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ORIGINAL ARTICLE

Algebraic lattices of solvably saturated formations and their applications

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Abstract

In each group G, we select a system of subgroups $\tau(G)$ and say that τ is a subgroup functor if $G \in \tau(G)$ for every group G, and for every epimorphism $\varphi : A \to B$ and any $H \in \tau(A)$ and $T \in \tau(B)$, we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$. We consider only subgroup functors τ such that for any group G all subgroups of $\tau(G)$ are subnormal in G. For any set of groups $\mathfrak{X}$, the symbol $s_{\tau}(\mathfrak{X})$ denotes the set of groups H such that $H \in \tau(G)$ for some group $G \in \mathfrak{X}$. A formation $\mathfrak{F}$ is τ-closed if $s_{\tau}(\mathfrak{F}) = \mathfrak{F}$. The Frattini subgroup $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G. A formation $\mathfrak{F}$ is said to be *solvably saturated* if it contains each group G with $G/\Phi(N) \in \mathfrak{F}$ for some solvable normal subgroup N of G. Composition formations are precisely solvably saturated formations. It is shown that the lattice of all τ-closed totally composition formations is algebraic.

Keywords Finite group · Subgroup functor · Formation of groups · Satellite of formation · Totally composition formation · Algebraic lattice of formations · Formal language · Hypergroup

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1 Introduction, definitions and the main result

A *variety* of groups may be defined as a nonempty class of groups closed under taking homomorphic images and subcartesian products [22], formations extend this notion.

Definition 1 [11] A *formation* is a class of finite groups $\mathfrak{F}$ satisfying the following two conditions:

(1) if $G \in \mathfrak{F}$, then $G/N \in \mathfrak{F}$, and
(2) if $G/N_1, G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$,

for any normal subgroups N, N_1 and N_2 of G.

The theory of saturated formations introduced by Gaschütz [13] in 1962 became a fundamental part of group theory by now. Further research showed that formations are of general algebraic nature and can be applied to the study of infinite groups, Lie algebras, universal algebras and even of general algebraic systems, see, e.g., the books [3, 11, 12, 15, 29, 42]. Recall that the Frattini subgroup $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G.

Definition 2 [11] A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$.

A well-known result states that any formation is saturated iff it is local [11, Gaschütz–Lubeseder–Schmid]. This nice circumstance makes saturated formations one of the most suitable classes for a better understanding of group structure. Composition (or Baer-local) formations form a broader than local formations family of classes. By Baer's theorem, composition formations are precisely solvably saturated formations [11, p. 373].

Definition 3 [15] A formation $\mathfrak{F}$ is said to be *solvably saturated* if it contains each group G with $G/\Phi(N) \in \mathfrak{F}$ for some solvable normal subgroup N of G. Any saturated formation is solvably saturated.

By $\mathbb{P}$, the set of all primes is denoted. For any $p \in \mathbb{P}$, the subgroup $C^p(G)$ of a group G is the intersection of the centralizers of all the abelian p-chief factors of G (we assume $C^p(G) = G$ if G has no abelian p-chief factors). For every set of groups $\mathfrak{X}$, we write $\mathrm{Com}(\mathfrak{X})$ to denote the class of all groups L such that L is isomorphic to some abelian composition factors of some group in $\mathfrak{X}$. If $\mathfrak{X}$ is the set of one group G, then we write $\mathrm{Com}(G)$ instead of $\mathrm{Com}(\mathfrak{X})$; $\pi(\mathfrak{X})$ is the set of all primes dividing the order of all groups $G \in \mathfrak{X}$. The symbol $R(G)$ denotes the product of all solvable normal subgroups of G.

Definition 4 [15] Consider a function f of the form

$$f : \mathbb{P} \cup \{0\} \to \{\text{formations of groups}\}, \qquad (*)$$

and the class of groups

$$\mathrm{CLF}(f) = (G \mid G/R(G) \in f(0);\; G/C^p(G) \in f(p) \text{ for all } p \in \pi(\mathrm{Com}(G))).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = \mathrm{CLF}(f)$ for a function f of the form $(*)$, then $\mathfrak{F}$ is said to be *composition* formation, and f is said to be a *composition satellite* of $\mathfrak{F}$.

