xi,s}^{(\alpha)} = \text{span}\{b_{p,q}^{(\alpha)} :\ p-q=\xi,\ \min\{p,q\} < s\}. \tag{34}$$

Indeed, by (32)

$$
\begin{aligned}
m_{p,q} = & \sqrt{\frac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(\alpha+q+1)p!\,q!}{\Gamma(\alpha+p+q+2)}}\, b_{p,q}^{(\alpha)} \\
& - \frac{p!\,q!}{\Gamma(\alpha+p+q+1)} \sum_{v=1}^{\min\{p,q\}} (-1)^v \frac{\Gamma(\alpha+p+q+1-v)}{v!\,(p-v)!\,(q-v)!}\, m_{p-v,q-v}.
\end{aligned} \tag{35}
$$

Proceeding by induction on s, we see that the monomials $m_{p,q}$ are linear combinations of $b_{p-s,q-s}^{(\alpha)}$ with $0 \le s \le \min\{p,q\}$. Therefore, formula (34) means that the first s elements in the diagonal ξ of the table $(b_{p,q}^{(\alpha)})_{p,q=0}^{\infty}$ generate the same subspace as the first s elements of the diagonal ξ in the table $(m_{p,q})_{p,q=0}^{\infty}$. For example

$$
\begin{aligned}
\mathcal{W}_{-2,3}^{(\alpha)} &= \operatorname{span}\{m_{0,2}, m_{1,3}, m_{2,4}\} = \operatorname{span}\{b_{0,2}^{(\alpha)}, b_{1,3}^{(\alpha)}, b_{2,4}^{(\alpha)}\}, \\
\mathcal{W}_{1,4}^{(\alpha)} &= \operatorname{span}\{m_{1,0}, m_{2,1}, m_{3,2}, m_{4,3}\} = \operatorname{span}\{b_{1,0}^{(\alpha)}, b_{2,1}^{(\alpha)}, b_{3,2}^{(\alpha)}, b_{4,3}^{(\alpha)}\}.
\end{aligned}
$$

In the following tables, we show generators of $\mathcal{W}_{1,4}^{(\alpha)}$ (light blue) and $\mathcal{W}_{-2,3}^{(\alpha)}$ (pink)

$m_{0,0}$	$m_{0,1}$	$m_{0,2}$	$m_{0,3}$	$m_{0,4}$	$m_{0,5}$	$\ddots$
$m_{1,0}$	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$	$m_{1,4}$	$m_{1,5}$	$\ddots$
$m_{2,0}$	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	$m_{2,4}$	$m_{2,5}$	$\ddots$
$m_{3,0}$	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	$m_{3,4}$	$m_{3,5}$	$\ddots$
$m_{4,0}$	$m_{4,1}$	$m_{4,2}$	$m_{4,3}$	$m_{4,4}$	$m_{4,5}$	$\ddots$
$m_{5,0}$	$m_{5,1}$	$m_{5,2}$	$m_{5,3}$	$m_{5,4}$	$m_{5,5}$	$\ddots$
$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

$b_{0,0}^{(\alpha)}$	$b_{0,1}^{(\alpha)}$	$b_{0,2}^{(\alpha)}$	$b_{0,3}^{(\alpha)}$	$b_{0,4}^{(\alpha)}$	$b_{0,5}^{(\alpha)}$	$\ddots$
$b_{1,0}^{(\alpha)}$	$b_{1,1}^{(\alpha)}$	$b_{1,2}^{(\alpha)}$	$b_{1,3}^{(\alpha)}$	$b_{1,4}^{(\alpha)}$	$b_{1,5}^{(\alpha)}$	$\ddots$
$b_{2,0}^{(\alpha)}$	$b_{2,1}^{(\alpha)}$	$b_{2,2}^{(\alpha)}$	$b_{2,3}^{(\alpha)}$	$b_{2,4}^{(\alpha)}$	$b_{2,5}^{(\alpha)}$	$\ddots$
$b_{3,0}^{(\alpha)}$	$b_{3,1}^{(\alpha)}$	$b_{3,2}^{(\alpha)}$	$b_{3,3}^{(\alpha)}$	$b_{3,4}^{(\alpha)}$	$b_{3,5}^{(\alpha)}$	$\ddots$
$b_{4,0}^{(\alpha)}$	$b_{4,1}^{(\alpha)}$	$b_{4,2}^{(\alpha)}$	$b_{4,3}^{(\alpha)}$	$b_{4,4}^{(\alpha)}$	$b_{4,5}^{(\alpha)}$	$\ddots$
$b_{5,0}^{(\alpha)}$	$b_{5,1}^{(\alpha)}$	$b_{5,2}^{(\alpha)}$	$b_{5,3}^{(\alpha)}$	$b_{5,4}^{(\alpha)}$	$b_{5,5}^{(\alpha)}$	$\ddots$
$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

As a consequence, $\mathcal{P} = \bigcup_{\xi\in\mathbb{Z}} \bigcup_{s\in\mathbb{N}} \mathcal{W}_{\xi,s}^{(\alpha)} = \operatorname{span}\{b_{p,q}^{(\alpha)} : p,q \in \mathbb{N}_0\}$.

Proposition 2 *The family* $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{N}_0}$ *is an orthonormal basis of* $L^2(\mathbb{D}, \mu_\alpha)$.

Proof The orthonormal property follows straightforwardly from (30) and (19):

$$
\begin{aligned}
\langle b_{p,q}^{(\alpha)}, b_{j,k}^{(\alpha)} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(p-q-j+k)\theta}\, \mathrm{d}\theta \int_0^1 \mathcal{J}_{\min\{p,q\}}^{(\alpha,|p-q|)}(t)\, \mathcal{J}_{\min\{j,k\}}^{(\alpha,|j-k|)}(t)\, \mathrm{d}t \\
&= \delta_{p-q,j-k} \cdot \delta_{\min\{p,q\},\min\{j,k\}} = \delta_{p,j} \cdot \delta_{q,k}.
\end{aligned}
$$

By the Stone–Weierstrass theorem, $\mathcal{P}$ is dense in $C(\operatorname{clos}(\mathbb{D}))$. In turn, by Luzin's theorem, the set $C(\operatorname{clos}(\mathbb{D}))|_{\mathbb{D}}$ is dense in $L^2(\mathbb{D}, \mu_\alpha)$, and for every $f \in C(\operatorname{clos}(\mathbb{D}))$, we have $\|f\| \le \max_{z\in\operatorname{clos}(\mathbb{D})} |f(z)|$. Now, it is easy to see that the set $\mathcal{P}$ is dense in $L^2(\mathbb{D}, \mu_\alpha)$, that is, the set of all linear combinations of elements of the family is dense in $L^2(\mathbb{D}, \mu_\alpha)$. For that reason, $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{N}_0}$ is a complete orthonormal family. □

Corollary 1 *Let* $\xi \in \mathbb{Z}$ *and* $s \in \mathbb{N}$. *Then,* $(b_{q+\xi,q}^{(\alpha)})_{q=\max\{0,-\xi\}}^{\max\{s-1,s-\xi-1\}}$ *is an orthonormal* *basis of* $\mathcal{W}^{(\alpha)}_{\xi,s}$.

