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ORIGINAL ARTICLE

Choi–Davis–Jensen's type trace inequalities for convex functions of self-adjoint operators in Hilbert spaces

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Abstract
Some Choi–Davis–Jensen's type trace inequalities for convex functions are proved. Also, we generalize these inequalities for any arbitrary operator mean via operator monotone decreasing functions. In particular, we present some new order among $\mathrm{tr}(\Phi(C)A)$ and $\mathrm{tr}(\Phi(C)A^{-1})$. New refinements of some power type trace inequalities via reverse and refinement of Young's inequality are established. Among our results, we obtain new versions of the Hölder type trace inequality for any arbitrary operator mean.

Keywords Choi–Davis–Jensen's inequality · Hölder operator inequality · Trace · Operator mean · Positive linear maps

Mathematics Subject Classification Primary 47A63 · Secondary 47A99

1 Introduction

For this purpose, let $\mathcal{B}(\mathcal{H})$ stand for the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $A \geq 0$. We write $A > 0$ if it is a positive

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invertible operator. Let $A, B \in \mathcal{B}(\mathcal{H})$ be two self-adjoint operators. The partial order $A \leq B$ is defined as $B - A \geq 0$. The absolute value of A is denoted by $|A|$, that is $|A| = (A^*A)^{\frac{1}{2}}$. A continuous real-valued function f defined on interval J is said to be operator monotone increasing (decreasing) if for every two positive operators A and B with spectral in J, the inequality $A \leq B$ implies $f(A) \leq f(B)$ $(f(A) \geq f(B))$, respectively. We recall that every operator monotone decreasing function is operator convex. An operator mean σ_f in the sense of Kubo–Ando [2] is defined by a positive operator monotone increasing function f on the half interval $(0, \infty)$ with $f(1) = 1$ as

$$A\sigma_f B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}},$$

for positive invertible operators $A, B \in \mathcal{B}(\mathcal{H})$. For $A, B > 0$ the weighted operator arithmetic, geometric and harmonic means, respectively, by

$$A \bigtriangledown_v B = (1 - v)A + vB, \quad A \sharp_v B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^v A^{\frac{1}{2}}$$

and $A !_v B = \left((1 - v)A^{-1} + vB^{-1}\right)^{-1}$, where $v \in [0, 1]$. A linear map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is called positive if $A \geq 0$ implies $\Psi(A) \geq 0$. It is said to be unital if $\Psi(I) = I$. Davis [9] and Choi [7] showed that if Ψ is a unital positive linear map on $\mathcal{B}(\mathcal{H})$ and if f is an operator convex function on an interval J, then so-called the Choi–Davis–Jensen inequality $f\left(\Psi(A)\right) \leq \Psi\left(f(A)\right)$ holds for every self-adjoint operator A on $\mathcal{H}$ whose spectrum is contained in J.

Let $\{e_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be trace class (denoted by $\mathcal{B}_1(\mathcal{H})$) if $\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty$. We define the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ to be

$$\mathrm{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle.$$

If $\|A\|_2 := \sum_{i \in I} \|Ae_i\|^2 < \infty$, A is said to be Hilbert–Schmidt operator (denotes $A \in \mathcal{B}_2(\mathcal{H})$). The definitions of $\|A\|_1$, $\|A\|_2$ and trace do not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$.

The Hölder's type inequality [21] is given as follows:

$$|\mathrm{tr}(AB)| \leq \mathrm{tr}(|AB|) \leq [\mathrm{tr}(|A|^{1/\alpha})]^{\alpha} [\mathrm{tr}(|B|^{1/(1-\alpha)})]^{1-\alpha},$$

where $\alpha \in (0, 1)$ and $A, B \in \mathcal{B}(\mathcal{H})$ with $|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in \mathcal{B}_1(\mathcal{H})$.

In particular, for $\alpha = \frac{1}{2}$, we get the Schwarz inequality

$$|\mathrm{tr}(AB)| \leq \mathrm{tr}(|AB|) \leq [\mathrm{tr}(|A|^2)]^{\frac{1}{2}} [\mathrm{tr}(|B|^2)]^{\frac{1}{2}}, \tag{1.1}$$

where $|A|^2, |B|^2 \in \mathcal{B}_1(\mathcal{H})$. We have the following Hölder type trace inequality for the weighted geometric mean [10]: if A, B are positive invertible operators, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $A^p, B^q \in \mathcal{B}_1(\mathcal{H})$, then $B^q \sharp_{\frac{1}{q}} A^p \in \mathcal{B}_1(\mathcal{H})$ and

$$\mathrm{tr}(B^q \natural_{\frac{1}{p}} A^p) \leq [\mathrm{tr}(A^p)]^{\frac{1}{p}} [\mathrm{tr}(B^q)]^{\frac{1}{q}}. \tag{1.2}$$

Also, if $C \in \mathcal{B}_1(\mathcal{H})$ and $C \geq 0$, then $CA^p, CB^q, C(B^q \natural_{\frac{1}{p}} A^p) \in \mathcal{B}_1(\mathcal{H})$ and

$$\mathrm{tr}(C(B^q \natural_{\frac{1}{p}} A^p)) \leq [\mathrm{tr}(CA^p)]^{\frac{1}{p}} [\mathrm{tr}(CB^q)]^{\frac{1}{q}}. \tag{1.3}$$

According to [13], if A and B are self-adjoint operators with $A \leq B$ and $P \in \mathcal{B}_1(\mathcal{H})$ with $P \geq 0$, then

$$\mathrm{tr}(PA) \leq \mathrm{tr}(PB). \tag{1.4}$$

Moreover, if $A \geq 0$, then $0 \leq \mathrm{tr}(PA) \leq \|A\| \mathrm{tr}(P)$, and

$$|\mathrm{tr}(PA)| \leq \mathrm{tr}(P|A|), \tag{1.5}$$

for a self-adjoint operator A and $P \in \mathcal{B}_1(\mathcal{H})$ with $P \geq 0$.

For the theory of trace functional and their applications, the reader is referred to [22]. Some classical trace inequalities investigated in [8, 20, 24], which are continuations of the work of Bellman [4].

We express the following results which is derived from [6, Sections 31.2.1 and 31.2.2].

Lemma 1.1 *Let $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ be a positive linear mapping such that*

$$\mathrm{tr}\big(\Phi(W)\big) \leq 1 \tag{1.6}$$

for all $W \in \mathcal{B}_1(\mathcal{K})$ with $W \geq 0$ and $\mathrm{tr}(W) = 1$, i.e., for all density operators W on Hilbert space $\mathcal{K}$. Then its dual map Φ^* is a linear map $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ which is well defined by

$$\mathrm{tr}\big(\Phi^*(B)A\big) = \mathrm{tr}\big(B\Phi(A)\big), \tag{1.7}$$

for all $B \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}_1(\mathcal{K})$.

Φ is continuous and $\Phi^*(I) \leq I$ is equivalent to $\mathrm{tr}\big(\Phi(W)\big) \leq 1$ for all density operators W on $\mathcal{K}$. A linear map $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ is positive if and only if its dual map Φ^* is positive.