If the values of composition satellites of some formation are themselves composition formations, then this circumstance leads to the following natural definition.

Definition 5 [31] Every formation is 0-multiply composition; for $n > 0$, a formation $\mathfrak{F}$ is called *n-multiply composition* if $\mathfrak{F} = \mathrm{CLF}(f)$, and all nonempty values of f are $(n-1)$-multiply composition formations. (For $n = 1$ we deal with the case of composition formations.) A formation is called *totally composition* if it is n-multiply composition for all positive integers n. The most well-known solvably saturated formations are totally composition. In particular, the formations $\emptyset$ and (1) are totally composition.

The concept of subgroup functor turned out to be useful in group theory, see, e.g., [4, 19, 30] .

Definition 6 [30] In each group G we select a system of subgroups $\tau(G)$ and say that τ is a *subgroup functor* if

1. $G \in \tau(G)$ for every group G;
2. for every epimorphism $\varphi : A \to B$ and any $H \in \tau(A)$ and $T \in \tau(B)$, we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

If $\tau(G) = \{G\}$ then the functor τ is called *trivial*.

For any set of groups $\mathfrak{X}$, the symbol s_τ denotes the set of groups H such that $H \in \tau(G)$ for some group $G \in \mathfrak{X}$. A class of groups $\mathfrak{F}$ is called *τ-closed* if $s_\tau(\mathfrak{F}) = \mathfrak{F}$. For instance, $\mathfrak{F}$ is called s-closed [11] (or hereditary) if it contains all the subgroups of $G \in \mathfrak{F}$ (i.e., $\tau(\mathfrak{F}) = s(\mathfrak{F})$), and s_n-closed [11] (or normally hereditary) if it contains all the normal subgroups of $G \in \mathfrak{F}$ (i.e., $\tau(\mathfrak{F}) = s_n(\mathfrak{F})$). A formation $\mathfrak{F}$ is *τ -closed* if $\tau(G) \subseteq \mathfrak{F}$ for every group G of $\mathfrak{F}$. In the present paper, we consider only subgroup functors τ such that for any group G all subgroups of $\tau(G)$ are subnormal in G.

Definition 7 A *complete lattice* is a partially ordered set in which all subsets have both a supremum (join) and an infimum (meet). Let L be a complete lattice and let a be an element of L. Then a is called *compact* [14] iff $a \leqslant \vee X$ for some $X \subseteq L$ implies that $a \leqslant \vee Y$ for some finite $Y \subseteq X$. Compact elements are important in domain theory [1] which has major applications for functional programming languages. A complete lattice is *algebraic* iff every element is the join of compact elements.

Properties of (algebraic) lattices of classes of finite groups were investigated especially in the recent thirty years in a large number of papers of many authors. Skiba [30, Question 4.4.6] posed the following question: *Let τ be a subgroup functor (in Skiba's sence), is the lattice of all τ-closed totally local formations algebraic?* Safonov [25] gave a positive answer on this question. It is already known by now that the lattices of all solvable totally local formations [44] and all τ-closed n-multiply ω-composition formations are algebraic [43], and results in [9, 27, 28, 33–35, 37] confirm the importance of studying of algebraic lattices of formations. For instance, recently it was shown that both the lattice of all n-multiply σ-local formations [9] and the lattice of all τ-closed totally ω-saturated formations [27] are algebraic. The lattice of all n-multiply composition formations is algebraic [31, 43]. The following problem was solved in [38]: *Is the lattice of all totally composition formations algebraic* (see [31, Problem 1])? Subgroup functors are closely related to classes of algebraic systems [19, 30], and our theorem extends the mentioned result for functor-closed classes of finite groups.

