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# Cospherical classes in some iterated loop spaces on spheres

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## Abstract

For a nice topological space $X$, working at the prime $p = 2$, we consider the 'unstable Boardman map' (homomorphism if $k > 0$)

$$b : \pi^{m+k}\Sigma^k X \simeq [X, \Omega^k S^{m+k}] \longrightarrow \mathrm{Hom}_{\mathbb{Z}/2(H^*\Omega^k S^{m+k}, H^*X)}$$

defined by $b(f) = f^*$ where $k \geq 0$ and $m \geq 0$. We use classic maps, such as the Kahn–Priddy map, to provide examples of $X$ so that $b$ is nonzero in many dimensions. We also consider the case of $X = \Omega^l S^{n+l}$, with particular interest in the cases with $0 \leq k < l \leq +\infty$, and consider the problem of computing the image of

$$b : \pi^m \Omega^l S^{n+l} \simeq [\Omega^l S^{n+l}, S^m] \longrightarrow \mathrm{Hom}_{\mathbb{Z}/2(H^*S^m, H^*\Omega^l S^{n+l})}.$$

Our results concern with the extreme values of $k$ given by $k = 0, l$. For $k = l$, a simple interpretation of well known facts about James-Hopf maps shows that the image of $b$ when $m = 2n$ is always nontrivial; we have not completely determined the image of $b$ in this case. For $k = 0$ we completely determine the image of $b$ in the following cases: (1) $m = n$ and $l > 0$ arbitrary; (2) $m > n$ and $l = 1$. We observe that in most of the cases the image is trivial with the exceptions corresponding to the cases when either there is a (commutative) $H$-space structure on $S^n$ or there is a Hopf invariant one element.

**Keywords** Hurewicz homomorphism · Boardman homomorphism · Loop spaces

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## 1 Introduction and statement of results

Throughout the paper, we work 2-locally and all notation is to be understood accordingly unless otherwise specified. In particular, $H^*$ (resp. $H_*$) denotes $H^*(-;\mathbb{Z}/2)$ (resp. $H^*(-;\mathbb{Z}/2)$). We shall write $P$ (resp. $\mathbb{C}P$) for the infinite dimensional real (resp. complex) projective space, and $P^n$ for its $n$-skeleton (resp. $2n$-skeleton). For a pointed space $X$, we write $QX = \mathrm{colim}\ \Omega^i\Sigma^i X$. For a map $f : X \to \Omega^n Y$ we shall write $\mathrm{ad}_i(f) : \Sigma^i X \to \Omega^{n-i} Y$ for its $i$-th adjoint where $i \leq n$.

Recall that a cospherical class in $H^*X$ is determined by a map $f : X \to S^m$ such that $f^* \neq 0$ (see for example [42]). A generalised cospherical class is obtained if we replace $S^m$ with a loop space on a sphere.

**Definition 1.1** A generalised cospherical class in $H_*X$ is determined by a map $f : X \to \Omega^k S^{m+k}$, $k \geq 0$, so that $f_* \neq 0$. A reduced generalised cospherical class is determined by a map $f : X \to \Omega^k S^{m+k}$ with $k > 0$ so that $f_* : H_i X \to H_i \Omega^k S^{m+k}$ is nontrivial for some $i > m$.

Note that over $\mathbb{Z}/2$, $H_*X$ and $H^*X$ are dual vector spaces and the dual of $f^*$ is $f_*$. The above definition then would be a reasonable definition. Note that one may replace $H^*(-;\mathbb{Z}/2)$ with any other homology theory $E$ and study cospherical classes in $E^*X$ (see [32] for an example with $E = KO$). Obviously $0 \in H_*X$ is cospherical. Hence, we shall focus on nonzero cospherical classes.

***Remark 1.2*** Recall that $x \in H_m X$ is called spherical if there exists $f : S^m \to X$ such that $f_* x_m = x$ where $x_m \in \widetilde{H}_m S^m$ is a generator. It should be noted that there exist spaces which contain classes which are not spherical, neither cospherical. We call these non-spherical classes. For example, note that $H_2 P$ contains no spherical classes as the generator $a_2 \in H_2 P$ is not $A$-annihilated. Moreover, if $f : P \to S^2$ is given with $f_*$ nonzero, then $f^* \neq 0$. The naturality of the cup-squaring operation shows that $x_2^2 \neq 0$ in $H^*S^2$ which is a contradiction. So, $a_2 \in H_2 P$ is a non-spherical class.

Often, we do not put much restrictions on $X$. However, in the case of applications to bordism theory as we will discuss in a sequel, it is useful to assume that $X$ is a finite dimensional CW-complex. In this case we have the following.

**Proposition 1.3** *Suppose $X$ is a finite dimensional CW-complex. Then a generalised cospherical class in $H^*X$ is determined by, and determines, an element $f \in \pi_s^m X$ viewed as a map $X \to QS^m$ with $f_* \neq 0$.*

***Proof*** In one direction, suppose $f : X \to \Omega^k S^{m+k}$ determines a cospherical class in $H^*X$. Recall that the stabilisation map $E : \Omega^k S^{m+k} \to QS^m = \mathrm{colim}\ \ \Omega^i S^{m+i}$ induces a monomorphism in homology [44, Proposition 3.1]. Therefore, $(E \circ f)_* \neq 0$ and the image of $E \circ f$ is a nontrivial element in $[X, QS^m] \simeq \pi_s^m X$. Conversely, suppose $g : X \to QS^m$ is given with $g_* \neq 0$. Since $X$ is finite dimensional then there exists $k$

and $f : X \to \Omega^k S^{m+k}$ so that $g = E \circ f$. In particular, $f_* \neq 0$. This completes the proof. □

Note that the problem of computing generalised cospherical classes in $H^*X$ also contributes to the study of the unstable Boardman homomorphism (see Sect. 2 for explanations on the terminology)

$$b : \pi^{m+k}\Sigma^k X \simeq [X, \Omega^k S^{m+k}] \longrightarrow \mathrm{Hom}_{\mathbb{Z}/2(H^*\Omega^k S^{m+k}, H^*X)}$$

defined by $h(f) = f^*$. Therefore, we may interpret our results in terms of image of $b$ as well. We record results in two opposite directions. In one direction, we provide example of cases where nonzero generalised cospherical classes do exist. On the opposite direction, we provide examples of spaces $X$ where $H_*X$ consists of no nonzero (generalised) cospherical classes. First, we wish to record the following existence result which would follow from classic computations of Kahn and Priddy [26] as well as well known facts on various transfer maps [33].

**Theorem 1.4** *Suppose $G = \mathbb{Z}/2, S^1, S^3$ and $BG^{[n]}$ is the n-skeleton of BG. There exists a map $\lambda_n^G : \Sigma^{\dim \mathfrak{g}} BG^{[n]} \to \Omega^{m+1}S^{m+1}$ which is nontrivial in homology. In particular, for any $n > 0$ there exists a map $P^n \to \Omega^{n+1}S^{n+1}$ which is nontrivial in homology in every dimension $0 < i \leq n$.*

There are possible way of extending the above list of groups. For instance, by Kahn–Priddy theorem [26] (see also [11, Lemma 2.3]), for a connected CW-complex $X$, a map $f : X \to Q_0S^0$ factors through the Kahn–Priddy map $\lambda : QP \to Q_0S^0$. If $f : X \to Q_0S^0$ satisfies $f_* \neq 0$ then any pull back $\widetilde{f} : X \to QP$ satisfies $\widetilde{f}_* \neq 0$. If $\widetilde{f}$ further pulls back to $P$, hence corresponding to a nontrivial element in $H^1(X; \mathbb{Z}/2)$, then we might expect $\lambda \circ \widetilde{f}$ to be nonzero in homology. As another example, one may consider the Segal type decomposition [36, Corollary] $QP = BO \times F$ and look for generators of KO($X$), represented by a map $X \to BO$, and see if they are nonzero in homology or not. For $X = BG$, these seem to provide ways of extending the above list further.

**Corollary 1.5** *For $i < n + 1$, there exist maps $\Sigma^i P^n \to \Omega^{(n+1)-i} S^{n+1}$ which are nontrivial in $\mathbb{Z}/2$-homology in dimensions less than $n + i$ and greater that $2i - 1$.*

Let's note that $\mathrm{ad}_{n+1}(\lambda_n) : \Sigma^{n+1}P^n \to S^{n+1}$ is trivial for obvious dimensional reasons as $\Sigma^{n+1}P$ has its bottom cell in dimension $n + 2$. So, we may ask whether for $i \geq 0$ and $n > 1$, there exists $m$ and $f : \Sigma^i P^n \to S^m$ which is nontrivial in homology. Note that once we find $g : P^n \to S^{m-i}$ with $g_* \neq 0$ then $f = \Sigma^i g$ would have the desired property. So, we may ask whether if there exists $f : P^n \to S^m$ so that $f_* \neq 0$? For dimensional reason, we need $1 \leq m \leq n$. Moreover, the projection onto the top cell $P^n \to S^n$ is nontrivial in homology. Hence, we may restrict to the cases $1 \leq m \leq n - 1$. We have the following.

**Lemma 1.6** *Suppose $f : P^n \to S^m$ is a map of spaces/suspension spectra and $1 \leq m < n \leq +\infty$.*

(i) *If $m$ is odd then $f_* = 0$.*
(ii) *For $n = m+1$ and $m$ even, there exists a map $f : P^{m+1} \to S^m$ with $f_* \neq 0$. Moreover, for any such map the restriction $f|_{P^m}$ is homotopic to the projection onto the top cell.*
(iii) *If $n \geq m+2$ and $m \equiv 2 \bmod 4$ then $f_* = 0$.*
(iv) *If $m \equiv 0 \bmod 4$ then for $i < 4$ there exists a map $f : P^{m+i} \to S^m$ with $f_* \neq 0$.*
(v) *If $m \equiv 0 \bmod 8$ then for $i < 7$ there exists a map $f : P^{m+i} \to S^m$ with $f_* \neq 0$.*

We have some comments in order. First, the above lemma does not offer a complete solution to the problem of determining those element of $\pi^m P^n$ which are nontrivial in homology. Second, the problem seems very much related to the famous vector field problem, but this relation is not clear to the author. Thirdly, if we knew that a secondary operation corresponding to the Adem relation for $\mathrm{Sq}^4\mathrm{Sq}^4$ acts trivially on $H^8P$ then the assertion of part (v) would be valid for all $i < 8$. Finally, notice that, in the case of $m = 1$, since $H^1(P^n; \mathbb{Z}) \simeq \mathbb{Z}/2$ then choosing a generator of this group represented as a map $P^m \to S^1$ we obtain a map which is nontrivial in $\mathbb{Z}$-homology. Part (i) of the above lemma then implies that any map $P \to S^1$ which is nontrivial in $\mathbb{Z}$-homology must be of even degree. Note that the case of $n = 1$ is well known as it forces $m = 1$, hence $f \in \pi_1 S^1$.