Remark 1 By Proposition 2 and formula (35)

$$\langle m_{\xi+k,k}, b^{(\alpha)}_{\xi+q,q}\rangle = \begin{cases} \sqrt{\dfrac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(\alpha+q+1)p!\,q!}{\Gamma(\alpha+p+q+2)\Gamma(\alpha+p+q+1)}}, & k=q; \\ 0, & \max\{0,-\xi\}\le k<q. \end{cases} \tag{36}$$

The table of basic functions can be expressed as follows:

$$\begin{aligned} b^{(\alpha)}_{0,0}(z) &= h^{(\alpha)}_{0,0}(|z|^2), \\ b^{(\alpha)}_{1,0}(z) &= z\, h^{(\alpha)}_{1,0}(|z|^2), \\ b^{(\alpha)}_{2,0}(z) &= z^2\, h^{(\alpha)}_{2,0}(|z|^2), \end{aligned}$$

where $h^{(\alpha)}_{p,q}(t) := \frac{c^{(\alpha,|p-q|)}_{\min\{p,q\}}}{\sqrt{\alpha+1}} Q^{(\alpha,|p-q|)}_{\min\{p,q\}}(t)$. Below, we show explicitly some elements of this basis

$$\begin{aligned} b^{(\alpha)}_{0,0}(z) &= 1, \qquad b^{(\alpha)}_{1,0}(z) = \sqrt{\alpha+2}\,z, \quad b^{(\alpha)}_{2,0}(z) = \sqrt{\frac{(\alpha+3)(\alpha+2)}{2}}\,z^2, \\ b^{(\alpha)}_{1,1}(z) &= \sqrt{\frac{\alpha+3}{\alpha+1}}\big((\alpha+2)z\overline{z}-1\big), \\ b^{(\alpha)}_{2,1}(z) &= \sqrt{\frac{2(\alpha+3)(\alpha+2)}{\alpha+1}}\left(\frac{\alpha+3}{2}z^2\overline{z}-z\right), \\ b^{(\alpha)}_{2,2}(z) &= \sqrt{\frac{\alpha+5}{\alpha+1}}\left(\frac{(\alpha+4)(\alpha+3)}{2}z^2\overline{z}^2-2(\alpha+3)z\overline{z}+\frac{1}{2}\right). \end{aligned}$$

Now, for every ξ in $\mathbb{Z}$, we introduce the subspace $\mathcal{W}^{(\alpha)}_{\xi}$ associated with the "frequency" ξ or, equivalently, to the diagonal ξ in the tables $(m_{p,q})_{p,q\in\mathbb{N}_0}$ and $(b^{(\alpha)}_{p,q})_{p,q\in\mathbb{Z}}$

$$\mathcal{W}^{(\alpha)}_{\xi} := \operatorname{clos}(\operatorname{span}\{m_{p,q} : p-q=\xi\}) = \operatorname{clos}\left(\bigcup_{s\in\mathbb{N}} \mathcal{W}^{(\alpha)}_{\xi,s}\right). \tag{37}$$

Corollary 2 *The sequence* $(b^{(\alpha)}_{\xi+q,q})_{q=\max\{0,-\xi\}}^{\infty}$ *is an orthonormal basis of the Hilbert space* $\mathcal{W}^{(\alpha)}_{\xi}$.

The space $\mathcal{W}^{(\alpha)}_{\xi}$ can be naturally identified with L^2 over (0, 1), providing (0, 1) with various weights.

Proposition 3 *Each one of the following linear operators is an isometric isomorphism of Hilbert spaces:*

(1) $L^2((0,1),(\alpha+1)\,(1-t)^\alpha\,\mathrm{d}t) \to \mathcal{W}_\xi^{(\alpha)}, \quad h \mapsto f,$

$$f(z) := \operatorname{sgn}^\xi(z) h(z\bar{z}), \quad \text{i.e.,} \quad f(r\tau) := \tau^\xi h(r^2), \tag{38}$$

where $z \in \mathbb{D}$, $0 \le r < 1$, $\tau \in \mathbb{T}$;

(2) $L^2((0,1),(\alpha+1)\,t^{|\xi|}(1-t)^\alpha) \to \mathcal{W}_\xi^{(\alpha)}, \quad h \mapsto f,$

$$f(z) := \begin{cases} z^\xi h(z\bar{z}), & \xi \ge 0, \\ \bar{z}^\xi h(z\bar{z}), & \xi < 0, \end{cases} \quad \text{i.e.,} \quad f(r\tau) := \tau^\xi r^{|\xi|} h(r^2); \tag{39}$$

(3) $L^2((0,1)) \to \mathcal{W}_\xi^{(\alpha)}, \quad h \mapsto f,$

$$f(z) := \operatorname{sgn}^\xi(z) \frac{(1-z\bar{z})^{-\alpha/2}}{\sqrt{\alpha+1}} h(z\bar{z}) = \tau^\xi \frac{(1-r^2)^{-\alpha/2}}{\sqrt{\alpha+1}} h(r^2). \tag{40}$$

Proof In each case, the isometric property is verified directly using polar coordinates, and the surjective property is justified with the help of the orthonormal basis of $\mathcal{W}_\xi^{(\alpha)}$ (Corollary 2). The function $\operatorname{sgn} : \mathbb{C} \to \mathbb{C}$ is defined by $\operatorname{sgn}(z) := z/|z|$ for $z \ne 0$ and $\operatorname{sgn}(0) := 0$. □

Corollary 3 *The space $L^2(\mathbb{D}, \mathrm{d}\mu_\alpha)$ is the orthogonal sum of the subspaces $\mathcal{W}_\xi^{(\alpha)}$:*

$$L^2(\mathbb{D}, \mathrm{d}\mu_\alpha) = \bigoplus_{\xi \in \mathbb{Z}} \mathcal{W}_\xi^{(\alpha)}. \tag{41}$$

The result of Corollary 3 can be seen as the *Fourier decomposition* of the space $L^2(\mathbb{D}, \mathrm{d}\mu_\alpha)$, and each space $\mathcal{W}_\xi^{(\alpha)}$ corresponds to the *frequency* ξ.

Here, we show the generators of $\mathcal{W}_0^{(\alpha)}$ (pink) and $\mathcal{W}_{-1}^{(\alpha)}$ (light blue)

$$\begin{array}{ccccc} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & \ddots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & \ddots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & \ddots \\ m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{array} \qquad \begin{array}{ccccc} b_{0,0}^{(\alpha)} & b_{0,1}^{(\alpha)} & b_{0,2}^{(\alpha)} & b_{0,3}^{(\alpha)} & \ddots \\ b_{1,0}^{(\alpha)} & b_{1,1}^{(\alpha)} & b_{1,2}^{(\alpha)} & b_{1,3}^{(\alpha)} & \ddots \\ b_{2,0}^{(\alpha)} & b_{2,1}^{(\alpha)} & b_{2,2}^{(\alpha)} & b_{2,3}^{(\alpha)} & \ddots \\ b_{3,0}^{(\alpha)} & b_{3,1}^{(\alpha)} & b_{3,2}^{(\alpha)} & b_{3,3}^{(\alpha)} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{array}$$

4 Weighted mean value property of polyanalytic functions

It is known [4, Section 1.1] that any n-analytic function can be expressed as a "polynomial" of degree $n-1$ in the variable $\overline{z}$ with 1-analytic coefficients, that is, for any $f \in \mathcal{A}_n(\mathbb{D})$, there exist analytic functions $g_0, g_1, \ldots, g_{n-1}$ in $\mathbb{D}$, such that

$$f(z) = \sum_{k=0}^{n-1} g_k(z)\overline{z}^k \qquad (z \in \mathbb{D}).$$

Replacing every g_k by its Taylor series, we get another classic form of n-analytic functions: there exist coefficients $\lambda_{j,k}$ in $\mathbb{C}$, such that

$$f(z) = \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \lambda_{j,k} z^j \overline{z}^k \qquad (z \in \mathbb{D}). \tag{42}$$

The following weighted mean value property was proved in [14] using a slightly different method. The mean value property for solutions of more general elliptic equations was studied in [39].

