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ORIGINAL ARTICLE

# Semi-almost periodic Fourier multipliers on rearrangement-invariant spaces with suitable Muckenhoupt weights

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## Abstract

Let $X(\mathbb{R})$ be a separable rearrangement-invariant space and $w$ be a suitable Muckenhoupt weight. We show that for any semi-almost periodic Fourier multiplier $a$ on $X(\mathbb{R}, w) = \{f : fw \in X(\mathbb{R})\}$ there exist uniquely determined almost periodic Fourier multipliers $a_l, a_r$ on $X(\mathbb{R}, w)$, such that

$$a = (1 - u)a_l + ua_r + a_0,$$

for some monotonically increasing function $u$ with $u(-\infty) = 0$, $u(+\infty) = 1$ and some continuous and vanishing at infinity Fourier multiplier $a_0$ on $X(\mathbb{R}, w)$. This result extends previous results by Sarason (Duke Math J 44:357–364, 1977) for $L^2(\mathbb{R})$ and by Karlovich and Loreto Hernández (Integral Equ Oper Theor 62:85–128, 2008) for weighted Lebesgue spaces $L^p(\mathbb{R}, w)$ with weights in a suitable subclass of the Muckenhoupt class $A_p(\mathbb{R})$.


**Keywords** Rearrangement-invariant Banach function space · Boyd indices · Muckenhoupt weight · Almost periodic function · Semi-almost periodic function · Fourier multiplier

**Mathematics Subject Classification** Primary 42A45 · Secondary 46E30

Dedicated to Professor Yuri I. Karlovich on the occasion of his 70th birthday.

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## 1 Introduction

Let $C(\overline{\mathbb{R}})$ be the $C^*$-algebra of all continuous functions on the two-point compactification of the real line $\overline{\mathbb{R}} = [-\infty, +\infty]$ and

$$C(\dot{\mathbb{R}}) = \{f \in C(\overline{\mathbb{R}}) : f(-\infty) = f(+\infty)\},$$

where $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the one-point compactification of the real line. Let $APP$ denote the set of all almost periodic polynomials, that is, finite sums of the form $\sum_{\lambda \in \Lambda} c_\lambda e_\lambda$, where

$$e_\lambda(x) := e^{i\lambda x}, \quad x \in \mathbb{R},$$

$c_\lambda \in \mathbb{C}$ and $\Lambda \subset \mathbb{R}$ is a finite subset of $\mathbb{R}$. The smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains $APP$ is denoted by $AP$ and called the algebra of (uniformly) almost periodic functions. Sarason [36] introduced the algebra of semi-almost periodic functions as the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains $AP$ and $C(\overline{\mathbb{R}})$:

$$SAP := \mathrm{alg}_{L^\infty(\mathbb{R})}\{AP, C(\overline{\mathbb{R}})\}.$$

It is not difficult to see that $AP$ and $SAP$ are $C^*$-subalgebras of $L^\infty(\mathbb{R})$.

**Theorem 1.1** (*Sarason* [36], *see also* [10, *Theorem* 1.21]) *Let $u \in C(\overline{\mathbb{R}})$ be any function for which $u(-\infty) = 0$ and $u(+\infty) = 1$. If $a \in SAP$, then there exist $a_l, a_r \in AP$ and $a_0 \in C(\dot{\mathbb{R}})$ such that $a_0(\infty) = 0$ and*

$$a = (1-u)a_l + ua_r + a_0. \tag{1.1}$$

*The functions $a_l, a_r$ are uniquely determined by $a$ and independent of the particular choice of $u$. The maps $a \mapsto a_l$ and $a \mapsto a_r$ are $C^*$-algebra homomorphisms of $SAP$ onto $AP$.*

The uniquely determined function $a_l$ (resp. $a_r$) is called the left (resp. right) almost periodic representative of the semi-almost periodic function $a$.

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform:

$$(\mathcal{F}f)(x) := \widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{itx}\,\mathrm{d}t, \quad x \in \mathbb{R},$$

and let $\mathcal{F}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of $\mathcal{F}$,

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)e^{-itx}\,\mathrm{d}x, \quad t \in \mathbb{R}.$$

It is well known that the Fourier convolution operator

$$W^0(a) := \mathcal{F}^{-1}a\mathcal{F} \tag{1.2}$$

is bounded on the space $L^2(\mathbb{R})$ for every $a \in L^\infty(\mathbb{R})$.

Let $X(\mathbb{R})$ be a separable Banach function space (see Sect. 2.1 for the definition and some properties of Banach function spaces). Then $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ (see, e.g., [15, Lemma 2.1]). A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $X(\mathbb{R})$ if the convolution operator $W^0(a)$ defined by (1.2) maps the set $L^2(\mathbb{R}) \cap X(\mathbb{R})$ into the space $X(\mathbb{R})$ and extends to a bounded linear operator on $X(\mathbb{R})$. The function $a$ is called the symbol of the Fourier convolution operator $W^0(a)$. The set $\mathcal{M}_{X(\mathbb{R})}$ of all Fourier multipliers on $X(\mathbb{R})$ is a unital normed algebra under pointwise operations and the norm:

$$\|a\|_{\mathcal{M}_{X(\mathbb{R})}} := \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))},$$

where $\mathcal{B}(X(\mathbb{R}))$ denotes the Banach algebra of all bounded linear operators on the space $X(\mathbb{R})$.

Note that the Lebesgue spaces $L^p(\mathbb{R})$, $1 \le p \le \infty$, constitute the simplest example of Banach function spaces. Motivated by the work of Duduchava and Saginashvili [14], Karlovich and Spitkovsky [29] (see also [10, Section 19.1]) introduced the algebra $SAP_{L^p(\mathbb{R})}$ of semi-almost periodic Fourier multipliers on the Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, and proved an analogue of Sarason's Theorem 1.1 for $SAP_{L^p(\mathbb{R})}$ (see [29, Lemma 3.1(iv)] and [10, Proposition 19.3]).

We should mention that, after Sarason's pioneering paper [36], various classes of Toeplitz and convolution type operators involving semi-almost periodic functions were studied on various function spaces, for instance, by Saginashvili [35], Grudsky [19]; Böttcher et al. [3–6, 8–10]; Nolasco and Castro [32, 33]; Bogveradze and Castro [2]; the second author and Spitkovsky [25].