***Remark 1.7*** Notice that the validity of Lemma 1.6 for stable maps, allows us to decide about the existence of maps of spaces $f : \Sigma^i P^n \to S^m$ with nontrivial homology. For any map as such, one gets a (stable) map $P^n \to S^{m-i}$ which then allows to apply Lemma 1.6. We leave it to the reader to investigate this case further.

Next, we return to the case when $X$ is a loop space on sphere and consider

$$b : [\Omega^l S^{n+l}, \Omega^k S^{m+k}] \to \mathrm{Hom}_{\mathbb{Z}/2(H^*\Omega^k S^{m+k}, H^*\Omega^l S^{n+l})}$$

with a particular interest in the case of $k < l$. Note that for $k > 0$ the source of $b$ is the 'unstable' group $[\Omega^l S^{n+l}, \Omega^k S^{m+k}]$ whose complete computation needs a suitable unstable Adams spectral sequence (ASS). However, we do not attempt working with any unstable ASS and instead we try to use available geometric techniques as well as unstable invariants to study the image of this homomorphism. Our main observations in this paper are about the 'extreme' values of $k$ with $k = 0, l$. In the case of $k = l$ there exist examples where the image of $b$ is nontrivial. We have the following.

**Theorem 1.8** *For $n > 0$ and $1 \leq i \leq n$ there are maps $f_i : \Omega^i S^{n+1} \to \Omega^i S^{2n+1}$ such that $f_i^* \neq 0$. Consequently, the image of*

$$b : [\Omega^l S^{(n+1-l)+l}, \Omega^l S^{(2n+1-l)+l}] \to \mathrm{Hom}_{\mathbb{Z}/2(H^*\Omega^l S^{(2n+1-l)+l}, H^*\Omega^l S^{(n+1-l)+l})}$$

*is nontrivial.*

The proof is immediate once we choose $f_i = \Omega^{i-1} H$ with $H : \Omega S^{n+1} \to \Omega S^{2n+1}$ being the second James-Hopf map. It is also possible to use odd primary James–Hopf map to produce examples at odd primes.

The following provides a partial answer when $k=0$.

**Theorem 1.9** *Suppose all spaces are localised at the prime 2 and let $k=0$. The following statements hold.*

(i) *If $m=n=0$ then for any $l>0$ the image of $b$ is isomorphic to $\mathbb{Z}/2\{\imath\}$ where $\imath\in H^1QS^1$ is the fundamental class.*
(ii) *If $m=n>0$ and $l=1$ then the image of $b$ is nontrivial if and only if $n\in\{1,3,7\}$ and in this case the image is isomorphic to $\mathbb{Z}/2\{\partial\}$ where $\partial:\Omega S^{n+1}\to S^n$ is the boundary map in the Barratt–Puppe sequence for one of the Hopf maps $\eta,\nu,\sigma$.*
(iii) *If $m=n>1$ and $l>1$ then the image of $b$ is trivial.*
(iv) *If $m=n=1$ and any $l>1$ then the image of $b$ is isomorphic to $\mathbb{Z}/2\{\theta_{S^1}\}$ where $\theta_{S^1}:QS^1\to S^1$ corresponds to the structure map of $S^1$ as an infinite loop space.*

Next we consider the cases with $f:\Omega^lS^{n+1}\to S^m$. In this case, we have only results for $l=1$. Note that by James splitting [41, Chapter VII] if $f:\Omega S^{n+1}\to S^m$ satisfies $f_*\neq 0$ then $m=tn$ for some $t$. We have the following.

**Theorem 1.10** *Suppose $f:\Omega S^{n+1}\to S^m$ with $m=tn>n$.*

(i) *For $m>n$ with $m=tn$ if $n+1$ is odd or both of $n+1$ and $t$ are even then the image of $b$ is trivial.*
(ii) *For $m>n$ with $m=tn$ if $n+1$ is even, $t$ is odd, and $tn\notin\{3,7\}$ then the image of $b$ is trivial.*
(iii) *For $tn\in\{3,7\}$ there exists a map $f:\Omega S^2\to S^t$ with $H_t(f)\neq 0$.*

We have excluded the case with $tn=1$ as it implies that $t=n=1$. In this case, it is impossible to have $1=m>n=1$. In this case, we have $f:\Omega S^{1+1}\to S^1$ is one of the cases studied by the previous theorem.

*Remark on the integral results* Some of the tools and results that we use such as the existence of Hopf bundles $\eta,\nu,\sigma$ or James' splitting $\Sigma\Omega\Sigma X\simeq\bigvee_{r=1}^{+\infty}\Sigma X^{\wedge r}$, for $X$ path connected, are integral results. Therefore, one might expect that some of our results, at least the existence results, also have integral counterparts. On the other hand, some of obstructions that we have used to prove nonexistence results are local, so we do not know whether the integral results hold. We have tried to highlight the integral results when it is possible.

## 2 Motivation

The main motivation for this work is to study the image of the Hurewicz homomorphism $h:\pi_*X\to H_*X$ for certain cases $X$, and in particular the homomorphisms

$$h:\pi_*\Omega^l S^{n+l}\to H_*\Omega^l S^{n+l},\ h:\pi_*QS^n\to H_*QS^n,$$

where $n\geq 0$ and $l\geq 0$. The problem of determining spherical classes in $H_*X$ is not always an easy problem, e.g. in the case of $X=QS^0=\text{colim}\ \Omega^i S^i$ it is open problem (see for example [13, 39, 43]). The problem of determining spherical classes in finite loop spaces $\Omega^l S^{n+l}$ is also open, although some progress for small values of $l$ has been made where we have achieved complete classification of these classes (see [44] and [45]). We follow the philosophy that, at least on the level of algebra, the Hurewicz and Boardman homomorphisms are dual (see Sect. 3 for further discussions) and sometimes the dual problem might be easier to tackle. Following this philosophy, we are interested in looking at the dual problem and studying the image of Boardman homomorphisms

$$b:[\Omega^l S^{n+l},\Omega^k S^{m+k}]\to \text{Hom}(H_*\Omega^l S^{n+l},H_*\Omega^k S^{m+k}),$$

where $k\leq\infty$ with the convention $\Omega^\infty\Sigma^\infty X=QX$. To set up a more general framework, let's recall that for a spaces $X$ with base point, we have isomorphisms of groups

$$\pi_n^s X\simeq \pi_n QX\simeq [S^n,QX],\ \pi_s^n X\simeq [X,QS^n]$$

provided by the adjointness between $\Omega^\infty$ and $\Sigma^\infty$ functors where $QX=\text{colim}\ \ \Omega^i\Sigma^i X$. The evaluation map $\Sigma^\infty QX\to\Sigma^\infty X$ induces the stable homology suspension $\sigma_*^\infty:H_*QX\to H_*X$ and the stable cohomology suspension $\sigma_\infty^*$ : $H^*X\to H^*QX$ which fit into commutative diagrams as

$$\begin{array}{ccc} \pi_n^s X & \xrightarrow{h} & \text{Hom}(H_nS^n,H_nQX) \\ & \searrow^{h^s} & \downarrow \sigma_*^\infty \\ & & \text{Hom}(H_nS^n,H_nX) \end{array} \qquad \begin{array}{ccc} \pi_s^n X & \xrightarrow{b^s} & \text{Hom}(H^nX,H^nS^n) \\ & \searrow^{b} & \downarrow \sigma^*_\infty \\ & & \text{Hom}(H^nQX,H^nS^n) \end{array}$$

Next, notice that we have inclusion maps $\Omega^i\Sigma^i X\to QX$ which in the case of $h$ provides commutative diagram as

$$\begin{array}{ccc} \pi_n\Omega^i\Sigma^i X & \xrightarrow{h} & \text{Hom}(H_nS^n,H_n\Omega^i\Sigma^i X) \\ \downarrow & & \downarrow \\ \pi_n^s X\simeq\pi_n QX & \xrightarrow{h} & \text{Hom}(H_nS^n,H_nQX). \end{array}$$

Also note that working at a prime $p$, the duality $H^nX\simeq\text{Hom}_{\mathbb{Z}/p(H_nX,\mathbb{Z}/p)}$ between homology and cohomology provided by the Universal Coefficient Theorem, allows to consider $b^s$ and $b$ as homomorphisms

$$b^s:\pi_s^n X\longrightarrow \text{Hom}(H_nX,H_nS^n),\ b:\pi_s^n X\longrightarrow\text{Hom}(H_nX,H_nQS^n),$$

which send $f$ to $f_*$. For $b$, this provides a commutative diagram as

$$\begin{array}{ccccccc} \pi^{n+i}\Sigma^i X & \xrightarrow{\simeq} & [X,\Omega^i\Sigma^i S^n] & \xrightarrow{b} & \mathrm{Hom}(H^n\Omega^i S^{n+i},H^nX) & \xrightarrow{\simeq} & \mathrm{Hom}(H_nX,H_n\Omega^i S^{n+i}) \\ & & \downarrow & & \uparrow & & \downarrow \\ \pi_s^n X & \xrightarrow{\simeq} & [X,QS^n] & \xrightarrow{b} & \mathrm{Hom}(H^nQS^n,H^nX) & \xrightarrow{\simeq} & \mathrm{Hom}(H_nX,H_nQS^n) \end{array}$$

These observations motivate one to study the unstable Boardman homomorphisms

$$b:[X,\Omega^iS^{n+i}]\longrightarrow \mathrm{Hom}(H_nX,H_n\Omega^iS^{n+i}),$$

where $i\geq 0$.