Theorem *The lattice c_∞^τ of all τ-closed totally composition formations of finite groups is algebraic.*

The paper is organized as follows—basic concepts are introduced in the second section, used lemmas are given in the third section, and the proof of the main result is discussed in the fourth section. In particular, it is shown that τ-closed one-generated totally composition formations are compact elements of the lattice c_∞^τ. Applications of this result and some ideas for further research are presented in the last fifth section. We consider only finite groups in this paper, notations and terminologies are standard and borrowed from the books [11, 15, 30, 31, 42].

2 Preliminaries

We write $(\mathfrak{X})$ to denote the intersection of all classes of groups containing a set of groups $\mathfrak{X}$. Let G be a group. Then for any nonempty formation $\mathfrak{F}$, we denote by $G^{\mathfrak{F}}$ the $\mathfrak{F}$-residual of G [11] (i.e., the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{F}$).

The class

$$\mathfrak{M}\mathfrak{F} = \{G \mid G^{\mathfrak{F}} \in \mathfrak{M}\}$$

is the *product of formations* [11] $\mathfrak{M}$ and $\mathfrak{F} \neq \emptyset$. Consider an example. Let $\mathfrak{M} = \mathfrak{N}^n\mathfrak{H}$ and $\mathfrak{F} = \mathfrak{N}_p\mathfrak{M}$, where the formation $\mathfrak{H} \neq \emptyset$ is not saturated. By [30, Example 1.3.3] and [31, Corollary 4], formations $\mathfrak{M}$ and $\mathfrak{F}$ are n-multiply composition.

Let Θ be a set of formations. A formation in Θ is called a *Θ-formation*. If the intersection of every set of Θ-formations belongs to Θ and there is a Θ-formation $\mathfrak{F}$ such that $\mathfrak{M} \subseteq \mathfrak{F}$ for every other Θ-formation $\mathfrak{M}$, then Θ is called a *complete lattice of formations* (see [30, p. 151]). Every complete lattice of formations is a complete lattice in the ordinary sense.

Remark 1 [38] We note that various sets of formations are complete lattices. For instance, such is the set of all saturated formations [30, p. 151], and the set of all composition formations [29, p. 97]. Moreover, for all positive integers n, the set of all n-multiply composition formations c_n, and the set of all totally composition formations $c_\infty = \bigcap_{n=1}^{\infty} c_n$ are complete lattices of formations; see [31, p. 904]. Then using [43, Lemma 3.1], we conclude that the set of all τ-closed totally composition formations c_{∞}^{τ} is a complete lattices of formations, too. Moreover, it was proved in [32] that the lattice of all saturated formations is a complete sublattice of the lattice of all composition formations.

For a complete lattice Θ, $\Theta\mathrm{form}\,\mathfrak{X}$ is the intersection of all Θ-formations containing a set of groups $\mathfrak{X}$, and $c_{\infty}^{\tau}\mathrm{form}\,\mathfrak{X}$ is the intersection of all τ-closed totally composition formations containing a set of groups $\mathfrak{X}$. Given a arbitrary set of Θ-formations $\{\mathfrak{F}_i \mid i \in I\}$, we denote $\bigvee_{\Theta}(\mathfrak{F}_i \mid i \in I) = \Theta\mathrm{form}(\bigcup_{i\in I}\mathfrak{F}_i)$. In particular, we write $\bigvee_{\infty}^{\tau}(\mathfrak{F}_i \mid i \in I) = c_{\infty}^{\tau}\mathrm{form}(\bigcup_{i\in I}\mathfrak{F}_i)$. If $\mathfrak{M}$ and $\mathfrak{H}$ are Θ-formations, then $\mathfrak{M}\bigcap\mathfrak{H}$ is the greatest lower bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in a complete lattice Θ; and $\mathfrak{M}\bigvee_{\Theta}\mathfrak{H}$ is the least upper bound for $\{\mathfrak{M}, \mathfrak{H}\}$ in Θ.