Let $\mathfrak{M}(\mathbb{R})$ denote the set of all measurable complex-valued Lebesgue measurable functions on $\mathbb{R}$. As usual, we identify two functions on $\mathbb{R}$ which are equal almost everywhere. A measurable function $w : \mathbb{R} \to [0, \infty]$ is called a weight if the set $w^{-1}(\{0, \infty\})$ has measure zero. For $1 < p < \infty$, the Muckenhoupt class $A_p(\mathbb{R})$ is defined as the class of all weights $w : \mathbb{R} \to [0, \infty]$ such that $w \in L^p_{\mathrm{loc}}(\mathbb{R})$, $w^{-1} \in L^{p'}_{\mathrm{loc}}(\mathbb{R})$ and

$$\sup_I \left(\frac{1}{|I|}\int_I w^p(x)\,\mathrm{d}x\right)^{1/p} \left(\frac{1}{|I|}\int_I w^{-p'}(x)\,\mathrm{d}x\right)^{1/p'} < \infty, \tag{1.3}$$

where $1/p + 1/p' = 1$ and the supremum is taken over all intervals $I \subset \mathbb{R}$ of finite length $|I|$. Since $w \in L^p_{\mathrm{loc}}(\mathbb{R})$ and $w^{-1} \in L^{p'}_{\mathrm{loc}}(\mathbb{R})$, the weighted Lebesgue space

$$L^p(\mathbb{R}, w) := \{f \in \mathfrak{M}(\mathbb{R}) : fw \in L^p(\mathbb{R})\}$$

is a separable Banach function space (see, e.g., [26, Lemma 2.4]) with the norm:

$$\|f\|_{L^p(\mathbb{R},w)} := \left(\int_{\mathbb{R}} |f(x)|^p w^p(x)\,\mathrm{d}x\right)^{1/p}.$$

Note that if $w \in A_p(\mathbb{R})$, then it may happen that the function $e_\lambda$ does not belong to $\mathcal{M}_{L^p(\mathbb{R},w)}$ for some $\lambda \in \mathbb{R}$. Hence,order to generalize Theorem 1.1 to the setting of weighted Lebesgue spaces $L^p(\mathbb{R}, w)$, one has to restrict the study to a narrower class of weights. For $1<p<\infty$, let

$$A_p^0(\mathbb{R}) := \left\{ w \in A_p(\mathbb{R}) \ : \ v_\lambda = \frac{w(\cdot + \lambda)}{w(\cdot)} \in L^\infty(\mathbb{R}) \text{ for all } \lambda \in \mathbb{R} \right\}.$$

For a weight $w \in A_p^0(\mathbb{R})$, Karlovich and Loreto Hernández defined the algebra $SAP_{L^p(\mathbb{R},w)}$ of semi-almost periodic Fourier multipliers on the weighted Lebesgue space $L^p(\mathbb{R}, w)$ and proved an analogue of Theorem 1.1 in this setting (see [27, Theorem 3.1]). The aim of this paper is to extend this result to the setting of separable rearrangement-invariant Banach function spaces with suitable Muckenhoupt weights.

It is well known that the Lebesgue spaces $L^p(\mathbb{R})$, $1 \le p \le \infty$, fall in the class of rearrangement-invariant Banach function spaces. Other classical examples of rearrangement-invariant Banach function spaces are Orlicz spaces $L^\Phi(\mathbb{R})$ and Lorentz spaces $L^{p,q}(\mathbb{R})$, $1 \le p, q \le \infty$. For a rearrangement-invariant Banach function space $X(\mathbb{R})$, its Boyd indices $\alpha_X, \beta_X$ are important interpolation characteristics. In particular, $\alpha_{L^p} = \beta_{L^p} = 1/p$ for $1 \le p \le \infty$. In general, $0 \le \alpha_X \le \beta_X \le 1$ and it may happen that $\alpha_X < \beta_X$. We postpone formal definitions of rearrangement-invariant Banach function spaces and their Boyd indices until Sects. 2.2–2.3 and refer to [1, Chap. 3] and [30, Chap. 2] for the detailed study of these concepts.

Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices $\alpha_X, \beta_X$ satisfying $0 < \alpha_X, \beta_X < 1$. Suppose that a weight $w$ belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Then

$$X(\mathbb{R}, w) := \{f \in \mathfrak{M}(\mathbb{R}) : fw \in X(\mathbb{R})\}$$

is a separable Banach function space (see Lemma 2.3(b) below). Suppose that $a : \mathbb{R} \to \mathbb{C}$ is a function of finite total variation $V(a)$ given by

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over all partitions of $\mathbb{R}$ of the form

$$-\infty < x_0 < x_1 < \cdots < x_n < +\infty$$

with $n \in \mathbb{N}$. The set $V(\mathbb{R})$ of all functions of finite total variation on $\mathbb{R}$ with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a)$$

is a unital non-separable Banach algebra. It follows from [21, Corollary 2.2] that there exists a constant $c_{X(\mathbb{R},w)} \in (0, \infty)$ such that for all $a \in V(\mathbb{R})$,

$$\|a\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le c_{X(\mathbb{R},w)} \|a\|_{V(\mathbb{R})}. \tag{1.4}$$

This inequality is usually called a Stechkin-type inequality (see, e.g., [13, Theorem 2.11] and [10, Theorem 17.1] for the case of Lebesgue spaces and Lebesgue spaces with Muckenhoupt weights, respectively). Let $C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$ and $C_{X(\mathbb{R},w)}(\overline{\mathbb{R}})$ denote the closures of $C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ and $C(\overline{\mathbb{R}}) \cap V(\mathbb{R})$ with respect to the norm of $\mathcal{M}_{X(\mathbb{R},w)}$, respectively.

If $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$, then $APP \subset \mathcal{M}_{X(\mathbb{R},w)}$ (see Corollary 5.2 below). Because of this observation, we will refer to $A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$ as the class of suitable Muckenhoupt weights. By $AP_{X(\mathbb{R},w)}$ we denote the closure of $APP$ with respect to the norm of $\mathcal{M}_{X(\mathbb{R},w)}$. Finally, let $SAP_{X(\mathbb{R},w)}$ be the smallest closed subalgebra of $\mathcal{M}_{X(\mathbb{R},w)}$ that contains the algebras $AP_{X(\mathbb{R},w)}$ and $C_{X(\mathbb{R},w)}(\overline{\mathbb{R}})$:

$$SAP_{X(\mathbb{R},w)} = \operatorname{alg}_{\mathcal{M}_{X(\mathbb{R},w)}}\left\{AP_{X(\mathbb{R},w)}, C_{X(\mathbb{R},w)}(\overline{\mathbb{R}})\right\}.$$

In this paper we present a self-contained proof of the following result.

**Theorem 1.2** (*Main result*) *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X, \beta_X < 1$. Suppose that $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$. Let $u \in C(\overline{\mathbb{R}})$ be any real-valued monotonically increasing function such that $u(-\infty) = 0$ and $u(+\infty) = 1$. Then for every function $a \in SAP_{X(\mathbb{R},w)}$ there exist functions $a_l, a_r \in AP_{X(\mathbb{R},w)}$ and a function $a_0 \in C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$ such that $a_0(\infty) = 0$ and (1.1) holds. The functions $a_l$, $a_r$ are uniquely determined by the function a and are independent of the particular choice of the function u. The maps $a \mapsto a_l$ and $a \mapsto a_r$ are continuous Banach algebra homomorphisms of $SAP_{X(\mathbb{R},w)}$ onto $AP_{X(\mathbb{R},w)}$ of norm 1.*

The paper is organized as follows. In Sect. 2, we collect definitions and properties of rearrangement-invariant Banach functions spaces and their Boyd indices $\alpha_X, \beta_X$. Further, we discuss properties of weighted rearrangement-invariant spaces $X(\mathbb{R}, w)$ and state several results about general Fourier multipliers on $X(\mathbb{R}, w)$ for weights $w$ belonging to the intersection of the Muckenhoupt classes $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$.