Proposition 4 *Let $f \in \mathcal{A}_n(\mathbb{D})$, such that*

$$\int_{\mathbb{D}} |f(z)|\,(1-|z|^2)^{\alpha}\,\mathrm{d}\mu(z) < +\infty.$$

Then

$$f(0) = \frac{\alpha+1}{\pi} \int_{\mathbb{D}} f(z) R_{n-1}^{(\alpha,0)}(|z|^2)\,(1-|z|^2)^{\alpha}\,\mathrm{d}\mu(z). \tag{43}$$

Proof For $0 < r \le 1$, consider

$$I(r) := \frac{\alpha+1}{\pi} \int_{r\mathbb{D}} f(w) R_{n-1}^{(\alpha,0)}(|w|^2)\,(1-|w|^2)^{\alpha}\,\mathrm{d}\mu(w).$$

Then, combining (42) with the polar decomposition, we get that

$$I(r) = \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \lambda_{j,k} \left((\alpha+1) \int_0^r u^{j+k} R_{n-1}^{(\alpha,0)}(u^2)(1-u^2)^{\alpha}\, 2u\,\mathrm{d}u \right) \times \left(\frac{1}{2\pi} \int_0^{2\pi} e^{i(j-k)\theta} \mathrm{d}\theta \right).$$

The terms in the inner series vanish whenever $j \ne k$. Setting t instead of u^2 in the integral, we have

$$I(r)=\sum_{k=0}^{n-1}\lambda_{k,k}(\alpha+1)\int_0^{r^2} t^k R_{n-1}^{(\alpha,0)}(t)(1-t)^\alpha \mathrm{d}t.$$

Take limits in both sides when r tends to 1. Apply Lebesgue's dominated convergence theorem to get the integral over $\mathbb{D}$ in the left-hand side and the integral over [0, 1) in the right-hand side. By formula (22)

$$I(1)=\sum_{k=0}^{n-1}\lambda_{k,k}(\alpha+1)\int_0^{1} t^k R_{n-1}^{(\alpha,0)}(t)(1-t)^\alpha \mathrm{d}t=\sum_{k=0}^{n-1}\lambda_{k,k}\,\delta_{k,0}=\lambda_{0,0}=f(0),$$

which proves (43). □

For $\alpha=0$, Proposition 4 reduces to the following mean value property that appeared in [23] and [27].

Corollary 4 *Let $z\in\mathbb{C}$, $r>0$, and $f\in\mathcal{A}_n(z+r\mathbb{D})$, such that*

$$\int_{z+r\mathbb{D}}|f(w)|\,\mathrm{d}\mu(w)<+\infty.$$

Then

$$f(z)=\frac{\alpha+1}{\pi r^2}\int_{z+r\mathbb{D}} f(w)R_{n-1}^{(\alpha,0)}\left(\frac{|w-z|^2}{r^2}\right)\mathrm{d}\mu(w). \tag{44}$$

Proof Denote by φ the linear change of variables $\varphi(w):=rw+z$. If $f\in\mathcal{A}_n(z+r\mathbb{D})$, then $f\circ\varphi\in\mathcal{A}_n(\mathbb{D})$. Applying (43) to $f\circ\varphi$, we obtain (44). □

4.1 Weighted Bergman spaces of polyanalytic functions on general complex domains

Given n in $\mathbb{N}$, an open subset Ω of $\mathbb{C}$ and a continuous function $W:\Omega\to(0,+\infty)$, we denote by $\mathcal{A}_n^2(\Omega,W)$ the space of n-analytic functions belonging to $L^2(\Omega,W)$ and provided with the norm of $L^2(\Omega,W)$. The mean value property (44) implies that the evaluation functionals in $\mathcal{A}_n^2(\Omega,W)$ are bounded (moreover, they are uniformly bounded on compacts), and $\mathcal{A}_n^2(\Omega,W)$ is an RKHS. Here, are proofs of these facts.

Lemma 1 *Let K be a compact subset of Ω. Then, there exists a number $C_{n,W,K}>0$, such that for every f in $\mathcal{A}_n^2(\Omega,W)$ and every z in K*

$$|f(z)|\le C_{n,W,K}\|f\|_{\mathcal{A}_n^2(\Omega,W)}. \tag{45}$$

Proof Let r_1 be the distance from K to $\mathbb{C}\setminus\Omega$. Since K is compact and $\mathbb{C}\setminus\Omega$ is closed, $r_1>0$. Put $r:=\min\{r_1/2,1\}$, $K_1:=\{w\in\mathbb{C}:\ d(w,K)\le r\}$

$$C_1 := \left(\max_{0\le t\le 1} |R_{n-1}^{(\alpha,0)}(t)|\right)\left(\max_{w\in K_1}\frac{1}{\sqrt{W(w)}}\right).$$

For every z in K, we estimate $|f(z)|$ from above applying (44) and Schwarz inequality

$$\begin{aligned}
|f(z)| &\le \frac{1}{\pi r^2}\int_{z+r\mathbb{D}} |f(w)| \left| R_{n-1}^{(\alpha,0)}\left(\frac{|w-z|^2}{r^2}\right)\right| \mathrm{d}\mu(w)\\
&\le \frac{C_1}{\pi r^2}\int_{z+r\mathbb{D}} |f(w)| \sqrt{W(w)}\,\mathrm{d}\mu(w)\\
&\le \frac{C_1}{\pi r^2}\left(\int_{z+r\mathbb{D}} |f(w)|^2 W(w)\,\mathrm{d}\mu(w)\right)^{1/2}\left(\int_{z+r\mathbb{D}} 1\,\mathrm{d}\mu(w)\right)^{1/2}\\
&\le \frac{C_1}{\sqrt{\pi r^2}}\,\|f\|_{\mathcal{A}_n^2(\Omega,W)}.
\end{aligned}$$

Therefore, (45) is fulfilled with $C_{n,W,K} = \frac{C_1}{\sqrt{\pi r^2}}$. □

Proposition 5 *$\mathcal{A}_n^2(\Omega, W)$ is an RKHS.*

Proof Given a Cauchy sequence in $\mathcal{A}_n^2(\Omega, W)$, for every compact K, it converges uniformly on K by Lemma 1. The pointwise limit of this sequence is also polyanalytic by [4, Corollary 1.8], and it coincides a.e. with the limit in $L^2(\Omega, W)$. Lemma 1 also assures the boundedness of the evaluation functionals and thereby the existence of the reproducing kernel. See similar proofs in [26, Proposition 3.3]. □

We denote by $\mathcal{A}_{(n)}^2(\Omega, W)$ the orthogonal complement of $\mathcal{A}_{n-1}^2(\Omega, W)$ in $\mathcal{A}_n^2(\Omega, W)$.

Corollary 5 *$\mathcal{A}_{(n)}^2(\Omega, W)$ is an RKHS.*

5 Weighted Bergman spaces of polyanalytic functions on the unit disk

In the rest of the paper, we suppose that $n \in \mathbb{N}$ and $\alpha > -1$. Given z in $\mathbb{D}$, denote by $K_{n,z}^{(\alpha)}$ the reproducing kernel of $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$ at the point z and by $K_{(n),z}^{(\alpha)}$ the reproducing kernel of $\mathcal{A}_{(n)}^2(\mathbb{D}, \mu_\alpha)$ at the point z. Hachadi and Youssfi [14] computed the reproducing kernel of $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$

$$K_{n,z}^{(\alpha)}(w) = \frac{(1-\overline{w}z)^{n-1}}{(1-\overline{z}w)^{n+1}} R_{n-1}^{(\alpha,0)}\left(\left|\frac{z-w}{1-\overline{z}w}\right|^2\right). \tag{46}$$

Their method uses (43) and a generalization of the unitary operator constructed by Pessoa [27]. Formula (46) implies an exact expression for the norm of $K_{n,z}^{(\alpha)}$, which is also the norm of the evaluation functional at the point z

$$\|K_{n,z}^{(\alpha)}\| = \sqrt{(n+\alpha)\binom{n+\alpha-1}{n-1}}\,\frac{1}{1-|z|^2}. \tag{47}$$

Obviously, the reproducing kernel of $\mathcal{A}^2_{(n)}(\mathbb{D},\mu_\alpha)$ can be written as

$$K_{(n),z}^{(\alpha)}(w) = K_{n,z}^{(\alpha)}(w) - K_{n-1,z}^{(\alpha)}(w). \tag{48}$$

Unfortunately, we were unable to obtain a simpler formula for $K_{(n),z}^{(\alpha)}$.