Corollary 1.2 *For a positive linear mapping $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ the following statements are equivalent*:

(i) $\mathrm{tr}\big(\Phi(W)\big) \leq 1$ *for all density operators W on $\mathcal{K}$*;
(ii) Φ *is continuous and* $\Phi^*(I) \leq I$.

Theorem 1.3 *A linear mapping $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ is positive if and only if $\Phi^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive.*

For a comprehensive account on positive linear maps see [5, 6, 23].

Dragomir in [11–14] proved Jensen's type trace inequalities for convex functions.

In Sect. 2, we first extend the Choi–Davis–Jensen inequality for trace. In particular, with the help of this we obtain some power type trace inequalities for positive linear maps. Also we generalize Choi–Davis–Jensen's type trace inequalities for any arbitrary operator mean via operator monotone decreasing functions. As an application of these, we prove some new order among $\mathrm{tr}(\Phi(C)A)$ and $\mathrm{tr}(\Phi(C)A^{-1})$, where $0 < mI \leq A \leq MI$, $C \in \mathcal{B}_1(\mathcal{K})$, $C > 0$ and Φ is a positive linear mapping satisfying (1.6). Next, we present reverses of the Choi–Davis–Jensen inequality for trace.

In Sect. 3 by applying a recent reverse and refinement of Young's inequality, we establish some power type trace inequalities. Also we show new versions of the Hölder type trace inequality for any arbitrary operator mean.

Some examples for the power function and logarithm are presented in Sect. 4.

2 Choi–Davis–Jensen's type trace inequalities

We recall the gradient inequality for the convex function $f : [m, M] \to \mathbb{R}$, namely

$$f(s) - f(t) \geq \delta_f(t)(s - t) \tag{2.1}$$

for any $s, t \in (m, M)$, where $\delta_f(t) \in [f'_-(t), f'_+(t)]$.

Now, we are ready to present our first result.

Theorem 2.1 *Let $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ be a positive linear mapping satisfying (1.6), whose adjoint is Φ^*, A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on [m, M] and $B \in \mathcal{B}_1(\mathcal{K}) \setminus \{0\}$ is a strictly positive operator, then we have $\frac{\mathrm{tr}\left(B(\Phi^*(A)\right)}{\mathrm{tr}\left(\Phi(B)\right)} \in [m, M]$ and the Choi–Davis–Jensen inequality*

$$f\left(\frac{\mathrm{tr}\left(B\Phi^*(A)\right)}{\mathrm{tr}\left(\Phi(B)\right)}\right) \leq \frac{\mathrm{tr}\left(B\Phi^*(f(A)\right)}{\mathrm{tr}\left(\Phi(B)\right)}. \tag{2.2}$$

Proof According to the hypothesis, we have

$$\begin{aligned} m\left\langle (\Phi(B))^{\frac{1}{2}}e_i, (\Phi(B))^{\frac{1}{2}}e_i \right\rangle &\leq \left\langle A(\Phi(B))^{\frac{1}{2}}e_i, (\Phi(B))^{\frac{1}{2}}e_i \right\rangle \\ &\leq M\left\langle (\Phi(B))^{\frac{1}{2}}e_i, (\Phi(B))^{\frac{1}{2}}e_i \right\rangle \end{aligned} \tag{2.3}$$

for any $i \in I$, which by summation (2.3), we get

$$m\mathrm{tr}\left(\Phi(B)\right) \leq \mathrm{tr}\left(\left(\Phi(B)\right)^{\frac{1}{2}}A\left(\Phi(B)\right)^{\frac{1}{2}}\right) \leq M\mathrm{tr}\left(\Phi(B)\right). \tag{2.4}$$

If follows from the properties of trace and equality (1.7) that

$$\begin{aligned}\operatorname{tr}\Big(\big(\Phi(B)\big)^{\frac{1}{2}}A(\Phi(B))^{\frac{1}{2}}\Big) &= \operatorname{tr}(A\Phi(B)) \\ &= \operatorname{tr}(\Phi^*(A)B) = \operatorname{tr}(B\Phi^*(A)).\end{aligned} \tag{2.5}$$

By inequality (2.4) and equality (2.5), we conclude that $\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \in [m, M]$. Utilising the gradient inequality (2.1), we have

$$\delta_f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\left(s - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right) \leq f(s) - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right) \tag{2.6}$$

for any $s \in [m, M]$, where

$$\delta_f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right) \in \left[f'_-\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right), f'_+\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\right].$$

Inequality (2.6) implies in the operator order of $\mathcal{B}(\mathcal{H})$ that

$$\delta_f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\left(A - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \cdot 1_{\mathcal{H}}\right) \leq f(A) - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right) 1_{\mathcal{H}}$$

which can be written as

$$\begin{aligned}&\delta_f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\left(\langle Ay, y\rangle - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\langle y, y\rangle\right) \\ &\qquad \leq \langle f(A)y, y\rangle - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\langle y, y\rangle\end{aligned} \tag{2.7}$$

for any $y \in \mathcal{H}$. If we take in (2.7), $y = \left(\Phi(B)\right)^{\frac{1}{2}} e_i$ and summing over $i \in I$, we obtain

$$\begin{aligned}&\delta_f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right) \\ &\times\left(\sum_{i\in I}\langle\left(\Phi(B)\right)^{\frac{1}{2}} A\left(\Phi(B)\right)^{\frac{1}{2}} e_i, e_i\rangle - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\sum_{i\in I}\langle\Phi(B)e_i, e_i\rangle\right) \\ &\leq \sum_{i\in I}\langle\left(\Phi(B)\right)^{\frac{1}{2}} f(A)\left(\Phi(B)\right)^{\frac{1}{2}} e_i, e_i\rangle - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\sum_{i\in I}\langle\Phi(B)e_i, e_i\rangle.\end{aligned}$$

Thus, using the properties of trace and equality (1.7) to the above inequality, we have that

$$\begin{aligned}&\delta_f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\left(\operatorname{tr}\left(B\Phi^*(A)\right) - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\operatorname{tr}\left(\Phi(B)\right)\right) \\ &\qquad \leq \operatorname{tr}\Big(B\Phi^*\big(f(A)\big)\Big) - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\operatorname{tr}\left(\Phi(B)\right),\end{aligned}$$

and inequality (2.2) is thus proved. □

As an application of Theorem 2.1, we have the following inequality.

