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ORIGINAL ARTICLE

Locally integrable functions and their indefinite integrals

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Abstract
Consider a locally determined positive measure space (Ω, Σ, μ) and a function $F : \Omega \rightarrow X$ taking values in a Banach space. When F is locally (Pettis or Bochner) integrable with respect to μ, a vector measure ν_F with density F defined on a δ-ring is obtained. We study the vector measure ν_F and its properties. We present the relation between the Banach spaces of integrable functions with respect to ν_F and the spaces of Dunford, Pettis or Bochner integrable functions.

Keywords Locally determined measure · Locally (Pettis or Bochner) integrability · Vector measure on δ-rings · L^1-spaces

Mathematics Subject Classification 46G10 · 28B05

1 Introduction

Let us consider a Banach space X and a σ-algebra Σ on Ω. The integral of X-valued functions with respect to a positive finite measure defined in Σ was introduced by B. J. Pettis and S. Bochner in the thirties of the last century and then this theory has been studied in depth by several authors for instance [5] and [7]. Later, the theory of integration of scalar valued functions with respect to X-valued measures defined in Σ, which are called vector measures, begins to be developed. As one might expect the Pettis and the Bochner integrals define vector measures, these ones were studied among other by J. Diestel and J. J. Uhl in [5]. Otherwise in [6] N. Dinculeanu and J. J. Uhl considered a locally σ-finite measure defined on a δ-ring $\mathcal{R}$ of subset of Ω and they introduced the concept of $\mathcal{R}$-locally Pettis or Bochner integrable function, namely a weakly (strongly) μ-measurable function $F : \Omega \rightarrow X$ such that the function

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$\chi_B F$ is Pettis (Bochner) integrable, for all $B \in \mathcal{R}$. It turns out that these integrals define vector measures now on the δ-ring $\mathcal{R}$. At the same time D. R. Lewis begins to develop the theory of integration with respect to vector measures defined on δ-rings in [8]. Subsequently, other authors study in depth these kind of measures and their inherent spaces of integrable functions; for the concepts and notations used in this note one should look [3].

Given any positive measure μ on Σ we can define the Dunford, Pettis and Bochner integrals of functions with values in X in analogous way to the case in which μ is finite. Also, we can obtain the spaces consisting of the Dunford and Pettis integrable functions which turn out to be normed spaces and the space of the Bochner integrable functions is a Banach space. On the other hand, the collection Σ^f consisting of those subsets in Σ which have μ finite measure is a δ-ring. When the measure μ is locally determined, it turns out that if μ is restricted to Σ^f, a locally σ-finite measure is obtained; so makes sense to consider vector functions which are Σ^f-locally Pettis or Bochner integrable, these functions will be simply called *locally Pettis and locally Bochner integrable* and the vector spaces obtained will be denoted by $\mathbb{P}(\mu, X)^{loc}$ and $\mathbb{B}(\mu, X)^{loc}$, respectively .

We begin this note recalling the basic concepts relative to the Dunford, the Pettis and the Bochner integrals with respect to a positive measure and also the main results about vector measures defined on δ-rings, in Sect. 2. In Sect. 3 we study briefly the vector measure ν_F defined on Σ^f by the integral of a locally Pettis integrable function $F : \Omega \to X$ over each $B \in \Sigma^f$ and a description of its corresponding semivariation is given. If additionally the function F is locally Bochner integrable we provide a characterization of the variation of the measure ν_F. Finally, in Sect. 4, we present the existing connection between the integrable functions with respect to ν_F and the Dunford, Pettis or Bochner integrable functions as well as their corresponding integrals. In this way Theorems 8 and 13 established by G.F. Stefansson in [11] are generalized in two directions, namely the positive measure is no longer necessarily finite but locally determined and the function F is now locally integrable. Besides we obtain some conditions to determine whether a locally Pettis integrable function is in fact Pettis integrable.

2 Preliminary results

2.1 Bochner and Pettis integrals

Throughout the paper Ω will be a non empty set and X stands for a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). We denote by X^* and B_X its dual space and its unit ball respectively. Let us consider a σ-algebra Σ on Ω and a positive measure $\mu : \Sigma \to [0, \infty]$. The set Σ^f consists of the subsets $B \in \Sigma$ such that $\mu(B) < \infty$ and $\mathcal{N}_0(\mu)$ is the collection of μ-null sets. Recall that the measure μ is said to be *semi-finite* if for each set $A \in \Sigma$ such that $\mu(A) > 0$, there exists a subset $B \in \Sigma^f$ satisfying that $B \subset A$ and $0 < \mu(B)$. The measure μ is *locally determined* if it is semi-finite and $\Sigma = \{A \subset \Omega \mid A \cap B \in \Sigma, \ \forall \ B \in \Sigma^f\}$.