In Sect. 3, we show that, under the assumption $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$, the set of continuous Fourier multipliers vanishing at infinity on the space $X(\mathbb{R}, w)$ coincides with the closure of the set of all smooth compactly supported functions with respect to the norm of $\mathcal{M}_{X(\mathbb{R},w)}$.

Relying on the results of the previous section, in Sect. 4, we show that $C_{X(\mathbb{R},w)}(\dot{\mathbb{R}}) = C_{X(\mathbb{R},w)}(\overline{\mathbb{R}}) \cap C(\dot{\mathbb{R}})$ and that the algebra $C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$ is contained in the algebra $SO_{X(\mathbb{R},w)}$ of slowly oscillating Fourier multipliers (see [21]).

In Sect. 5, we show that if $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$, then the set of almost periodic polynomials $APP$ is contained in $\mathcal{M}_{X(\mathbb{R},w)}$. We give an example of a nontrivial weight in $A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$ (based on an example from [27]). Further, we show that the product of an almost periodic Fourier multiplier and a continuous Fourier multiplier vanishing at infinity is a continuous Fourier multiplier vanishing at infinity.

Section 6 is devoted to the proof of the main result. We show that the set $\mathcal{A}_u$ of functions of the form (1.1) with $a_l, a_r \in AP_{X(\mathbb{R},w)}$ and $a_0 \in C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$ such that $a_0(\infty) = 0$ forms an algebra, and that the mappings $a \mapsto a_l$ and $a \mapsto a_r$ are algebraic homomorphisms of $\mathcal{A}_u$ onto $AP_{X(\mathbb{R},w)}$. We prove that

$$\|a_l\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}, \quad \|a_r\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}, \quad a \in \mathcal{A}_u, \tag{1.5}$$

which implies that the algebra $\mathcal{A}_u$ is closed. Since the closure of $\mathcal{A}_u$ with respect to the norm of $\mathcal{M}_{X(\mathbb{R},w)}$ coincides with $SAP_{X(\mathbb{R},w)}$, we conclude that $\mathcal{A}_u$ is equal to $SAP_{X(\mathbb{R},w)}$. Moreover, inequalities (1.5) mean that $a \mapsto a_l$ and $a \mapsto a_r$ are Banach algebra homomorphisms of $SAP_{X(|R,w)}$ onto $AP_{X(\mathbb{R},w)}$ of norm 1.

## 2 Preliminaries

### 2.1 Banach function spaces

Let $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{S} \in \{\mathbb{R}_+, \mathbb{R}\}$. The set of all Lebesgue measurable complex-valued functions on $\mathbb{S}$ is denoted by $\mathfrak{M}(\mathbb{S})$. Let $\mathfrak{M}^+(\mathbb{S})$ be the subset of functions in $\mathfrak{M}(\mathbb{S})$ whose values lie in $[0, \infty]$. The Lebesgue measure of a measurable set $E \subset \mathbb{S}$ is denoted by $|E|$ and its characteristic function is denoted by $\chi_E$. Following [1, Chap. 1, Definition 1.1], a mapping $\rho : \mathfrak{M}^+(\mathbb{S}) \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n$ $(n \in \mathbb{N})$ in $\mathfrak{M}^+(\mathbb{S})$, for all constants $a \ge 0$, and for all measurable subsets $E$ of $\mathbb{S}$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\quad \rho(af) = a\rho(f), \quad \rho(f+g) \le \rho(f) + \rho(g)$,
(A2) $0 \le g \le f$ a.e. $\Rightarrow \rho(g) \le \rho(f)$ (the lattice property),
(A3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),
(A4) $|E| < \infty \Rightarrow \rho(\chi_E) < \infty$,
(A5) $|E| < \infty \Rightarrow \int_E f(x)\,\mathrm{d}x \le C_E \rho(f)$

with $C_E \in (0, \infty)$ which may depend on $E$ and $\rho$ but is independent of $f$. When functions differing only on a set of measure zero are identified, the set $X(\mathbb{S})$ of all functions $f \in \mathfrak{M}(\mathbb{S})$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathbb{S})$, the norm of $f$ is defined by

$$\|f\|_{X(\mathbb{S})} := \rho(|f|).$$

Under the natural linear space operations and under this norm, the set $X(\mathbb{S})$ becomes a Banach space (see [1, Chap. 1, Theorems 1.4 and 1.6]). If $\rho$ is a Banach function norm, its associate norm $\rho'$ is defined on $\mathfrak{M}^+(\mathbb{S})$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{S}} f(x)g(x)\,dx \ : \ f \in \mathfrak{M}^+(\mathbb{S}),\ \rho(f) \le 1 \right\}, \quad g \in \mathfrak{M}^+(\mathbb{S}).$$

It is a Banach function norm itself [1, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{R})$ determined by the Banach function norm $\rho'$ is called the associate space (Köthe dual) of $X(\mathbb{S})$. The associate space $X'(\mathbb{S})$ is naturally identified with a subspace of the (Banach) dual space $[X(\mathbb{S})]^*$.

### 2.2 Rearrangement-invariant Banach function spaces

Suppose that $\mathbb{S} \in \{\mathbb{R}, \mathbb{R}_+\}$. Let $\mathfrak{M}_0(\mathbb{S})$ and $\mathfrak{M}_0^+(\mathbb{S})$ be the classes of a.e. finite functions in $\mathfrak{M}(\mathbb{S})$ and $\mathfrak{M}^+(\mathbb{S})$, respectively. The distribution function $\mu_f$ of a function $f \in \mathfrak{M}_0(\mathbb{S})$ is given by

$$\mu_f(\lambda) := \left|\{x \in \mathbb{S} : |f(x)| > \lambda\}\right|, \quad \lambda \geq 0.$$

Two functions $f, g \in \mathfrak{M}_0(\mathbb{S})$ are said to be equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$. The non-increasing rearrangement of $f \in \mathfrak{M}_0(\mathbb{S})$ is the function defined by

$$f^*(t) := \inf\{\lambda : \mu_f(\lambda) \leq t\}, \quad t \geq 0.$$

We here use the standard convention that $\inf \emptyset = +\infty$.

