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ORIGINAL ARTICLE

On a paper of Dressler and Van de Lune

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Abstract
If $z \in \mathbb{C}$ and $1 \leq n$ is a natural number then

$$\sum_{d_1 d_2 = n} (1 - z^{p_1}) \cdots (1 - z^{p_m}) z^{q_1 e_1 + \cdots + q_i e_i} = 1,$$

where $d_1 = p_1^{r_1} \ldots p_m^{r_m}$, $d_2 = q_1^{e_1} \ldots q_i^{e_i}$ are the prime decompositions of d_1, d_2. This is one of the identities involving arithmetic functions that we prove using ideas from the paper of Dressler and van de Lune [3].

Keyword Arithmetic functions · Identities · Zeta function

Mathematics Subject Classification 11A25 · 11MXX

1 Introduction and results

Recall that the arithmetic function $\omega(n)$ is the number of distinct prime divisors of a positive integer n and $\Omega(n)$ is the total number of prime divisors of n. In other words if for a natural number $n \geq 2$ we write $n = p_1^{r_1} \ldots p_m^{r_m}$ with p_i distinct primes (a notation that we keep throughout this paper where p always denotes a prime) then $\omega(n) = m$, $\Omega(n) = r_1 + \cdots + r_m$ and $\Omega(1) = \omega(1) = 0$. Write $\zeta(s)$ for the Riemann–Zeta function.

One knows that $\omega(n) = O(\log n / \log \log n)$ which easily implies that $\sum_{n=1}^{\infty} z^{\omega(n)}/n^s$ is an entire function of z when $\Re\mathfrak{s} = \sigma >$. Also

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$$\sum_{n=1}^{\infty} \frac{z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \frac{z}{p^s} + \frac{z^2}{p^{2s}} + \cdots\right) = \prod_p \frac{1}{1 - z/p^s},$$

is an analytic function of z if $1 < \sigma$ and $|z| < 2^\sigma$.

In [3] the following remarkable duality relation was proved.

Theorem 1 *(R. Dressler and J. van de Lune) If $|z| < 2^\sigma$ and $\Re s = \sigma >$ then*

$$\left(\sum_{n=1}^{\infty} \frac{(1-z)^{\omega(n)}}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{z^{\Omega(n)}}{n^s}\right) = \zeta(s).$$

The aim of this note is to obtain similar formulas using the methods of [3].

To state our results we need some definitions: let $\omega_o(n)$ ($\omega_e(n)$ respectively) be the number of primes in the decomposition of n with odd (even respectively) exponent. Thus $\omega_o(2^2 3^5 5^6) = 1$, $\omega_e(2^2 3^5 5^6) = 2$. Note that $\omega_e(n) + \omega_o(n) = \omega(n)$. $\lfloor x \rfloor$ is the floor function and μ is the Möbius function. The radical of a number n is defined as $rad(n) = p_1 \cdots p_m$.

Ramanujan's tau function is defined by (see [2], p. 136)

$$z \prod_{n=1}^{\infty} (1 - z^n)^{24} = \sum_{n=1}^{\infty} \tau(n) z^n,$$

and its associated Dirichlet series is

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}. \tag{1}$$

One has the bound

$$|\tau(p)| \leq 2p^{11/2}. \tag{2}$$

This result was conjectured by Ramanujan and it was proved by Deligne [2].

The main contribution of this note is the following theorem. Note: in formulas (a)–(g) below it is assumed that in all the sums the term with $n = 1$ is equal to 1.

Theorem 2

(a) *If $1 < \sigma = \Re s$, $|z| \leq$ then*

$$\left(\sum_{n=1}^{\infty} \frac{(1 - z^{p_1}) \cdots (1 - z^{p_m})}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{z^{p_1 r_1 + \cdots + p_m r_m}}{n^s}\right) = \zeta(s).$$

(b) *If $1 < \sigma$, $|z| \leq 1$ then*

$$\sum_{n=1}^{\infty}\frac{z^{p_1r_1+\cdots+p_mr_m}}{n^s}=\zeta(s)\sum_{n=1}^{\infty}\frac{(z^{p_1}-1)\cdots(z^{p_m}-1)z^{(r_1-1)p_1+\cdots+(r_m-1)p_m}}{n^s}.$$

(c) *If* $0<|z|<2^{\sigma}$ *and* $1<\sigma$ *then*

$$\sum_{n=1}^{\infty}\frac{z^{\Omega(n)-\omega(n)}(1+z)^{\omega(n)}}{n^s}=\frac{\zeta(s)}{\zeta(2s)}\left(\sum_{n=1}^{\infty}\frac{z^{\Omega(n)}}{n^s}\right).$$