We denote by $St(\mu,X)$ the vector space of X-valued simple functions whose support has finite measure. An X-valued function $F:\Omega\to X$ is said to be *strongly* μ *-measurable* if there exists a sequence $\{S_n\}\subset St(\mu,X)$, which converges pointwise to F μ-a.e. and to be *weakly* μ *-measurable* if $\langle F,x^*\rangle:\Omega\to\mathbb{R}$ is μ-measurable for any $x^*\in X^*$. Clearly each strongly μ-measurable function is a weakly μ-measurable function. We say that two functions $F,G:\Omega\to X$ are *weakly equal* μ-a.e. if $\langle F,x^*\rangle=\langle G,x^*\rangle$ μ-a.e. for all $x^*\in X^*$. We will denote by $L^0(\mu,X)$ the vector space that consists of the equivalence classes that are obtained by identifying strongly μ-measurable functions if they are equal μ- a.e. and $L^0_w(\mu,X)$ the vector spaces formed by the equivalence classes that are obtained when we identified weakly μ-measurable functions if they are weakly equal μ-a.e. We write $\mathbb{B}(\mu,X)$ to indicate the Banach space of the *Bochner integrable* functions, namely the functions $F\in L^0(\mu,X)$ such that $\|F\|_X\in L^1(\mu)$, with the norm defined by $\|F\|_1=\int_\Omega\|F\|_X d\mu$. On the other hand a function $F\in L^0_w(\mu,X)$ is *Dunford integrable* when $\langle F,x^*\rangle\in L^1(\mu)$, $\forall\ x^*\in X^*$, if additionally for each $A\in\Sigma$ there exists a vector $x_A\in X$ such that $\int_A\langle F,x^*\rangle d\mu=\langle x_A,x^*\rangle$; $\forall\ x^*\in X^*$, the function F is *Pettis integrable* and the vector x_A is called the *Pettis integral* of F over A and it is denoted by $\mathbb{P}-\int_A Fd\mu$. We write $\mathbb{D}(\mu,X)$ and $\mathbb{P}(\mu,X)$ for the vector spaces consisting of the Dunford and Pettis integrable functions respectively.

Lemma 1 *The space $\mathbb{D}(\mu,X)$ is a normed space with the norm given by*

$$\|F\|_{\mathbb{D}}=\sup_{x^*\in B_{X^*}}\int_\Omega|\langle F,x^*\rangle|d\mu,\ \forall\ F\in\mathbb{D}(\mu,X).$$

Proof Observe that to obtain the conclusion it only remains to establish that $\|F\|_{\mathbb{D}}<\infty$, $\forall\ F\in\mathbb{D}(\mu,X)$. So, let us fix $F\in\mathbb{D}(\mu,X)$ and define $T:X^*\to L^1(\mu)$ by $T(x^*)=\langle F,x^*\rangle,\ \forall\ x^*\in X^*$. Clearly T is a well defined linear operator. Now take $\{x_n^*\}\subset X^*$ and $x^*\in X^*$ such that $x_n^*\to x$. Let us assume that there exists $g\in L^1(\mu)$ satisfying that $Tx_n^*\to g$ in $L^1(\mu)$. Proceeding as in [4, p.46] we get a subsequence $\{x_{n_k}^*\}\subset\{x_n^*\}$ such that $\langle F,x_{n_k}^*\rangle=Tx_{n_k}^*\to g$ μ-a.e. On the other hand $\langle F(t),x_n^*\rangle\to\langle F(t),x^*\rangle$, $\forall\ t\in\Omega$. Thus, $T(x^*)=\langle F(t),x^*\rangle=g$, μ-a.e. By the Closed Graph Theorem we get that T is bounded. Therefore

$$\|F\|_{\mathbb{D}}=\sup_{x^*\in B_{X^*}}\int_\Omega|\langle F,x^*\rangle|d\mu=\sup_{x^*\in B_{X^*}}\|Tx^*\|_{L^1(\mu)}\le\|T\|.$$

□

Since $\mathbb{P}(\mu,X)\subset\mathbb{D}(\mu,X)$ it turns out that $\mathbb{P}(\mu,X)$ is also a normed space with the same norm $\|\cdot\|_{\mathbb{D}}$ which will be denoted by $\|\cdot\|_{\mathbb{P}}$ in this case. It is well know that $\mathbb{B}(\mu,X)\subset\mathbb{P}(\mu,X)$ with $\|F\|_{\mathbb{P}}\le\|F\|_1$ and $\mathbb{B}-\int_A Fd\mu=\mathbb{P}-\int_A Fd\mu$, $\forall\ F\in\mathbb{B}(\mu,X)$.

2.2 Integration with respect to measures defined on δ-rings

A family $\mathcal{R}$ of subsets of Ω is a *δ -ring* if $\mathcal{R}$ is a ring which is closed under countable intersections. From now on in this paper $\mathcal{R}$ will be a δ-ring. We denote by $\mathcal{R}^{loc}$ the σ-algebra of all sets $A \subset \Omega$ such that $A \cap B \in \mathcal{R}$, $\forall\ B \in \mathcal{R}$. Given $A \in \mathcal{R}^{loc}$ we indicate by $\mathcal{R}_A$ the δ-ring $\{B \subset A : B \in \mathcal{R}\}$ and by π_A the collection of finite families of pairwise disjoint sets in $\mathcal{R}_A$. Note that if $\Omega \in \mathcal{R}$, then $\mathcal{R}$ is a σ-algebra, and in this case we have that $\mathcal{R}^{loc} = \mathcal{R}$. Moreover, for each $B \in \mathcal{R}$ it turns out that $\mathcal{R}_B$ is a σ-algebra.