Lemma 1 [31, *Lemma* 2] *Let* $\mathfrak{F} = \bigcap_{i\in I}\mathfrak{F}_i$, *where* $\mathfrak{F}_i = \mathrm{CLF}(f_i)$. *Then* $\mathfrak{F} = \mathrm{CLF}(f)$, *where* $f = \bigcap_{i\in I} f_i$.

Let $\{f_i \mid i \in I\}$ be a set of Θ-valued functions of the form $(*)$. Then by the join $\bigvee_{\Theta}(f_i \mid i \in I)$ we denote a function f such that

$$f(a) = \Theta\mathrm{form}\left(\bigcup_{i\in I} f_i(a)\right)$$

for all $a \in \mathbb{P} \cup \{0\}$. Let $\{f_i \mid i \in I\}$ be the set of all composition c_{∞}^{τ}-valued satellites of a formation $\mathfrak{F}$. Since c_{∞}^{τ} is a complete lattice, using Lemma 1, we conclude that $f = \bigcap_{i\in I} f_i$ is a composition c_{∞}^{τ}-valued satellite of $\mathfrak{F}$. This satellite is called *minimal* [15].

3 Lemmata

To obtain proof of our main result, we cite here some lemmata.

Lemma 2 *Let f_i be the minimal c_{∞}^{τ}-valued composition satellite of a formation $\mathfrak{F}_i$, where $i \in I$. Then $f = \bigvee_{\infty}^{\tau}(f_i \mid i \in I)$ is the minimal c_{∞}-valued composition satellite of formation $\mathfrak{F} = \bigvee_{\infty}^{\tau}(\mathfrak{F}_i \mid i \in I)$.*

Proof We obtain the assertion by direct calculation. □

Lemma 3 *Let $\mathfrak{X}$ be a nonempty set of groups, $\mathfrak{F} = c_{\infty}^{\tau}\mathrm{form}\,\mathfrak{X}$, and $\pi = \pi(\mathrm{Com}(\mathfrak{X}))$, and let f be the minimal c_{∞}^{τ}-valued composition satellite of $\mathfrak{F}$. Then the following statements hold*:

1) $f(0) = c_{\infty}^{\tau}\mathrm{form}(G/R(G) \mid G \in \mathfrak{X})$;
2) $f(p) = c_{\infty}^{\tau}\mathrm{form}(G/C^p(G) \mid G \in \mathfrak{X})$ *for all* $p \in \pi$;
3) $f(p) = \varnothing$ *for all* $p \in \mathbb{P}\backslash\pi$;
4) *if* $\mathfrak{F} = \mathrm{CLF}(h)$ *and the satellite* h *is* c_{∞}^{τ}*-valued, then for all* $p \in \pi$ *we have*

$$f(p) = c_\infty^\tau\text{form}(G \mid G \in h(p) \cap \mathfrak{F} \text{ and } O_p(G) = 1), \text{ and}$$
$$f(0) = c_\infty^\tau\text{form}(G \mid G \in h(0) \cap \mathfrak{F} \text{ and } R(G) = 1).$$

Proof See the proof of Lemma 8 in [41] and Corollary 2.1 in [36]. □

Recall that a group G is called *monolithic* if it has a unique minimal normal subgroup (i.e., a monolith of G), and this is contained in every nontrivial normal subgroup of G. The *socle* $\text{Soc}(G)$ of $G \neq 1$ is the product of all minimal normal subgroups of G. A *semiformation* is a class of groups closed under taking homomorphic images.

Lemma 4 [41, *Lemma* 9] *Let A be a monolithic group with a nonabelian socle R, $\mathfrak{M}$ be a semiformation and $A \in c_n^\tau\text{form}\,\mathfrak{M}$, $n \geqslant 0$. Then $A \in \mathfrak{M}$.*

The symbol $\mathfrak{N}_p$ denotes the class of all p-groups, and $O_p(G)$ is the $\mathfrak{N}_p$-radical of G, where p is a prime.