(d) *If* $z\in\mathbb{C}$ *and* $1<\sigma$ *then*

$$\sum_{n=1}^{\infty}\frac{(z+2)^{\omega(n)}}{n^s}=\frac{\zeta(s)^2}{\zeta(2s)}\left(\sum_{n=1}^{\infty}\frac{(-1)^{\Omega(n)-\omega(n)}z^{\omega(n)}}{n^s}\right).$$

(e) *If* $0<|z|\le 1$ and $1<\sigma$ *then*

$$\sum_{n=1}^{\infty}\frac{z^{\Omega(n)-\omega(n)}(z-1/p_1)\cdots(z-1/p_m)}{n^s}=\frac{\zeta(s)}{\zeta(s+1)}\left(\sum_{n=1}^{\infty}\frac{z^{\Omega(n)-\omega(n)}(z-1)^{\omega(n)}}{n^s}\right).$$

(f) *If* $0<|z|\le 1$ *and* $1<\sigma$ *then*

$$\frac{\zeta(s)\zeta(2s)\zeta(3s)}{\zeta(6s)}=\left(\sum_{n=1}^{\infty}\frac{(-1)^{\omega_o(n)}(1+z)^{\omega_e(n)}z^{\sum_{i=1}^{m}\left\lfloor\frac{r_i-1}{2}\right\rfloor}}{n^s}\right)\times\left(\sum_{n=1}^{\infty}\frac{\{(r_1+1)-(r_1-1)z\}\cdots\{(r_m+1)-(r_m-1)z\}}{n^s}\right).$$

(g) *If* $1<\Re\lambda$ *and* $1<\sigma$ *then*

$$\sum_{n=1}^{\infty}\frac{1}{rad(n)^{\lambda}n^s}=\zeta(s)\left\{\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\left(1-\frac{1}{p_1^{\lambda}}\right)\cdots\left(1-\frac{1}{p_m^{\lambda}}\right)\right\}.$$

(h) *Assume that* $\{\epsilon_p\}$ *is any sequence of complex numbers defined on the set of primes with* $|\epsilon_p|\le 1$ *for all* p. *Write* $n=n_1n_2$, *where* $n_1=p_1\ldots p_t$, $n_2=p_{t+1}^{r_{t+1}}\ldots p_m^{r_m}$ *with* $2\le\min\{r_{t+1},\ldots,r_m\}$ *and define*

$$a_n := \prod_1 \prod_2,$$
$$\prod_1 := \prod_{p \in n_1} \left\{ \epsilon_p p^{11/2} - \tau(p) \right\},$$
$$\prod_2 := n_2^{11/2} rad(n_2)^{-11} \prod_{k=t+1}^{m} \left\{ \left(1 + \epsilon_{p_k}^2\right) p_k^{11} - \epsilon_{p_k} p_k^{11/2} \tau(p_k) \right\} \epsilon_{p_k}^{r_k - 2}.$$
$$A(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$
$$B(s) := \sum_{n=1}^{\infty} \frac{\mu(n) \epsilon_{p_1} \cdots \epsilon_{p_m}}{n^{s-11/2}}.$$

Then if $13/2 < \sigma$

$$\left(\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \right)^{-1} = A(s)B(s).$$

Note: it is understood that the first summand is 1 in any of the last two sums. Also $\prod_i = 1$ if $n_i = 1$.

We remark that the formula in the abstract follows from formula (a) where it is understood there that $(1 - z^{p_1}) \cdots (1 - z^{p_m}) \equiv 1$ if $d_1 = 1$ and $z^{q_1 e_1 + \cdots + q_i e_i} \equiv 1$ if $d_2 = 1$.

Observe that formula (b) yields, on setting $z = 0$, the well-known relation

$$1 = \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Also, the above zeta quotients appearing in formulas (c)–(f) are well-known: if $\phi(n)$ is the number of numbers less than n and prime to n and the arithmetic function $\kappa(n)$ is defined by $\kappa(1) = 1$ and $\kappa(p_1^{r_1} \ldots p_m^{r_m}) = r_1 \ldots r_m$ then

$$\frac{\zeta(s)^2}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s},$$
$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s},$$
$$\frac{\zeta(s)\zeta(2s)\zeta(3s)}{\zeta(6s)} = \sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s},$$
$$\frac{\zeta(s)}{\zeta(s+1)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^{s+1}}.$$

See [1] p. 247, [4] formulas (1,2,7), (1.2.8) and (1.2.12).