A Banach function norm $\rho : \mathfrak{M}^+(\mathbb{S}) \to [0, \infty]$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in \mathfrak{M}_0^+(\mathbb{S})$ the equality $\rho(f) = \rho(g)$ holds. In that case, the Banach function space $X(\mathbb{S})$ generated by $\rho$ is said to be a rearrangement-invariant Banach function space (or simply rearrangement-invariant space). Lebesgue, Orlicz, and Lorentz spaces are classical examples of rearrangement-invariant Banach function spaces (see, e.g., [1] and the references therein). By [1, Chap. 2, Proposition 4.2], if a Banach function space $X(\mathbb{S})$ is rearrangement-invariant, then its associate space $X'(\mathbb{S})$ is rearrangement-invariant, too.

### 2.3 Boyd indices

Suppose $X(\mathbb{R})$ is a rearrangement-invariant Banach function space generated by a rearrangement-invariant Banach function norm $\rho$. In this case, the Luxemburg representation theorem [1, Chap. 2, Theorem 4.10] provides a unique rearrangement-invariant Banach function norm $\overline{\rho}$ over the half-line $\mathbb{R}_+$ equipped with the Lebesgue measure, defined by

$$\overline{\rho}(h) := \sup\left\{\int_{\mathbb{R}_+} g^*(t)h^*(t)\,\mathrm{d}t : \ \rho'(g) \leq 1\right\},$$

and such that $\rho(f) = \overline{\rho}(f^*)$ for all $f \in \mathfrak{M}_0^+(\mathbb{R})$. The rearrangement-invariant Banach function space generated by $\overline{\rho}$ is denoted by $\overline{X}(\mathbb{R}_+)$.

For each $t > 0$, let $E_t$ denote the dilation operator defined on $\mathfrak{M}(\mathbb{R}_+)$ by

$$(E_t f)(s) = f(st), \quad 0 < s < \infty.$$

With $X(\mathbb{R})$ and $\overline{X}(\mathbb{R}_+)$ as above, let $h_X(t)$ denote the operator norm of $E_{1/t}$ as an operator on $\overline{X}(\mathbb{R}_+)$. By [1, Chap. 3, Proposition 5.11], for each $t > 0$, the operator

$E_t$ is bounded on $\overline{X}(\mathbb{R}_+)$ and the function $h_X$ is increasing and submultiplicative on $(0, \infty)$. The Boyd indices of $X(\mathbb{R})$ are the numbers $\alpha_X$ and $\beta_X$ defined by

$$\alpha_X := \sup_{t\in(0,1)} \frac{\log h_X(t)}{\log t}, \quad \beta_X := \inf_{t\in(1,\infty)} \frac{\log h_X(t)}{\log t}.$$

By [1, Chap. 3, Proposition 5.13], $0 \le \alpha_X \le \beta_X \le 1$. The Boyd indices are said to be nontrivial if $\alpha_X, \beta_X \in (0, 1)$. The Boyd indices of the Lebesgue space $L^p(\mathbb{R})$, $1 \le p \le \infty$, are both equal to $1/p$. Note that the Boyd indices of a rearrangement-invariant space may be different [1, Chap. 3, Exercises 6, 13].

The next theorem follows from the Boyd interpolation theorem [11, Theorem 1] for quasi-linear operators of weak types $(p, p)$ and $(q, q)$. Its proof can also be found in [1, Chap. 3, Theorem 5.16] and [30, Theorem 2.b.11].

**Theorem 2.1** *Let $1 \le q < p \le \infty$ and $X(\mathbb{R})$ be a rearrangement-invariant Banach function space with the Boyd indices $\alpha_X, \beta_X$ satisfying $1/p < \alpha_X$, $\beta_X < 1/q$. Then there exists a constant $C_{p,q} \in (0, \infty)$ such that if a linear operator $T : \mathfrak{M}(\mathbb{R}) \to \mathfrak{M}(\mathbb{R})$ is bounded on the Lebesgue spaces $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, then it is also bounded on the rearrangement-invariant Banach function space $X(\mathbb{R})$ and*

$$\|T\|_{\mathcal{B}(X(\mathbb{R}))} \le C_{p,q} \max\left\{\|T\|_{\mathcal{B}(L^p(\mathbb{R}))}, \|T\|_{\mathcal{B}(L^q(\mathbb{R}))}\right\}. \tag{2.1}$$

Notice that estimate (2.1) is not stated explicitly in [1, 11, 30]. However, it can be extracted from the proof of the Boyd interpolation theorem.

### 2.4 Weighted Banach function spaces

Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm $\rho$. We say that $f \in X_{\mathrm{loc}}(\mathbb{R})$ if $f\chi_E \in X(\mathbb{R})$ for any measurable set $E \subset \mathbb{R}$ of finite measure.

**Lemma 2.2** [26, *Lemma* 2.4] *Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm $\rho$, let $X'(\mathbb{R})$ be its associate space, and let $w : \mathbb{R} \to [0, \infty]$ be a weight. Suppose that $w \in X_{\mathrm{loc}}(\mathbb{R})$ and $1/w \in X'_{\mathrm{loc}}(\mathbb{R})$. Then*

$$\rho_w(f) := \rho(fw), \quad f \in \mathfrak{M}^+(\mathbb{R}),$$

*is a Banach function norm and*

$$X(\mathbb{R}, w) := \{f \in \mathfrak{M}(\mathbb{R}) : fw \in X(\mathbb{R})\}$$

*is a Banach function space generated by the Banach function norm $\rho_w$. The space $X'(\mathbb{R}, w^{-1})$ is the associate space of $X(\mathbb{R}, w)$.*

### 2.5 Density of nice functions in separable rearrangement-invariant Banach function spaces with Muckenhoupt weights

Recall that the (noncentered) Hardy–Littlewood maximal function $Mf$ of a function $f \in L^1_{\rm loc}(\mathbb{R})$ is defined by

$$(Mf)(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy, \quad x \in \mathbb{R},$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ of finite length containing the point $x$.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing smooth functions and let us denote by $\mathcal{S}_0(\mathbb{R})$ the set of all functions $f \in \mathcal{S}(\mathbb{R})$ such that their Fourier transforms $\mathcal{F}f$ have compact supports.

**Lemma 2.3** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space and $X'(\mathbb{R})$ be its associate space. Suppose that the Boyd indices of $X(\mathbb{R})$ satisfy $0<\alpha_X, \beta_X<1$ and $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Then*

- *(a) $w \in X_{\rm loc}(\mathbb{R})$ and $1/w \in X'_{\rm loc}(\mathbb{R})$;*
- *(b) the Banach function space space $X(\mathbb{R}, w)$ is separable;*
- *(c) the Hardy-Littlewood maximal operator $M$ is bounded on the Banach function space $X(\mathbb{R}, w)$ and on its associate space $X'(\mathbb{R}, w^{-1})$;*
- *(d) the set $\mathcal{S}_0(\mathbb{R})$ is dense in the Banach function space $X(\mathbb{R}, w)$.*

***Proof*** Parts (a) and (c) are proved in [21, Section 4.3]. Part (b) follows from part (a), Lemma 2.2 and [26, Lemmas 2.7 and 2.11]. Part (d) is a consequence of parts (b), (c) and [16, Theorem 4]. □

### 2.6 The Banach algebra $\mathcal{M}_{X(\mathbb{R},w)}$ of Fourier multipliers

The following result plays an important role in this paper.