Lemma 5 [31, *Lemma* 4] *Let $\mathfrak{F} = \text{CLF}(f)$. If $G/O_p(G) \in f(p) \cap \mathfrak{F}$ for some prime p, then $G \in \mathfrak{F}$.*

Let G be a group, and p be a prime. We use $Z_p \wr G$ to denote the regular wreath product of groups Z_p and G; see [11, p. 66].

Lemma 6 [15, *Lemma* 1.3, *p.* 250] *Let Z_p be a group of a prime order p, and G be a group with $O_p(G) = 1$. Suppose that $T = Z_p \wr G = [K]G$ is the regular wreath product, where K is the base group of T. Then $K = C^p(T) = O_p(T)$.*

We recall that $\mathfrak{S}_\pi$ is the class of all solvable π-groups. (If π is an empty set, then $\mathfrak{S}_\emptyset = (1)$.) This formation is saturated.

Lemma 7 *Let $\mathfrak{F}$ be a nonempty τ-closed formation. Then $\mathfrak{S}_\pi\mathfrak{F}$ is a τ-closed totally composition formation, where $\pi(\mathfrak{F}) \subseteq \pi \subseteq \mathbb{P}$.*

Proof The lemma can be obtained by Theorem 6 in [31] and the proof of Lemma 12 in [26]. □

Lemma 8 *Let $\mathfrak{F} = \bigvee_\infty^\tau(\mathfrak{F}_i \mid i \in I)$, where $\mathfrak{F}_i \in c_\infty^\tau$ for all $i \in I$, and let A be a monolithic $\mathfrak{F}$-group with a nonabelian socle R. Then $A \in \bigcup_{i \in I} \mathfrak{F}_i$.*

Proof Let $\pi = \pi(\mathfrak{F})$. Lemma 7 implies

$$\mathfrak{F} \subseteq \mathfrak{M} = \mathfrak{S}_\pi c_0^\tau\text{form}\left(\bigcup_{i \in I} \mathfrak{F}_i\right).$$

Thus, $A \in \mathfrak{M}$. Note that

$$A \in c_0^\tau \mathrm{form}\left(\bigcup_{i \in I} \mathfrak{F}_i\right)$$

since $R = \mathrm{Soc}(A)$ is nonabelian, and A belongs to $\bigcup_{i\in I} \mathfrak{F}_i$ by Lemma 4. Q.E.D. □

4 Compact elements of the lattice c_∞^τ

Proposition 1 *For any group G, the one-generated totally composition formation $\mathfrak{F} = c_\infty^\tau \mathrm{form}\, G$ is a compact element of the lattice c_∞^τ.*

Proof Here we adopt a procedure similar to that used in [25, 38], proceeding by induction on $|G|$. Suppose that A is a counterexample of the minimal order. Let $\mathfrak{F} = c_\infty^\tau \mathrm{form}\, A \subseteq \mathfrak{M} = c_\infty^\tau \mathrm{form}(\bigcup_{i\in I} \mathfrak{F}_i) = \bigvee_\infty^\tau (\mathfrak{F}_i \mid i \in I)$, where $\mathfrak{F}_i \in c_\infty^\tau$ for all $i \in I$. We wish to show that A is monolithic. Consider the following cases:

(i) Let M_1, M_2 be two distinct minimal normal subgroups of A. Suppose that $\mathfrak{M}_j = c_\infty^\tau \mathrm{form}(A/M_j)$ for $j = 1, 2$. Therefore, $|A/M_j| < |A|$. Thus, by induction, $\mathfrak{M}_j \subseteq \mathfrak{M}$, which implies

$$\mathfrak{M}_1 \subseteq c_\infty^\tau \mathrm{form}\left(\mathfrak{F}_{i_1} \bigcup \cdots \bigcup \mathfrak{F}_{i_t}\right),$$
$$\mathfrak{M}_2 \subseteq c_\infty^\tau \mathrm{form}\left(\mathfrak{F}_{i_{t+1}} \bigcup \cdots \bigcup \mathfrak{F}_{i_s}\right)$$