5.1 Orthonormal basis in $\mathcal{A}^2_n(\mathbb{D},\mu_\alpha)$

Proposition 6 *$(b_{p,q}^{(\alpha)})_{p\in\mathbb{N}_0,q<n}$ is an orthonormal basis of $\mathcal{A}^2_n(\mathbb{D},\mu_\alpha)$.*

Proof It is clear that this family is contained in $\mathcal{A}^2_n(\mathbb{D},\mu_\alpha)$, and by Proposition 2 is orthonormal. Using ideas of Ramazanov [29, proof of Theorem 2], we will show the total property. Suppose that $f\in\mathcal{A}^2_n(\mathbb{D},\mu_\alpha)$ and $\langle f, b_{p,q}^{(\alpha)}\rangle = 0$ for every p in $\mathbb{N}_0$ and $q<n$. For $r>0$, using expansion (42) and the orthogonality of the Fourier basis on $\mathbb{T}$, we easily obtain

$$\int_{r\mathbb{D}} f\,\overline{b_{p,q}^{(\alpha)}}\,\mathrm{d}\mu_\alpha = \sum_{k=0}^{n-1}\lambda_{k+p-q,k}\int_{r\mathbb{D}} m_{k+p-q,k}\overline{b_{p,q}^{(\alpha)}}\,\mathrm{d}\mu_\alpha.$$

The dominated convergence theorem allows us to pass to integrals over $\mathbb{D}$, because $f\,\overline{b_{p,q}^{(\alpha)}}$ and $m_{k+p-q,k}\,\overline{b_{p,q}^{(\alpha)}}$ belong to $L^1(\mathbb{D},\mu_\alpha)$. Now, the assumption $f\perp b_{p,q}^{(\alpha)} = 0$ yields

$$\sum_{k=0}^{n-1}\langle m_{k+p-q,k}, b_{p,q}^{(\alpha)}\rangle\lambda_{k+p-q,k} = 0 \qquad (p\in\mathbb{N}_0,\ 0\le q<n). \tag{49}$$

For a fixed ξ in $\mathbb{Z}$ with $\xi > -n$, we put $s=\min\{n, n+\xi\}$. The vector $[\lambda_{k+\xi,k}]_{k=\max\{0,-\xi\}}^{n-1}$ satisfies the homogeneous linear system (49) with the $s\times s$ matrix

$$\left[\langle m_{\xi+k,k}, b_{\xi+q,q}^{(\alpha)}\rangle\right]_{q,k=\max\{0,-\xi\}}^{n-1}.$$

By (36), this is an upper triangular matrix with nonzero diagonal entries; hence, the unique solution of (49) is zero. □

Corollary 6 *$(b_{p,n-1}^{(\alpha)})_{p\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{A}^2_{(n)}(\mathbb{D},\mu_\alpha)$.*

For example, $(b_{p,q}^{(\alpha)})_{p\in\mathbb{N}_0,q<4}$ is an orthonormal basis of $\mathcal{A}^2_4(\mathbb{D},\mu_\alpha)$, and $(b_{p,3}^{(\alpha)})_{p\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{A}^2_{(4)}(\mathbb{D},\mu_\alpha)$

$$\begin{array}{cccccc} b_{0,0}^{(\alpha)} & b_{0,1}^{(\alpha)} & b_{0,2}^{(\alpha)} & b_{0,3}^{(\alpha)} & b_{0,4}^{(\alpha)} & \cdots \\ b_{1,0}^{(\alpha)} & b_{1,1}^{(\alpha)} & b_{1,2}^{(\alpha)} & b_{1,3}^{(\alpha)} & b_{1,4}^{(\alpha)} & \cdots \\ b_{2,0}^{(\alpha)} & b_{2,1}^{(\alpha)} & b_{2,2}^{(\alpha)} & b_{2,3}^{(\alpha)} & b_{2,4}^{(\alpha)} & \cdots \\ b_{3,0}^{(\alpha)} & b_{3,1}^{(\alpha)} & b_{3,2}^{(\alpha)} & b_{3,3}^{(\alpha)} & b_{3,4}^{(\alpha)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \qquad \begin{array}{cccccc} b_{0,0}^{(\alpha)} & b_{0,1}^{(\alpha)} & b_{0,2}^{(\alpha)} & b_{0,3}^{(\alpha)} & b_{0,4}^{(\alpha)} & \cdots \\ b_{1,0}^{(\alpha)} & b_{1,1}^{(\alpha)} & b_{1,2}^{(\alpha)} & b_{1,3}^{(\alpha)} & b_{1,4}^{(\alpha)} & \cdots \\ b_{2,0}^{(\alpha)} & b_{2,1}^{(\alpha)} & b_{2,2}^{(\alpha)} & b_{2,3}^{(\alpha)} & b_{2,4}^{(\alpha)} & \cdots \\ b_{3,0}^{(\alpha)} & b_{3,1}^{(\alpha)} & b_{3,2}^{(\alpha)} & b_{3,3}^{(\alpha)} & b_{3,4}^{(\alpha)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We denote by $P_n^{(\alpha)}$ and $P_{(n)}^{(\alpha)}$ the orthogonal projections acting in $L^2(\mathbb{D},\mu_\alpha)$, whose images are $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ and $\mathcal{A}_{(n)}^2(\mathbb{D},\mu_\alpha)$, respectively. They can be computed in terms of the corresponding reproducing kernels

$$(P_n^{(\alpha)}f)(z) = \langle f, K_{n,z}^{(\alpha)}\rangle, \qquad (P_{(n)}^{(\alpha)}f)(z) = \langle f, K_{(n),z}^{(\alpha)}\rangle.$$

5.2 Decomposition of $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ into subspaces corresponding to different "frequencies"

We will use the following elementary fact about orthonormal bases in Hilbert spaces. In the next proposition, we treat them like sets rather than families.

Proposition 7 *Let H_1 be a Hilbert space and $\mathcal{B}_1 \subseteq H_1$ be an orthonormal basis of H_1. Suppose that $\mathcal{B}_2$ and $\mathcal{B}_3$ are some subsets of $\mathcal{B}_1$. Denote by H_2 and H_3 the closed subspaces of H_1 generated by $\mathcal{B}_2$ and $\mathcal{B}_3$, respectively. Then, $\mathcal{B}_2 \cap \mathcal{B}_3$ is an orthonormal basis of $H_2 \cap H_3$.*

Applying Proposition 7 to the Hilbert space $L^2(\mathbb{D},\mu_\alpha)$ and thinking in terms of orthonormal bases (see Propositions 2, 6, and Corollaries 1, 2), we easily find the intersection of $\mathcal{W}_\xi^{(\alpha)}$ and $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$

$$\mathcal{W}_\xi^{(\alpha)} \cap \mathcal{A}_n^2(\mathbb{D},\mu_\alpha) = \begin{cases} \mathcal{W}_{\xi,\min\{n,n+\xi\}}^{(\alpha)}, & \xi \geq -n+1; \\ \{0\}, & \xi < -n+1. \end{cases} \tag{50}$$

Here is a description of the subspaces $\mathcal{W}_{\xi,m}^{(\alpha)}$ in terms of the polar coordinates.