**Theorem 2.4** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0<\alpha_X, \beta_X<1$. Suppose that a weight $w$ belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. If $a \in \mathcal{M}_{X(\mathbb{R},w)}$, then*

$$\|a\|_{L^\infty(\mathbb{R})} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}. \tag{2.2}$$

*The constant 1 on the right-hand side of (2.2) is best possible.*

This theorem follows from Lemma 2.3(b) and [15, Theorem 2.4] (which was deduced from [24, Corollary 4.2]).

Inequality (2.2) was established earlier in [22, Theorem 1] with some constant on the right-hand side that depends on the space $X(\mathbb{R}, w)$.

Since (2.2) is available, an easy adaptation of the proof of [18, Proposition 2.5.13] leads to the following (we refer to the proof of [22, Corollary 1] for details).

**Corollary 2.5** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0<\alpha_X,\ \beta_X<1$. Suppose that a weight $w$ belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Then the set of the Fourier multipliers $\mathcal{M}_{X(\mathbb{R},w)}$ is a Banach algebra under pointwise operations and the norm $\|\cdot\|_{\mathcal{M}_{X(\mathbb{R},w)}}$.*

As usual, we denote by $C_c^\infty(\mathbb{R})$ the set of all infinitely differentiable functions with compact support.

**Theorem 2.6** *Suppose that a non-negative even function $\varphi \in C_c^\infty(\mathbb{R})$ satisfies the condition*

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1 \tag{2.3}$$

*and the function $\varphi_\delta$ is defined for $\delta > 0$ by*

$$\varphi_\delta(x) := \delta^{-1}\varphi(x/\delta), \quad x \in \mathbb{R}. \tag{2.4}$$

*Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0<\alpha_X,\ \beta_X<1$. Suppose that a weight $w$ belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. If $a \in \mathcal{M}_{X(\mathbb{R},w)}$, then for every $\delta > 0$,*

$$\|a * \varphi_\delta\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}. \tag{2.5}$$

***Proof*** The proof is analogous to the proof of [23, Theorem 2.6]. It follows from Lemma 2.3(c) and [26, Theorems 3.8(a) and 3.9(c)] that if the weight $w$ belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$, then

$$\sup_{-\infty<a<b<\infty} \frac{1}{b-a}\|\chi_{(a,b)}\|_{X(\mathbb{R},w)}\|\chi_{(a,b)}\|_{X'(\mathbb{R},w^{-1})} < \infty.$$

Therefore, by [24, Lemma 1.3], the Banach function space $X(\mathbb{R},w)$ satisfies the hypotheses of [24, Theorem 1.3]. It is shown in its proof (see [24, Section 4.2]) that for every $\delta > 0$ and every $f \in \mathcal{S}(\mathbb{R}) \cap X(\mathbb{R},w)$,

$$\|\mathcal{F}^{-1}(a*\varphi_\delta)\mathcal{F}f\|_{X(\mathbb{R},w)} \le \sup\left\{\frac{\|\mathcal{F}^{-1}a\mathcal{F}f\|_{X(\mathbb{R},w)}}{\|f\|_{X(\mathbb{R},w)}} : f \in X_{\mathcal{S}}(\mathbb{R},w)\right\}\|f\|_{X(\mathbb{R},w)},$$

where

$$X_{\mathcal{S}}(\mathbb{R},w) := (\mathcal{S}(\mathbb{R}) \cap X(\mathbb{R},w))\setminus\{0\}.$$

Then, for every $\delta > 0$,

$$\sup\left\{\frac{\|\mathcal{F}^{-1}(a*\varphi_\delta)\mathcal{F}f\|_{X(\mathbb{R},w)}}{\|f\|_{X(\mathbb{R},w)}} : f \in X_{\mathcal{S}}(\mathbb{R},w)\right\} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}. \tag{2.6}$$

By Lemma 2.3(b), the Banach function space $X(\mathbb{R},w)$ is separable. Then it follows from [1, Chap. 1, Corollary 5.6] and [24, Theorems 2.3 and 6.1] that for every $\delta > 0$, the left-hand side of inequality (2.6) coincides with the multiplier norm $\|a * \varphi_\delta\|_{\mathcal{M}_{X(\mathbb{R},w)}}$, which completes the proof of inequality (2.5). $\square$

## 3 Continuous Fourier multipliers vanishing at infinity

### 3.1 The case of Lebesgue spaces with Muckenhoupt weights

The closure of a subset $\mathfrak{S}$ of a Banach space $\mathcal{E}$ in the norm of $\mathcal{E}$ will be denoted by $\mathrm{clos}_{\mathcal{E}}(\mathfrak{S})$.

Let $C_0(\mathbb{R})$ be the set of all functions $f \in C(\dot{\mathbb{R}})$ such that $f(\infty) = 0$.

**Lemma 3.1** *Let $1<p<\infty$ and $w \in A_p(\mathbb{R})$. Then*

$$C_0(\mathbb{R}) \cap V(\mathbb{R}) \subset \mathrm{clos}_{\mathcal{M}_{L^p(\mathbb{R},w)}}\big(C_c^\infty(\mathbb{R})\big).$$

***Proof*** The idea of the proof is borrowed from [20, Theorem 1.16] (see also [23, Theorem 3.1]). If $w \in A_p(\mathbb{R})$, then $w^{1+\delta_2} \in A_{p(1+\delta_1)}(\mathbb{R})$ whenever $|\delta_1|$ and $|\delta_2|$ are sufficiently small (see, e.g., [7, Theorem 2.31]). If $p \geq 2$, then one can find sufficiently small $\delta_1, \delta_2 > 0$ and a number $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p(1+\delta_1)}, \quad w = 1^{1-\theta} w^{(1+\delta_2)\theta}, \quad w^{1+\delta_2} \in A_{p(1+\delta_1)}(\mathbb{R}). \tag{3.1}$$

If $1<p<2$, then one can find a sufficiently small number $\delta_2 > 0$, a number $\delta_1 < 0$ with sufficiently small $|\delta_1|$, and a number $\theta \in (0,1)$ such that all conditions in (3.1) are fulfilled.

