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ORIGINAL ARTICLE

Canonical isometric embeddings of projective spaces into spheres

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Abstract
We define inductively isometric embeddings of $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ (with their canonical metrics conveniently scaled) into the standard unit sphere, which present the former as the restriction of the latter to the set of real points. Our argument parallels the telescopic construction of $\mathbb{P}^\infty(\mathbb{R})$, $\mathbb{P}^\infty(\mathbb{C})$, and $\mathbb{S}^\infty$ in that, for each n, it extends the previous embedding to the attaching cell, which after a suitable renormalization makes it possible for the result to have image in the unit sphere.

Keywords Immersions · Embeddings · Minimal · Canonical embedding

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1 Isometric embeddings into spheres

We recall that if (M^n, g) is a Riemannian manifold isometrically immersed into the standard unit sphere $(\mathbb{S}^N, \tilde{g})$ in Euclidean space $(\mathbb{R}^{N+1}, \|\ \|^2)$, if α and H are the second fundamental form and mean curvature vector of the embedding, the scalar curvature s_g of g relates to the extrinsic quantities as

$$s_g = n(n-1) + \tilde{g}(H,H) - \tilde{g}(\alpha,\alpha). \tag{1}$$

A *canonical embedding* will be one that is a critical point of the functional

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$$\Pi(M) = \int_M \tilde{g}(\alpha, \alpha)\, \mathrm{d}\mu, \tag{2}$$

under deformations of the embedding [6]. If one such is also minimal, (1) implies that the embedding is a critical point of the total scalar curvature among metrics on M that can be realized by isometric embeddings into $\mathbb{S}^N$, and so if N is sufficiently large, by the Nash isometric embedding theorem [5], we conclude that the metric g on M is Einstein; conversely, if (M, g) is an Einstein Riemannian manifold isometrically embedded into $\mathbb{S}^N$ as a minimal submanifold, then the embedding is canonical.

2 Canonical minimal embeddings of projective spaces

On a circle centered at the origin, let us consider the map

$$(x_0 : x_1) \mapsto \imath_1(x) = (2x_0x_1, (x_0^2 - x_1^2)) \in \mathbb{R}^2. \tag{3}$$

Antipodal points are sent to the same image, and since the map underlies the local degree map $\theta \mapsto 2\theta$, different sets of antipodal points are mapped to different images. Since

$$\|\imath_1(x)\|^2 = (x_0^2 + x_1^2)^2,$$

and the components functions are all homogeneous harmonic polynomials of degree two, we obtain a 2-to-1 minimal immersion

$$\imath_1 : \mathbb{S}^1 \to \mathbb{S}^1,$$

into the unit circle, of codimension zero. Hence, with the metric on $\mathbb{P}^1(\mathbb{R})$ induced by that on $\mathbb{S}^1$, we obtain a minimal isometric embedding identification

$$\mathbb{P}^1(\mathbb{R}) \hookrightarrow \mathbb{S}^1,$$

between the domain circle $\mathbb{P}^1(\mathbb{R})$ of length π and the range circle with its metric of length 2π.

We now proceed by induction. Let us assume that we have defined an isometric 2-to-1 minimal immersion $\imath_{n-1}$ of the sphere $\mathbb{S}^{n-1}(r_{n-1})$ of radius r_{n-1} into $\mathbb{S}^{N_{n-1}} \subset \mathbb{R}^{N_{n-1}+1}$, which descends to an isometric embedding of the quotient $\mathbb{P}^{n-1}(\mathbb{R})$ with the induced metric, and is such that the Euclidean $\mathbb{R}^{N_{n-1}+1}$ norm of $\imath_{n-1}(x)$ satisfies

$$\|\imath_{n-1}(x)\|^2 = \frac{1}{r_{n-1}^4}(x_0^2 + \cdots + x_{n-1}^2)^2.$$

We set $x = (x', x_n)$, where $x' = (x_0, \ldots, x_{n-1})$. Then, if

$$r_n^4 = \frac{(n+1)(n^2-1)}{n^2} r_{n-1}^4, \quad b^2 = \frac{1}{(n^2-1)r_{n-1}^4}, \quad a^2 = 2n(n+1)b^2, \tag{4}$$

respectively, we consider the map

$$\imath_n(x) = \frac{1}{\sqrt{n+1}}(\imath_{n-1}(x'), ax_nx_0, \ldots, ax_nx_{n-1}, b(x_0^2 + \cdots + x_{n-1}^2 - nx_n^2)) \in \mathbb{R}^{N_{n-1}+n+2}. \tag{5}$$