A *scalar measure* is a function $\lambda : \mathcal{R} \to \mathbb{K}$ satisfying that if $\{B_n\} \subset \mathcal{R}$, is a family of pairwise disjoint sets such that $\bigcup_{n=1}^{\infty} B_n \in \mathcal{R}$, then $\sum_{n=1}^{\infty} \lambda(B_n) = \lambda\big(\bigcup_{n=1}^{\infty} B_n\big)$. The *variation of* λ is the countably additive measure $|\lambda| : \mathcal{R}^{loc} \to [0,\infty]$ defined by $|\lambda|(A) := \sup\Big\{\sum_{j=1}^{n} |\lambda(A_j)| : \{A_j\} \in \pi_A\Big\}$. A function $f \in L^0(\mathcal{R}^{loc})$ is *λ -integrable* if $f \in L^1(|\lambda|)$. We denote by $L^1(\lambda)$ the vector space consisting of the equivalence classes of λ-integrable functions when we identify two functions if they are equal $|\lambda|$-a.e.

Let X be a Banach space. A set function $\nu : \mathcal{R} \to X$ is a *vector measure* if for any collection $\{B_n\} \subset \mathcal{R}$ of pairwise disjoint sets satisfying that $\bigcup_{n=1}^{\infty} B_n \in \mathcal{R}$, we have that $\sum_{n=1}^{\infty} \nu(B_n) = \nu(\bigcup_{n=1}^{\infty} B_n)$. A vector measure ν is called *strongly additive* if $\nu(B_n) \to 0$ whenever $\{B_n\}$ is a disjoint sequence in $\mathcal{R}$. The *variation of* ν is the positive measure $|\nu|$ defined in $\mathcal{R}^{loc}$ by $|\nu|(A) := \sup\Big\{\sum_j \|\nu(A_j)\|_X : \{A_j\} \in \pi_A\Big\}$. The *semivariation of* ν is the function $\|\nu\| : \mathcal{R}^{loc} \to [0,\infty]$ given by $\|\nu\|(A) := \sup\{|\langle \nu, x^*\rangle|(A) : x^* \in B_{X^*}\}$, where $|\langle \nu, x^*\rangle|$ is the variation of the scalar measure $\langle \nu, x^*\rangle : \mathcal{R} \to \mathbb{K}$, defined by $\langle \nu, x^*\rangle(B) = \langle \nu(B), x^*\rangle$, $\forall\ B \in \mathcal{R}$. The semivariation of ν is finite in $\mathcal{R}$ and $\|\nu(B)\| \le \|\nu\|(B)$ if $B \in \mathcal{R}$, moreover for any $A \in \mathcal{R}^{loc}$ is satisfied $\|\nu\|(A) \le |\nu|(A)$. A set $A \in \mathcal{R}^{loc}$ is said to be ν-null if $\|\nu\|(A) = 0$. We will denote by $\mathcal{N}_0(\nu)$ the collection of ν-null sets. It turns out that $\mathcal{N}_0(\nu) = \mathcal{N}_0(|\nu|)$. Moreover $A \in \mathcal{N}_0(\nu)$ if and only if $\nu(B) = 0$, $\forall\ B \in \mathcal{R}_A$. We say that two functions $f, g \in L^0(\mathcal{R}^{loc})$ are equal ν-a.e. if they are equal outside of a set in $\mathcal{N}_0(\nu)$.

We define $L^0(\nu)$ as the space of equivalence classes of functions in $L^0(\mathcal{R}^{loc})$, where two functions are identified when they are equal ν-a.e.

A function $f \in L^0(\mathcal{R}^{loc})$ is *weakly ν -integrable*, if $f \in L^1(\langle \nu, x^*\rangle)$, for each $x^* \in X^*$. We will denote by $L^1_w(\nu)$ the subspace of $L^0(\nu)$ of all weakly ν-integrable functions. With the norm given by

$$\|f\|_\nu := \sup\left\{ \int_\Omega |f| d|\langle \nu, x^*\rangle| : x^* \in B_{X^*} \right\},$$

$L^1_w(\nu)$ is a Banach space.

A function $f \in L^1_w(\nu)$ is *ν -integrable*, if for each $A \in \mathcal{R}^{loc}$ there exists a vector $x_A \in X$, such that

$$\langle x_A, x^* \rangle = \int_A f d\langle \nu, x^* \rangle, \ \forall\, x^* \in X^*. \tag{1}$$

In this case the vector x_A is denoted by $\int_A f d\nu$. With the norm $\|\cdot\|_\nu$ the subset of all ν-integrable functions is a closed subspace of $L^1_w(\nu)$ and it will be denoted by $L^1(\nu)$. Therefore $L^1(\nu)$ is also a Banach space. We indicate by $S(\mathcal{R})$ the collection of simple functions in $L^0(\mathcal{R}^{loc})$ which have support in $\mathcal{R}$. It turns out that $S(\mathcal{R})$ is a dense subspace of $L^1(\nu)$. Finally the integral operator $I_\nu : L^1(\nu) \to X$ defined by $I_\nu(f) = \int_\Omega f d\nu$, is linear and bounded.

3 Vector valued indefinite integral

Recall that (Ω, Σ, μ) is a positive measure space and X a Banach space. Given a vector valued function $F \in \mathbb{P}(\mu, X)$ let us define the set function $\tilde{\nu}_F : \Sigma \to X$ by

$$\tilde{\nu}_F(A) = \mathbb{P} - \int_A F d\mu, \quad \forall\, A \in \Sigma. \tag{2}$$

In the case when μ is finite it is well known that $\tilde{\nu}_F$ is a vector measure [5, Thm. II. 3.5]. The next result generalizes this fact, it can be established in the similar way, using the Orlicz-Pettis Theorem [5, Cor. I.4.4].