## 3 Some generalities

Suppose $E$ is a nice ring spectrum with the identity. Write $A_E=E^*E$ for the algebra of $E$-cohomology operations and $A_E^{\mathrm{op}}$ for its opposite algebra. Furthermore, suppose for any space $X$ there is a Kronecker pairing $E^nX\otimes E_nX\to E_0$ reflecting a duality between the vector spaces $E^nX$ and $E_nX$. Also modules in this section are left modules. For any space $X$, $E^*X$ is a left $A_E$-module whereas $E_*X$ is a left $A_E^{\mathrm{op}}$-module where the existence of this structure comes from the existence of the Kronecker pairing.

For spaces $X$ and $Y$, one may consider Hurewicz and Boardman maps

$$h:[X,Y]\longrightarrow \mathrm{Hom}_{A_E^{\mathrm{op}}-\mathrm{mod}}(E_*X,E_*Y),\ \ b:[X,Y]\longrightarrow \mathrm{Hom}_{A_E-\mathrm{mod}}(E^*Y,E^*X)$$

which are defined by $h(f)=f_*$ and $b(f)=f^*$, respectively. If we fix $X$ (resp. $Y$) then upon being provided with a choice of a fixed element $x_E\in E_*X$ (resp. $y^E\in E^*Y$) the composition with the evaluation maps yields the usual Hurewicz and Boardman maps

$$h:[X,Y]\longrightarrow E_*Y,\ \ b:[X,Y]\longrightarrow E^*X.$$

The main examples of such homomorphisms are the classic Hurewicz and Boardman homomorphisms with $X$ or $Y$ being a sphere. One may replace $[X, Y]$ with the group of stable maps $\Sigma^\infty X\to\Sigma^\infty Y$, denoted by $\{X,Y\}$, which provides stable Hurewicz and Boardman homomorphisms (instead of maps)

$$h^s:\{X,Y\}\longrightarrow \mathrm{Hom}_{A_E^{\mathrm{op}}-\mathrm{mod}}(E_*X,E_*Y),$$
$$b^s:\{X,Y\}\longrightarrow \mathrm{Hom}_{A_E-\mathrm{mod}}(E^*Y,E^*X).$$

Choosing either $X$ or $Y$ to be a sphere, we have the stable Hurewicz and Boardman homomorphisms, respectively. The first one is the stable Hurewicz homomorphism

$$h_E^s:\pi_n^sY\to E_nY.$$

Taking $X=S^n$ we have $h^s:\{X,Y\}=\pi_n^sY\to H_*Y.$ which sends $f$ to $(E_*f)x_n$ where $x_n^E\in E_n(S^n)$ is a generator provided by the unit $S^0\to E$. This homomorphism has been studied extensively for many important spectra such as $E = H\mathbb{F}_p, \mathrm{MU}, \mathrm{BP}, K, \mathrm{KO}, \mathrm{tmf}$ as there exists complete description of the stable Hurewicz homomorphism $h_E^s : \pi_*^s S^0 \to E_* S^0$ for these spectra [3, 20, 35, 19]. In fact computing the image of this homomorphism is so important that one tends to find a spectrum $E$ so that $h_E^s : \pi_*^s S^0 \to E_* S^0$ is much closer to becoming an isomorphism and detects more and more elements in $\pi_*^s S^0$. From this point of view, there is an attempt to find bounds on the dimension/exponents of kernel and cokernel of this homomorphism (see [5, 34]). The second homomorphism is the stable Boardman homomorphism

$$b_E^s : \pi_s^n X \to E^n X$$

which is defined by $b^s(f) = (E^* f) x_E^n$ where $x_E^n \in E^n X$ is a generator provided by the unit $S^0 \to E$. This homomorphism also has been studied in detail [22, 6].

These two homomorphisms are dual, possibly up to some degree shift, over $\pi_* E$ in a suitable sense (see [38, Chapter 13] for a detailed discussion). However, up to our knowledge, despite existence of some explicit relation among $h_E^s$ and $b_E^s$, there is no dictionary of the results about $h_E^s$ and $b_E^s$. We also note that this duality is not one that is induced by means of the $S$-duality in the stable homotopy category in the sense that for a given $f : S^n \to X$ then $D(f)$ is not necessarily, up to finite number of suspensions, a map $X \to S^n$. However, knowing that these homomorphisms are dual in a suitable sense and the philosophy that sometimes solving a ‘dual’ problem could be easier tempts one to study one of these homomorphisms to justify some of claims about the other one. This duality could be used to obtain some information on the algebra. For instance, one may try to relate the rank of the image of $h_E^s$ to the rank of the kernel of $b_E^s$, etc.

The aim of this work is to apply and explore this idea in the presence of the destabilisation functor $Q := \Omega^\infty \Sigma^\infty$. Note that there are commutative diagrams of groups and their homomorphisms

$$\begin{array}{ccc}
[X, QY] & \xrightarrow{h} & \mathrm{Hom}_{A_E^{\mathrm{op}}-\ \mathrm{mod}}(E_*X, E_*QY) \\
\simeq\downarrow & & \downarrow \mathrm{Hom}(1_{E_*X}, \sigma_*^\infty) \\
\{X, Y\} & \xrightarrow{h^s} & \mathrm{Hom}_{A_E^{\mathrm{op}}-\ \mathrm{mod}}(E_*X, E_*Y)
\end{array}$$

and

$$\begin{array}{ccc}
\{X, Y\} & \xrightarrow{b^s} & \mathrm{Hom}_{A_E-\ \mathrm{mod}}(E^*Y, E^*X) \\
\simeq\downarrow & & \downarrow \mathrm{Hom}(\sigma_\infty^*, 1_{E^*X}) \\
[X, QY] & \xrightarrow{b} & \mathrm{Hom}_{A_E-\ \mathrm{mod}}(E^*QY, E^*X)
\end{array}$$

where $\sigma_*^\infty$ and $\sigma_\infty^*$ are induced by the evaluation map $e : \Sigma^\infty QY \to Y$ in $E$-homology and $E$-cohomology, respectively. Here, we refer to $h$ and $b$ as unstable Hurewicz and Boardman homomorphisms, respectively, in order to distinguish them from their stable counterparts.

## 4 Examples arising from the Kahn–Priddy type maps

The aim of this section is to prove Theorem 1.4 and Corollary 1.5. We show that the Becker–Schultz–Mann–Miller–Miller transfer maps give rise to maps satisfying Theorem 1.4 and Corollary 1.5. In particular, we observe that maps such as the Kahn–Priddy map offer a geometric constructions for such transfer maps with explicitly known homology [26, Theorem 3.1] (see also [33, Theorem A and its Corollary]).

Recall that for a compact Lie group $G$, an embedding of a closed subgroup $i : K \to G$, and a twisting (virtual) vector bundle $\alpha \to \mathrm{BG}$ there exists a transfer map, twisted by $\alpha$, which is a map of Thom spectra

$$\mathrm{BG}^{\mathrm{ad}_G \oplus \alpha} \to \mathrm{BK}^{\mathrm{ad}_K + \alpha|_{\mathrm{BK}}},$$

where $\mathrm{ad}_G$ is the vector bundle $\mathrm{EG} \times_G \mathfrak{g} \to \mathrm{BG}$ with $\mathfrak{g}$ being the Lie algebra of $G$ on which $G$ acts through the adjoint representation and $\alpha|_{\mathrm{BK}} = (\mathrm{Bi})^*\alpha$ is the pull back of $\alpha$ by the induced map between classifying spaces $Bi$ [9] (see also [33] as well as [27, Section 2.3]). In general, for any Lie group $G$ if we choose $\alpha = -\mathrm{ad}_G$ then corresponding to the embedding of the trivial group $1 \to G$ we have a transfer map

$$\Sigma^{\dim \mathfrak{g}} \mathrm{BG}_+ \to S^0.$$

Finally, notice that for any space the Becker–Gottlieb transfer map [8] associated with the trivial bundle $X \to *$ provides us with a stable splitting (in fact just after one suspension) $X_+ \simeq X \vee S^0$. Consequently, we may consider reduced transfer maps

$$\Sigma^{\dim \mathfrak{g}} \mathrm{BG} \to \Sigma^{\dim \mathfrak{g}} \mathrm{BG}_+ \to S^0.$$

The composition with the inclusion of $\mathrm{BG}^{[n]}$, the $n$-skeleton of BG, in BG yields a stable map

$$\Sigma^{\dim \mathfrak{g}} \mathrm{BG}^{[n]} \to S^0.$$

The stable adjoint of this map is a map of spaces

$$\Sigma^{\dim \mathfrak{g}} \mathrm{BG}^{[n]} \to \mathrm{QS}^0.$$

By Freudenthal's theorem, noting that the source has top cell in dimension $\dim \mathfrak{g} + n$, this map factors through $\Omega^{m+1} S^{m+1}$ with $m = \dim \mathfrak{g} + n$.

***Proof*** (Proof of Theorem 1.4) We begin with the specific example of $G = \mathbb{Z}/2$. In this case, a geometrically constructed representation for the reduced transfer map $P \to S^0$ is provided by the Kahn–Priddy map [26]. It is known that this map induces an epimorphism on ${}_2\pi_*^s$, the 2-component of the stable homotopy. The stable adjoint of this map provides a map $\lambda : QP \to Q_0 S^0$ where $Q_0 S^0$ is the base point component of $QS^0$ corresponding to $0 \in \pi_0 QS^0 \simeq \pi_0^s$. In fact, Kahn–Priddy theorem provides a composition

$$Q_0S^0 \to QP \to Q_0S^0$$

which is a 2-local equivalence [30, Corollaery 2.14]. Consequently, $\lambda$ induces an epimorphism in $H_*(-;\mathbb{Z}/2)$. On the other hand, notice that for a path connected space $X$, the inclusion $X \to QX$ induces a monomorphism in homology. Therefore, the composition

$$B\mathbb{Z}/2 = P \to QP \to Q_0S^0$$

induces an epimorphism in $\mathbb{Z}/2$-homology. Finally, notice that the inclusion of the $n$-skeleton $P^n \to P$ induces a monomorphism in homology in every positive dimension less than $n+1$. Hence, the composition $P^n \to P \to Q_0S^0$ induces an epimorphism in positive dimensions less than $n+1$. Finally, either by construction of the unstable Kahn–Priddy map (see Proof of Corollary 1.5) or by Freudenthal's theorem, this latter map factors through $\Omega^{n+1}S^{n+1}$ which induces an epimorphism in homology in positive dimensions less than $n+1$. Since $H_iP^n$ is nonzero for $0<i\leq n$, therefore our claim about the map $\lambda_n$ being nonzero follows automatically.