Proposition 8 *For every ξ in $\mathbb{Z}$ and every s in $\mathbb{N}$, the space $\mathcal{W}_{\xi,s}^{(\alpha)}$ consists of all functions of the form*

$$f(r\tau) = \tau^\xi r^{|\xi|} Q(r^2) \qquad (r \geq 0,\ \tau \in \mathbb{T}),$$

where Q is a polynomial of degree $\leq s-1$. Moreover

$$\|f\| = \|Q\|_{L^2([0,1),(\alpha+1)(1-t)^\alpha t^{|\xi|}\,\mathrm{d}t)}.$$

Proof The result follows directly by Proposition (3) and formula (39). □

The decomposition of $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ into a direct sum of the "truncated frequency subspaces" shown below follows from Proposition 6 and Corollary 1, and plays a crucial role in the study of radial operators. It can be seen as the "Fourier series decomposition" of $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$:

Proposition 9

$$\mathcal{A}_n^2(\mathbb{D},\mu_\alpha) = \bigoplus_{\xi=-n+1}^{\infty} \mathcal{W}^{(\alpha)}_{\xi,\min\{n,n+\xi\}}. \tag{51}$$

Let us illustrate Proposition 9 for $n=3$ with a table (we have marked in different shades of blue the basic functions that generate each truncated diagonal)

$$\begin{array}{ccccc}
b^{(\alpha)}_{0,0} & b^{(\alpha)}_{0,1} & b^{(\alpha)}_{0,2} & b^{(\alpha)}_{0,3} & \cdots \\
b^{(\alpha)}_{1,0} & b^{(\alpha)}_{1,1} & b^{(\alpha)}_{1,2} & b^{(\alpha)}_{1,3} & \cdots \\
b^{(\alpha)}_{2,0} & b^{(\alpha)}_{2,1} & b^{(\alpha)}_{2,2} & b^{(\alpha)}_{2,3} & \cdots \\
b^{(\alpha)}_{3,0} & b^{(\alpha)}_{3,1} & b^{(\alpha)}_{3,2} & b^{(\alpha)}_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Define: $U_n^{(\alpha)} : \mathcal{A}_n^2(\mathbb{D},\mu_\alpha) \to \bigoplus_{\xi=-n+1}^{\infty} \mathbb{C}^{\min\{n,n+\xi\}}$,

$$(U_n^{(\alpha)} f)_{\xi,q} := \langle f, b^{(\alpha)}_{q+\xi,q}\rangle \qquad (\xi \geq -n+1,\ \max\{0,-\xi\} \leq q \leq n-1). \tag{52}$$

Here, for $-n+1 \leq \xi < 0$, the components of vectors in $\mathbb{C}^{n+\xi}$ are enumerated from $-\xi$ to $n-1$.

Proposition 10 *The operator $U_n^{(\alpha)}$ is an isometric isomorphism of Hilbert spaces.*

Proof Follows from Proposition 6 or, even easier, from Proposition 9 and the fact that $(b^{(\alpha)}_{q+\xi,q})_{q=\max\{0,-\xi\}}^{n-1}$ is an orthonormal basis of $\mathcal{W}^{(\alpha)}_{\xi,\min\{n,n+\xi\}}$ (see Corollary 1). □

An analog of the upcoming fact for the unweighted poly-Bergman space was proved by Vasilevski [42, Section 4.2]. We obtain it as a corollary from Proposition 2 and Corollary 6.

Corollary 7 *The space $L^2(\mathbb{D},\mu_\alpha)$ is the orthogonal sum of the subspaces $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$, $m \in \mathbb{N}$*

$$L^2(\mathbb{D},\mu_\alpha) = \bigoplus_{m\in\mathbb{N}} \mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha).$$

6 The set of Toeplitz operators is not weakly dense

Given a Hilbert space H, we denote by $\mathcal{B}(H)$ the algebra of all bounded operators acting in H. If H is an RKHS naturally embedded into $L^2(\mathbb{D},\mu_\alpha)$ and $S\in\mathcal{B}(H)$, then the *Berezin transform* of S is defined by

$$\operatorname{Ber}_H(S)(z) := \frac{\langle SK_z,K_z\rangle_H}{\langle K_z,K_z\rangle_H},\qquad \text{i.e.,}\qquad \operatorname{Ber}_H(S)(z) = \frac{(SK_z)(z)}{K_z(z)}.$$

The Berezin transform can be considered as a bounded linear operator $\mathcal{B}(H)\to L^\infty(\mathbb{D})$. Stroethoff proved [36] that Ber_H is injective for various RKHS of analytic functions, in particular, for $H=\mathcal{A}_1^2(\mathbb{D})$. Engliš noticed [10, Section 2] that Ber_H is not injective for various RKHS of harmonic functions. The idea of Engliš can be applied without any changes to various spaces of polyanalytic and polyharmonic functions. For clarity of presentation, we state the result of Engliš for $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$, $n\geq 2$, and repeat his proof.

Proposition 11 *Let $H=\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ with $n\geq 2$. Then, the Berezin transform* Ber_H *is not injective.*

Proof Let $f\in H$, such that $\overline{f}\in H$ and the functions $f,\overline{f}$ are linearly independent. For example, $f(z):=z$. Following the idea from [10, Section 2], consider the operator:

$$Sh := \langle h,f\rangle_H f - \langle h,\overline{f}\rangle_H \overline{f}.$$

Then, $S\neq 0$, but $\langle SK_z,K_z\rangle_H = |f(z)|^2-|f(z)|^2=0$ for every z in $\mathbb{D}$. Therefore, $\operatorname{Ber}_H(S)$ is the zero constant. □

Given a function g in $L^\infty(\mathbb{D})$, let M_g be the multiplication operator defined on $L^2(\mathbb{D},\mu_\alpha)$ by $M_g f := gf$. If H is a closed subspace of $L^2(\mathbb{D},\mu_\alpha)$, then the *Toeplitz operator* $T_{H,g}$ is defined on H by

$$T_{H,g}(f) := P_H(gf) = P_H M_g f.$$

For $H=\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ and $H=\mathcal{A}_{(n)}^2(\mathbb{D},\mu_\alpha)$, we write just $T_{n,g}^{(\alpha)}$ and $T_{(n),g}^{(\alpha)}$, respectively. The proof of the following fact is the same as the proof of [26, Proposition 3.18] or the proof of [7, Theorem 4].

Proposition 12 *If $g\in L^\infty(\mathbb{D})$ and $T_{n,g}^{(\alpha)}=0$, then $g=0$ a.e. In other words, the function $g\mapsto T_{n,g}^{(\alpha)}$, acting from $L^\infty(\mathbb{D})$ to $\mathcal{B}(\mathcal{A}_n^2(\mathbb{D},\mu_\alpha))$, is injective.*

Inspired by the idea of Engliš explained in the proof of Proposition 11, we will prove that for $n\geq 2$, the set of Toeplitz operators is not weakly dense in $\mathcal{B}(\mathcal{A}_n^2(\mathbb{D},\mu_\alpha))$. First, let us prove an auxiliary fact from linear algebra: bounded quadratic forms separate linearly independent vectors.

Lemma 2 *Let H be a Hilbert space and f, g be two linearly independent vectors in H. Then, there exists S in $\mathcal{B}(H)$, such that*

$$\langle Sf, f\rangle_H \neq \langle Sg, g\rangle_H.$$

Proof Without lost of generality, we will suppose that $\|f\|_H = 1$. Decompose g into the linear combination $g = \lambda_1 f + \lambda_2 h$, with $\lambda_1, \lambda_2 \in \mathbb{C}$, $\|h\|_H = 1$, $h \perp f$. More explicitly

$$\lambda_1 := \langle g, f\rangle_H, \qquad w := g - \lambda_1 f, \qquad \lambda_2 := \|w\|_H, \qquad h := \frac{1}{\lambda_2} w.$$

Define S as the orthogonal projection onto h

$$Sv := \langle v, h\rangle_H h \qquad (v \in H).$$

Then, $Sf = 0$ and $Sg = \lambda_2 h$, and hence, $\langle Sf, f\rangle_H = 0$ and $\langle Sg, g\rangle_H = \lambda_2^2 > 0$. □

Theorem 1 *Let $H = \mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$ with $n \geq 2$. Then, the set of the Toeplitz operators with bounded symbols is not weakly dense in $\mathcal{B}(H)$.*

Proof Let $f \in H$, such that $\overline{f} \in H$ and the functions $f, \overline{f}$ are linearly independent. For example, $f(z) := z$. The set

$$W := \{S \in \mathcal{B}(H) : \ \langle Sf, f\rangle_H = \langle S\overline{f}, \overline{f}\rangle_H\}$$

is a weakly closed subspace of $\mathcal{B}(H)$. By Lemma 2, $W \neq \mathcal{B}(H)$. On the other hand, for every a in $L^\infty(\mathbb{D})$

$$\langle T_{n,a}^{(\alpha)} f, f\rangle_H = \int_X a\, |f|^2 \, \mathrm{d}\mu_\alpha = \langle T_{n,a}^{(\alpha)} \overline{f}, \overline{f}\rangle_H,$$

i.e., $\{T_{n,a}^{(\alpha)} : \ a \in L^\infty(\mathbb{D})\} \subseteq W$. □

Remark 2 An analog of Theorem 1 is true for the space of μ_α-square-integrable n-harmonic functions on $\mathbb{D}$, with $n \geq 1$.