Theorem 1 *The set function $\tilde{\nu}_F$ defined on (2) is a vector measure with semivariation*

$$\|\tilde{\nu}_F\|(A) = \sup_{x^* \in B_{X^*}} \int_A |\langle F, x^* \rangle| d\mu, \ \forall\, A \in \Sigma. \tag{3}$$

Proof Let us fix $x^* \in X^*$ and take a pairwise disjoint countable collection $\{A_n\} \in \Sigma$, then

$$\begin{aligned} \left\langle \mathbb{P} - \int_{\bigcup_n A_n} F d\mu, x^* \right\rangle &= \int_{\bigcup_n A_n} \langle F, x^* \rangle d\mu \\ &= \sum_{n=1}^{\infty} \int_{A_n} \langle F, x^* \rangle d\mu = \sum_{n=1}^{\infty} \left\langle \mathbb{P} - \int_{A_n} F d\mu, x^* \right\rangle. \end{aligned}$$

So, $\tilde{\nu}_F$ is weakly σ-additive, by the Orlicz-Pettis Theorem [5, Cor. I.4.4] $\tilde{\nu}_F$ is a vector measure. On the other hand since $\langle \tilde{\nu}_F, x^* \rangle(A) = \int_A \langle F, x^* \rangle$, $\forall\, A \in \Sigma$, μ is a positive measure and $\langle F, x^* \rangle \in L^1(\mu)$ from [10, Thm. 6.13] we get that

$$|\langle \tilde{\nu}_F, x^* \rangle|(A) = \int_A |\langle F, x^* \rangle| d\mu.$$

It is follows (3). □

Since $\mathbb{B}(\mu, X) \subset \mathbb{P}(\mu, X)$ we have the following consequence. In order to get (4) we can proceed as in [5, Thm. II.2.4 iv)].

Corollary 1 *Let $F \in \mathbb{B}(\mu, X)$. Then $\tilde{\nu}_F$ defined on (2) is a vector measure with bounded variation such that*

$$|\tilde{\nu}_F|(A) = \int_A \|F\|_X d\mu, \ \forall \ A \in \Sigma. \tag{4}$$

Hereafter we will consider a locally determined positive measure μ on Σ. Then Σ^f is a δ-ring such that $(\Sigma^f)^{loc} = \Sigma$. Let us denote the restriction of μ to Σ^f by λ. Since μ is a semi-finite and locally determined positive measure, it turns out that λ is a scalar measure such that $|\lambda| = \mu$ ([1, Lemma 4.3]).

Now we will study a kind of vector functions that include vector measures having Pettis or Bochner functions as density functions.

Definition 1 Let $F : \Omega \to X$ be a vector function.

a) The function F is *locally Pettis integrable* if F is weakly μ-measurable function and $\chi_B F \in \mathbb{P}(\mu, X)$, $\forall \ B \in \Sigma^f$. The collection of equivalence classes obtained by identifying locally Pettis integrable functions if they are weakly equal μ-a.e. will be denoted by $\mathbb{P}(\mu, X)^{loc}$.
b) Analogously F is *locally Bochner integrable* if F is strongly μ-measurable function and $\chi_B F \in \mathbb{B}(\mu, X)$, $\forall \ B \in \Sigma^f$. The collection of equivalence classes obtained by identifying locally Bochner integrable functions if they are equal μ-a.e. will be denoted by $\mathbb{B}(\mu, X)^{loc}$.

Remark 1 Observe that $\mathbb{P}(\mu, X)^{loc}$ and $\mathbb{B}(\mu, X)^{loc}$ are vector spaces. Additionally we have that

$$\mathbb{B}(\mu, X) \subset \mathbb{B}(\mu, X)^{loc} \subset \mathbb{P}(\mu, X)^{loc}.$$

The following examples show that the containment $\mathbb{B}(\mu, X) \subset \mathbb{B}(\mu, X)^{loc}$ and $\mathbb{P}(\mu, X) \subset \mathbb{P}(\mu, X)^{loc}$ can be proper.

Example 1 Let us fix $x \in X$ and assume that $f : \Omega \to \mathbb{R}$ is such that $\chi_B f \in L^1(\mu)$, $\forall \ B \in \Sigma^f$ (c.f. [9, Def. 2.14 c)]). Now define $F : \Omega \to X$ by

$$F(t) := f(t)x. \tag{5}$$

Let us see that $\chi_B F \in \mathbb{B}(\mu, X)$, $\forall \ B \in \Sigma^f$. Since $f\chi_B \in L^0(\Sigma)$, for all $B \in \Sigma^f$, it turns out that $f \in L^0(\Sigma)$. Take $\{s_n\} \subset S(\Sigma)$ such that $s_n \to f$ and $|s_n| \le |f|$, $\forall \ n \in \mathbb{N}$. Fix $B \in \Sigma^f$. For each $n \in \mathbb{N}$, define $S_n : \Omega \to X$ by $S_n(t) = \chi_B s_n(t)x$. Since $\chi_B f \in L^1(\mu)$, we have that $\chi_B s_n \in L^1(\mu)$. And so $S_n \in St(\mu, X)$ and $S_n(t) \to \chi_B f(t)x = \chi_B F(t)$, $\forall$

$t \in \Omega$ indicating that $\chi_B F$ is strongly μ-measurable.

Further

$$\int_B \|F\|_X d\mu = \int_B |f| \|x\|_X d\mu = \left(\int_B |f| d\mu \right) \|x\|_X.$$

Since $\chi_B f \in L^1(\mu)$, it follows that $\int_B \|F\|_X d\mu < \infty$, and so $\chi_B F \in \mathbb{B}(\mu, X)$. Thus $F \in \mathbb{B}(\mu, X)^{loc}$. Finally observe that $F \in \mathbb{B}(\mu, X)$ if and only if $f \in L^1(\mu)$.