Corollary 2.2 *If A is a self-adjoint operator on the Hilbert space $\mathcal{H}$ satisfying $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 \le m < M$ and $B \in \mathcal{B}_1(\mathcal{H}) \setminus \{0\}$, $B > 0$. Then for every positive linear map $\Phi^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which is adjoint of a continuous linear map with $\Phi^*(I) \le I$ and every $p \ge 1$*

$$\Big(\mathrm{tr}\big(B\Phi^*(A)\big)\Big)^p \le \mathrm{tr}\big(B\Phi^*(A^p)\big)\big(\mathrm{tr}(B)\big)^{p-1}.$$

Proof Let Φ^* be the adjoint of the continuous linear map $\Phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$, then Φ is positive and $\mathrm{tr}\big(\Phi(W)\big) \le 1$ for any density operators W on Hilbert space $\mathcal{H}$ by Theorem 1.3 and Corollary 1.2. Since $\mathrm{tr}\big(\Phi(B)\big) = \mathrm{tr}\big(B\Phi^*(I)\big)$ and $\Phi^*(I) \le I$; hence, using inequality (1.4), we have $\mathrm{tr}\big(\Phi(B)\big) \le \mathrm{tr}(B)$. Therefore, by convexity of the function $f(x) = x^p$ for $p \ge 1$ and applying inequality (2.2), the proof is complete. □

In [17], the authors proved that for two positive operators $0 < mI \le A, B \le MI$ and $!_v \le \sigma_1, \sigma_2 \le \nabla_v$,

$$g(\Psi(A))\sigma_1 g(\Psi(B)) \le kg(\Psi(A\sigma_2 B)), \tag{2.8}$$

where $g : (0, \infty) \to (0, \infty)$ is an operator monotone increasing, Ψ is a positive unital linear map and k stands for the known Kantorovich constant $k = \frac{(M+m)^2}{4mM}$.

It is well known that for positive invertible operators $A, B \in \mathcal{B}(\mathcal{H})$, if σ_v is a symmetric operator mean, then

$$A!_v B \le A\sigma_v B \le A \nabla_v B. \tag{2.9}$$

Furuichi et al. [16] showed the following new reverse inequalities of (2.9): if $0 < mI \le A, B \le MI$. Then

$$\frac{m\sharp_\lambda M}{m \nabla_\lambda M} A \nabla_v B \le A\sharp_v B \le \frac{m\sharp_\mu M}{m \nabla_\mu M} A!_v B, \tag{2.10}$$

where $\lambda = \min\{v, 1-v\}$, $\mu = \max\{v, 1-v\}$ and $v \in [0, 1]$.

In the following result by Theorem 2.1, we extend the reverse of inequality (2.8) for a positive operator monotone decreasing function on $(0, \infty)$ involving trace.

Theorem 2.3 *Let $0 < mI \le A, B \le MI$. Then for every positive linear map Ψ on $\mathcal{B}(\mathcal{H})$ and σ_1, σ_2 between ∇_v and $!_v$*

$$\Psi\big(f(A\sigma_2 B)\big) \le \frac{(m \nabla_\lambda M)(m(\sharp_\mu M)}{(m\sharp_\lambda M)(m!_\mu M)} \Psi\big(f(A)\big)\sigma_1 \Psi\big(f(B)\big), \tag{2.11}$$

where $f : (0, \infty) \to (o, \infty)$ is an operator monotone decreasing function,

$\lambda = \min\{v, 1-v\}$, $\mu = \max\{v, 1-v\}$ and $v \in [0,1]$.

Moreover, if $v = \frac{1}{2}$, *then for any* $! \le \sigma_1, \sigma_2 \le \nabla$,

$$\Psi\big(f(A\sigma_2 B)\big) \le k\Psi\big(f(A)\big)\sigma_1\Psi\big(f(B)\big).$$

Proof Since $0 < mI \le A, B \le MI$, we can write

$$m = m(I\sigma_2 I) \le m\sigma_2 m \le A\sigma_2 B \le M\sigma_2 M \le M(I\sigma_2 I) = M.$$

By virtue of [3, Remark 2.6], it follows that

$$f(A)\sigma_1 f(B) \ge f(A)!_v f(B) \ge f(A \nabla_v B). \tag{2.12}$$

Using the left-hand side of inequality (2.10) and the fact that for $\alpha \ge 1, f(\alpha t) \ge \frac{1}{\alpha} f(t)$ when f is an operator monotone decreasing we have

$$f(A \nabla_v B) \ge f\left(\frac{m \nabla_\lambda M}{m\sharp_\lambda M} A\sharp_v B\right) \ge \frac{m\sharp_\lambda M}{m \nabla_\lambda M} f(A\sharp_v B), \tag{2.13}$$

and similarly from the right-hand side of inequality (2.10) we obtain

$$\begin{aligned} f(A\sharp_v B) &\ge f\left(\frac{m\sharp_\mu M}{m!_\mu M} A!_v B\right) \ge \frac{m!_\mu M}{m\sharp_\mu M} f(A!_v B) \\ &\ge \frac{m!_\mu M}{m\sharp_\mu M} f(A\sigma_2 B). \end{aligned} \tag{2.14}$$

It follows from (2.12), (2.13) and (2.14) that

$$f(A\sigma_2 B) \le \frac{(m \nabla_\lambda M)(m\sharp_\mu M)}{(m\sharp_\lambda M)(m!_\mu M)} f(A)!_v f(B).$$

Applying positive linear map Ψ and [1, Theorem 3] and inequality (2.12), we get the required inequality (2.11). □

Since the power function x^p on $(0, \infty)$ is operator monotone decreasing for $p \in [-1, 0]$, we get the following result.

Corollary 2.4 *Let* $0 < mI \le A, B \le MI$, Ψ *be a positive linear map and operator means* σ_1, σ_2 *between* ∇_v *and* $!_v$ *and* $-1 \le p \le 0$. *Then*

$$\Psi\big((A\sigma_2 B)^p\big) \le \frac{(m \nabla_\lambda M)(m(\sharp_\mu M)}{(m\sharp_\lambda M)(m!_\mu M)} \Psi(A^p)\sigma_1\Psi(B^p),$$

where $\lambda = \min\{v, 1-v\}$, $\mu = \{v, 1-v\}$ *and* $v \in [0,1]$.

In the next result, we present new versions of Choi–Davis–Jensen's type trace inequalities.

Proposition 2.5 *Let* $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ *be a positive linear mapping satisfying (1.6), whose adjoint is* Φ^*, $0 < mI \le A, B \le MI$ *and* $C \in \mathcal{B}_1(\mathcal{K}) \setminus \{0\}$, $C > 0$. *If f is a positive operator monotone decreasing on [m, M] and* $!_v \le \sigma_1, \sigma_2 \le \nabla_v$, *then*

$$f\left(\frac{\mathrm{tr}\Big(C\big(\Phi^*(A\sigma_2 B)\big)\Big)}{\mathrm{tr}(\Phi(C))}\right) \le \frac{(m \nabla_\lambda M)(m(\sharp_\mu M)}{(m\sharp_\lambda M)(m!_\mu M)} \frac{\mathrm{tr}\Big(C\Big(\Phi^*\big(f(A)\big)\sigma_1\Phi^*\big(f(B)\big)\Big)\Big)}{\mathrm{tr}(\Phi(C))}, \tag{2.15}$$

where $\lambda = \min\{v, 1-v\}$, $\mu = \max\{v-1, v\}$ *and* $v \in [0, 1]$.