Next, consider the case of $G=S^1$. In this case, we have the complex transfer map

$$\lambda^{S^1}: \Sigma\mathbb{C}P_+ \to S^0.$$

It is known that the map of spaces $Q\Sigma\mathbb{C}P_+ \to QS^0$ yields an epimorphism when restricted to the submodule of primitive elements [16, Theorem 7.8]. Note that in homology of $\Sigma\mathbb{C}P$, because of the existence of a suspension, every nontrivial class would be primitive. Consequently, the composition

$$\Sigma\mathbb{C}P \to Q\Sigma\mathbb{C}P_+ \to Q_0S^0$$

induces an epimorphism in $\mathbb{Z}/2$ homology which is nontrivial in every odd degree. As above, one can show that the composition $\Sigma\mathbb{C}P^n \to Q_0S^0$ factors through $\Omega^{2n+2}S^{2n+2}$ and the map $\Sigma\mathbb{C}P^n \to \Omega^{2n+2}S^{2n+2}$ is nonzero in homology in every odd degree less than $2n+2$.

Similarly, for the case of $G=S^3$ one can show that the reduced twisted transfer map $\Sigma^3\mathbb{H}P \to S^0$ gives rise to maps

$$\Sigma^3\mathbb{H}P^n \to \Omega^{4n+4}S^{4n+4}$$

are nonzero (at least at some positive dimensions). We leave the details to the reader. □

Next, we wish to offer a proof of Corollary 1.5. We do this by choosing a geometrically constructed representative for the transfer $\mathbb{R}P \to S^0$ and use the computations of [26].

***Proof of Corollary 1.5*** The proof of Corollary 1.5 immediately follows from the above theorem if we give more details on homology. As there is a more geometric way of proving this Corollary using the Kahn and Priddy map [26] (see also [14, Section 4]) we begin with recalling this map and computations regarding its homology. For $n > 1$, given $L \in P^n$ determines a hyperplane $L^{\perp}$ and reflection with respect to $L^{\perp}$ gives a linear map $r_{L^{\perp}} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of determinant $-1$. After one point compactification one obtains a map $S^{n+1} \to S^{n+1}$ of degree $-1$. The loop sum with a map of degree one (class of the identity) gives a map $S^{n+1} \to S^{n+1}$ of degree 0. This defines the unstable Kahn–Priddy map $\lambda_n : P^n \to \Omega_0^{n+1} S^{n+1}$ where the subindex 0 denotes the path component of $\Omega^{n+1} S^{n+1}$ corresponding to $0 \in \pi_0 \Omega^{n+1} S^{n+1} \simeq \mathbb{Z}$ which is the basepoint component of $\Omega^{n+1} S^{n+1}$. The maps are compatible as $n$ varies and we have commutative diagrams such as

$$\begin{array}{ccc} P^n & \xrightarrow{\lambda_n} & \Omega_0^{n+1} S^{n+1} \\ \downarrow & & \downarrow \\ P^{n+1} & \xrightarrow{\lambda_{n+1}} & \Omega_0^{n+2} S^{n+2} \end{array}$$

for any $n$. Here, the downward arrows are inclusions. These give rise to a map $\lambda : P \to Q_0 S^0$ where $Q_0 S^0$ denotes the base component of $QS^0$ corresponding to $0 \in \pi_0 QS^0 \simeq \pi_0^s$ so that the following diagram commutes

$$\begin{array}{ccc} P^n & \xrightarrow{\lambda_n} & \Omega_0^{n+1} S^{n+1} \\ \downarrow & & \downarrow \\ P & \xrightarrow{\lambda} & Q_0 S^0. \end{array}$$

We note that taking the component of $QS^0$ in which $\lambda$ lands is important in writing the homological computations. The homology of the space $Q_0 S^0$ is known to be a polynomial algebra [15, Page 86. Corollary 2] (see also [12, Part I. Lemma 4.10]) given by

$$H_* Q_0 S^0 \simeq \mathbb{Z}/2[Q^I[1] * [-2^{l(I)}] : I \text{ admissible }, i_1 \geq i_2 + \cdots + i_r].$$

Here, $I = (i_1, \ldots, i_r)$ is called admissible if it is a sequence of positive integers and $i_j \leq 2i_{j+1}$ for all $j \in \{1, \ldots, r-1\}$. Also, $Q^I = Q^{i_1} \cdots Q^{i_r}$ is the iterated Kudo-Araki operation first defined in [28] and [29] (see also [15] and [12]). If $a_i \in H_i P$ is a generator then the relation

$$\lambda_*(a_i) = Q^i[1] * [-2]$$

describes homology of $\lambda$ [26, Theorem 3.1](see also [37, Chapter 1, Proof of Theorem 5.6]). Moreover, recall that $H_* \Omega^{n+1} S^{n+1}$ sits monomorphically inside

$H_*Q_0S^0$ as a subalgebra [44, Proposition] and is described by [12, Part III] (see also [39])

$$H_*\Omega^{n+1}S^{n+1} \simeq \mathbb{Z}/2[Q^I[1] * [-2^{l(I)}] : I \text{ admissible }, i_1 \geq i_2 + \cdots + i_r, i_r < n+1].$$

The commutative diagram above, relating $\lambda_n$ and $\lambda$, then implies that in homology

$$(\lambda_n)_*(a_i) = Q^i[1] * [-2]$$

for all $i < n+1$. Moreover, by [12, Part I, Theorem 1.1(7)] as well as [12, Page 47, First Line], the iterated homology suspension $\sigma_*^k : H_*QS^0 \to H_{*+k}QS^k$ satisfies $\sigma_*^k(Q^i[1] * [-2]) = Q^i x_k$ where $x_k \in H_kS^k$ is a generator. Also, by properties of Kudo-Araki operations [12, Part I. Theorem 1.1] we have $Q^k x_k = x_k^2$ and $Q^i x_k = 0$ if $i < k$. Moreover, the definition of $QX$ and the fact that $\Omega$ and colim functors commute show that $\Omega^k Q\Sigma^k X = QX$ for any $k \geq 0$. In particular, this allows to consider adjoints $\lambda : P \to QS^0$. Consequently, for the $k$-adjoint of $\lambda$ as $\mathrm{ad}_k(\lambda) : \Sigma^k P \to QS^k$ we have

$$(\mathrm{ad}_k(\lambda))_*(\Sigma^k a_i) = \begin{cases} Q^i x_k \neq 0 & \text{if } i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Again, by the commutativity of the above square, noting that $\Omega_0^{n+1}S^{n+1} \to QS^0$ is an $(n+1)$-fold loop map, it is easily verified that we have similar relations in place for the $k$-th adjoint maps $\mathrm{ad}_k(\lambda_n) : \Sigma^k P^n \to \Omega^{n-k}S^n$ which is defined as far as $k \leq n$ which proves the Corollary. ☐

***Remark 4.1*** It is easy to show that $\lambda_n$ does not lift to a map $P^n \to \Omega^n S^n$. To see this consider the EHP-sequence $S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$ and loop it $n$-times. It is easy to show that the composition $(\Omega^n H) \circ \lambda_n : P^n \to \Omega^{n+1}S^{n+1} \to \Omega^{n+1}S^{2n+1}$ satisfies $((\Omega^n H) \circ \lambda_n)_* a_n = x_n$ where $x_n \in H_n\Omega^{n+1}S^{2n+1}$ is a generator corresponding to the bottom cell. Hence, $(\Omega^n H) \circ \lambda_n$ is essential, consequently $\lambda_n$ does not lift to $\Omega^n S^n$.

We can use the maps $\lambda_n$ to provide further examples of generalised cospherical classes. Note that by universal property of iterated loop spaces, the map $\lambda_n$ extends to an $(n+1)$-fold loop map $\Omega^{n+1}\Sigma^{n+1}P^n \to \Omega^{n+1}S^{n+1}$ which we keep denoting by $\lambda_n$. Likewise, the mapping $\lambda$ extends to an infinite loop map $QP \to QS^0$. By the construction of these extension maps, with the convention $\lambda_\infty = \lambda$, we have the following commutative diagrams

$$\begin{array}{ccc} P^n & \xrightarrow{\lambda_n} & \Omega^{n+1}S^{n+1} \\ \downarrow & & \downarrow \\ \Omega^{n+1}\Sigma^{n+1}P^n & \xrightarrow{\lambda_n} & \Omega^{n+1}S^{n+1} \end{array}$$

which shows that $\lambda_n : \Omega^{n+1}\Sigma^{n+1}P^n \to \Omega^{n+1}S^{n+1}$ is also nontrivial in homology in many dimensions.

## 5 Proof of Lemma 1.6

We proof Lemma 1.6 in a few steps. Notice that we are dealing with the case $n > m$. We first eliminate the cases with $m$ being odd.

**Lemma 5.1** *Suppose $m$ is odd and $n > m$. Then, for any $f : P^n \to S^m$ we have $f_* = 0$.*

***Proof*** Since $m$ is odd then

$$a_{m+1} = Sq^1 a_m = Sq^1 f^*(x_m) = f^*(Sq^1 x_m) = 0$$

in $H^*P^n$ which is a contradiction. □

Hence, from now on we consider the cases of $m$ being even. First, let $n = m + 1$.

**Lemma 5.2** *Suppose $m \geq 1$ is even. Then, the projection onto the top cell $P^m \to S^m$ extends to a map $P^{m+1} \to S^m$ which is nontrivial in homology. Moreover, for any map $f : P^{m+1} \to S^m$ with $f_* \neq 0$ the restriction $f|_{P^m} : P^m \to S^m$ is homotopic to the projection onto the top cell.*

***Proof*** For the quotient map $q_m : S^m \to P^m$ and the projection onto the top cell $p_m : P^m \to S^m$ it is known that the composition $p_m q_m : S^m \to S^m$ has degree 0 for $m$ even. Hence, $p_m$ admits an extension over the cofibre of $q_m$ which is $P^{m+1}$, say $\overline{p_m} : P^{m+1} \to S^m$ which is nonzero in $H_m(-;\mathbb{Z}/2)$. Finally, note that for any $f : P^{m+1} \to S^m$ with $f_* \neq 0$ the restriction $f|_{P^m} = f \circ i$ where $i : P^m \to P^{m+1}$ is the inclusion. In particular, $(f \circ i)_* \neq 0$ shows that $i^*(f) = f \circ i \neq 0$ in $\pi^m P^m$ where $i^* : \pi^m P^{m+1} \to \pi^m P^m$. Note that for $m \geq 2$ even, $\pi^m P^m \simeq \mathbb{Z}/2$ is generated by the projection onto the top cell $P^m \to S^m$ [40, Theorem 4.1]. Since $f \circ i \neq 0$ hence $f|_{P^m}$ must be homotopic to the projection onto the top cell. This completes the proof. □

From now on, we consider the cases with $n - m \geq 2$ and $n$ being even. Suppose $f : P^n \to S^m$ is given with $f_* \neq 0$ in homology. Since we work over $\mathbb{Z}/2$ then $f_* \neq 0$ if and only if $f^* \neq 0$. We abuse the notation to use the same symbols to denote homology and cohomology generators.