Let us use the following abbreviations:

$$\mathcal{M}_p := \mathcal{M}_{L^p(\mathbb{R},w)}, \quad \mathcal{M}_{p_\theta} := \mathcal{M}_{L^{p(1+\delta_1)}(\mathbb{R},w^{1+\delta_2})},$$
$$\mathcal{B}_p := \mathcal{B}(L^p(\mathbb{R},w)), \quad \mathcal{B}_{p_\theta} := \mathcal{B}(L^{p(1+\delta_1)}(\mathbb{R},w^{1+\delta_2})).$$

For $n \in \mathbb{N}$, let

$$\psi_n(x) := \begin{cases} 1 & \text{if} \quad |x| \leq n, \\ n+1-|x| & \text{if} \quad n<|x|<n+1, \\ 0 & \text{if} \quad |x| \geq n+1. \end{cases} \tag{3.2}$$

Then $\psi_n$ has compact support and $\|\psi_n\|_{V(\mathbb{R})} = 3$. By the Stechkin-type inequality (1.4),

$$\|\psi_n\|_{\mathcal{M}_{p_\theta}} \leq c_\theta,$$

where $c_\theta$ is three times $c_{L^{p(1+\delta_1)}(\mathbb{R},w^{1+\delta_2})}$, and the latter constant is the constant from (1.4).

Let $a \in C_0(\mathbb{R}) \cap V(\mathbb{R})$. Fix $\varepsilon > 0$. For $n \in \mathbb{N}$, take $b_n := a\psi_n$. Then

$$\lim_{n\to\infty} \|a - b_n\|_{L^\infty(\mathbb{R})} = 0 \tag{3.3}$$

and $b_n \in C_0(\mathbb{R})$ has compact support. Taking into account the Stechkin-type inequality (1.4), we get

$$\|a - b_n\|_{\mathcal{M}_{p_\theta}} \le \|a\|_{\mathcal{M}_{p_\theta}}(1 + \|\psi_n\|_{\mathcal{M}_{p_\theta}}) \le (1 + c_\theta)c_\theta\|a\|_{V(\mathbb{R})} \tag{3.4}$$

and

$$\|b_n\|_{\mathcal{M}_{p_\theta}} \le \|a\|_{\mathcal{M}_{p_\theta}}\|\psi_n\|_{\mathcal{M}_{p_\theta}} \le c_\theta^2\|a\|_{V(\mathbb{R})}. \tag{3.5}$$

It follows from (3.1) and the Stein–Weiss interpolation theorem (see, e.g., [1, Chap. 3, Theorem 3.6]) that

$$\begin{aligned} \|a - b_n\|_{\mathcal{M}_p} &= \|W^0(a - b_n)\|_{\mathcal{B}_p} \\ &\le \|W^0(a - b_n)\|^{1-\theta}_{\mathcal{B}(L^2(\mathbb{R}))}\|W^0(a - b_n)\|^{\theta}_{\mathcal{B}_{p_\theta}} \\ &= \|a - b_n\|^{1-\theta}_{L^\infty(\mathbb{R})}\|a - b_n\|^{\theta}_{\mathcal{M}_{p_\theta}}. \end{aligned} \tag{3.6}$$

Combining (3.3), (3.4) and (3.6), we see that there exists $n_0 \in \mathbb{N}$ such that

$$\|a - b_{n_0}\|_{\mathcal{M}_p} < \varepsilon/2. \tag{3.7}$$

Let $\varphi \in C_c^\infty(\mathbb{R})$ be a non-negative even function satisfying (2.3) and for $\delta > 0$ let the function $\varphi_\delta$ be defined by (2.4). By Theorem 2.6 and inequality (3.5), for every $\delta > 0$,

$$\|b_{n_0} * \varphi_\delta\|_{\mathcal{M}_{p_\theta}} \le \|b_{n_0}\|_{\mathcal{M}_{p_\theta}} \le c_\theta^2\|a\|_{V(\mathbb{R})}. \tag{3.8}$$

It follows from [12, Propositions 4.18, 4.20–4.21] that $b_{n_0} * \varphi_\delta \in C_c^\infty(\mathbb{R})$ and

$$\lim_{\delta\to 0^+} \|b_{n_0} * \varphi_\delta - b_{n_0}\|_{L^\infty(\mathbb{R})} = 0. \tag{3.9}$$

In view of (3.1) and the Stein-Weiss interpolation theorem (see, e.g., [1, Chap. 3, Theorem 3.6]), we see that

$$\begin{aligned} &\|b_{n_0} * \varphi_\delta - b_{n_0}\|_{\mathcal{M}_p} \\ &\quad = \|W^0(b_{n_0} * \varphi_\delta - b_{n_0})\|_{\mathcal{B}_p} \\ &\quad \le \|W^0(b_{n_0} * \varphi_\delta - b_{n_0})\|^{1-\theta}_{\mathcal{B}(L^2(\mathbb{R}))}\|W^0(b_{n_0} * \varphi_\delta - b_{n_0})\|^{\theta}_{\mathcal{B}_{p_\theta}} \\ &\quad = \|b_{n_0} * \varphi_\delta - b_{n_0}\|^{1-\theta}_{L^\infty(\mathbb{R})}\|b_{n_0} * \varphi_\delta - b_{n_0}\|^{\theta}_{\mathcal{M}_{p_\theta}} \\ &\quad \le \|b_{n_0} * \varphi_\delta - b_{n_0}\|^{1-\theta}_{L^\infty(\mathbb{R})}(\|b_{n_0} * \varphi_\delta\|_{\mathcal{M}_{p_\theta}} + \|b_{n_0}\|_{\mathcal{M}_{p_\theta}})^{\theta}. \end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10), we conclude that there exists $\delta_0 > 0$ such that

$$\|b_{n_0} * \varphi_{\delta_0} - b_{n_0}\|_{\mathcal{M}_p} < \varepsilon/2. \tag{3.11}$$

Hence, it follows from (3.7) and (3.11) that for every function $a$ in the intersection $C_0(\mathbb{R}) \cap V(\mathbb{R})$ and every $\varepsilon > 0$ there exists a function $b_{n_0} * \varphi_{\delta_0} \in C_c^\infty(\mathbb{R})$ such that $\|a - b_{n_0} * \varphi_{\delta_0}\|_{\mathcal{M}_p} < \varepsilon$. Therefore, $a \in \mathrm{clos}_{\mathcal{M}_p}\big(C_c^\infty(\mathbb{R})\big)$. □

### 3.2 The case of rearrangement-invariant spaces with Muckenhoupt weights

The following lemma is an extension of the previous result to the case of rearrangement-invariant Banach function spaces with Muckenhoupt weights.