If $v = \frac{1}{2}$, *then for any* $! \le \sigma_1, \sigma_2 \le \nabla$,

$$f\left(\frac{\mathrm{tr}(C(\Phi^*(A\sigma_2 B)))}{\mathrm{tr}(\Phi(C))}\right) \le k \frac{\mathrm{tr}\big(C\big(\Phi^*\big(f(A)\big)\sigma_1\Phi^*(f(B))\big)\big)}{\mathrm{tr}(\Phi(C))}. \tag{2.16}$$

Proof Due to Theorems 2.1, 2.3 and inequality (1.4), we obtain (2.15). □

In the following remark, we can obtain the relation between $\mathrm{tr}(\Phi(C)A)$ and $\mathrm{tr}(\Phi(C)A^{-1})$.

Remark 2.6 Let $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ be a positive linear mapping satisfying (1.6), $A, B \in \mathcal{B}(\mathcal{H})$ such that $0 < mI \le A, B \le MI$ and $C \in \mathcal{B}_1(\mathcal{K}) \setminus \{0\}$, $C > 0$. If we put $\sigma_1 = \sigma_2 = \nabla$ and $f(t) = t^{-1}$ in (2.16), we have

$$[\mathrm{tr}(\Phi(C))]^2[\mathrm{tr}(\Phi(C)(A+B))]^{-1} \le k\mathrm{tr}\big[\Phi(C)(A^{-1}+B^{-1})\big]. \tag{2.17}$$

If in (2.17) we take $A = B$, then we get

$$[\mathrm{tr}(\Phi(C))]^2[\mathrm{tr}(\Phi(C)A)]^{-1} \le k\mathrm{tr}\big[\Phi(C)A^{-1}\big].$$

To present our next result, we will need the following lemma.

Lemma 2.7 *Let T be a self-adjoint operator such that* $\alpha 1_{\mathcal{H}} \le T \le \beta 1_{\mathcal{H}}$ *for some real constant* $\beta \ge \alpha$ *and assume that* $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ *be a positive linear mapping satisfying* (1.6), *whose adjoint is* Φ^*. *Then for any strictly positive operator* $S \in \mathcal{B}_1(\mathcal{K}) \setminus \{0\}$ *we have*

$$\begin{aligned}0 &\leq \frac{\operatorname{tr}(S\Phi^*(T^2))}{\operatorname{tr}(\Phi(S))} - \left(\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\right)^2 \\ &\leq \frac{1}{2}(\beta-\alpha)\frac{1}{\operatorname{tr}(\Phi(S))}\operatorname{tr}\left(S\Phi^*\left(\left|T - \frac{(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}.1_{\mathcal{H}}\right|\right)\right) \\ &\leq \frac{1}{2}(\beta-\alpha)\left[\frac{\operatorname{tr}(S\Phi^*(T^2))}{\operatorname{tr}(\Phi(S))} - \left(\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\beta-\alpha)^2.\end{aligned} \tag{2.18}$$

Proof The first inequality follows from Choi–Davis Jensen's inequality (2.2) for the convex function $f(t) = t^2$. Now observe that

$$\begin{aligned}&\frac{1}{\operatorname{tr}(\Phi(S))}\operatorname{tr}\left(S\Phi^*\left(\left[T - \frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\right]\left[T - \frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right]\right)\right) \\ &\quad = \frac{1}{\operatorname{tr}(\Phi(S))}\operatorname{tr}\left(S\Phi^*\left(T\left[T - \frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right]\right)\right) \\ &\qquad - \frac{\beta+\alpha}{2}\frac{1}{\operatorname{tr}(\Phi(S))}\operatorname{tr}\left(S\Phi^*\left(\left[T - \frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right]\right)\right) \\ &\quad = \frac{\operatorname{tr}(S\Phi^*(T^2))}{\operatorname{tr}(\Phi(S))} - \left(\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\right)^2.\end{aligned} \tag{2.19}$$

Since $\operatorname{tr}(S\Phi^*(1_{\mathcal{H}})) = \operatorname{tr}(\Phi^*(1_{\mathcal{H}})S) = \operatorname{tr}(\Phi(S))$, we have

$$\frac{1}{\operatorname{tr}(\Phi(S))}\operatorname{tr}\left(S\Phi^*\left(\left[T - \frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right]\right)\right) = 0.$$

Now, since $\alpha 1_{\mathcal{H}} \leq T \leq \beta 1_{\mathcal{H}}$, we infer

$$\left|T - \frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\right| \leq \frac{1}{2}(\beta-\alpha)\cdot 1_{\mathcal{H}},$$

which implies that

$$\left\|T - \frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\right\| \leq \frac{1}{2}(\beta-\alpha). \tag{2.20}$$

Taking the modulus in (2.19) and using the property (1.5), we obtain (2.21)

$$\begin{aligned}&\frac{\operatorname{tr}\big(S\Phi^*(T^2)\big)}{\operatorname{tr}\big(\Phi(S)\big)}-\left(\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\right)^2\\&\quad\le\frac{1}{\operatorname{tr}\big(\Phi(S)\big)}\operatorname{tr}\left(\Phi(S)\left|\Big(T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\Big)\Big(T-\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big)\right|\right).\end{aligned}\tag{2.21}$$

Put $v=T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}$ and $u=T-\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}$. Let $vu=w'|vu|$ be the polar decomposition of vu, where w' is a unique partial isometry on $\mathcal{H}$, and $w''=w'^*vw'$. Then $|vu|=w'^*vu=w''|u|$. Hence,

$$|vu|^2=|u|w''^*w''|u|\le|u|^2\|w''\|^2\le|u|^2\|v\|^2,$$

so

$$|vu|\le|u|\|v\|.\tag{2.22}$$

Now by (2.22), (2.20) and using the property (1.4), we get

$$\begin{aligned}&\frac{1}{\operatorname{tr}\big(\Phi(S)\big)}\operatorname{tr}\left(\Phi(S)\left|\Big(T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\Big)\Big(T-\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big)\right|\right)\\&\quad\le\frac{1}{2}(\beta-\alpha)\frac{1}{\operatorname{tr}\big(\Phi(S)\big)}\operatorname{tr}\left(\Phi(S)\left|T-\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right),\end{aligned}\tag{2.23}$$

this proves the first part of (2.18).