Its components are all quadratic harmonic polynomials, and we have that

$$\|\imath_n(x)\|^2 = \frac{1}{r_n^4}(x_0^2 + \cdots + x_n^2)^2.$$

Theorem 1 *Let r_n and N_n be the sequences*

$$r_n^4 = \left(\frac{n+1}{2}\right)^2 (n-1)!, \quad N_n = \frac{1}{2}n(n+3) - 1, \tag{6}$$

respectively. Then, the map given inductively by (3), (5) above defines an isometric 2-to-1 minimal immersion

$$\imath_n : \mathbb{S}^n(r_n) \to \mathbb{S}^{N_n}$$

that maps the fibers of

$$\begin{array}{ccc} & & \mathbb{P}^n(\mathbb{R}) \\ & & \uparrow \\ \mathbb{Z}/2 & \hookrightarrow & \mathbb{S}^n(r_n) \end{array}$$

injectively into the image, and with the Einstein metric on $\mathbb{P}^n(\mathbb{R})$ induced by that of the sphere $\mathbb{S}^n(r_n)$, the map descends to an isometric minimal embedding

$$\imath_n : \mathbb{P}^n(\mathbb{R}) \hookrightarrow \mathbb{S}^{N_n} \subset \mathbb{R}^{N_n+1},$$

and the diagram of isometric immersions

$$\begin{array}{ccccc} & & \mathbb{P}^n(\mathbb{R}) & \overset{\imath_n}{\hookrightarrow} & \mathbb{S}^{N_n} \\ & & \uparrow & \nearrow & \\ \mathbb{Z}/2 & \hookrightarrow & \mathbb{S}^n(r_n) & & \end{array} \tag{7}$$

commutes.

Remark 2 For $n = 2$, we have that

$$(x_0 : x_1 : x_2) \mapsto \imath_2(x) = \frac{1}{\sqrt{3}}(\imath_1(x_0, x_1), 2x_0x_2, 2x_1x_2, (x_0^2 + x_1^2 - 2x_2^2)/\sqrt{3}) \in \mathbb{R}^5, \tag{8}$$

and we get an isometric minimal embedding $\imath_2 : \mathbb{P}^2(\mathbb{R}) \hookrightarrow \mathbb{S}^4$, where the metric g on $\mathbb{P}^2(\mathbb{R})$ is Einstein of scalar curvature 4/3 induced by the metric on $\mathbb{S}^2(\sqrt{3/2})$. The second fundamental form α is such that $\|\alpha\|^2 = 2/3$ pointwise. Thus,

$$\frac{1}{4\pi}\int_{\mathbb{P}^2(\mathbb{R})} s_g d\mu_g = 1 = \chi(\mathbb{P}^2(\mathbb{R})),$$

and

$$\int_{\mathbb{P}^2(\mathbb{R})} \|\alpha\|^2 d\mu_g = 2\pi = \Pi(\mathbb{P}^2(\mathbb{R})),$$

respectively. This is the Veronese surface of [2]; its canonical property is proven in [3] by showing that the Euler–Lagrange equations for the functional (2), developed in [6] with complete generality, are satisfied for the sphere background. The same argument proves that all of the embeddings above, in any dimension, are canonical; all of them are given by quadratic harmonic polynomials inducing eigenfunctions of the Laplacian on projective space for the first nonzero eigenvalue [4, 7].