7 Von Neumann algebras of radial operators

7.1 Set of operators diagonalized by a family of subspaces

The theory of von Neumann algebras and their decompositions is well developed. For our purposes, it is sufficient to use the following elementary scheme from [26]. This scheme is similar to ideas from [12, 28, 46].

Definition 1 Let H be a Hilbert space, $\mathcal{U}$ be a self-adjoint subset of $\mathcal{B}(H)$, and $(W_j)_{j\in J}$ be a finite or countable family of nonzero closed subspaces of H, such that

$H = \bigoplus_{j\in J} W_j$. We say that this family *diagonalizes* $\mathcal{U}$ if the following two conditions are satisfied.

1. For each j in J and each U in $\mathcal{U}$, there exists $\lambda_{U,j}$ in $\mathbb{C}$, such that $W_j \subseteq \ker(\lambda_{U,j} I - U)$, i.e., $U(v) = \lambda_{U,j} v$ for every v in W_j.
2. For every j, k in J with $j \neq k$, there exists U in $\mathcal{U}$, such that $\lambda_{U,j} \neq \lambda_{U,k}$.

Proposition 13 *Let H, $\mathcal{U}$, and $(W_j)_{j\in J}$ be as in Definition 1. Denote by $\mathcal{A}$ the commutant of $\mathcal{U}$. Then, $\mathcal{A}$ consists of all bounded linear operators that act invariantly on each of the subspaces W_j, with $j \in J$*

$$\mathcal{A} = \{S \in \mathcal{B}(H) : \quad \forall j \in J \quad S(W_j) \subseteq W_j\}. \tag{53}$$

Furthermore, $\mathcal{A}$ is isometrically isomorphic to $\bigoplus_{j\in J} \mathcal{B}(W_j)$, and the von Neumann algebra generated by $\mathcal{U}$ is isometrically isomorphic to $\bigoplus_{j\in J} \mathbb{C} I_{W_j}$.

Example 1 Let $j_1, \ldots, j_m \in J$, $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$, and $u_{j_k}, v_{j_k} \in W_{j_k}$ for every k in $\{1, \ldots, m\}$. Then, the operator $S : H \to H$ defined by

$$Sf := \sum_{k=1}^{m} \lambda_k \langle f, u_{j_k} \rangle v_{j_k}, \tag{54}$$

belongs to $\mathcal{A}$. Moreover, every operator of finite rank, belonging to $\mathcal{A}$, can be written in this form. See the proof of [26, Corollary 5.7] for a similar situation.

Proposition 14 *Let H, $\mathcal{U}$, and $(W_j)_{j\in J}$ be as in Definition 1, and H_1 be a closed subspace of H invariant under $\mathcal{U}$. For every U in $\mathcal{U}$, denote by $U|_{H_1}^{H_1}$ the compression of U onto the invariant subspace H_1, and put*

$$\mathcal{U}_1 := \left\{ U|_{H_1}^{H_1} : \ U \in \mathcal{U} \right\}, \qquad J_1 := \{j \in J : \ W_j \cap H_1 \neq \{0\}\}.$$

Then

$$H_1 = \bigoplus_{j\in J_1} (W_j \cap H_1), \tag{55}$$

and the family $(W_j \cap H_1)_{j\in J}$ diagonalizes $\mathcal{U}_1$.

Example 2 The operators of finite rank, commuting with $U|_{H_1}^{H_1}$ for every U in $\mathcal{U}$, are of the form (54), but with $u_{j_k}, v_{j_k} \in W_{j_k} \cap H_1$.

7.2 Radial operators in $L^2(\mathbb{D}, \mu_\alpha)$

For each τ in $\mathbb{T}$, we denote by $\rho^{(\alpha)}(\tau)$ the rotation operator acting in $L^2(\mathbb{D}, \mu_\alpha)$ by the rule

$$(\rho^{(\alpha)}(\tau) f)(z) := f(\tau^{-1} z). \tag{56}$$

It is easy to see that $\rho^{(\alpha)}(\tau_1 \tau_2) = \rho^{(\alpha)}(\tau_1)\rho^{(\alpha)}(\tau_2)$, the operators $\rho^{(\alpha)}(\tau)$ are unitary, and for every f in $L^2(\mathbb{D},\mu_\alpha)$, the mapping $\tau\mapsto\rho^{(\alpha)}(\tau)f$ is continuous (this is easy to check first for the case when f is a continuous function with compact support). Therefore, $(\rho^{(\alpha)}, L^2(\mathbb{D},\mu_\alpha))$ is a unitary representation of the group $\mathbb{T}$. The operators commuting with $\rho^{(\alpha)}(\tau)$ for every τ in $\mathbb{T}$ are called *radial operators*. We denote the set of all radial operators in $L^2(\mathbb{D},\mu_\alpha)$ by: $\mathcal{R}^{(\alpha)}$:

$$\mathcal{R}^{(\alpha)} := \{S\in\mathcal{B}(L^2(\mathbb{D},\mu_\alpha)) : \quad \forall\tau\in\mathbb{T}\quad \rho^{(\alpha)}(\tau)S = S\rho^{(\alpha)}(\tau)\}.$$

Since $\{\rho^{(\alpha)}(\tau) : \tau\in\mathbb{T}\}$ is a self-adjoint subset of $\mathcal{B}(L^2(\mathbb{D},\mu_\alpha))$, its commutant $\mathcal{R}^{(\alpha)}$ is a von Neumann algebra [34].

Recall that the subspaces $\mathcal{W}_\xi^{(\alpha)}$ are defined by (37).

Lemma 3 *The family* $(\mathcal{W}_\xi^{(\alpha)})_{\xi\in\mathbb{Z}}$ *diagonalizes the collection* $\{\rho^{(\alpha)}(\tau) :\ \tau\in\mathbb{T}\}$ *in the sense of Definition 1.*

Proof 1. Let $\tau\in\mathbb{T}$. For every $p,q\in\mathbb{Z}$ with $p-q=\xi$, formula (29) implies

$$\rho^{(\alpha)}(\tau)b_{p,q}^{(\alpha)} = \tau^{q-p}b_{p,q}^{(\alpha)} = \tau^{-\xi}b_{p,q}^{(\alpha)}, \tag{57}$$

i.e., $b_{p,q}^{(\alpha)}\in\ker(\tau^{-\xi}I-\rho^{(\alpha)}(\tau))$. By Corollary 2, the functions $b_{p,q}^{(\alpha)}$ with $p-q=\xi$ form an orthonormal basis of $\mathcal{W}_\xi^{(\alpha)}$. Therefore

$$\mathcal{W}_\xi^{(\alpha)} \subseteq \ker(\tau^{-\xi}I-\rho^{(\alpha)}(\tau)). \tag{58}$$

2. Let $\xi_1,\xi_2\in\mathbb{Z}$ and $\xi_1\neq\xi_2$. Put $\tau=\exp\frac{\mathrm{i}\pi}{\xi_1-\xi_2}$. Then $\tau^{-\xi_1}\neq\tau^{-\xi_2}$. □

Proposition 15 *The von Neumann algebra* $\mathcal{R}^{(\alpha)}$ *consists of all operators that act invariantly on* $\mathcal{W}_\xi^{(\alpha)}$ *for every* ξ *in* $\mathbb{Z}$, *and is isometrically isomorphic to* $\bigoplus_{\xi\in\mathbb{Z}}\mathcal{B}(\mathcal{W}_\xi^{(\alpha)})$.