This permits, using Dirichlet convolution, to obtain the following corollary.

Corollary 1 *If $z \in \mathbb{C}$ in (i), (ii) and (iii) then*

(i)

$$z^{\Omega(n)-\omega(n)}(1+z)^{\omega(n)} = \sum_{d|n} |\mu(n/d)| z^{\Omega(d)}.$$

(ii)

$$(z+2)^{\omega(n)} = \sum_{d|n} 2^{\omega(n/d)}(-1)^{\Omega(d)}(-z)^{\omega(d)}.$$

(iii)

$$n(z-1/p_1)\cdots(z-1/p_m)z^{\Omega(n)-\omega(n)} = \sum_{d|n} d\ \phi(n/d) z^{\Omega(d)-\omega(d)}(z-1)^{\omega(d)}.$$

(iv) *If $\lambda \in \mathbb{C}$ then*

$$rad(n)^{\lambda} = \sum_{d|n} \mu(d)\big(1-q_1^{\lambda}\big)\cdots\big(1-q_i^{\lambda}\big).$$

(Note: here $d = q_1^{e_1}\cdots q_i^{e_i}$ is the prime decomposition of d and if $d=1$ in the last sum then summand is understood to be 1.)

Proof Formulas (i), (ii), (iii), (iv) follow from formulas (c), (d), (e), (g) respectively and analytic continuation. □

2 Proof

Recall that if $2 \le n$ is an integer then we write $n = p_1^{r_1}\cdots p_m^{r_m}$ with p_i different primes. We record the following formal formulas:

$$\begin{aligned}\prod_p\left(1+\frac{g(p)}{p^s-z}\right) &= \prod_p\left(1+\frac{g(p)}{p^s}+\frac{zg(p)}{p^{2s}}+\frac{z^2g(p)}{p^{3s}}+\cdots\right)\\ &= \sum_{n=1}^{\infty}\frac{z^{\Omega(n)-\omega(n)}g(p_1)\cdots g(p_m)}{n^s},\end{aligned} \tag{3}$$

$$\begin{aligned}\prod_p\frac{p^s}{p^s-g(p)} &= \prod_p\left(1+\frac{g(p)}{p^s}+\frac{g(p)^2}{p^{2s}}+\frac{g(p)^3}{p^{3s}}+\cdots\right)\\ &= \sum_{n=1}^{\infty}\frac{g(p_1)^{r_1}\cdots g(p_m)^{r_m}}{n^s}.\end{aligned} \tag{4}$$

If we set $g(p)=z$, $z=1$ in the first equation and $g(p)=z$ in the second equation one gets:

$$\prod_p\left(1+\frac{z}{p^s-1}\right)=\sum_{n=1}^{\infty}\frac{z^{\omega(n)}}{n^s}, \qquad (5)$$

$$\prod_p\frac{p^s}{p^s-z}=\sum_{n=1}^{\infty}\frac{z^{\Omega(n)}}{n^s}. \qquad (6)$$

Using this two formulas in

$$\zeta(s)=\prod_p\frac{p^s}{p^s-1}=\prod_p\left(\frac{p^s}{p^s-z}\right)\left(1+\frac{1-z}{p^s-1}\right)$$

yields Theorem 1.

The values of z and σ depend on each formula in such way that the products or sums involved are absolutely convergent. For example assume that $|z|\leq 1$ and $1<\sigma$. Then if $g(p)=1-z^p$, $z=1$ in (3) and $g(p)=z^p$ in (4) one has

$$\prod_p\left(1+\frac{1-z^p}{p^s-1}\right)=\sum_{n=1}^{\infty}\frac{(1-z^{p_1})\cdots(1-z^{p_m})}{n^s}, \qquad (7)$$

$$\prod_p\frac{p^s}{p^s-z^p}=\sum_{n=1}^{\infty}\frac{z^{p_1r_1+\cdots+p_mr_m}}{n^s}. \qquad (8)$$

The product of (7) and (8) is equal to $\zeta(s)$. This proves formula (a) of Theorem 2.