For $n = 3$, $r_3 = 2^{\frac{3}{4}}$, and the metric on $\mathbb{P}^3(\mathbb{R})$ is Einstein of scalar curvature $s_g = \frac{1}{2^{\frac{3}{2}}}6$, and volume $\mu_g = \pi^2(2^{\frac{3}{4}})^3 = \frac{1}{2}\omega_3(2^{\frac{3}{4}})^3$. Thus,

$$\frac{\int s_g d\mu_g}{\left(\int d\mu_g\right)^{1/3}} = 6\pi^{\frac{4}{3}} = \frac{6\omega_3^{\frac{2}{3}}}{2^{\frac{2}{3}}},$$

the sigma constant of $\mathbb{P}^3(\mathbb{R})$ [1].

We regard $\mathbb{P}^n(\mathbb{R})$ as a real n-dimensional submanifold of $\mathbb{P}^n(\mathbb{C})$, and extend the embeddings above, so that they fit as the restriction of the canonical embeddings of their complex alter egos to the set of real points.

We write a point in $\mathbb{S}^3 \subset \mathbb{C}^2$ as $z = (z_0, z_1)$. The Fubini–Study metric g on $\mathbb{P}^1(\mathbb{C})$ is defined to make of the fibration

$$\begin{array}{ccc} \mathbb{S}^1 & \hookrightarrow & \mathbb{S}^3 \\ & & \downarrow \\ & & \mathbb{P}^1(\mathbb{C}), \end{array} \tag{9}$$

a Riemannian submersion, the action of $\mathbb{S}^1$ on $\mathbb{S}^3$ is given by $e^{i\theta}\cdot z = (e^{i\theta}z_0, e^{i\theta}z_1)$. The sectional curvature of a normalized section is given by $K_g(e_1, e_2) = 1 + 3|\langle \pi^* e_1, J\pi^* e_2\rangle|^2$, where $\{\pi^* e_1, J\pi^* e_2\}$ is a horizontal lift of the section to the fiber, and J is the complex structure in $\mathbb{C}^2$.

We consider the map

$$(z_0 : z_1) \mapsto \iota_1(z) = (2z_0\bar{z}_1, (z_0\bar{z}_0 - z_1\bar{z}_1)) \in \mathbb{C}\times\mathbb{R} = \mathbb{R}^3, \tag{10}$$

All points on an $\mathbb{S}^1$ orbit are mapped onto the same image, and different orbits are mapped to different points. Using the Euclidean norm in the range, we have that

$$\|\iota_1(z)\|^2 = (|z_0|^2 + |z_1|^2)^2,$$

and passing to the quotient, we obtain an isometric embedding identification

$$\imath_1 : \mathbb{P}^1(\mathbb{C}) \hookrightarrow \mathbb{S}^2,$$

between $\mathbb{P}^1(\mathbb{C})$ with the Fubini–Study metric of volume π, and $\mathbb{S}^2$ with volume 4π. It is minimal of codimension zero, and by construction, it restricts to give the isometric embedding (3) of $\mathbb{P}^1(\mathbb{R})$ as the set of totally geodesic real points of $\mathbb{P}^1(\mathbb{C})$ embedded into its image,

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{C}) & \hookrightarrow & \mathbb{S}^2 \\ \cup & & \cup \\ \mathbb{P}^1(\mathbb{R}) & \hookrightarrow & \mathbb{S}^1 \end{array}.$$

Notice that the composition of the projection (9) and $\imath_1$ is the Hopf map $H : \mathbb{S}^3 \to \mathbb{S}^2$ generator of the homotopy group $\pi_3(\mathbb{S}^2)$. Indeed, if $z = (z_0, z_1)$ with $z_0 = x_0 + iy_0$, $z_1 = x_1 + iy_1$, we have

$$H(z) = \begin{cases} (2(x_0x_1 + y_0y_1), 2(y_0x_1 - x_0y_1), x_0^2 + y_0^2 - x_1^2 - y_1^2) & \text{if } (x_1, y_1) \neq (0,0), \\ (0,0,1) & \text{if } (x_1, y_1) = (0,0). \end{cases}$$