**Lemma 3.2** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X, \beta_X < 1$. Suppose that a weight $w$ belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Then*

$$C_0(\mathbb{R}) \cap V(\mathbb{R}) \subset \mathrm{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}\big(C_c^\infty(\mathbb{R})\big).$$

***Proof*** Since $\alpha_X, \beta_X \in (0,1)$ and $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$, it follows from [7, Theorem 2.31] that there exist $p$ and $q$ such that

$$1 < q < 1/\beta_X \le 1/\alpha_X < p < \infty, \quad w \in A_p(\mathbb{R}) \cap A_q(\mathbb{R}). \tag{3.12}$$

Let $C_{p,q} \in (0,\infty)$ be the constant from estimate (2.1). Fix $\varepsilon > 0$ and take a function $a \in C_0(\mathbb{R}) \cap V(\mathbb{R})$. As in the proof of inequality (3.7) (see the proof of Lemma 3.1), it can be shown that there exists $n_0 \in \mathbb{N}$ such that

$$\|a - b_{n_0}\|_{\mathcal{M}_{L^p(\mathbb{R},w)}} < \frac{\varepsilon}{2C_{p,q}}, \quad \|a - b_{n_0}\|_{\mathcal{M}_{L^q(\mathbb{R},w)}} < \frac{\varepsilon}{2C_{p,q}}, \tag{3.13}$$

where $b_n = a\psi_n$ and $\psi_n$ is given by (3.2) for every $n \in \mathbb{N}$. It follows from (3.12), (3.13) and Theorem 2.1 that

$$\begin{aligned}
\|a - b_{n_0}\|_{\mathcal{M}_{X(\mathbb{R},w)}} &= \|W^0(a - b_{n_0})\|_{\mathcal{B}(X(\mathbb{R},w))} \\
&= \|wW^0(a - b_{n_0})w^{-1}I\|_{\mathcal{B}(X(\mathbb{R}))} \\
&\le C_{p,q} \max\Big\{\|wW^0(a - b_{n_0})w^{-1}I\|_{\mathcal{B}(L^p(\mathbb{R}))}, \|wW^0(a - b_{n_0})w^{-1}I\|_{\mathcal{B}(L^q(\mathbb{R}))}\Big\} \\
&= C_{p,q} \max\Big\{\|W^0(a - b_{n_0})\|_{\mathcal{B}(L^p(\mathbb{R},w))}, \|W^0(a - b_{n_0})\|_{\mathcal{B}(L^q(\mathbb{R},w))}\Big\} \\
&= C_{p,q} \max\Big\{\|a - b_{n_0}\|_{\mathcal{M}_{L^p(\mathbb{R},w)}}, \|a - b_{n_0}\|_{\mathcal{M}_{L^q(\mathbb{R},w)}}\Big\} < \varepsilon/2.
\end{aligned} \tag{3.14}$$

As in the proof of inequality (3.11) (see the proof of Lemma 3.1), it can be shown that there exists $\delta_0 > 0$ such that

$$\|b_{n_0} * \varphi_{\delta_0} - b_{n_0}\|_{\mathcal{M}_{L^p(\mathbb{R},w)}} < \frac{\varepsilon}{2C_{p,q}}, \quad \|b_{n_0} * \varphi_{\delta_0} - b_{n_0}\|_{\mathcal{M}_{L^q(\mathbb{R},w)}} < \frac{\varepsilon}{2C_{p,q}}, \tag{3.15}$$

where $\varphi \in C_c^\infty(\mathbb{R})$ is a non-negative even function satisfying (2.3) and the functions $\varphi_\delta$ are defined for all $\delta > 0$ by (2.4). Arguing as in the proof of (3.14), we deduce from (3.12), (3.15) and Theorem 2.1 that

$$\|b_{n_0} * \varphi_{\delta_0} - b_{n_0}\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon/2. \tag{3.16}$$

It follows from (3.14) and (3.16) that for every function $a$ in the intersection $C_0(\mathbb{R}) \cap V(\mathbb{R})$ and every $\varepsilon > 0$ there exists a function $b_{n_0} * \varphi_{\delta_0} \in C_c^\infty(\mathbb{R})$ such that $\|a - b_{n_0} * \varphi_{\delta_0}\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon$. Therefore, $a \in \operatorname{clos}_{\mathcal{M}_{X(R,w)}}\big(C_c^\infty(\mathbb{R})\big)$. □

Now we are in a position to prove the main result of this section.

**Theorem 3.3** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X, \beta_X < 1$. Suppose that a weight $w$ belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Consider the set*

$$C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}) := \big\{a \in C_{X(\mathbb{R},w)}(\dot{\mathbb{R}}) \ : \ a(\infty) = 0\big\}. \tag{3.17}$$

*Then*

$$C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}) = \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}\big(C_c^\infty(\mathbb{R})\big). \tag{3.18}$$

***Proof*** Let $a \in C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$ be such that $a(\infty) = 0$. Fix $\varepsilon > 0$. By the definition of the algebra $C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$, there exists a function $b \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ such that

$$\|a - b\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon/3. \tag{3.19}$$

It follows from this observation and the continuous embedding of $\mathcal{M}_{X(\mathbb{R},w)}$ into $L^\infty(\mathbb{R})$ (see Theorem 2.4) that

$$|b(\infty)| = |a(\infty) - b(\infty)| \le \|a - b\|_{L^\infty(\mathbb{R})} \le \|a - b\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon/3. \tag{3.20}$$

Take $c = b - b(\infty) \in C_0(\mathbb{R}) \cap V(\mathbb{R})$. By Lemma 3.2, there exists a function $d \in C_c^\infty(\mathbb{R}) \subset \mathcal{M}_{X(\mathbb{R},w)}$ such that

$$\|c - d\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon/3. \tag{3.21}$$

Combining inequalities (3.19)–(3.21), we see that

$$\|a - d\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \|a - b\|_{\mathcal{M}_{X(\mathbb{R},w)}} + |b(\infty)| + \|c - d\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon.$$

Hence

$$C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}) \subset \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}\left(C_c^\infty(\mathbb{R})\right). \tag{3.22}$$

Let us prove the reverse embedding. Take $a \in \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}\left(C_c^\infty(\mathbb{R})\right)$. Then there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|a_n - a\|_{\mathcal{M}_{X(\mathbb{R},w)}} = 0.$$

Since $C_c^\infty(\mathbb{R}) \subset C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$, the above equality and the continuous embedding of the algebra $\mathcal{M}_{X(\mathbb{R},w)}$ into the algebra $L^\infty(\mathbb{R})$ (see Theorem 2.4) imply that $a \in C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$ and

$$\begin{aligned}|a(\infty)| &= \lim_{n\to\infty} |a_n(\infty) - a(\infty)| \le \lim_{n\to\infty} \|a_n - a\|_{L^\infty(\mathbb{R})} \\ &\le \lim_{n\to\infty} \|a_n - a\|_{\mathcal{M}_{X(R,w)}} = 0.\end{aligned}$$

Thus

$$\operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}\left(C_c^\infty(\mathbb{R})\right) \subset C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}). \tag{3.23}$$

Combining (3.22) and (3.23), we arrive at (3.18). □

# 4 Continuous and slowly oscillating Fourier multipliers

## 4.1 Continuous Fourier multipliers on one and two-point compactifications of the real line

For a function $f \in C(\overline{\mathbb{R}})$, let

$$J_f(x) := \begin{cases} f(-\infty) & \text{if} \quad x \in (-\infty, -1), \\ \dfrac{1}{2}\left[f(-\infty)(1-x) + f(+\infty)(1+x)\right] & \text{if} \quad x \in [-1, 1], \\ f(+\infty) & \text{if} \quad x \in (1, +\infty). \end{cases} \tag{4.1}$$

It is easy to see that

$$\|J_f\|_{V(\mathbb{R})} = \max\left\{|f(-\infty)|, |f(+\infty)|\right\} + |f(+\infty) - f(-\infty)|. \tag{4.2}$$

Therefore $J_f \in C(\overline{\mathbb{R}}) \cap V(\mathbb{R})$ and $f - J_f \in C_0(\mathbb{R})$.