Also

$$\prod_p\left(\frac{p^s-1}{p^s-z^p}\right)=\prod_p\left(\left\{1-\frac{1}{p^s}\right\}\left\{1+\frac{z^p}{p^s}+\frac{z^{2p}}{p^{2s}}+\cdots\right\}\right)$$
$$=\sum_{n=1}^{\infty}\frac{(z^{p_1}-1)\cdots(z^{p_m}-1)z^{(r_1-1)p_1+\cdots+(r_m-1)p_m}}{n^s}.$$

Formula (b) follows using (8) and observing that $\prod_p\frac{p^s}{p^s-z^p}\left(\frac{p^s-1}{p^s-z^p}\right)^{-1}=\zeta(s)$.

If $g(p)=1/p^{\lambda}$, $z=1$ in (3) then

$$\prod_p\left(1+\frac{1/p^{\lambda}}{p^s-1}\right)=\sum_{n=1}^{\infty}\frac{1}{rad(n)^{\lambda}n^s}.$$

Also

$$\prod_p \left(\frac{p^s - 1 + 1/p^\lambda}{p^s}\right)^{-1} = \prod_p \left(1 - \frac{1}{p^s}\left\{1 - \frac{1}{p^\lambda}\right\}\right)^{-1}$$
$$= \left\{\sum_{n=1}^\infty \frac{\mu(n)}{n^s}\left(1 - \frac{1}{p_1^\lambda}\right)\cdots\left(1 - \frac{1}{p_m^\lambda}\right)\right\}^{-1}.$$

The product of this two formulas yields formula (g).

In a similar way using (3) one has that

$$\prod_p \left(1 + \frac{z}{p^s - u}\right) = \sum_{n=1}^\infty \frac{u^{\Omega(n)-\omega(n)} z^{\omega(n)}}{n^s}, \tag{9}$$

which can be used to prove that

$$\frac{\zeta(s)^2}{\zeta(2s)} = \prod_p \frac{p^s + 1}{p^s - 1} = \prod_p \frac{p^s + z + 1}{p^s - 1}\frac{p^s + 1}{p^s + z + 1}$$
$$= \prod_p \left(1 + \frac{z + 2}{p^s - 1}\right)\left(1 + \frac{z}{p^s + 1}\right)^{-1}$$
$$= \left(\sum_{n=1}^\infty \frac{(z + 2)^{\omega(n)}}{n^s}\right)\left(\sum_{n=1}^\infty \frac{(-1)^{\Omega(n)-\omega(n)} z^{\omega(n)}}{n^s}\right)^{-1},$$

which is formula (d).

Also

$$\frac{\zeta(s)}{\zeta(s+1)} = \prod_p \frac{p^s - 1/p}{p^s - 1} = \prod_p \left(\frac{p^s - 1}{p^s - z}\right)^{-1}\left(\frac{p^s - 1/p}{p^s - z}\right)$$
$$= \left(\sum_{n=1}^\infty \frac{z^{\Omega(n)-\omega(n)}(z - 1)^{\omega(n)}}{n^s}\right)^{-1}$$
$$\times \left(\sum_{n=1}^\infty \frac{z^{\Omega(n)-\omega(n)}(z - 1/p_1)\cdots(z - 1/p_m)}{n^s}\right),$$

which is formula (e).

Formula (c) follows from (using (9) and (6))

$$\frac{\zeta(s)}{\zeta(2s)} = \prod_p \frac{p^s + 1}{p^s} = \prod_p \left(1 + \frac{1 + z}{p^s - z}\right)\left(\frac{p^s}{p^s - z}\right)^{-1}$$
$$= \left(\sum_{n=1}^\infty \frac{z^{\Omega(n)-\omega(n)}(1 + z)^{\omega(n)}}{n^s}\right)\left(\sum_{n=1}^\infty \frac{z^{\Omega(n)}}{n^s}\right)^{-1}.$$

To prove formula (f) observe that after simplification using $\zeta(s) = \prod_p \frac{p^s}{p^s - 1}$ one has

$$\frac{\zeta(s)\zeta(2s)\zeta(3s)}{\zeta(6s)}=\prod_p\frac{p^{2s}-p^s+1}{(p^s-1)^2}.$$