**Lemma 5.3** *Suppose $2m \leq n$. Then for any $f : P^n \to S^m$ we have $f_* = 0$.*

***Proof*** In this case, $f^* \neq 0$ implies that $f^*(x_m) = a_m$. This leads to the equation

$$a_{2m} = a_m^2 = Sq^m a_m = Sq^m f^*(x_m) = f^* Sq^m x_m = 0$$

in $H^*P^n$ which is a contradiction. This proves our claim. □

Hence, we need to deal with the case of $n < 2m$. Note that we have resolved the cases with $n = m, m + 1$. Hence, we may assume $m + 2 \leq n < 2m$.

**Lemma 5.4** *Suppose $n - m \geq 2$ and $m$ even such that $m \equiv 2^t \bmod 2^{t+1}$ and $m + 2^t \leq n$. Then, for any map $f : P^n \longrightarrow S^m$ we have $f_* = 0$.*

***Proof*** If $f_* \neq 0$ then $f^* \neq 0$. Suppose $m \equiv 2^t \bmod 2^{t+1}$. Then,

$$a_{m+2^t} = Sq^{2^t} a_m = Sq^{2^t} f^*(x_m) = f^*(Sq^{2^t} x_m) = 0$$

which is a contradiction in $H^*P^n$. Hence, $f_* = 0$. □

Note that we already have $n - m \geq 2$. By Lemma 5.4 if $m \equiv 2 \bmod 4$ then for any $f : P^n \longrightarrow S^m$ we have $f_* = 0$. Hence, we only need to deal with the case of $m \equiv 0 \bmod 4$. We show that in this case, there is some $n$ such that there exist maps $f : P^n \longrightarrow S^m$ with $f_* \neq 0$. Note that for $n \geq m$, by pinching $P^{m-1}$ to a point, any map $f : P^n \longrightarrow S^m$ extends to a map $P^n_m \longrightarrow S^m$, and vice versa and map $P^n_m \longrightarrow S^m$ gives a map $P^n \longrightarrow S^m$ by composition. Hence, it is enough to show that there exists a map $P^n_m \longrightarrow S^m$ which is nonzero in homology. This latter is equivalent to saying that there is a 2-local splitting

$$P^n_m \simeq S^m \vee P^n_{m+1}.$$

The following would be well known to experts. But, we record a proof.

**Lemma 5.5** *Suppose $m \geq 4$ with $m \equiv 0 \bmod 4$. Then, there is a 2-local splitting of spaces*

$$P^{m+2}_m \simeq S^m \vee P^{m+2}_{m+1}.$$

*There is also a 2-local splitting of spaces*

$$P^{m+3}_m \simeq S^m \vee P^{m+3}_{m+1}.$$

Now, by taking the projection map onto the $S^m$ summand the following is immediate.

**Corollary 5.6** *Suppose $m \geq 4$ and $m \equiv 0 \bmod 4$. Then, there exists a map $P^{m+2} \longrightarrow S^m$ with $f_* \neq 0$. There is exists also a map $P^{m+3} \longrightarrow S^m$ which is nontrivial in $\mathbb{Z}/2$-homology.*

We shall use James periodicity to do proof Lemma 5.5. Recall that, by James periodicity [4, Theorem 6] (see also [35, Page 31]), there is a homotopy equivalence

$$\Sigma^{2^{\varphi(k)}} P^{m+(k-1)}_m \longrightarrow P^{m+(k-1)+2^{\varphi(k)}}_{m+2^{\varphi(k)}},$$

where

$$\varphi(k) = \#\{r : 0 < r < k, r \equiv 0, 1, 2, 4 \bmod 8\}.$$

***Proof of Lemma 5.5*** By James periodicity, $P^{m+2}_m \equiv \Sigma^4 P^{m-2}_{m-4}$ which by iterating implies that $P^{m+2}_m$ is homeomorphic to $\Sigma^{m-4} P^6_4$. So, it is enough to show $P^6_4$ splits as

$S^4 \vee P_5^6$. To see this notice that for $P_4^5$ the attaching map of the 5-cell is the composition of the projection map $S^4 \to P^4$ and the pinch map $P^4 \to P^4/P^3 = S^4$. It is known that this composition is of degree 0, hence null. Hence, $P_4^5 \simeq S^4 \vee S^5$. For the attaching map of the 6-cell of $P_4^6$, say $S^5 \to P_4^5$ one could process similarly, noting that the component $S^5 \to S^4$ belongs to $\pi_1^s \simeq \mathbb{Z}/2\{\eta\}$. If it is nontrivial, then there must be a nontrivial action of $Sq^2$ on $H^*P_4^6$ which there is none. Hence, this component is trivial. The other is of degree 2, also seen by the nontrivial action of $Sq^1$. This shows that as spaces

$$P_4^6 \simeq S^4 \vee P_5^6.$$

Regarding the splitting $P_m^{m+3}$ notice that by James periodicity $P_m^{m+3} \simeq \Sigma^{m-4} P_4^7$ and we need to prove the splitting for $P_4^7$. For $P_4^7 \simeq S^4 \vee P_4^7$ notice that the Adams Hopf invariant one theorem (see Lemma 6.1) together with the existence of the Hopf invariant one element $\sigma \in \pi_7^s$ imply that $P^7$ stably splits as $P^6 \vee S^7$. This implies that stably $P_4^7 \simeq P_4^6 \vee S^7$. On the other hand, notice that the attaching map of the 7-cell of $P_4^7$ is a map $S^6 \to P_4^6 \simeq S^4 \vee P_5^6$. To prove the claimed splitting of spaces, we need to show that the component $S^6 \to S^4$ is null. However, this component belongs to $\pi_2^s \simeq \mathbb{Z}/2\{\eta^2\}$. If this map is not null then, by naturality of the secondary operations, there must be a nontrivial secondary operation $\Phi$ in $P_4^7$ with $\Phi(a_4) = a_7$. The stability of the secondary operations then would not allow a stable splitting of $P_4^7$ implied by the existence of the Hopf invariant one elements as demonstrated above. This is a contradiction. Hence, the component $S^6 \to S^4$ is null and as spaces we have

$$P_4^7 \simeq S^4 \vee P_5^7.$$

□

The above lemmata prove Theorem 1.6 in the case $f : P^n \to S^m$ is a map of spaces. But, notice that all the techniques we have used, are ‘stable’ and apply stably, hence the result also holds for stable maps. In fact, in some specific examples, the main point of Lemma 5.5 is to provide an unstable splitting.

As it seems the existence of Hopf invariant one elements plays a role here. We have the following example as well.

**Lemma 5.7** *There is a 2-local splitting of spaces as*

$$P_8^{14} \simeq S^8 \vee P_9^{14}.$$

*Consequently, whenever* $m \equiv 0 \bmod 8$, *for* $i<7$, *there is a map* $f : P^{m+i} \to S^m$ *such that* $f_* \neq 0$.

***Proof*** By Lemma 6.1(iv) the existence of the Hopf invariant one element $\sigma \in \pi_7^s$ implies that there is a map $\widetilde{\sigma} : S^7 \to P^7$ which is nonzero in homology. If $D$ denotes the $S$-duality functor then $DP_q^n = \Sigma P_{-n-1}^{-q-1}$ [7]. Since $S^n$ and $P^n$ are finite CW complexes then their suspension spectra are finite spectra, consequently $g_* \neq 0$ if and only if $D(g)_* \neq 0$ by [31, Lemma A.3]. Consequently, $(\widetilde{\sigma})_* \neq 0$ if and only if

$D(\widetilde{\sigma}) : \Sigma P_{-8}^{-2} \to S^{-7}$ is nonzero in homology or equivalently $\Sigma^{-1}D(\widetilde{\sigma}) : P_{-8}^{-2} \to S^{-8}$ is nonzero in homology. Using James periodicity, let

$$f = \Sigma^{15}D(\widetilde{\sigma}) : P_8^{14} \to S^8.$$

Obviously, $f_* \neq 0$. However, the connectivity arguments shows that $f$ can be realised as a map of spaces. Moreover, the inclusion of the bottom cell $S^8 \to P_8^{14}$ is non-trivial in homology. Hence, as spaces

$$P_8^{14} \simeq S^8 \vee P_9^{14}.$$

By iterated application of the James periodicity, for any $k > 0$, we obtain a map

$$P_{8(k+1)}^{8(k+1)+6} \to S^{8(k+1)}$$

which is nonzero in homology. The composition of the pinching map and the above map as

$$P^{8(k+1)+6} \to P_{8(k+1)}^{8(k+1)+6} \to S^{8(k+1)}$$

is obviously nontrivial in homology. For $i<7$ the composition of the inclusion $P^{8(k+1)+i} \to P^{8(k+1)+6}$ with the above map yields a map

$$P^{8(k+1)+i} \to S^{8(k+1)}$$

which is nonzero in homology. This completes the proof. □

## 6 Proof of Theorem 1.9: case of $m=n>0$

### 6.1 Case of $l=1$

For $l = 1$ there are examples at hand which are provided by Hopf fibrations, namely maps $\Omega S^{n+1} \to S^n$ for $n = 1, 3, 7$. The existence of these maps also provides a decomposition

$$\Omega S^{n+1} \simeq S^n \times \Omega S^{2n+1}.$$

We show these are the only possible cases (at least modulo 2). The following formulation of Adams' Hopf invariant one element result is well known and the equivalence of the various statements are to be found in various papers of James and Adams and others. For instance, the equivalence of (*ii*) and (*iii*) follows from [18, Proposition 6.1.5] when interpreted modulo 2, or the result of James [24, Theorem 1.5] can be seen as an unstable analogue of (*iii*) $\Rightarrow$ (*iv*) below (see also [25, Theorem 1.2]. Similarly, splittings similar to, or leading to, splittings as in (*iv*) could be found in [2, Theorem 1.2]. However, we record a proof for the sake of convenience and future reference.