Example 2 Let us consider $(\mathbb{N}, 2^N, \mu_0)$, where μ_0 is the counting measure and $X = c_0$. Clearly $\mathbb{B}(\mu_0, c_0)^{loc} = L^0(\mu_0, c_0)$, so $\mathbb{P}(\mu_0, c_0)^{loc} = \mathbb{B}(\mu_0, c_0)^{loc}$. Now let us consider the functions $F, G : \mathbb{N} \to c_0$ defined by $F(n) := \sum_{k=1}^n 2^k e_k$ and $G(n) := \frac{1}{n} e_n$. Then $F, G \in L^0(\mu_0, c_0)$. However, if we take $x^* = \{\frac{1}{2^n}\} \in \ell^1 = c_0^*$ we have that $\langle F, x^* \rangle : \mathbb{N} \to \mathbb{R}$ is the sequence $\{n\}$ which is not integrable with respect to μ_0, so F is not Pettis integrable. On the other hand, if $x^* = \{a_n\} \in \ell^1$ we have that $\langle G, x^* \rangle = \{\frac{a_n}{n}\}$, thus $G \in \mathbb{P}(\mu_0, c_0)$ but $\|G\|_\infty = \{\frac{1}{n}\}$, thereby G is not Bochner integrable. Hence

$$\mathbb{B}(\mu_0, c_0) \subsetneq \mathbb{P}(\mu_0, c_0) \subsetneq \mathbb{P}(\mu_0, c_0)^{loc} = \mathbb{B}(\mu_0, c_0)^{loc}.$$

As a consequence of Theorem 1 we obtain the following result.

Proposition 1 *Let $F \in \mathbb{P}(\mu, X)^{loc}$. Then the set function $\nu_F : \Sigma^f \to X$ defined by*

$$\nu_F(B) := \mathbb{P} - \int_B F d\mu, \tag{6}$$

is a vector measure such that

$$\|\nu_F\|(A) = \sup_{x^* \in B_{X^*}} \int_A |\langle F, x^* \rangle| d\mu, \ \forall\ A \in \Sigma. \tag{7}$$

Proof Let us show that ν_F is a vector measure. Let $\{B_n\} \subset \Sigma^f$ be a disjoint collection such than $B := \bigcup_{n=1}^\infty B_n \in \Sigma^f$. Since $\chi_B F \in \mathbb{P}(\mu, X)$, by Theorem 1 we have

$$\begin{aligned} \nu_F(B) &= \mathbb{P} - \int_B F d\mu = \mathbb{P} - \int_B \chi_B F d\mu \\ &= \sum_{n=1}^\infty \mathbb{P} - \int_{B_n} \chi_B F d\mu = \sum_{n=1}^\infty \mathbb{P} - \int_{B_n} F d\mu = \nu_F(B_n). \end{aligned}$$

Thus ν_F is a vector measure. Now fix $x^* \in X^*$. Since $F \in \mathbb{P}(\mu, X)^{loc}$ we have $\langle F, x^* \rangle \in L^1_{loc}(\lambda)$. From [9, Thm. 2.31] we have that the variation of the scalar measure $\mu_{\langle F, x^* \rangle} : \Sigma^f \to \mathbb{K}$ defined by $\mu_{\langle F, x^* \rangle}(B) = \int_B \langle F, x^* \rangle d\mu$, $\forall\ B \in \Sigma^f$ is given by

$$|\mu_{\langle F,x^*\rangle}|(A) = \int_A |\langle F, x^*\rangle| d\mu, \ \forall\, A \in \Sigma.$$

On the other hand notice that for each $B \in \Sigma^f$, we have

$$\langle \nu_F, x^*\rangle(B) = \left\langle \mathbb{P} - \int_B F d\mu, x^* \right\rangle = \int_B \langle F, x^*\rangle d\mu = \mu_{\langle F,x^*\rangle}(B). \tag{8}$$

Therefore

$$|\langle \nu_F, x^*\rangle|(A) = \int_A |\langle F, x^*\rangle| d\mu. \tag{9}$$

From this we get (7). □

Observe that from (7) we have that if $A \in \mathcal{N}_0(\mu)$, then $\|\nu_F\|(B) = 0$, $\forall\, B \in \Sigma^f_A$. So $\mathcal{N}_0(\mu) \subset \mathcal{N}_0(\nu_F)$. Also observe that $\|\nu_F\|(B) = \|\chi_B F\|_{\mathbb{P}}$, $\forall\, B \in \Sigma^f$. Moreover, from the Dunford integrability definition we get our next result.

Corollary 2 *Let $F \in \mathbb{P}(\mu, X)^{loc}$. Then F is Dunford integrable if and only if ν_F has bounded semivariation. In particular if $F \in \mathbb{P}(\mu, X)$, then $\|\nu_F\|(A) = \|\chi_A F\|_{\mathbb{P}}$.*

In the case that F is locally Bochner integrable the variation of ν_F has the same characterization that when F is Bochner integrable, as we can see in what follow.