Formula (f) follows multiplying

$$\begin{aligned}\prod_p\frac{p^{2s}-p^s+1}{p^{2s}-z}&=\prod_p\left(1-\frac{1}{p^s}+\frac{1}{p^{2s}}\right)\left(1+\frac{z}{p^{2s}}+\frac{z^2}{p^{4s}}+\cdots\right)\\&=\prod_p\left(1-\frac{1}{p^s}-\frac{z}{p^{3s}}-\frac{z^2}{p^{5s}}-\cdots+\frac{(1+z)}{p^{2s}}+\frac{z(1+z)}{p^{4s}}+\frac{z^2(1+z)}{p^{6s}}+\cdots\right)\\&=\sum_{n=1}^{\infty}\frac{(-1)^{\omega_o(n)}(1+z)^{\omega_e(n)}z^{\sum_{i=1}^{m}\lfloor\frac{r_i-1}{2}\rfloor}}{n^s},\end{aligned}$$

with

$$\begin{aligned}\prod_p\frac{p^{2s}-z}{(p^s-1)^2}&=\prod_p\left(1-\frac{z}{p^{2s}}\right)\left(1+\frac{2}{p^s}+\frac{3}{p^{2s}}+\cdots\right)\\&=\prod_p\left(1+\frac{2}{p^s}+\frac{3-z}{p^{2s}}+\frac{4-2z}{p^{3s}}+\cdots\right)\\&=\sum_{n=1}^{\infty}\frac{\{(r_1+1)-(r_1-1)z\}\cdots\{(r_m+1)-(r_m-1)z\}}{n^s}.\end{aligned}$$

Next we prove formula (h). One has the following formal formula

$$\begin{aligned}\prod_p\frac{p^s-\tau(p)+p^{11}/p^s}{p^s-g(p)}&=\prod_p\left(1-\frac{\tau(p)}{p^s}+\frac{p^{11}}{p^{2s}}\right)\left(1+\frac{g(p)}{p^s}+\frac{g(p)^2}{p^{2s}}+\cdots\right)\\&=\prod_p\left\{1+\frac{g(p)-\tau(p)}{p^s}+\sum_{j=2}^{\infty}\frac{g(p)^{j-2}\left\{g(p)^2-g(p)\tau(p)+p^{11}\right\}}{p^{js}}\right\}\\&=\sum_{n=1}^{\infty}\frac{\prod_{k=1;r_k=1}^{m}\{g(p_k)-\tau(p_k)\}\prod_{k=1;r_k\geq 2}^{m}\left\{g(p_k)^2-g(p_k)\tau(p_k)+p_k^{11}\right\}g(p_k)^{r_k-2}}{n^s}.\end{aligned}$$

Setting $g(p)=\epsilon_p p^{11/2}$ with $|\epsilon_p|\leq 1$ in the last formula one gets

$$A(s):=\prod_p\frac{p^s-\tau(p)+p^{11}/p^s}{p^s-\epsilon_p p^{11/2}}=\sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

where a_n is defined as follows. To ease the notation we write $n=n_1n_2$, where n_i are positive integers such that $n_1=p_1\ldots p_t$, $n_2=p_{t+1}^{r_{t+1}}\ldots p_m^{r_m}$ with $2\leq\min\{r_{t+1},\ldots,r_m\}$. Then

$$a_n := \prod_1 \prod_2,$$

$$\prod_1 := \prod_{p \in n_1} \Big\{ \epsilon_p p^{11/2} - \tau(p) \Big\},$$

$$\prod_2 := \prod_{k=t+1}^{m} \Big\{ (1+\epsilon_{p_k}^2) p_k^{11} - \epsilon_{p_k} p_k^{11/2} \tau(p_k) \Big\} \epsilon_{p_k}^{r_k - 2} p_k^{\frac{11}{2}(r_k - 2)}$$

$$= n_2^{11/2} rad(n_2)^{-11} \prod_{k=t+1}^{m} \Big\{ (1+\epsilon_{p_k}^2) p_k^{11} - \epsilon_{p_k} p_k^{11/2} \tau(p_k) \Big\} \epsilon_{p_k}^{r_k - 2}.$$

If one sets

$$B(s) := \prod_p \frac{p^s - \epsilon_p p^{11/2}}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n) \epsilon_{p_1} \cdots \epsilon_{p_m}}{n^{s-11/2}},$$

then

$$A(s)B(s) = \left(\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \right)^{-1},$$

which follows using (1). The proof is complete.

References

1. Apostol, T.: Introduction to Analytic Number Theory. Springer, New York (1976)
2. Apostol, T.: Modular Functions and Dirichlet Series in Number Theory. Springer, New York (1976)
3. Dressler, R.E., van de Lune, J.: Some remarks concerning the number theoretic functions $\omega(n)$ and $\Omega(n)$. Proc. Am. Math. Soc. **41**(2), 403–406 (1973)
4. Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function, 2nd edn. Oxford Univ. Press, New York (1986). (revised by D. R. Heath-Brown)