Using Schwarz's inequality (1.1), one can obtain

$$\begin{aligned}&\operatorname{tr}\left(\Phi(S)\left|T-\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right)\\&\quad\le\operatorname{tr}\left(\left|\Phi(S)\left|T-\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right|\right)\\&\quad=\operatorname{tr}\left(\left|(\Phi(S))^{\frac{1}{2}}(\Phi(S))^{\frac{1}{2}}\left|T-\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right|\right)\\&\quad\le\big[\operatorname{tr}\big(\Phi(S)\big)\big]^{\frac{1}{2}}\left[\operatorname{tr}\left(\left|(\Phi(S))^{\frac{1}{2}}\left|T-\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right|^2\right)\right]^{\frac{1}{2}}.\end{aligned}\tag{2.24}$$

Observe that

$$\begin{aligned}&\operatorname{tr}\left(\left|(\Phi(S))^{\frac{1}{2}}\left|T-\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right|\right|\right)^2\\&\quad=\operatorname{tr}\left(\left|T-\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right|(\Phi(S))^{\frac{1}{2}}(\Phi(S))^{\frac{1}{2}}\left|T-\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right|\right)\\&\quad=\left(\frac{\operatorname{tr}(\Phi(S)T^2)}{\operatorname{tr}(\Phi(S))}-\left(\frac{\operatorname{tr}(\Phi(S)T)}{\operatorname{tr}(\Phi(S))}\right)^2\right)\operatorname{tr}(\Phi(S)).\end{aligned}\tag{2.25}$$

By (2.24) and (2.25), we can write

$$\begin{aligned}&\frac{1}{\operatorname{tr}(\Phi(S))}\operatorname{tr}\left(S\Phi^*\left(\left|T-\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right|\right)\right)\\&\quad\le\left[\frac{\operatorname{tr}(S\Phi^*(T^2))}{\operatorname{tr}(\Phi(S))}-\left(\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\right)^2\right]^{\frac{1}{2}}\end{aligned}\tag{2.26}$$

and by (2.21) and (2.26), we have

$$\frac{\operatorname{tr}(S\Phi^*(T^2))}{\operatorname{tr}(\Phi(S))}-\left(\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\right)^2\le\frac{1}{2}(\beta-\alpha)\left[\frac{\operatorname{tr}(S\Phi^*(T^2))}{\operatorname{tr}(\Phi(S))}-\left(\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\right)^2\right]^{\frac{1}{2}}$$

which implies that

$$\left[\frac{\operatorname{tr}(S\Phi^*(T^2))}{\operatorname{tr}(\Phi(S))}-\left(\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\right)^2\right]^{\frac{1}{2}}\le\frac{1}{2}(\beta-\alpha).$$

Therefore, by (2.26), we get

$$\begin{aligned}&\frac{1}{\operatorname{tr}(\Phi(S))}\operatorname{tr}\left(S\Phi^*\left(\left|T-\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\cdot 1_{\mathcal{H}}\right|\right)\right)\\&\quad\le\left[\frac{\operatorname{tr}(S\Phi^*(T^2))}{\operatorname{tr}(\Phi(S))}-\left(\frac{\operatorname{tr}(S\Phi^*(T))}{\operatorname{tr}(\Phi(S))}\right)^2\right]^{\frac{1}{2}}\le\frac{1}{2}(\beta-\alpha)\end{aligned}$$

which proves the last part of (2.18). □

The following result provides reverses for the inequality (2.2).

Theorem 2.8 *Let* $\Phi:\mathcal{B}_1(\mathcal{K})\to\mathcal{B}_1(\mathcal{H})$ *be a positive linear mapping satisfying* (1.6), *whose adjoint is* Φ^*, A *be a self-adjoint operator on the Hilbert space* $\mathcal{H}$ *and assume that* $\operatorname{Sp}(A)\subseteq[m,M]$ *for some scalars* m, M *with* $m<M$.

If f *is a continuously differentiable convex function on* $[m,M]$ *and* $B\in\mathcal{B}_1(\mathcal{K})\setminus\{0\}$ *is a strictly positive operator, then we have*

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}\left(B\Phi^*\left(f(A)\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right) \\
&\le \frac{\operatorname{tr}\left(B\Phi^*\left(f'(A)A\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \cdot \frac{\operatorname{tr}\left(B\Phi^*\left(f'(A)\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \\
&=: L(\Phi, \Phi^*, f', B, A)
\end{aligned} \tag{2.27}
$$

and

$$
\begin{aligned}
&L(\Phi, \Phi^*, f', B, A) \\
&\overset{(i)}{\underset{(ii)}{\le}} \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \dfrac{\operatorname{tr}\left(B\Phi^*\left(\left|A - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \cdot 1_{\mathcal{H}}\right|\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \\ \frac{1}{2}[f'(M) - f'(m)] \left[\dfrac{\operatorname{tr}\left(B\Phi^*(A^2)\right)}{\operatorname{tr}\left(\Phi(B)\right)} - \left(\dfrac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)^2\right]^{\frac{1}{2}} \end{cases} \\
&\overset{(iii)}{\underset{(iv)}{\le}} \begin{cases} \frac{1}{2}(M - m) \dfrac{\operatorname{tr}\left(B\Phi^*\left(\left|f'(A) - \frac{\operatorname{tr}\left(B\Phi^*(f'(A))\right)}{\operatorname{tr}\left(\Phi(B)\right)} \cdot 1_{\mathcal{H}}\right|\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \\ \frac{1}{2}(M - m) \left[\dfrac{\operatorname{tr}\left(B\Phi^*\left([f'(A)]^2\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} - \left(\dfrac{\operatorname{tr}\left(B\Phi^*(f'(A))\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)^2\right]^{\frac{1}{2}} \end{cases} \\
&\le \frac{1}{4}[f'(M) - f'(m)](M - m).
\end{aligned} \tag{2.28}
$$

Proof By the gradient inequality we have

$$
f(s) - f(t) \le f'(s)(s - t)
$$

for any $s, t \in [m, M]$. This inequality implies in the operator order

$$
f(A) - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right) \cdot 1_{\mathcal{H}} \le f'(A)\left(A - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \cdot 1_{\mathcal{H}}\right),
$$

which is equivalent to

$$
\langle f(A)y, y\rangle - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\langle y, y\rangle \le \langle f'(A)Ay, y\rangle - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\langle f'(A)y, y\rangle \tag{2.29}
$$

for any $y \in \mathcal{H}$, which is of interest in itself as well. If we take $y = \left(\Phi(B)\right)^{\frac{1}{2}} e_i$ in (2.29), we obtain (2.27).

Since f is continuously convex on $[m, M]$, then f' is monotonic non-decreasing on $[m, M]$ and $f'(m) \le f'(t) \le f'(M)$ for any $t \in [m, M]$. We also observe that

$$\begin{aligned}
&\frac{1}{\operatorname{tr}(\Phi(B))}\operatorname{tr}\left(B\Phi^*\left(\left[f'(A) - \frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right]\left[A - \frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\cdot 1_{\mathcal{H}}\right]\right)\right)\\
&= \frac{1}{\operatorname{tr}(\Phi(B))}\operatorname{tr}\left(B\Phi^*\left(f'(A)\left[A - \frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\cdot 1_{\mathcal{H}}\right]\right)\right)\\
&\quad - \frac{f'(m)+f'(M)}{2}\frac{1}{\operatorname{tr}(\Phi(B))}\operatorname{tr}\left(B\Phi^*\left[A - \frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\cdot 1_{\mathcal{H}}\right]\right)\\
&= L(\Phi, \Phi^*, f', B, A).
\end{aligned} \tag{2.30}$$

Since

$$\left|f'(A) - \frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right| \le \frac{1}{2}[f'(M) - f'(m)]1_{\mathcal{H}},$$

we have

$$\left\|f'(A) - \frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right\| \le \frac{1}{2}[f'(M) - f'(m)]. \tag{2.31}$$