Proposition 2 *If $F \in \mathbb{B}(\mu, X)^{loc}$, then $|\nu_F|(A) = \int_A \|F\|_X d\mu$, $\forall\, A \in \Sigma$.*

Proof Take $B \in \Sigma^f$. Notice that $\nu_F(A \cap B) = \nu_{\chi_B F}(A)$, $\forall\; A \in \Sigma$. Since $\chi_B F \in \mathbb{B}(\mu, X)$ we have that

$$|\nu_F|(B) = |\nu_{\chi_B F}|(B) = \int_B \|F\|_X d\mu.$$

Hence

$$\begin{aligned} |\nu_F|(A) &= \sup_{B \in \Sigma^f_A} |\nu_F|(B) \\ &= \sup_{B \in \Sigma^f_A} \int_B \|F\|_X d\mu = \int_A \|F\|_X d\mu, \ \forall\, A \in \Sigma. \end{aligned} \tag{10}$$

□

Remark 2 Let us note that if $F : \Omega \to X$ is a strong μ-measurable function, by the previous result we obtain that:

$F \in \mathbb{B}(\mu, X)$ if and only if $F \in \mathbb{B}(\mu, X)^{loc}$ and ν_F has bounded variation.

Example 3 Let us return to the Example 1. It was shown there that F defined in (5) is locally Bochner integrable. In particular F is locally Pettis integrable. Let us obtain now the vector measure ν_F, its variation and its semivariation. See that for each $B \in \Sigma^f$

$$\nu_F(B) = \mathbb{B} - \int_B F d\mu = \left(\int_B f d\mu \right) x, \ \forall \ B \in \Sigma^f.$$

Now take $A \in \Sigma$, from (7) and (10)

$$\begin{aligned} \|\nu_F\|(A) &= \sup_{x^* \in B_{X^*}} \int_A |\langle F, x^* \rangle| d\mu = \sup_{x^* \in B_{X^*}} \int_A |f||\langle x, x^* \rangle| d\mu \\ &= \left(\int_A |f| d\mu \right) \sup_{x^* \in B_{X^*}} |\langle x, x^* \rangle| = \left(\int_A |f| d\mu \right) \|x\|_X \\ &= \int_A \|F\|_X d\mu = |\nu_F|(A). \end{aligned}$$

Therefore $\|\nu_F\| = |\nu_F|$ in this case.

4 The space of ν_F-integrable functions

When F is a locally Pettis or Bochner integrable function we have constructed the vector measure ν_F defined on the δ-ring Σ^f. In the present section we will study the spaces $L^0(\nu_F)$, $L^1_w(\nu_F)$, $L^1(\nu_F)$ and $L^1(|\nu_F)|)$ associated to this vector measure through the operator M_F which to each measurable function g assigns the function gF. The following lemmas allow us to conclude that $M_F : L^0(\nu_F) \to L^0_w(\mu, X)$ or $L^0(\mu, X)$ is well defined. Clearly M_F is a linear operator.

Lemma 2 *Let $F : \Omega \to X$ be a function and $g \in L^0(\Sigma)$.*

i) If F is strongly μ-measurable, then gF is strongly μ-measurable.
ii) If F is weakly μ-measurable, then gF is weakly μ-measurable.

Proof

(i) Observe that if $\varphi \in S(\Sigma)$ and $S \in St(\mu, X)$, then $\varphi S \in St(\mu, X)$. Let us assume that F is strongly μ-measurable. Take $\{\varphi_n\} \subset S(\Sigma)$ and $\{S_n\} \subset St(\mu, \Sigma)$ such that $\varphi_n \to g$ and $S_n \to F$, μ-a.e. Thus $\varphi_n S_n \in St(\mu, X)$, $\forall \ n \in \mathbb{N}$ and $\varphi_n S_n \to gF$, μ-a.e. It follows that gF is strongly μ-measurable.

(ii) By definition if F is weakly μ-measurable, we have that for each $x^* \in X^*$, the function $\langle F, x^* \rangle$ is strongly μ-measurable. Using (i) we obtain that $\langle gF, x^* \rangle = g\langle F, x^* \rangle \in L^0(\mu, X) \ \forall \ x^* \in X^*$.

Lemma 3 *Let $F \in \mathbb{P}(\mu, X)^{loc}$, $\{g_n\} \subset L^0(\Sigma)$ and $g, h \in L^0(\Sigma)$.*

(i) If $g = h$, ν_F-a.e., then $gF = hF$, weakly μ-a.e.
(ii) If $g_n \to g$, ν_F-a.e., then $\langle g_n F, x^* \rangle \to \langle gF, x^* \rangle$, μ-a.e., $\forall \ x^* \in X^*$.

Proof

(i) Choose $N \in \mathcal{N}_0(\nu_F)$ such that $g(t) = h(t)$, $\forall\ t \in N^c$. Then $g\chi_{N^c}F = h\chi_{N^c}F$; Moreover, $\chi_N F = 0$ weakly μ -a.e. implies that $g\chi_N F = h\chi_N F = 0$, weakly μ-a.e. Thus $gF = hF$, weakly μ-a.e.

(ii) Let $N \in \mathcal{N}_0(\nu_F)$ such that $g_n(t) \to g(t)$, $\forall\ t \in N^c$. So $g_n\chi_{N^c} \to g\chi_{N^c}F$ and $g_n\chi_N F = g\chi_N F = 0$, weakly μ-a.e. Then for each $x^* \in X^*$, $\langle g_n\chi_{N^c}F, x^*\rangle \to \langle g\chi_{N^c}F, x^*\rangle$ and $\langle g_n\chi_N F, x^*\rangle = \langle g\chi_N F, x^*\rangle = 0$, μ-a.e. Therefore $\langle g_n F, x^*\rangle \to \langle gF, x^*\rangle$, μ-a.e., $\forall\ x^* \in X^*$.