Proof Follows from Proposition 13 and Lemma 3. □

The *radialization transform* $\mathrm{Rad}^{(\alpha)} : \mathcal{B}(L^2(\mathbb{D},\mu_\alpha))\to\mathcal{B}(L^2(\mathbb{D},\mu_\alpha))$, introduced by Zorboska [46], acts by the rule:

$$\mathrm{Rad}^{(\alpha)}(S) := \int_{\mathbb{T}}\rho(\tau)S\rho(\tau^{-1})\,\mathrm{d}\mu_{\mathbb{T}}(\tau),$$

where $\mu_{\mathbb{T}}$ is the normalized Haar measure on $\mathbb{T}$, and the integral is understood in the weak sense. The condition $S\in\mathcal{R}^{(\alpha)}$ is equivalent to $\mathrm{Rad}^{(\alpha)}(S)=S$.

7.3 Radial operators in $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$

Proposition 16 *The space* $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ *is invariant under* $\rho^{(\alpha)}(\tau)$ *for every* τ *in* $\mathbb{T}$.

First proof The reproducing kernel of $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$, given by (46), is invariant under simultaneous rotations in both arguments

$$K_{n,\tau z}^{(\alpha)}(\tau w) = K_{n,z}^{(\alpha)}(w) \qquad (z,w\in\mathbb{D},\ \tau\in\mathbb{T}). \tag{59}$$

By [26, Proposition 4], this implies the invariance of the subspace. □

Second proof By (56), the elements of the basis $(b_{p,q}^{(\alpha)})_{p\in\mathbb{N}_0,0\le q<n}$ are eigenfunctions of $\rho^{(\alpha)}$. □

For every τ in $\mathbb{T}$, we denote by $\rho_n^{(\alpha)}(\tau)$ the compression of $\rho^{(\alpha)}(\tau)$ onto the space $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$. In other words, the operator $\rho_n^{(\alpha)}(\tau)$ acts in $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$ and is defined by (56). Therefore, $(\rho_n^{(\alpha)},\mathcal{A}_n^2(\mathbb{D},\mu_\alpha))$ is a unitary representation of $\mathbb{T}$. We denote by $\mathcal{R}_n^{(\alpha)}$ the commutant of this representation, i.e., the von Neumann algebra of all bounded linear radial operators acting in $\mathcal{A}_n^2(\mathbb{D},\mu_\alpha)$.

Denote by $\mathfrak{M}_n$ the following direct sum of matrix algebras:

$$\mathfrak{M}_n := \bigoplus_{\xi=-n+1}^{\infty}\mathcal{M}_{\min\{n,n+\xi\}} = \left(\bigoplus_{\xi=-n+1}^{-1}\mathcal{M}_{n+\xi}\right)\oplus\left(\bigoplus_{\xi=0}^{\infty}\mathcal{M}_n\right).$$

For example

$$\mathfrak{M}_3 = \underbrace{\mathcal{M}_1}_{\xi=-2}\oplus\underbrace{\mathcal{M}_2}_{\xi=-1}\oplus\underbrace{\mathcal{M}_3}_{\xi=0}\oplus\underbrace{\mathcal{M}_3}_{\xi=1}\oplus\underbrace{\mathcal{M}_3}_{\xi=2}\oplus\ldots$$

According to the definition of the direct sum (see [34, Definition 1.1.5]), $\mathfrak{M}_n$ consists of all matrix sequences of the form $A=(A_\xi)_{\xi=-n+1}^{\infty}$, where $A_\xi\in\mathcal{M}_{n+\xi}$ if $\xi<0$, $A_\xi\in\mathcal{M}_n$ if $\xi\ge 0$, and

$$\sup_{\xi\ge -n+1}\|A_\xi\|<+\infty.$$

Being a direct sum of W*-algebras, $\mathfrak{M}_n$ is a W*-algebra. We identify the elements of $\mathfrak{M}_n$ with bounded linear operators acting in $\bigoplus_{\xi=-n+1}^{\infty}\mathbb{C}^{\min\{n,n+\xi\}}$. Now, we are ready to describe the structure of $\mathcal{R}_n^{(\alpha)}$. Recall that $U_n^{(\alpha)}$ is given by (52).

Theorem 2 *Let $n\in\mathbb{N}$. Then, $\mathcal{R}_n^{(\alpha)}$ consists of all operators belonging to $\mathcal{B}(\mathcal{A}_n^2(\mathbb{D},\mu_\alpha))$ that act invariantly on each subspace $\mathcal{W}_{\xi,\min\{n,n+\xi\}}^{(\alpha)}$, for $\xi\ge -n+1$. Furthermore*

$$\mathcal{R}_n^{(\alpha)} \cong \bigoplus_{\xi=-n+1}^{\infty}\mathcal{B}(\mathcal{W}_{\xi,\min\{n,n+\xi\}}^{(\alpha)}), \tag{60}$$

and $\mathcal{R}_n^{(\alpha)}$ is spatially isomorphic to $\mathfrak{M}_n$

$$U_n^{(\alpha)}\mathcal{R}_n^{(\alpha)}(U_n^{(\alpha)})^* = \mathfrak{M}_n. \tag{61}$$

Proof We apply the scheme from Propositions 13, 14, with

$$W_j = \mathcal{W}_\xi^{(\alpha)}, \qquad \mathcal{U} = \{\rho^{(\alpha)}(\tau):\ \tau \in \mathbb{T}\},$$

and $H_1 = \mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$. By (50), we obtain

$$J_1 = \{\xi \in \mathbb{Z}:\ \xi \geq -n+1\}, \qquad \mathcal{A}_n^2(\mathbb{D}, \mu_\alpha) \cap \mathcal{W}_\xi^{(\alpha)} = \mathcal{W}_{\xi,\min\{n,n+\xi\}}^{(\alpha)}.$$

Thereby, we obtain (60). Using the orthonormal basis $(b_{\xi+k,k}^{(\alpha)})_{k=\max\{0,-\xi\}}^{n-1}$ of $\mathcal{W}_{\xi,\min\{n,n+\xi\}}^{(\alpha)}$, we represent linear operators on this space as matrices. Define $\Phi_n^{(\alpha)} : \mathcal{R}_n^{(\alpha)} \to \mathfrak{M}_n$ by

$$\Phi_n^{(\alpha)}(S) := \left(\left[\left\langle S b_{\xi+k,k}^{(\alpha)}, b_{\xi+j,j}^{(\alpha)} \right\rangle \right]_{j,k=\max\{0,-\xi\}}^{n-1} \right)_{\xi=-n+1}^{\infty}. \tag{62}$$

In other words, $\Phi_n^{(\alpha)}(S) = U_n^{(\alpha)} S (U_n^{(\alpha)})^*$, i.e., $\Phi_n^{(\alpha)}$ is an isometrical isomorphism of W*-algebras induced by the unitary operator $U_n^{(\alpha)}$. □

Radial operators of finite rank, acting in $\mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$, can be constructed as in Examples 1 and 2.

It is easy to verify (see a more general result in [26, Corollary 4.3]) that if $\mathcal{A}_n^2 = \mathcal{A}_n^2(\mathbb{D}, \mu_\alpha)$ and $S \in \mathcal{R}_n^{(\alpha)}$, then $\mathrm{Ber}_{\mathcal{A}_n^2}(S)$ is a radial function. For $n = 1$, the Berezin transform $\mathrm{Ber}_{\mathcal{A}_1^2}$ is injective. Therefore, if $S \in \mathcal{B}(\mathcal{A}_1^2(\mathbb{D}, \mu_\alpha))$ and the function $\mathrm{Ber}_{\mathcal{A}_1^2}(S)$ is radial, then the operator S is radial.

7.4 Radial operators in $\mathcal{A}_{(n)}^2(\mathbb{D}, \mu_\alpha)$

Let $n \in \mathbb{N}$. The space $\mathcal{A}_{(n)}^2(\mathbb{D}, \mu_\alpha)$ is invariant under the rotation $\rho^{(\alpha)}(\tau)$ for all τ in $\mathbb{T}$. The proof is similar to the proof of Proposition 16. Denote the compression of $\rho^{(\alpha)}(\tau)$ onto $\mathcal{A}_{(n)}^2(\mathbb{D}, \mu_\alpha)$ by $\rho_{(n)}(\tau)$. Let $\mathcal{R}_{(n)}^{(\alpha)}$ be the von Neumann algebra of all radial operators in $\mathcal{A}_{(n)}^2(\mathbb{D}, \mu_\alpha)$.