for some $i_1, \ldots, i_s$. Thus, $\mathfrak{F} = \mathfrak{M}_1 \bigvee_\infty^\tau \mathfrak{M}_2$ is a subformation of

$$c_\infty^\tau \mathrm{form}\left(\mathfrak{F}_{i_1} \bigcup \cdots \bigcup \mathfrak{F}_{i_t} \bigcup \mathfrak{F}_{i_{t+1}} \bigcup \cdots \bigcup \mathfrak{F}_{i_s}\right),$$

and we obtain a contradiction.

(ii) Suppose now $R = \mathrm{Soc}(A)$. When R is nonabelian, Lemma 8 implies $A \in \bigcup_{i\in I} \mathfrak{F}_i$. Thus, $A \in \mathfrak{M}$, and again we have a contradiction.

(iii) Assume R is an abelian p-group for some $p \in \pi(\mathrm{Com}(A))$. Then $A/\Phi(A) \in \mathrm{form}\, A$ implies $c_\infty^\tau \mathrm{form}(A/\Phi(A)) = c_\infty^\tau \mathrm{form}\, A$. But $|A/\Phi(A)| < |A|$; hence, by induction, we see that $R \not\subseteq \Phi(A)$.

Assume B is a subgroup of A, such that $R \cap B = 1$ with $O_p(B) = 1$. Thus, $A = Z_p \wr B = [R]B$. Lemma 6 implies $R = C^p(A) = O_p(A)$.

Let f_i, f, and m be minimal c_∞^τ-valued composition satellites of formations $\mathfrak{F}$, $\mathfrak{F}_i$, and $\mathfrak{M}$, respectively. By Lemma 2, $m = \bigvee_\infty^\tau (f_i \mid i \in I)$. Then by the properties of regular wreath products, we obtain $B \cong A/O_p(A) = A/R = A/C^p(A)$ belongs to $m(p)$.

Since $|B| < |A|$, it follows for some $j_1, j_2, \ldots, j_k \in J \subseteq I$, $B \cong A/C^p(A) \in f_{j_1}(p) \bigvee_\infty^\tau \cdots \bigvee_\infty^\tau f_{j_k}(p)$.

Lemma 2 guarantees that $m_3 = \bigvee_\infty^\tau (f_j \mid j \in J)$ is the minimal c_∞^τ-valued composition satellite of formation $\mathfrak{M}_3 = \bigvee_\infty^\tau (\mathfrak{F}_j \mid j \in J)$. Therefore,

$A/O_p(A) \cong B \in m_3(p)$. Now Lemma 5 implies that A belongs to formation $\mathfrak{M}_3$. Then $\mathfrak{F} = c_\infty^\tau \mathrm{form}\, A \subseteq \mathfrak{M}_3$, and it is a contradiction. Q.E.D. □

Proof of the theorem Let $\mathfrak{F}$ be a τ-closed totally composition formation. Obviously, we have $\mathfrak{F} = c_\infty^\tau \mathrm{form}(\bigcup_{i\in I} \mathfrak{F}_i) = \bigvee_\infty^\tau(\mathfrak{F}_i \mid i \in I)$, where $\mathfrak{F}_i = c_\infty^\tau \mathrm{form}\, G_i$ for some group G_i and $i \in I$. Then it suffices to show that each one-generated formation $\mathfrak{F}_i$ is a compact element of the lattice of all τ-closed totally composition formations, but it follows from Proposition 1. □

For trivial subgroup functor τ, we have

Corollary 1 [38] *The lattice of all totally composition formations of finite groups is algebraic.*

It is known that subgroup-closed Fitting classes of solvable groups are saturated formations; see [11, IV, (4.6)]. The lattices of such classes were studied in Reifferscheid [24]. In the universe of all solvable groups, we have

Corollary 2 *The lattice of all subgroup-closed Fitting classes of solvable groups is algebraic.*

5 Remarks

5.1 Inductive lattices of formations

Skiba [30] introduced the concept of an inductive lattice of formations to adapt lattice-theoretical methods for the investigation of saturated formations. This concept plays an important role in the research of the lattices of formations and their law systems (see Chapter 4 of the book [30], Chapter 4 of the book [42]). Both the lattice of all totally saturated formations [45] and the lattice of all τ-closed totally composition formations [36] are inductive.