□

Proposition [11, Prop.8] established by G. F. Stefansson for the case that $F \in \mathbb{P}(\mu, X)$ and μ is a finite positive measure defined on a σ-algebra is generalized in the next theorem.

Theorem 2 *For $F \in \mathbb{P}(\mu, X)^{loc}$ and $g \in L^0(\Sigma)$, we have that*

(i) $g \in L^1_w(\nu_F)$ if, and only if, $gF \in \mathbb{D}(\mu, X)$. Moreover, the restriction to $L^1_w(\nu_F)$ of the operator M_F is a linear isometry from $L^1_w(\nu_F)$ into $\mathbb{D}(\mu, X)$.

(ii) $g \in L^1(\nu_F)$ if, and only if, $gF \in \mathbb{P}(\mu, X)$. Moreover, $M_F : L^1(\nu_F) \to \mathbb{P}(\mu, X)$, the restriction of the operator M_F, is a linear isometry such that $I_{\nu_F} = I_{\mathbb{P}} \circ M_F$.

Proof Fix $x^* \in X^*$ and consider $s = \sum_{j=1}^{n} a_j\chi_{A_j} \in S(\Sigma)$. By hypothesis $\chi_B F \in \mathbb{P}(\mu, X)$, $\forall\ B \in \Sigma^f$. It follows that $sF \in \mathbb{P}(\mu, X)$. From (9) we obtain

$$\begin{aligned} \int_\Omega |s| d|\langle \nu_F, x^*\rangle| &= \sum_{j=1}^{n} |a_j| |\langle \nu_F, x^*\rangle|(A_j) = \sum_{j=1}^{n} |a_j| \int_{A_j} |\langle F, x^*\rangle| d\mu \\ &= \int_\Omega \sum_{j=1}^{n} |a_j|\chi_{A_j} |\langle F, x^*\rangle| d\mu = \int_\Omega |\langle sF, x^*\rangle| d\mu. \end{aligned} \tag{11}$$

Thus $s \in L^1_w(\nu_F)$ if and only if $sF \in \mathbb{D}(\mu, X)$.

Proceeding in the same way, it follows from (8) that

$$\int_\Omega s d\langle \nu_F, x^*\rangle = \int_\Omega \langle sF, x^*\rangle d\mu. \tag{12}$$

Now take $g \in L^0(\nu_F)^+$ and $\{s_n\} \subset S(\mathcal{R}^{loc})$ such that $0 \le s_n \uparrow g$, ν_F-a.e. From Lemma 3 we obtain $\langle s_n F, x^*\rangle \to \langle gF, x^*\rangle$, μ-a.e. Then, $|\langle s_n F, x^*\rangle| \uparrow |\langle gF, x^*\rangle|$, μ-a.e. By the Monotone Convergence Theorem and (11) it turns out that

$$\begin{aligned}\int_\Omega |\langle gF, x^*\rangle| d\mu &= \lim_{n\to\infty} \int_\Omega |\langle s_n F, x^*\rangle| d\mu \\ &= \lim_{n\to\infty} \int_\Omega s_n d|\langle \nu_F, x^*\rangle| = \int_\Omega |g| d|\langle \nu_F, x^*\rangle|,\end{aligned} \tag{13}$$

showing that $g \in L^1_w(\nu_F)$ if and only if $gF \in \mathbb{D}(\mu, X)$.

By the Dominate Convergence Theorem and (8)

$$\int_\Omega \langle gF, x^*\rangle d\mu = \lim_{n\to\infty} \int_\Omega \langle s_n F, x^*\rangle d\mu = \lim_{n\to\infty} \int_\Omega s_n d\langle \nu_F, x^*\rangle = \int_\Omega g d\langle \nu_F, x^*\rangle.$$

We conclude from here that $g \in L^1(\nu_F)$ if and only if $gF \in \mathbb{P}(\mu, X)$ and

$$\int_\Omega g d\nu_F = \mathbb{P} - \int_\Omega gF d\mu. \tag{14}$$

Since the involved sets are vector spaces and each $g \in L^0(\nu_F)$ is a linear combination of non negative functions, we obtain the first part in (i) and (ii).

Finally take $g \in L^1_w(\nu_F)$, since $|g| \geq 0$ we obtain equality (13) with a sequence $\{s_n\} \subset S(\Sigma)$ such that $0 \leq s_n \uparrow |g|$, ν_F-a.e. Taking the supremum over $x^* \in B_{X^*}$ it turns out that $\|g\|_{\nu_F} = \|gF\|_{\mathbb{D}}$. That is, M_F restricted to $L^1_w(\nu_F)$ is a linear isometry. Since $L^1(\nu_F)$ and $\mathbb{P}(\mu, X)$ are subspaces of $L^1_w(\nu_F)$ and $\mathbb{D}(\mu, X)$, respectively, we conclude that M_F restricted to $L^1(\nu_F)$ is also an isometry. Moreover, from (14) it follows that $I_{\nu_F} = I_{\mathbb{P}} \circ M_F$. □

Corollary 3 *Let $F \in L^0_w(\mu, X)$. Then $F \in \mathbb{P}(\mu, X)$ if and only if $F \in \mathbb{P}(\mu, X)^{loc}$ and ν_F is strongly additive.*

Proof Let assume that $F \in \mathbb{P}(\mu, X)$. Consider the vector measure $\tilde{\nu}_F : \mathcal{R}^{loc} \to X$ defined in (2). Since Σ is a σ-algebra, it turns out that $\tilde{\nu}_F$ is strongly additive. Observe that ν_F is the restriction of $\tilde{\nu}_F$ to Σ^f, so it follows that it is strongly additive.