Then by taking the modulus in (2.30) and using the property (1.5), we get the following inequality

$$\begin{aligned}
&0 \le L(\Phi, \Phi^*, f', B, A)\\
&= \frac{1}{\operatorname{tr}(\Phi(B))}\left|\operatorname{tr}\left(B\Phi^*\left(\left[f'(A) - \frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right]\left[A - \frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\cdot 1_{\mathcal{H}}\right]\right)\right)\right|\\
&\le \frac{1}{\operatorname{tr}(\Phi(B))}\operatorname{tr}\left(\Phi(B)\left|\left(f'(A) - \frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right)\left(A - \frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\cdot 1_{\mathcal{H}}\right)\right|\right)\\
&\le \frac{1}{2}[f'(M) - f'(m)]\frac{1}{\operatorname{tr}(\Phi(B))}\operatorname{tr}\left(B\Phi^*\left(\left|A - \frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\cdot 1_{\mathcal{H}}\right|\right)\right).
\end{aligned} \tag{2.32}$$

By (2.22) and (2.31), we have the part (i) of (2.28).

If follows from Lemma 2.7 that

$$\frac{1}{\operatorname{tr}(\Phi(B))}\operatorname{tr}\left(B\Phi^*\left(\left|A-\frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\cdot 1_{\mathcal{H}}\right|\right)\right)$$
$$\leq\left[\frac{\operatorname{tr}(B\Phi^*(A^2))}{\operatorname{tr}(\Phi(B))}-\left(\frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\right)^2\right]^{\frac{1}{2}} \tag{2.33}$$
$$\leq\frac{1}{2}(M-m).$$

By applying (2.32) and (2.33) we get the part (*ii*) of (2.28). We observe that $L(\Phi,\Phi^*,f',B,A)$ can be represented as

$$L(\Phi,\Phi^*,f',B,A)$$
$$=\frac{1}{\operatorname{tr}(\Phi(B))}\operatorname{tr}\left(B\Phi^*\left(\left[f'(A)-\frac{\operatorname{tr}\left(B\Phi^*\left(f'(A)\right)\right)}{\operatorname{tr}(\Phi(B))}\cdot 1_{\mathcal{H}}\right]\left[A-\frac{m+M}{2}1_{\mathcal{H}}\right]\right)\right).$$

Applying a similar argument as above for this representation, we get parts (*iii*) and (*iv*) of (2.28). □

Remark 2.9 For the convex function $f(t)=t^p$, $p\geq 1$, we get the following inequalities of interest:

$$0\leq\frac{\operatorname{tr}(B\Phi^*(A^p))}{\operatorname{tr}(\Phi(B))}-\left(\frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}\right)^p\leq\frac{1}{4}p\left(M^{p-1}-m^{p-1}\right)(M-m) \tag{2.34}$$

for some constants m, M with $M>m>0$ and $mI\leq A\leq MI$.

3 The Hölder type trace inequality for an operator mean

The main purpose of this section is to find new refinements of some power type trace inequalities via reverse and refinement of Young's inequality. Also we show new versions of the Hölder type trace inequality for an operator mean σ_f.

Let a and b be positive numbers. The famous Young inequality states that $a^{1-v}b^v\leq(1-v)a+vb$ for every $0\leq v\leq 1$. Liao et al. [18] obtained reverse of the Young inequality as

$$(1-v)a+vb\leq k^R(h)a^{1-v}b^v, \tag{3.1}$$

where $h=\frac{b}{a}$, $h>0$, $k(h)=\frac{(h+1)^2}{4h}$ is the Kantorovich constant and $R=\max\{1-v,v\}$. Note that the function k is decreasing on (0, 1) and increasing on $[1,\infty)$, $k(h)\geq 1$ and $k(h)=k\left(\frac{1}{h}\right)$ for any $h>0$. Recently, Moradi et al. [19] obtained the following refinement of inequality (3.1):

$$(1-v)a+vb\leq k^R(h)\exp\left(\left(\frac{v(1-v)}{2}-\frac{R}{4}\right)\left(\frac{a-b}{D}\right)^2\right)a^{1-v}b^v, \tag{3.2}$$

where $R = \max\{v, 1-v\}$ and $D = \max\{a, b\}$. In [15], it has been shown that if $0 < mI \le A \le MI$ and $B \in \mathcal{B}_1(\mathcal{H})$, $B > 0$, then

$$\left(\frac{\operatorname{tr}(BA^p)}{\operatorname{tr}(B)}\right)^{\frac{1}{p}} \le k^{\max\{\frac{1}{p},\frac{1}{q}\}}\left(\left(\frac{M}{m}\right)^p\right)\frac{\operatorname{tr}(BA))}{\operatorname{tr}(B)}, \tag{3.3}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In the following theorem, we present a refinement of inequality (3.3) for positive linear map.

Theorem 3.1 *Let $\Phi : \mathcal{B}_1(\mathcal{K}) \to \mathcal{B}_1(\mathcal{H})$ be a positive linear mapping satisfying (1.6), whose adjoint is Φ^*, $mI \le A \le MI$ for some constants m, M with $M > m > 0$ and $B \in \mathcal{B}_1(\mathcal{K})$, $B > 0$. Then for any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\begin{aligned}&\left(\frac{\operatorname{tr}(B\Phi^*(A^p))}{\operatorname{tr}(\Phi(B))}\right)^{\frac{1}{p}}\\ &\le k^{\max\{\frac{1}{p},\frac{1}{q}\}}\left(\left(\frac{M}{m}\right)^p\right)\exp\left(\left(\frac{1}{2pq} - \frac{\max\{\frac{1}{p},\frac{1}{q}\}}{4}\right)\left(\left(\frac{m}{M}\right)^p - 1\right)^2\right)\frac{\operatorname{tr}(B\Phi^*(A))}{\operatorname{tr}(\Phi(B))}.\end{aligned} \tag{3.4}$$

Proof Assume that $v \in (0,1)$, $a, b \in [m, M]$, $D = \max\{a, b\}$ and $R = \max\{v, 1-v\}$. Since

$$\frac{(a-b)^2}{D^2} = \frac{(a-b)^2}{\max^2\{a,b\}} = \left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2 \quad \text{and} \quad \frac{v(1-v)}{2} - \frac{R}{4} \le 0$$

for each $0 \le v \le 1$, we can write

$$\exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{a-b}{D}\right)^2\right) \le \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{m}{M} - 1\right)^2\right). \tag{3.5}$$

Let $0 < m < M$, then $\frac{m}{M} < 1 < \frac{M}{m}$ and $\frac{m}{M} \le \frac{a}{b} \le \frac{M}{m}$. Indeed, for $\frac{m}{M} \le \frac{a}{b} < 1$ and $1 < \frac{a}{b} \le \frac{M}{m}$ we have $k^R\left(\frac{b}{a}\right) = k^R\left(\frac{a}{b}\right) \le k^R\left(\frac{M}{m}\right)$. Thus, for any $a, b \in [m, M]$, it follows from (3.2) and (3.5) that

$$(1-v)a + vb \le k^R\left(\frac{M}{m}\right)\exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{m}{M} - 1\right)^2\right)a^{1-v}b^v. \tag{3.6}$$