**Lemma 6.1** *The following are equivalent.*

(i) *There is a map* $f : \Omega S^{n+1} \to S^n$ *with* $f_* \neq 0$ (*modulo* 2).
(ii) *There is a map* $g : S^{2n} \to \Omega S^{n+1}$ *with* $g_* \neq 0$ (*modulo* 2).
(iii) *There is a map* $h : S^{2n+1} \to S^{n+1}$ *of unstable Hopf invariant one, and* $n \in \{1, 3, 7\}$.
(iv) *There is a stable splitting* $P^n \simeq S^n \vee P^{n-1}$ *where* $P^n$ *is the n-dimensional real projective space.*

***Proof*** $(i) \Rightarrow (ii)$ : Let $i : S^n \to \Omega S^{n+1}$ be the inclusion adjoint the to identity $S^{n+1} \to S^{n+1}$. Then $(f \circ \imath)_* \neq 0$, hence (at $p = 2$) $f \circ \imath$ is homotopic to the identity. Together with James fibration $S^n \to \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$ it follows that

$$(f, H) : \Omega S^{n+1} \to S^n \times \Omega S^{2n+1}$$

is a homotopy equivalence. The inclusion $S^{2n} \to \Omega S^{2n+1}$ gives rise to a spherical class. Consequently, the composition

$$g : S^{2n} \to \Omega S^{2n+1} \to \Omega S^{n+1}$$

gives rise to a spherical class in $H_*(\Omega S^{2n+1}; \mathbb{Z}/2)$, so $g_* \neq 0$.

$(ii) \Rightarrow (iii)$ : If $g_* \neq 0$ then from James' description of $H_*\Omega S^{2n+1}$ we see that $g_*(x_{2n}) = x_n^2$. It is well known that the adjoint of $g$, say $h : S^{2n+1} \to S^{n+1}$ has unstable Hopf invariant one.

$(iii) \Rightarrow (i)$ : As noted above, (ii) and (iii) are equivalent. It follows that the adjoint of $h$, say $g : S^{2n} \to \Omega S^{n+1}$ maps nontrivially under $H_\# : \pi_{2n}\Omega S^{n+1} \to \pi_{2n}\Omega S^{2n+1}$. This implies that $H \circ g$ is homotopic to the identity map. The claimed decomposition follows immediately.

$(ii) \Rightarrow (iv)$: First recall that the projection on the top cell $P^n \to S^n$ is an unstable map which is nontrivial in $\mathbb{Z}/2$-homology. For $g : S^{2n} \to \Omega S^{n+1}$ being nontrivial in homology, consider the composition $S^{2n} \to \Omega S^{n+1} \to QS^n$ and write $g$ for this map as well. By Kahn-Priddy theorem, $g$ lifts to $Q\Sigma^n P$, and yields a map $S^{2n} \to Q\Sigma^n P$ which is nontrivial in homology. By taking adjoint, we obtain a map $\widetilde{g} : S^n \to QP$ which is nontrivial in homology. For dimensional reasons, this implies that $h(\widetilde{g}) = a_n$ where $h$ is the Hurewicz homomorphism $\pi_*QP \to H_*QP$. Moreover, by cellular approximation, we may restrict to a map $\widetilde{g} : S^n \to QP^n$ which satisfies $h(\widetilde{g}) = a_n$. This implies that the stable adjoint of this map as $S^n \to P^n$ is nontrivial in homology. Therefore the composition $S^n \to P^n \to S^n$ is nontrivial in $\mathbb{Z}/2$-homology. Noting that the cofibre of $P^n \to S^n$ is $\Sigma P^{n-1}$ gives the other stable piece, so $P^n \simeq S^n \vee P^{n-1}$.

$(iv) \Rightarrow (ii)$: Choose a splitting map $S^n \to P^{n-1} \vee S^n \simeq P^n$ given by the inclusion into the second summand. Then the adjoint $S^n \to QP^n$ is nontrivial in homology, so the composition $S^n \to QP$. For dimensional reasons, the $n$-adjoint of this map $S^{2n} \to Q\Sigma^n P$ is nontrivial in homology. After composition with the Kahn-Priddy map $Q\Sigma^n P \to QS^n$ we obtain a map $S^{2n} \to QS^n$ which is nontrivial in homology. Also note that the inclusion $\Omega S^{n+1} \to QS^n$ is a $2n$ equivalence, so we obtain a map $S^{2n} \to \Omega S^{n+1}$ which is nontrivial in homology. This completes the proof. □

This settles the case with $l = 1$. The case of $l > 1$ reduces to the case of $l = 1$ in the following sense.

### 6.2 Case of $l > 1$

Unlike the case of $l = 1$, for $l > 1$ the existence of maps $f : \Omega^l S^{n+l} \to S^n$ with $f_* \neq 0$ is not so immediate. For $n = 1$ we may choose $f : \Omega^l S^{n+l} \to S^1$ to be any representative of the identity element of $H^1(\Omega^l S^{n+l}; \mathbb{Z}) \simeq \mathbb{Z}$. It is immediate that $f$ is nonzero in $H_*(-; k)$ for $k = \mathbb{Z}, \mathbb{Z}/2$. There is another way to see existence of such maps. Since $S^1$ is an infinite loop space, let $\theta : QS^1 \to S^1$ be the structure map which has the property that the composition $S^1 \to QS^1 \to S^1$ is identity. In particular, $\theta_* \neq 0$. Now, the composition

$$f : \Omega^l S^{l+1} \to QS^1 \to S^1$$

satisfies $f_* \neq 0$.

For the remaining cases, we have the following nonexistence result. Let's note that if $f : \Omega^l S^{n+l} \to S^n$ exists with $f_* \neq 0$ then the composition $\Omega S^{n+1} \to \Omega^l S^{n+l} \to S^n$ also would be nontrivial in $H_n(-; k)$, hence by Lemma 6.1 we see that $n$ must be either 1, 3, or 7. As we considered the case of $n = 1$ above then we only need to resolve the cases $n \in \{3, 7\}$.

**Lemma 6.2**

(i) *Suppose* $n = 3$. *Then, there exists no map* $f : \Omega^2 S^5 \to S^3$ *with* $f_* \neq 0$. *Consequently, for* $2 \leq l \leq +\infty$ *there exists no map* $f : \Omega^l S^{l+3} \to S^3$ *with* $f_* \neq 0$.

(ii) *Suppose* $n = 7$. *Then, these exists no map* $f : \Omega^2 S^9 \to S^7$ *with* $f_* \neq 0$. *Consequently, for* $2 \leq l \leq +\infty$ *there exists no map* $f : \Omega^l S^{l+7} \to S^3$ *with* $f_* \neq 0$.

***Proof*** Proof of (i) and (ii) are similar. First note that the general case follows from our claim for double loop spaces as follows. For instance, note that for $l \geq 2$, the composition $\Omega^2 S^5 \to \Omega^l S^{l+3}$ is nonzero in $H_3(-; k)$ with $k = \mathbb{Z}, \mathbb{Z}/2$. Hence, existence of any map $f : \Omega^l S^{l+3} \to S^3$ with $f_* \neq 0$ would imply that the composition $\Omega^2 S^5 \to \Omega^l S^{l+3} \to S^3$ is nonzero in homology, giving the desired contradiction.

Now we show there is no map $f : \Omega^2 S^{n+2} \to S^n$ (with $n = 3, 7$) so that $f_* \neq 0$. Given a map $f : \Omega^2 \Sigma^2 X \to X$ we may define $\mu : X \times X \to X$ by the following composition

$$X \times X \to * \times_{\Sigma_2} (X \times X) \to F(\mathbb{R}^2, 2) \times_{\Sigma_2} (X \times X) \to \Omega^2 \Sigma^2 X \to X,$$

where the first map on the left is projection, second and third maps are inclusion, and the last map is $f$. This map is a commutative multiplication on $X$. Since the composition $S^n \to \Omega^2\Sigma^2 S^n \to S^n$ is nonzero in homology (we may assume it is multiplication of degree 1), hence it is homotopic to the identity. On the other hand, by construction, for a based path connected space $X$, the inclusion $X \to \Omega^2\Sigma^2 X$ can be decomposed as a composition

$$X \xrightarrow{\alpha} X \times X \to \Omega^2\Sigma^2 X,$$

where $\alpha$ can be taken as either $(1, *)$ or $(*, 1)$ with $*$ being the base point of $X$. This implies that, for $X = S^3, S^7$, $(X, \mu, *)$ is a commutative $H$ space in the sense of [17]. But this is a contradiction as according to James [23, Theorem 1.1] it is known that $S^3$ and $S^7$ do not admit any commutative $H$-space structure. This latter also can be seen from general result of Hubbuck [21, Theorem 1.1] as $S^3$ and $S^7$ do not have homotopy type of a torus. □

## 7 Case of $m = n = 0$

**Lemma 7.1** *For any $l > 0$ there exists a map $f : \Omega^l S^l \to S^0$ with $f_* \neq 0$. Moreover, for any such $f$ we have $f = \Omega\iota$ where $\iota : \Omega^{l-1}S^l \to P = K(\mathbb{Z}/2, 1)$ represents the fundamental class.*

***Proof*** Let $l > 0$, and take the fundamental class $\iota \in H^1 QS^1 \simeq \mathbb{Z}/2$ which can be realised as a map $\iota : QS^1 \to P$. Define $f$ to be the composition

$$\Omega^l S^l \to QS^0 \xrightarrow{\Omega\iota} \mathbb{Z}/2 = S^0.$$

Clearly, $f_* \neq 0$. On the other hand, if $f : \Omega^l S^l \to S^0 = \Omega P$ is given with $f_* \neq 0$, then the adjoint of $f$, say $\widetilde{f} : \Sigma\Omega^l S^l \to P$ is nontrivial in homology. Also, note we may consider the composition

$$\Sigma\Omega^l S^l \xrightarrow{e} \Omega^{l-1} S^l \xrightarrow{\iota} P$$

which is nontrivial in homology where $e$ is the evaluation map (adjoint to the identity). Since $\widetilde{f}, \iota \circ e \in [\Sigma\Omega^l S^l, P] \simeq \mathbb{Z}/2$ and both elements are nontrivial, therefore

$$\widetilde{f} = \iota \circ e \Rightarrow f = \Omega\iota \circ \widetilde{e} = \Omega\iota \circ 1 = \Omega\iota.$$

This completes the proof. □

## 8 Proof of Theorem 1.10

### 8.1 The cases of $m = tn$ with $n+1$ odd, or $n+1$ and $t$ even

We begin with the case $n = 0$. In this case, as $\Omega S^1 \simeq \mathbb{Z}$ it is not possible to find $m > 0$ so that $g : \Omega S^1 \to S^m$ is nontrivial in homology. Therefore, we consider the cases with $n > 0$. If $f : \Omega S^{n+1} \to S^m$ is given with $f_* \neq 0$ then $(\Sigma f)_* \neq 0$. In the light of James splitting, $\Sigma\Omega S^{n+l} \simeq \bigvee_{t=1} S^{tn+1}$, $m = tn$ for some $t > 1$ are the only possible choices for which $(\Sigma f)_* \neq 0$ and consequently $f_* \neq 0$ can happen. That is

**Proposition 8.1** *For $f : \Omega S^{n+1} \to S^m$ with $m \neq tn$ for all $t$, we have $f_* = 0$.*

Our main result in this section is the following.