Let Θ be a complete lattice of formations. A satellite f is called *Θ-valued* if all its values belong to Θ. We denote by Θ^c the set of all formations having a composition Θ-valued satellite. In [31, p. 901], it is shown that this set is a complete lattice of formations. A complete lattice Θ^c is called *inductive* if for any collection of formations $\{\mathfrak{F}_i = \mathrm{CLF}(f_i) \mid i \in I\}$, where f_i is an integrated satellite of $\mathfrak{F}_i \in \Theta^c$, the following equality holds: $\bigvee_{\Theta^c}(\mathfrak{F}_i \mid i \in I) = \mathrm{CLF}(\bigvee_\Theta(f_i \mid i \in I))$. The inductance of a lattice Θ^c, in fact, means that a research of the operation $\bigvee_{\Theta^c}$ on the set Θ^c can be reduced to a research of the operation $\bigvee_\Theta$ on the set Θ. Therefore, the inductance is one a very powerful property of the lattice Θ^c.

By Theorem, every composition formation is the join of some one-generated composition formations, and the inductance of the lattice c_∞^τ implies the following result.

Proposition 2 *For any groups G_i and formations*

$$\mathfrak{F}_i = \mathrm{CLF}(f_i) = c^{\tau}_{\infty}\mathrm{form}\, G_i,$$

where $i \in I$, *we have* $\bigvee^{\tau}_{\infty}(\mathfrak{F}_i \mid i \in I) = \mathrm{CLF}(\bigvee^{\tau}_{\infty}(f_i \mid i \in I))$. *Moreover, when* f_i *is the minimal composition satellite of* $\mathfrak{F}_i$ *and* $\pi_i = \pi(\mathrm{Com}(G_i))$, *we have*

$$f_i(p) = c^{\tau}_{\infty}\mathrm{form}(G_i/C^p(G_i))$$

for all $p \in \pi_i$, *and*

$$f_i(p) = \varnothing$$

for all $p \in \mathbb{P}\backslash\pi_i$.

5.2 Formations of group languages

Monoids are commonly used in computer science, both in its theoretical foundations and in programming. In computer programming, various abstract data types can be considered as monoids. Because the operation takes two values of a given type and returns a new value of the same type, it can be chained indefinitely, and associativity abstracts away the details of construction; see [40, Example 3.1]. A *list* (array) is a fine example of a monoid. The *identity* of a list is an *empty list*, and the associative operation is appending.

Languages are subsets of a certain type of monoid, the free monoid over an alphabet. Regular languages are precisely the behaviours of finite automata. A language is regular if its syntactic monoid is a finite monoid (a regular language is a group language if its syntactic monoid is a finite group). By A^* we denote a *free monoid* on a set A, i.e., the set of all words with letters from A. A *class* of regular languages $\mathcal{C}$ associates with each finite alphabet A a set $\mathcal{C}(A^*)$ of regular languages of A^*. We consider only finite monoids later on.

Definition 8 [5] A *formation of languages* is a class of regular languages $\mathcal{F}$ satisfying the following two conditions:

(1) for each alphabet A, $\mathcal{F}(A^*)$ is closed under Boolean operations and quotients, and
(2) if L is a language of $\mathcal{F}(B^*)$ and $\eta : B^* \to M$ denotes its syntactic morphism, then for each monoid morphism $\alpha : A^* \to B^*$ such that $\eta \circ \alpha$ is surjective, the language $\alpha^{-1}(L)$ belongs to $\mathcal{F}(A^*)$.