According to the hypothesis for $p > 1$ we have $m^pI \le A^p \le M^pI$. Now, applying functional calculus for $v = \frac{1}{p}$ in (3.6), we obtain

$$\left(1-\frac{1}{p}\right)t\langle y,y\rangle+\frac{1}{p}\langle A^p y,y\rangle$$
$$\leq k^{\max\{\frac{1}{p},\frac{1}{q}\}}\left(\left(\frac{M}{m}\right)^p\right)\exp\left(\left(\frac{1}{2pq}-\frac{\max\{\frac{1}{p},\frac{1}{q}\}}{4}\right)\left(\left(\frac{m}{M}\right)^p-1\right)^2\right)t^{1-\frac{1}{p}}\langle Ay,y\rangle, \tag{3.7}$$

for any $y\in\mathcal{H}$ and $t\in[m^p,M^p]$. If we take in (3.7) $y=\left(\Phi(B)\right)^{\frac{1}{2}}e_i$ and summing over $i\in I$, then by the properties of trace and equality (1.7), we get

$$\frac{\mathrm{tr}(B\Phi^*(A^p))}{\mathrm{tr}(\Phi(B))}\leq k^{\max\{\frac{1}{p},\frac{1}{q}\}}\left(\left(\frac{M}{m}\right)^p\right)\exp\left(\left(\frac{1}{2pq}-\frac{\max\{\frac{1}{p},\frac{1}{q}\}}{4}\right)\left(\left(\frac{m}{M}\right)^p-1\right)^2\right)$$
$$\times\,\mathrm{tr}\left(\frac{B\Phi^*(A^p)}{\mathrm{tr}\left(\Phi(B)\right)}\right)^{1-\frac{1}{p}}\frac{\mathrm{tr}\left(B\Phi^*(A)\right)}{\mathrm{tr}\left(\Phi(B)\right)},$$

that is, putting $t=\frac{\mathrm{tr}\left(B\Phi^*(A^p)\right)}{\mathrm{tr}\left(\Phi(B)\right)}\in[m^p,M^p]$, so we have inequality (3.4). □

Remark 3.2 To see that inequality (3.4) refines the inequality (3.3), let $p=q=2$, we have

$$\mathrm{tr}\left(B\Phi^*(A^2)\right)\mathrm{tr}\left(\Phi(B)\right)\leq k\left(\left(\frac{M}{m}\right)^2\right)\left[\mathrm{tr}\left(B\Phi^*(A)\right)\right]^2.$$

On the other hand, since $\left(\frac{1}{2pq}-\frac{\max\{\frac{1}{p},\frac{1}{q}\}}{4}\right)\leq 0$ for $p,q>2$ we have

$$\left(\frac{\mathrm{tr}(B\Phi^*(A^p))}{\mathrm{tr}(\Phi(B))}\right)^{\frac{1}{p}}$$
$$\leq k^{\max\{\frac{1}{p},\frac{1}{q}\}}\left(\left(\frac{M}{m}\right)^p\right)\exp\left(\left(\frac{1}{2pq}-\frac{\max\{\frac{1}{p},\frac{1}{q}\}}{4}\right)\left(\left(\frac{m}{M}\right)^p-1\right)^2\right)\frac{\mathrm{tr}\left(B\Phi^*(A)\right)}{\mathrm{tr}\left(\Phi(B)\right)}$$
$$\leq k^{\max\{\frac{1}{p},\frac{1}{q}\}}\left(\left(\frac{M}{m}\right)^p\right)\frac{\mathrm{tr}\left(B\Phi^*(A)\right)}{\mathrm{tr}\left(\Phi(B)\right)}.$$

In the next theorem, we extend inequalities (1.2) and (1.3) for any arbitrary operator mean σ_f.

Theorem 3.3 *Let $A,B,C\in\mathcal{B}(\mathcal{H})$ such that $A,B>0$.*

(i) *If $A,C\in\mathcal{B}_1(\mathcal{H})$, then $C(A\sigma_f B)$, $|C^*|^2A^{\frac{1}{2}}f^2(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}\in\mathcal{B}_1(\mathcal{H})$ and*

$$|\mathrm{tr}(C(A\sigma_f B))|^2\leq\mathrm{tr}\left(|C^*|^2A^{\frac{1}{2}}f^2(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}\right)\mathrm{tr}(A). \tag{3.8}$$

(ii) *If $A,f(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})\in\mathcal{B}_1(\mathcal{H})$, then $A\sigma_f B$, $A^{\frac{1}{2}}f^2(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}\in\mathcal{B}_1(\mathcal{H})$ and*

$$|\mathrm{tr}(A\sigma_f B)|^2 \leq \mathrm{tr}(A^{\frac{1}{2}} f^2 (A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}) \mathrm{tr}(A).$$

Proof (*i*) Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$ and F a finite part of I, then

$$\begin{aligned} |\langle (A\sigma_f B)Ce_i, e_i\rangle| &= |\langle A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} Ce_i, e_i\rangle| \\ &= |\langle f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} Ce_i, A^{\frac{1}{2}} e_i\rangle| \\ &\leq \| f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} Ce_i\| \|A^{\frac{1}{2}} e_i\| \\ &= \langle C^* A^{\frac{1}{2}} f^2 (A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} Ce_i, e_i\rangle^{\frac{1}{2}} \langle Ae_i, e_i\rangle^{\frac{1}{2}}, \end{aligned} \tag{3.9}$$

for any $i \in I$. Using the generalized triangle inequality for the modulus and the Cauchy–Bunyakovsky–Schwarz inequality for finite sums, we have from (3.9) that

$$\begin{aligned} \Big|\sum_{i\in F} \langle (A\sigma_f B)Ce_i, e_i\rangle\Big| &\leq \sum_{i\in F} |\langle (A\sigma_f B)Ce_i, e_i\rangle| \\ &\leq \sum_{i\in F} \langle C^* A^{\frac{1}{2}} f^2 (A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} Ce_i, e_i\rangle^{\frac{1}{2}} \langle Ae_i, e_i\rangle^{\frac{1}{2}} \\ &\leq \left[\sum_{i\in F} \left(\langle C^* A^{\frac{1}{2}} f^2 (A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} Ce_i, e_i\rangle^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \left[\sum_{i\in F} (\langle Ae_i, e_i\rangle^{\frac{1}{2}})^2\right]^{\frac{1}{2}} \\ &= \left[\sum_{i\in F} \langle C^* A^{\frac{1}{2}} f^2 (A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} Ce_i, e_i\rangle\right]^{\frac{1}{2}} \left[\sum_{i\in F} \langle Ae_i, e_i\rangle\right]^{\frac{1}{2}}, \end{aligned} \tag{3.10}$$

for any F a finite part of I.