**Theorem 8.2**

(i) *Suppose $n+1$ is odd, $n > 0$, and $t > 1$. Then for any $f : \Omega S^{n+1} \to S^{tn}$ we have $f_* = 0$ in $H_*(-;k)$ with $k = \mathbb{Z}, \mathbb{Z}/2$.*

(ii) *Suppose $n+1$, $n \geq 1$, and $t > 1$ are even. Then for any $f : \Omega S^{n+1} \to S^{tn}$ we have $f_* = 0$ in $H_*(-;k)$ with $k = \mathbb{Z}, \mathbb{Z}/2$.*

***Proof*** The proof in both cases is by contradiction. By abuse of notation, we write $x_i$ for a generator of $H^i S^i$. Let's recall that

$$H^*(\Omega S^{n+1};\mathbb{Z}) \simeq \left\{ H^*(\Omega S^{n+1};\mathbb{Z}) \simeq \left\{ \Gamma_{\mathbb{Z}(x_n)} \text{if } n+1 \text{ is odd} \Gamma_{\mathbb{Z}(x_{2n})\otimes\mathbb{Z}E_{\mathbb{Z}(x_n)}} \text{if } n+1 \text{ is even} \right.\right.$$

where $\Gamma_{\mathbb{Z}}$ and $E_{\mathbb{Z}}$ denote divided power algebra and exterior algebra functors over $\mathbb{Z}$ respectively. We first prove the integral result.

(i) Suppose $m = tn$ with $t > 1$ and $f : \Omega S^{n+1} \to S^{tn}$ is given with $f_* \neq 0$ in $\mathbb{Z}$-homology. For $n > 1$ with $n+1$ odd, and $t > 1$, for dimensional reasons, for $X = \Omega S^{n+1}, S^{tn}$ there is a duality between homology and cohomology with $H^*X \simeq \mathrm{Hom}_{\mathbb{Z}(H_*X,\mathbb{Z})}$. It follows that $f^* \neq 0$. If $n+1$ is odd then having $f^*(x_{tn}) \neq 0$ we see that $(f^*(x_{tn}))^2 \neq 0$ in $\Gamma_{\mathbb{Z}(x_n)}$ which implies that $x_{tn}^2 \neq 0$ in $H^*S^{tn}$ which is an obvious contradiction. Hence, $f_* = 0$.

(ii) First suppose $n > 1$. Let's note that for $n > 1$, $H_*\Omega S^{n+1}$ has nontrivial homology only in dimensions $kn$, with $k > 0$, by James' splitting. So, if $H_i\Omega S^{n+1} \neq 0$ then $H_{i+\epsilon}\Omega S^{n+1} \simeq 0$ for $\epsilon \in \{-1, 1\}$. This property is essential while appealing to the Universal Coefficient Theorem. Obviously, $H_*\Omega S^2$ does not have this property. We just note that choosing $t$ even forces $f^*(x_{tn}) \in \Gamma_{\mathbb{Z}(x_{2n})}$. The rest of the proof goes exactly as part (i) and we leave the details to the reader.

Next, suppose either $n+1$ is odd with $n > 0$ and $t > 1$ or $n+1$ and $t$ both are even with $t, n > 1$. Assume $f$ is given with $f_* \neq 0$ in $\mathbb{Z}/2$-homology. By the duality between homology and cohomology over $\mathbb{Z}/2$ we see that $f^* \neq 0$ in $\mathbb{Z}/2$-cohomology. For dimensional reasons, implied by our choices of $n$ and $t$, together with the naturality of the Universal Coefficient Theorem, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{tn-1}(S^{tn};\mathbb{Z}),\mathbb{Z}/2) & \longrightarrow & H^{tn}(S^{tn};\mathbb{Z}/2) & \longrightarrow & \mathrm{Hom}(H_{tn}(S^{tn};\mathbb{Z}),\mathbb{Z}/2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f^* & & \downarrow \mathrm{Hom}(f_*,\mathbb{Z}/2) & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{tn-1}(\Omega S^{n+1};\mathbb{Z}),\mathbb{Z}/2) & \longrightarrow & H^{tn}(\Omega S^{n+1};\mathbb{Z}/2) & \longrightarrow & \mathrm{Hom}(H_{tn}(\Omega S^{n+1};\mathbb{Z}),\mathbb{Z}/2) & \longrightarrow & 0 \end{array}$$

The Ext terms do vanish for dimensional reasons, hence yielding a commutative diagram as

$$\begin{array}{ccc} H^{tn}(S^{tn};\mathbb{Z}/2) & \xrightarrow{\simeq} & \mathrm{Hom}(H_{tn}(S^{tn};\mathbb{Z}),\mathbb{Z}/2) \\ \downarrow f^* & & \downarrow \mathrm{Hom}(f_*,\mathbb{Z}/2) \\ H^{tn}(\Omega S^{n+1};\mathbb{Z}/2) & \xrightarrow{\simeq} & \mathrm{Hom}(H_{tn}(\Omega S^{n+1};\mathbb{Z}),\mathbb{Z}/2). \end{array}$$

We then conclude that the non-vanishing of $f^*$ in $\mathbb{Z}/2$-cohomology implies the non-vanishing of $\mathrm{Hom}(f_*,\mathbb{Z}/2)$ and consequently, non-vanishing of $f_*$ in $\mathbb{Z}$-homology. The result now follows by appealing to the integral case.

Finally, $n=1$ and $t$ even, notice that by the Hopf invariant one result $\Omega S^2 \simeq \Omega S^3 \times S^1$ and the above decomposition of cohomology is becomes a result of this splitting. The fact that $t$ is even implies that the composition $\Omega S^3 \xrightarrow{\Omega\eta} \Omega S^2 \to S^t$ is nonzero in homology where $t$ is even. But, this leads to a contradiction by part (i). □

## 8.2 Attaching by whitehead products

The content of this subsection is probably well known, but we include an exposition for further reference. The idea is that given a *CW*-complex which splits after finite number of suspensions, or stably splits, certain attaching maps are obtained from Whitehead products. The main purpose of this subsection is to make such statements more precise. In the case of splitting after finite number of suspensions, it is enough to deal with the case of splitting after one suspension. During this section and afterwards, for a CW-complex $X$, write $X^k$ for its $k$-skeleton. We write $X_k = X/X^{k-1}$ and $X_n^m = X^m/X^{n-1}$. We have the following.

**Lemma 8.3** *Suppose X is a CW-complex with finite number of cells in each dimension. Suppose X has only one cell in dimensions n and* $2n$. *Suppose* $\Sigma X^{2n} \simeq S^{2n+1} \vee \Sigma X^{2n-1}$ *then the attaching map of the* $2n$*-cell in X, resulting in the* $S^{2n+1}$ *summand of* $\Sigma X^{2n}$, *is "obtained" by a Whitehead product on the n-cell. More precisely, write* $\alpha$ *for the attaching map* $S^{2n-1} \to X^{2n-1}$, $p : X^{2n-1} \to X_n^{2n-1}$ *for the pinching map, and* $i : S^n \to X_n^{2n-1}$ *for the inclusion of the bottom cell. If* $p \circ \alpha \neq 0$ *then up to multiplication by nonzero integer k*

$$p \circ \alpha = i_{\#}[\iota_n, \iota_n].$$

*In particular, $k$ could be chosen an odd number if the splitting is 2-local and $p \circ \alpha \neq 0$ holds 2-locally.*

Note that the above lemma says that the attaching map after composition with a suitable pinching map is image of a Whitehead product. In particular, it does not claim that $\alpha$ is always a Whitehead product. Also, notice that the lemma does not make any claim about the essentiality of $\alpha$. If $\alpha$ is null then $i_{\#}[\iota_n, \iota_n] = p \circ \alpha = 0$ which only says that $[\iota_n, \iota_n] \in \ker(i_{\#})$. In fact such an equation is useful and required in proving Theorem 8.4.