Following [5], we associate with each formation of monoids $\mathfrak{M}$ the class of languages $\mathcal{F}(\mathfrak{M})$ as follows. For each alphabet A, by $\mathcal{F}(\mathfrak{M})(A^*)$ we denote the set of languages of A^* fully recognised by some monoid of $\mathfrak{M}$ (or, equivalently, whose syntactic monoid belongs to $\mathfrak{M}$). Given a formation of languages $\mathcal{F}$, we denote by $\mathfrak{M}(\mathcal{F})$ the formation of monoids generated by the syntactic monoids of the languages of $\mathcal{F}$.

The Formation Theorem [5] states that the correspondences $\mathfrak{M} \to \mathcal{F}(\mathfrak{M})$ and $\mathcal{F} \to \mathfrak{M}(\mathcal{F})$ are two mutually inverse, order preserving, bijections between formations of monoids and formations of languages. In particular, we have a one-to-one correspondence between one-generated composition formations of finite groups and some formations of languages. *That makes formations useful for the study of abstract machines and automata, which commonly appear in theory of computation, compiler construction, artificial intelligence, parsing, formal verification and other aspects of theoretical computer science.*

Languages corresponding to saturated formations of finite groups were studied in [6, 39]. Naturally rises the following question: *How to describe the languages corresponding to τ-closed (one-generated) totally composition formations of finite groups?*

Fuzzy sets introduced by Zadeh [47] and Klaua [18] became applied in fields such as pattern recognition, machine learning and data mining [7, 16]. Wee [46] introduced the fuzzy automaton as a model of learning systems. This model is the natural fuzzification of the classical finite automaton: a *fuzzy automaton* is a tuple $\mathcal{A} = (A, X, \mu)$, where A is a finite set of states, X is a finite set of input symbols and μ is a fuzzy subset of $A \times X \times A$ representing the transition mapping, which can be represented as a collection of matrices with entries from [0, 1]. Let X^* be a free monoid, then a *fuzzy language* over an alphabet X is a fuzzy subset of X^*. A fuzzy language is *regular* if it is recognizable by a fuzzy automaton. Fuzzy languages have a wide range of applications [8, 17, 23].

The Eilenberg theorem establishes that there exists a bijection between the set of all varieties of regular languages and the set of all varieties of finite monoids. Petković [21] proved a counterpart of Eilenberg's theorem for varieties of fuzzy languages. The Formation Theorem is an Eilenberg-like theorem for formations. The mentioned results give a motivation for studying formations of fuzzy languages. In particular, the following problem is of special interest.

Problem. *Prove a counterpart of Eilenberg's theorem for formations of fuzzy languages.*

5.3 Classes of hypergroups

In 1934, at the eight Congress of Scandinavian Mathematics, Marty [20] introduces a concept of algebraic hyperstructure, which naturally generalizes classical algebraic structures such as groups and rings. As mentioned in [10], the first example of hypergroups, which motivates the introduction of this structure, is the quotient of a finite group by arbitrary (not necessarily normal) subgroup, i.e., if the subgroup is not normal, then the quotient is not a group, but it is always a hypergroup with respect to a certain hyperoperation. Keeping in mind this idea, we can introduce a concept of hyperformation, assuming that subgroups in Definition 1 are not necessarily normal.

Definition 9 A *hyperformation* is a class of hypergroups $\mathfrak{F}$ satisfying the following two conditions:

(1) if $H \in \mathfrak{F}$, then $H/N \in \mathfrak{F}$, and
(2) if H/N_1, $H/N_2 \in \mathfrak{F}$, then $H/N_1 \cap N_2 \in \mathfrak{F}$,

for any subhypergroups N, N_1, N_2 of H.

It will be interesting to study the relation between classical one-generated (saturated, composition) formations and hyperformations. Finally, we note that some examples of hypergroups associated with models of biological inheritance were considered recently in [2], and connections between hypergroups and fuzzy sets were discussed in Chapter 5 of the book [10].

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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