Theorem 3 $\mathcal{R}_{(n)}^{(\alpha)}$ *consists of all operators belonging to* $\mathcal{B}(\mathcal{A}_{(n)}^2(\mathbb{D}, \mu_\alpha))$ *that are diagonal with respect to the orthonormal basis* $(b_{p,n-1}^{(\alpha)})_{p=0}^{\infty}$. *Furthermore*

$$\mathcal{R}_{(n)}^{(\alpha)} \cong \ell^\infty(\mathbb{N}_0).$$

Proof Corollaries 2 and 6 give

$$\mathcal{W}_{\xi}^{(\alpha)} \cap \mathcal{A}_{(n)}^{2}(\mathbb{D}, \mu_{\alpha}) = \begin{cases} \mathbb{C} b_{\xi+n-1,n-1}^{(\alpha)}, & \xi \geq -n+1, \\ \{0\}, & \xi < -n+1. \end{cases} \tag{63}$$

By Propositions 13, 14 and formula (63), $\mathcal{R}_{(n)}^{(\alpha)}$ consists of the operators that act invariantly on $\mathbb{C} b_{\xi+n-1,n-1}^{(\alpha)}$, $\xi \geq -n+1$, i.e., are diagonal with respect to the basis $(b_{p,n-1}^{(\alpha)})_{p=0}^{\infty}$. Therefore, the function $\Phi_{(n)}^{(\alpha)} : \mathcal{R}_{(n)}^{(\alpha)} \to \ell^{\infty}(\mathbb{N}_0)$, defined by

$$\Phi_{(n)}^{(\alpha)}(S) = \left(\langle S b_{p,n-1}^{(\alpha)}, b_{p,n-1}^{(\alpha)} \rangle\right)_{p=0}^{\infty}, \tag{64}$$

is an isometric isomorphism. □

8 Radial Toeplitz operators in polyanalytic Bergman spaces

This section is similar to [26, Section 6], but here we use Jacobi polynomials instead of the generalized Laguerre polynomials.

8.1 Radial functions

Given g in $L^{\infty}(\mathbb{D})$, define $\operatorname{rad}(g) : \mathbb{D} \to \mathbb{C}$ by

$$\operatorname{rad}(g)(z) := \int_{\mathbb{T}} g(\tau z) \, d\mu_{\mathbb{T}}(\tau). \tag{65}$$

Given a in $L^{\infty}([0,1))$, define $\widetilde{a} : \mathbb{D} \to \mathbb{C}$ by

$$\widetilde{a}(z) := a(|z|) \qquad (z \in \mathbb{D}).$$

The proof of the following criterion is a simple exercise.

Proposition 17 *Given g in $L^{\infty}(\mathbb{D})$, the following conditions are equivalent:*

(a) for every τ in $\mathbb{T}$, the equality $g(\tau z) = g(z)$ is true for a.e. z in $\mathbb{D}$;
(b) for every τ in $\mathbb{T}$, the equality $\rho^{(\alpha)}(\tau) g = g$ is true a.e.;
(c) $\operatorname{rad}(g) = g$ a.e.;
(d) there exists a in $L^{\infty}([0,1))$, such that $g = \widetilde{a}$ a.e.

8.2 Radial multiplication operators in $L^2(\mathbb{D}, \mu_{\alpha})$

Proposition 18 *Let $g \in L^{\infty}(\mathbb{D})$. Then,* $\operatorname{Rad}^{(\alpha)}(M_g) = M_{\operatorname{rad}(g)}^{(\alpha)}$.

Proof It is verified easily by Fubini theorem. □

Given a in $L^{\infty}([0,1))$, we define the numbers $\beta_{a,\alpha,\xi,j,k}$ by

$$\beta_{a,\alpha,\xi,j,k} := \int_0^1 a(\sqrt{t})\mathcal{J}^{(\alpha,|\xi|)}_{\min\{j,j+\xi\}}(t)\mathcal{J}^{(\alpha,|\xi|)}_{\min\{k,k+\xi\}}(t)\,\mathrm{d}t. \tag{66}$$

Proposition 19 *Let $a \in L^\infty([0,1))$. Then, $M_{\widetilde{a}} \in \mathcal{R}^{(\alpha)}$, and*

$$\langle M_{\widetilde{a}} b^{(\alpha)}_{p,q}, b^{(\alpha)}_{j,k}\rangle = \langle \widetilde{a} b^{(\alpha)}_{p,q}, b^{(\alpha)}_{j,k}\rangle = \delta_{p-q,j-k}\beta_{a,\alpha,p-q,q,k}. \tag{67}$$

Proof Since $\widetilde{a}$ is invariant under rotations, it follows directly from definitions that $M^{(\alpha)}_{\widetilde{a}}$ commutes with $\rho^{(\alpha)}(\tau)$ for every τ. This is a particular case of [26, Lemma 4.4]. Formula (67) is obtained directly using polar coordinates. □

8.3 Radial Toeplitz operators in $\mathcal{A}^2_n(\mathbb{D},\mu_\alpha)$

Proposition 20 *Let $g \in L^\infty(\mathbb{D})$. Then, $T^{(\alpha)}_{n,g}$ is radial if and only if the function g is radial.*

Proof Follows from Proposition 12 and [26, Corollaries 4.6, 4.7]. □

Given a in $L^\infty([0,1))$, we denote by $\gamma^{(\alpha)}_n(a)$ the sequence of matrices $[\gamma^{(\alpha)}_n(a)_\xi]_{\xi=-n+1}^\infty$, where $\gamma^{(\alpha)}_n(a)_\xi \in \mathcal{M}_{\min\{n+\xi,n\}}$ is given by

$$\gamma^{(\alpha)}_n(a)_\xi := \left[\beta_{a,\alpha,\xi,j,k}\right]_{j,k=\max\{0,-\xi\}}^{n-1}. \tag{68}$$

Recall that $\Phi^{(\alpha)}_n : \mathcal{R}^{(\alpha)}_n \to \mathfrak{M}_n$ is defined by (62).

Proposition 21 *Let $a \in L^\infty([0,1))$. Then, $T^{(\alpha)}_{n,\widetilde{a}} \in \mathcal{R}^{(\alpha)}_n$ and*

$$\Phi_n(T^{(\alpha)}_{n,\widetilde{a}}) = \gamma^{(\alpha)}_n(a).$$

Proof Apply Propositions 19 and 20. □

8.4 Radial Toeplitz operators in $\mathcal{A}^2_{(n)}(\mathbb{D},\mu_\alpha)$

Proposition 22 *Let $a \in L^\infty([0,1))$. Then, $T^{(\alpha)}_{(n),\widetilde{a}} \in \mathcal{R}^{(\alpha)}_{(n)}$, the operator $T^{(\alpha)}_{(n),\widetilde{a}}$ is diagonal with respect to the orthonormal basis $(b^{(\alpha)}_{p,n-1})_{p=0}^\infty$, and the corresponding eigenvalues can be computed by*

$$\lambda_{a,\alpha,n}(p) = \int_0^1 a(\sqrt{t})\left(\mathcal{J}^{(\alpha,|p-n+1|)}_{\min\{p,n-1\}}(t)\right)^2\mathrm{d}t \qquad (p \in \mathbb{N}_0). \tag{69}$$

Proof From Proposition 20, we get $T^{(\alpha)}_{(n),\widetilde{a}} \in \mathcal{R}^{(\alpha)}_{(n)}$. Due to Proposition 19 and Theorem 3

$$\lambda_{a,\alpha,n}(p) = (\Phi_{(n)}(T^{(\alpha)}_{(n),\widetilde{a}}))_p = \langle T^{(\alpha)}_{(n),\widetilde{a}} b^{(\alpha)}_{p,n-1}, b^{(\alpha)}_{p,n-1}\rangle = \beta_{a,p-n+1,n-1,n-1},$$

which is equivalent to (69). □

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