***Proof*** Suppose $\alpha : S^{2n-1} \to X^{2n-1}$ is the attaching map of the desired $2n$-cell. We may compose $\alpha$ with the projection $p : X^{2n-1} \to X_n^{2n-1}$. The fact that $X^{2n} = X^{2n-1} \cup_\alpha e^{2n}$ after one suspension splits as $\Sigma X^{2n} \simeq S^{2n+1} \vee \Sigma X^{2n-1}$ is the same as saying that $\alpha$ belongs to the kernel of the suspension map $E : \pi_{2n-1} X^{2n-1} \to \pi_{2n-1}\Omega\Sigma X^{2n-1}$. The splitting of $\Sigma X^{2n}$ also implies that $\Sigma X_n^{2n} \simeq \Sigma X_n^{2n-1} \vee S^{2n+1}$. Hence, $E(p \circ \alpha) = 0$ for $E : \pi_{2n-1} X_n^{2n-1} \to \pi_{2n-1}\Omega\Sigma X_n^{2n-1}$. The space $X_n^{2n-1}$ has its bottom cell in dimension $n$, so we may use the EHP-sequence in the meta-stable range. In particular, consider the following portion of the EHP-sequence

$$\pi_{2n-1}\Omega^2\Sigma(X_n^{2n-1} \wedge X_n^{2n-1}) \xrightarrow{P} \pi_{2n-1} X_n^{2n-1} \xrightarrow{E} \pi_{2n-1}\Omega\Sigma X_n^{2n-1}.$$

By exactness of the above sequence from $E(p \circ \alpha) = 0$ we conclude that $p \circ \alpha = P\beta$ for some $\beta \in \pi_{2n-1}\Omega^2\Sigma(X_n^{2n-1} \wedge X_n^{2n-1})$ and in particular $\beta \neq 0$ if $\alpha \neq 0$. Writing $i : S^n \to X_n^{2n-1}$ for the inclusion we have the following commutative diagram

$$\begin{array}{ccccc}
\pi_{2n-1}\Omega^2 S^{2n+1} & \xrightarrow{P} & \pi_{2n-1} S^n & \xrightarrow{E} & \pi_{2n-1}\Omega\Sigma S^n \\
\downarrow{\scriptstyle (\Omega^2\Sigma(i\wedge i))_{\#}} & & \downarrow{\scriptstyle i_{\#}} & & \downarrow{\scriptstyle (\Omega\Sigma i)_{\#}} \\
\pi_{2n-1}\Omega^2\Sigma(X_n^{2n-1} \wedge X_n^{2n-1}) & \xrightarrow{P} & \pi_{2n-1} X_n^{2n-1} & \xrightarrow{E} & \pi_{2n-1}\Omega\Sigma X_n^{2n-1}.
\end{array}$$

Since $X$ has only one cell in dimension $n$ then the space $\Sigma(X_n^{2n-1} \wedge X_n^{2n-1})$ has its bottom cell in dimension $2n+1$, and $(\Omega^2\Sigma(i \wedge i))_{\#}$ is an epimorphism in the above diagram. Hence, there exists a nonzero $\beta' \in \pi_{2n-1}\Omega^2 S^{2n+1}$ with $(\Omega^2\Sigma(i \wedge i))_{\#}\beta' = \beta$. Since $\pi_{2n-1}\Omega^2 S^{2n+1} \simeq \mathbb{Z}$ then $\beta' = k\iota_{2n+1}$ where $k \in \mathbb{Z}$ is nonzero and $\iota_{2n+1} \in \pi_{2n+1} S^{2n+1} \simeq \pi_{2n-1}\Omega^2 S^{2n+1}$ is the generator. It follows that

$$p \circ \alpha = i_{\#}P(k\iota_{2n+1}) = k i_{\#}P\iota_{2n+1}.$$

However, it is known that $P : \pi_{2n-1}\Omega^2 S^{2n+1} \to \pi_{2n-1} S^n$ is the Whitehead product, that is $P(\iota_{2n+1}) = [\iota_n, \iota_n]$. Hence, up to a multiplication by a nonzero integer we have

$$p \circ \alpha = i_{\#}[\iota_n, \iota_n].$$

In the case of 2-local splitting, choosing $k$ to be even implies that $p \circ \alpha = 0$ which contradicts the assumption. This completes the proof. □

It is possible to say more if we have more information on the attaching map of the $(n+1)$-cell of $X$. For instance, if $X$ has no cell in dimension $n+1$ or the attaching map is null then $\pi_{2n+1}\Sigma(X_n \wedge X_n) \simeq \pi_{2n+1}S^{2n+1}$ induced by the inclusion of the bottom cell. In this case, $(\Omega^2\Sigma(i \wedge i))_{\#}$ would be an isomorphism and $\beta'$ must be a unique element. On the other hand, writing $a : S^n \to X^n$ for the attaching map of the $(n+1)$-cell, if $p \circ a : S^n \to X^n_n = S^n$ is essential then $\pi_n X_n \simeq H_n(X_n; \mathbb{Z})$ would be the cyclic group $\mathbb{Z}/^\circ(p \circ a)$, hence $H_{2n}(X_n \wedge X_n; \mathbb{Z})$ and consequently $\pi_{2n+1}\Sigma(X_n \wedge X_n) \simeq H_{2n+1}(\Sigma(X_n \wedge X_n); \mathbb{Z})$ would be cyclic groups too. In this case, up to multiplication by an odd number, we may choose $k = 1$.

## 8.3 The case of $n+1$ even and $t$ odd

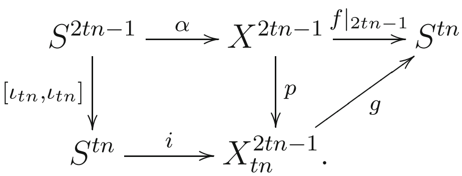
We localise at the prime 2.

**Theorem 8.4** *Suppose $f : \Omega S^{n+1} \to S^{tn}$ where $t$ is odd and $n+1$ is even. If $tn \notin \{1, 3, 7\}$ then $f_* = 0$.*

***Proof*** The proof is by contradiction, and we wish to apply Lemma 8.3 to prove the Theorem. The space $X = \Omega S^{n+1}$ satisfies conditions of Lemma 8.3. Suppose $tn \notin \{1, 3, 7\}$ and $f_* \neq 0$. Let $i : S^{tn} \to X^{2tn-1}_{tn}$ denote the inclusion of the bottom cell, $\alpha : S^{2tn-1} \to X^{2tn-1}$ the attaching map of the $2tn$-cell, and $p : X^{2tn-1} \to X^{2tn-1}_{tn}$ the pinching map which collapses $X^{tn-1}$. Then, by Lemma 8.3 we have

$$i_{\#}[\iota_{tn}, \iota_{tn}] = p \circ \alpha.$$

Moreover, for dimensional reasons, the restriction of $f$ to the $(2tn-1)$-cell of $X$, say $f|_{2tn-1} : X^{2tn-1} \to S^{tn}$ extends to a map $g : X^{2tn-1}_{tn} \to S^{tn}$ so that $gp = f|_{2tn-1}$. These data yield a commutative diagram as follows




On the other hand $f|_{2tn-1}$ has to extend over the $2tn$-skeleton of $X$ to a map $f|_{2tn} : X^{2tn}_{tn} \to S^{tn}$ where the necessity being implied by the existence of $f : X \to S^{tn}$. The existence of $f|_{2tn}$ is the same as saying that $f|_{2tn-1} \circ \alpha = 0$. On the other hand $i$ and $g$ are both nonzero in $H_{tn}(-; \mathbb{Z}/2)$ and working at the prime 2 this implies that we may take $g \circ i$ is the identity (over $\mathbb{Z}$ this implies that $g \circ i$ is an odd multiple of the identity). It follows that

$$[\iota_{tn}, \iota_{tn}] = g_\# \circ i_\#[\iota_{tn}, \iota_{tn}] = f|_{2tn-1} \circ \alpha = 0.$$

It is known that for $m$ odd the Whitehead product $[\iota_m, \iota_m] \in \pi_{2m-1}S^m$ vanishes if and only if there is a Hopf invariant one element in $\pi_{2m-1}S^m$ [10, Corollary 1.3, Remark 1.4] which together with Adams Hopf invariant one result means that $m \in \{1, 3, 7\}$ [1]. This shows that if $f$ extends over the $2tn$-skeleton then $tn \in \{1, 3, 7\}$. This gives the desired contradiction. Hence, $f_* = 0$. □

It remains to decide about the cases with $tn \in \{1, 3, 7\}$ with $t$ odd and $n + 1$ even with $n > 0$. However, note that we have required $t > 1$, so $t \geq 3$. Also, $n$ has to be odd. This leave us with the cases where $n = 1$ and $t = 3, 7$. That is we have to decide about the existence of maps $\Omega S^2 \to S^t$ with $t \in \{3, 7\}$ such that $f_* \neq 0$. We have the following existence result and we note that its proof is also valid for the case of $\Omega S^2 \to S^1$.

**Theorem 8.5** *For $t \in \{1, 3, 7\}$ there exists a map $f : \Omega S^2 \to S^t$ with $f_* \neq 0$.*

***Proof*** First, notice that the existence of Hopf invariant one elements in $\pi_t^s$ is equivalent to existence of a decomposition $\Omega S^{t+1} \simeq \Omega S^{2t+1} \times S^t$. By Lemma 6.1(i) this is the same as existence of a map $\partial : \Omega S^{t+1} \to S^t$ with $\partial_* \neq 0$. Second, recall that for a path connected space $X$, James' splitting $\Sigma\Omega\Sigma X \simeq \bigvee_{t=1}^{+\infty} \Sigma X^{\wedge t}$ provides us with projection maps $p_t : \Sigma\Omega\Sigma X \to \Sigma X^{\wedge t}$ whose adjoint

$$\widetilde{p_t} : \Omega\Sigma X \to \Omega\Sigma X^{\wedge t}$$

is the $t$-th James–Hopf invariant, sometimes denoted by $j_t$. In particular, it is known that $j_t$ induces nonzero map in $t$-th homology. Now, define $f$ to be the composition

$$\Omega S^2 \xrightarrow{j_t} \Omega S^{t+1} \xrightarrow{\partial} S^t.$$

Obviously $f_* \neq 0$. This proves the Theorem. □

Notice that the map $\partial$, which can be chosen to be the boundary map in the fibration sequence corresponding to Hopf bundles, as well as James' splitting both exist integrally. This allows to show that the above theorem also holds in $\mathbb{Z}$-homology.

## 9 Notes on the case with $l > 1$ and $m > n$

We can use some geometry to prove

**Lemma 9.1** *For any $l > 1$ and $f : \Omega^l S^{n+l} \to S^{2n}$ we have $f_* = 0$.*

***Proof*** The inclusion $i : \Omega S^{n+1} \to \Omega^l S^{n+l}$ is a $2n$-equivalence. Hence, assuming $f : \Omega^2 S^{n+2} \to S^{2n}$ is nontrivial in homology, $g : \Omega S^{n+1} \xrightarrow{i} \Omega^l S^{n+l} \xrightarrow{f} S^{2n}$ satisfies $g_* \neq 0$. This contradicts Theorem 8.2. □

## Compliance with ethical standards

**Conflict of interest** The authors declare that they do not have conflict of interests.

**Ethical standard** This research complies with ethical standards.

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