To proceed by induction, and since we are to regard the real embeddings (5) as the restriction of the embeddings, we are about to define to the set of real points of $\mathbb{P}^n(\mathbb{C})$, we begin by observing that the scales defined by the constants in (4) are already fixed. So, let us assume that we have defined a map

$$\imath_{n-1} : \mathbb{S}^{2(n-1)+1}(r_{n-1}) \to \mathbb{S}^{M_{n-1}} \subset \mathbb{R}^{M_{n-1}+1},$$

that descends to an embedding of the base of the Riemannian submersion

$$\begin{array}{ccccc} \mathbb{S}^1 & \hookrightarrow & \mathbb{S}^{2(n-1)+1}(r_{n-1}) & & \\ & & \downarrow & & \\ & & \mathbb{P}^{n-1}(\mathbb{C}) & \overset{\imath_{n-1}}{\hookrightarrow} & \mathbb{S}^{M_{n-1}} \end{array} \tag{11}$$

into $\mathbb{S}^{M_{n-1}}$ with the desired properties. If $z = (z', z_n)$, where $z' = (z_0, \ldots, z_{n-1}) \in \mathbb{C}^{2(n+1)}$, we consider the map

$$\imath_n(z) = \frac{1}{\sqrt{n+1}}(\imath_{n-1}(z'), a\bar{z}_n z_0, \ldots, a\bar{z}_n z_{n-1}, b(|z_0|^2 + \cdots + |z_{n-1}|^2 - n|z_n|^2)) \in \mathbb{R}^{M_{n-1}+2n+2}. \tag{12}$$

Then, we have that

$$\|\imath_n(z)\|^2 = \frac{1}{r_n^4}(|z_0|^2 + \cdots + |z_n|^2)^2,$$

and we obtain a map of spheres

$$\iota_n : \mathbb{S}^{2n+1}(r_n) \to \mathbb{S}^{M_n} \subset \mathbb{R}^{M_{n-1}+2n+2},$$

$M_n = M_{n-1} + 2n + 1$, which descends to define an isometric minimal embedding of the base of the Riemannian submersion

$$\begin{array}{ccccc} \mathbb{S}^1 & \hookrightarrow & \mathbb{S}^{2n+1}(r_n) & & \\ & & \downarrow & & \\ & & \mathbb{P}^n(\mathbb{C}) & \stackrel{\iota_n}{\hookrightarrow} & \mathbb{S}^{M_n} \end{array} \tag{13}$$

into $\mathbb{S}^{M_n}$.

Theorem 3 *Let M_n be the sequence*

$$M_n = (n+1)^2 - 2\,,$$

and r_n and N_n be the sequences in (6). Then, the map

$$\iota_n : \mathbb{S}^{2n+1}(r_n) \to \mathbb{S}^{M_n} \subset \mathbb{R}^{M_n+1},$$

given inductively by (10), (12) above, maps the fibers of the fibration (13) injectively into the image, and with the scaled Fubini–Study metric on $\mathbb{P}^n(\mathbb{C})$ induced by the metric on the sphere $\mathbb{S}^{2n+1}(r_n)$, the map descends to an isometric minimal embedding

$$\iota_n : \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{S}^{M_n} \subset \mathbb{R}^{M_n+1}\,,$$

which restricts to the embedding given inductively by (3), (5), on the totally geodesic subset of real points $\mathbb{P}^n(\mathbb{R})$, and the diagram

$$\begin{array}{ccccc} \mathbb{S}^1 & \hookrightarrow & \mathbb{S}^{2n+1}(r_n) & & \\ & & \downarrow & \searrow & \\ & & \mathbb{P}^n(\mathbb{C}) & \stackrel{\iota_n}{\hookrightarrow} & \mathbb{S}^{M_n} \\ & & \cup & & \cup \\ & & \mathbb{P}^n(\mathbb{R}) & \stackrel{\iota_n}{\hookrightarrow} & \mathbb{S}^{N_n} \\ & & \uparrow & \nearrow & \\ \mathbb{Z}/2 & \hookrightarrow & \mathbb{S}^n(r_n) & & \end{array} \tag{14}$$

commutes.

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