By the properties of trace and (3.10), we obtain (3.8). □

Letting $f(t) = t^{\frac{1}{2}}$, in Theorem 3.3 we get

Corollary 3.4 *Let $A, B, C \in \mathcal{B}(\mathcal{H})$ such that $A, B > 0$. If $A, C \in \mathcal{B}_1(\mathcal{H})$, then*

$$|\mathrm{tr}(C(A\sharp B))|^2 \leq \mathrm{tr}(|C^*|^2 B)\mathrm{tr}(A).$$

In particular, for $A, B \in \mathcal{B}_1(\mathcal{H})$

$$|\mathrm{tr}(A\sharp B)|^2 \leq \mathrm{tr}(B)\mathrm{tr}(A).$$

4 Some examples

We start this section with a well-known theorem.

Theorem 4.1 [6, *First Representation Theorem of Kraus*] *Given an operator $\Phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$, there exists a finite or countable family $\{A_j : j \in J\}$ of bounded linear operators on $\mathcal{H}$, satisfying*

$$\sum_{j\in J_0} A_j^* A_j \le I, \tag{4.1}$$

for all finite $J_0 \subset J$, *such that for every* $A \in \mathcal{B}_1(\mathcal{H})$ *and every* $B \in \mathcal{B}(\mathcal{H})$ *one has*

$$\Phi(A) = \sum_{j\in J} A_j A A_j^*, \tag{4.2}$$

respectively,

$$\Phi^*(B) = \sum_{j\in J} A_i^* B A_j \tag{4.3}$$

and

$$\Phi^*(I) = \sum_{j\in J} A_j^* A_j. \tag{4.4}$$

Conversely, if a countable family $\{A_j : j \in J\}$ *of bounded linear operators on* $\mathcal{H}$ *is given which satisfies* (4.1) *then* Eq. (4.2) *defines an operation* Φ *whose adjoint* Φ^* *is given by* (4.3) *and* $\Phi^*(I)$ *defines by* (4.4).

Employing the above theorem and Theorem 2.8 for some examples of convex functions, we can present new versions of trace inequalities.

Example 4.2 Consider the power function $f : (0,\infty) \to (0,\infty)$, $f(t) = t^r$ with $t \in \mathbb{R} \setminus \{0\}$. For $r \in (-\infty, 0) \cup [1,\infty)$, f is convex and for $r \in (0,1)$, f is concave. Let $r \ge 1$ and A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $0 \le m < M$. If $B \in \mathcal{B}_1(\mathcal{H}) \setminus \{0\}$ is a strictly positive operator and $\{A_j : j \in J\}$ is a countable family of bounded linear operators on $\mathcal{H}$ which satisfies (4.1), then by Theorem 2.8 and equality (1.7) respectively, we get

$$\begin{aligned}
0 &\le \frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A^r A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \left(\frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right)^r \\
&\le r\left[\frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A^r A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{r\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A^{r-1} A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right] \\
&\le \begin{cases}
\dfrac{1}{2} r(M^{r-1} - m^{r-1}) \dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* \left|A - \dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \cdot 1_{\mathcal{H}}\right| A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \\
\dfrac{1}{2} r(M^{r-1} - m^{r-1}) \left[\dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A^2 A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \left(\dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right)^2\right]^{\frac{1}{2}}
\end{cases}
\end{aligned}$$

$$\leq \begin{cases} \dfrac{1}{2}r(M-m)\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*\left|A-\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*A^{r-1}A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\cdot 1_{\mathcal{H}}\right|A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)} \\ \dfrac{1}{2}r(M-m)\left[\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*A^{2(r-1)}A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}-\left(\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*A^{r-1}A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\right)^2\right]^{\frac{1}{2}} \end{cases} \quad (4.5)$$

$$\leq \frac{1}{4}r(M^{r-1}-m^{r-1})(M-m).$$

Example 4.3 Consider the convex function $f:(0,\infty)\to(0,\infty), f(t)=-\ln t$ and let A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with $0<m<M$. If $B\in\mathcal{B}_1(\mathcal{H})\setminus\{0\}$ is a strictly positive operator and $\{A_j: j\in J\}$ is a countable family of bounded linear operators on $\mathcal{H}$ which satisfies (4.1), then by Theorem 2.8 and equality (1.7), respectively, we get

$$0\leq \ln\left(\frac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*AA_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\right)-\frac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*\ln AA_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}$$

$$\leq \frac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*AA_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\cdot\frac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*A^{-1}A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}-1$$

$$\leq \begin{cases} \dfrac{M-m}{2mM}\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*\left|A-\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*AA_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\cdot 1_{\mathcal{H}}\right|A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)} \\ \dfrac{M-m}{2mM}\left[\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*A^2A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}-\left(\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*AA_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\right)^2\right]^{\frac{1}{2}} \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{2}(M-m)\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*\left|A^{-1}-\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*A^{-1}A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}a_JBA_j^*\right)}\cdot 1_{\mathcal{H}}\right|A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)} \\ \dfrac{1}{2}(M-m)\left[\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*A^{-2}A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}-\left(\dfrac{\operatorname{tr}\left(B\sum_{j\in J}A_j^*A^{-1}A_j\right)}{\operatorname{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\right)^2\right]^{\frac{1}{2}} \end{cases}$$

$$\leq \frac{(M-m)^2}{4mM}.$$

Example 4.4 Consider the convex function $f(t) = t\ln t$ and let A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\mathrm{Sp}(A) \subseteq [m, M]$ for some m, M with $0 < m < M$. If $B \in \mathcal{B}_1(\mathcal{H}) \setminus \{0\}$ is a strictly positive operator and $\{A_j : j \in J\}$ is a countable family of bounded linear operators on $\mathcal{H}$ which satisfies (4.1), then by Theorem 2.8 and equality (1.7), respectively, we have

$$\begin{aligned}
0 &\leq \frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A\ln A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \ln\left(\frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right) \\
&\leq \frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A\ln A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \cdot \frac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* \ln(eA) A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \\
&\leq \begin{cases} \dfrac{1}{2}\ln\left(\dfrac{M}{m}\right) \dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* \left| A - \dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \cdot 1_{\mathcal{H}} \right| A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \\[2ex] \dfrac{1}{2}\ln\left(\dfrac{M}{m}\right) \left[\dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A^2 A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \left(\dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right)^2\right]^{\frac{1}{2}} \end{cases} \\
&\leq \begin{cases} \dfrac{1}{2}(M-m) \dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* \left| \ln(eA) - \dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* \ln(eA) A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \cdot 1_{\mathcal{H}} \right| A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \\[2ex] \dfrac{1}{2}(M-m) \left[\dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* [\ln(eA)]^2 A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \left(\dfrac{\mathrm{tr}\left(B\sum_{j\in J} A_j^* \ln(eA) A_j\right)}{\mathrm{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right)^2\right]^{\frac{1}{2}} \end{cases} \\
&\leq \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).
\end{aligned}$$

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