Now assume that $F \in \mathbb{P}(\mu, X)^{loc}$ and that ν_F is strongly additive. From [3, Cor. 3.2] we obtain that $\chi_\Omega \in L^1(\nu_F)$. So, by the previous theorem $F = \chi_\Omega F \in \mathbb{P}(\mu, X)$. □

Corollary 4 *Let $F \in \mathbb{P}(\mu, X)^{loc}$. If X does not contain any subspace isomorphic to c_0 and ν_F is bounded, then $F \in \mathbb{P}(\mu, X)$.*

Proof Since X does not contain any subspace isomorphic to c_0 and ν_F is bounded it turns out that ν_F is strongly additive [4, p. 36]. Then by the previous corollary $F \in \mathbb{P}(\mu, X)$. □

The following result gives us the connection between the spaces $L^1(|\nu_F|)$ and $\mathbb{B}(\mu, X)$ through the operator M_F in case that $F \in \mathbb{B}(\mu, X)^{loc}$. We will show that, as it occurs when $F \in \mathbb{P}(\mu, X)^{loc}$, M_F is a linear isometry in this case.

Proposition 3 *Consider $F \in \mathbb{B}(\mu, X)^{loc}$ and $g \in L^0(|\nu_F|)$. Then $g \in L^1(|\nu_F|)$ if and only if $gF \in \mathbb{B}(\mu, X)$. Moreover, $M_F : L^1(|\nu_F|) \to \mathbb{B}(\mu, X)$ is a linear isometry such that $I_{\nu_F}(g) = I_{\mathbb{B}} \circ M_F(g)$, $\forall\ g \in L^1(|\nu_F|)$.*

Proof Clearly M_F is a linear operator, we will see that its image is a subset of $\mathbb{B}(\mu, X)$. By Lemma 3 we have that the restriction $M_F : L^1(|\nu_F|) \to \mathbb{B}(\mu, X)$ is well defined.

Since the norms in $L^1(|\nu_F|)$ and $\mathbb{B}(\mu, X)$ are different from those in $L^1(\nu_F)$ and $\mathbb{P}(\mu, X)$, respectively, we need to establish that, under these norms, M_F is also an isometry.

By hypothesis $F \in \mathbb{B}(\mu, X)^{loc}$, then from (10) it follows that $|\nu_F|(B) < \infty$, $\forall\ B \in \Sigma^f$. So, $S(\Sigma^f) \subset L^1(|\nu_F|)$. Further for each $s = \sum_{j=1}^n a_j \chi_{A_j} \in S(\Sigma)$ we have that

$$\begin{aligned}\int_\Omega |s| d|\nu_F| &= \int_\Omega \sum_{j=1}^n |a_j| |\nu_F|(A_j) = \sum_{j=1}^n |a_j| \int_{A_j} \|F\|_X d\mu \\ &= \int_\Omega \sum_{j=1}^n |a_j| \chi_{A_j} \|F\|_X d\mu = \int_\Omega \|sF\|_X d\mu.\end{aligned}$$

Therefore $s \in L^1(|\nu_F|)$ if and only if $sF \in \mathbb{B}(\mu, X)$. Now consider $g \in L^0(|\nu_F|)$ and take $\{s_n\} \subset S(\Sigma)$ such that $0 \leq s_n \uparrow |g|$, ν_F-a.e. Then $\|s_n F\|_X \uparrow \|gF\|_X$, μ-a.e. By the Monotone Convergence Theorem

$$\int_\Omega \|gF\|_X d\mu = \lim_{n\to\infty} \int_\Omega \|s_n F\|_X d\mu = \lim_{n\to\infty} \int_\Omega |s_n| d|\nu_F| = \int_\Omega |g| d|\nu_F|.$$

Thus we have that $gF \in \mathbb{B}(\mu, X)$ if and only if $g \in L^1(|\nu_F|)$. Moreover, $\|g\|_{|\nu_F|} = \|gF\|_1$.

The equality between the operators follows from Proposition 2. □

Example 4 Consider again the function F defined in (5). As we see in Example 1 $F \in \mathbb{B}(\mu, X)^{loc}$.

Take $g \in L^1(\nu_F)$, from Proposition 2

$$\int_\Omega g d\nu_F = \mathbb{P} - \int_\Omega gF d\mu = \left(\int_\Omega gf d\mu\right) x,$$

then $gf \in L^1(\mu)$. And so,

$$\int_\Omega \|gF\|_X d\mu = \int_\Omega |gf| \|x\|_X d\mu < \infty. \tag{15}$$

By Lemma 2 gF is strongly μ-measurable. Thus we have that $gF \in \mathbb{B}(\mu, X)$ and by Proposition 3, $g \in L^1(|\nu_F|)$. We conclude that $L^1(|\nu_F|) = L^1(\nu_F)$. And from [2, Prop. 5.4] it follows that $L^1(|\nu_F|) = L^1(\nu_F) = L^1_w(\nu_F)$.

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Compliance with ethical standards

Conflict of interest The author declare that she has no conflict of interest.

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