The next lemma extends [29, Lemma 3.1(i)] from the setting of Lebesgue spaces to the setting of rearrangement-invariant Banach function spaces with Muckenhoupt weights.

**Lemma 4.1** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X, \ \beta_X < 1$. Suppose that $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Then*

$$C_{X(\mathbb{R},w)}(\dot{\mathbb{R}}) = C_{X(\mathbb{R},w)}(\overline{\mathbb{R}}) \cap C(\dot{\mathbb{R}}). \tag{4.3}$$

***Proof*** The proof is analogous to the proof of [29, Lemma 3.1(i)] (see also [23, Lemma 3.2]). It is obvious that $C_{X(\mathbb{R},w)}(\dot{\mathbb{R}}) \subset C_{X(\mathbb{R},w)}(\overline{\mathbb{R}})$. On the other hand, it follows from Theorem 2.4 that $C_{X(\mathbb{R},w)}(\dot{\mathbb{R}}) \subset C(\dot{\mathbb{R}})$. Therefore,

$$C_{X(\mathbb{R},w)}(\dot{\mathbb{R}}) \subset C_{X(\mathbb{R},w)}(\overline{\mathbb{R}}) \cap C(\dot{\mathbb{R}}). \tag{4.4}$$

To prove the opposite embedding, let us consider an arbitrary function $a \in C_{X(\mathbb{R},w)}(\overline{\mathbb{R}})$ such that $a(+\infty) = a(-\infty)$. Let $\{a_n\}_{n\in\mathbb{N}} \subset C(\overline{\mathbb{R}}) \cap V(\mathbb{R})$ be a sequence such that $\|a_n - a\|_{\mathcal{M}_{X(\mathbb{R},w)}} \to 0$ as $n \to \infty$. According to Theorem 2.4, the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to $a$ uniformly on $\mathbb{R}$. Hence, in particular, $a_n(\pm\infty) \to a(\infty)$ as $n \to \infty$. Let the functions $b_n := J_{a_n - a(\infty)}$ be defined by (4.1) with $a_n - a(\infty)$ in place of $f$. By the Stechkin-type inequality (1.4) and equality (4.2), we have

$$\begin{aligned}\|b_n\|_{\mathcal{M}_{X(\mathbb{R},w)}} &\le c_{X(\mathbb{R},w)} \|J_{a_n - a(\infty)}\|_{V(\mathbb{R})} \\ &= c_{X(\mathbb{R},w)} \max\left\{|a_n(-\infty) - a(\infty)|, |a_n(+\infty) - a(\infty)|\right\} \\ &\quad + c_{X(\mathbb{R},w)} |a_n(+\infty) - a_n(-\infty)|.\end{aligned}$$

Therefore, $\|b_n\|_{\mathcal{M}_{X(\mathbb{R},w)}} \to 0$ as $n \to \infty$ and thus,

$$\lim_{n\to\infty} \|a_n - b_n - a\|_{\mathcal{M}_{X(\mathbb{R},w)}} = 0.$$

Since $a_n - b_n \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$, the latter equality implies that $a \in C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$. Thus

$$C_{X(\mathbb{R},w)}(\overline{\mathbb{R}}) \cap C(\dot{\mathbb{R}}) \subset C_{X(\mathbb{R},w)}(\dot{\mathbb{R}}). \tag{4.5}$$

Combining embeddings (4.4)–(4.5), we arrive at equality (4.3). □

### 4.2 Embedding of the algebra $C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$ into the algebra $SO_{X(\mathbb{R},w)}$ of slowly oscillating Fourier multipliers

Let $C_b(\mathbb{R}) := C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For a bounded measurable function $f : \mathbb{R} \to \mathbb{C}$ and a set $J \subset \mathbb{R}$, let

$$\operatorname{osc}(f, J) := \operatorname*{ess\,sup}_{x,y\in J} |f(x) - f(y)|.$$

Let $SO$ be the $C^*$-algebra of all slowly oscillating functions at $\infty$ defined by

$$SO := \left\{ f \in C_b(\mathbb{R}) : \lim_{x\to+\infty} \operatorname{osc}(f, [-x, -x/2] \cup [x/2, x]) = 0 \right\}.$$

Consider the differential operator $(Df)(x) = xf'(x)$ and its iterations defined by $D^0 f = f$ and $D^j f = D(D^{j-1} f)$ for $j \in \mathbb{N}$. Let

$$SO^3 := \left\{ a \in SO \cap C^3(\mathbb{R}) : \lim_{x\to\infty} (D^j a)(x) = 0,\ j = 1,2,3 \right\},$$

where $C^3(\mathbb{R})$ denotes the set of all three times continuously differentiable functions. It is easy to see that $SO^3$ is a commutative Banach algebra under pointwise operations and the norm

$$\|a\|_{SO^3} := \sum_{j=0}^{3} \frac{1}{j!} \|D^j a\|_{L^\infty(\mathbb{R})}.$$

It follows from [21, Corollary 2.6] that if $X(\mathbb{R})$ is a separable rearrangement-invariant Banach function space with the Boyd indices $\alpha_X, \beta_X$ such that $0 < \alpha_X,\ \beta_X < 1$ and $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$, then there exists a constant $c_{X(\mathbb{R},w)} \in (0,\infty)$ such that for all $a \in SO^3$,

$$\|a\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le c_{X(\mathbb{R},w)} \|a\|_{SO^3}.$$

The continuous embedding $SO^3 \subset \mathcal{M}_{X(\mathbb{R},w)}$ allows us to define the algebra $SO_{X(\mathbb{R},w)}$ of slowly oscillating Fourier multipliers as the closure of $SO^3$ with respect to the multiplier norm:

$$SO_{X(\mathbb{R},w)} := \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}} \left( SO^3 \right).$$

The following result is analogous to [28, Lemma 3.6].

**Lemma 4.2** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X,\ \beta_X < 1$. Suppose that a weight $w$ belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Then $C_{X(\mathbb{R},w)}(\dot{\mathbb{R}}) \subset SO_{X(\mathbb{R},w)}$.*

***Proof*** Let $a \in C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$. Fix $\varepsilon > 0$. Then there exists $b \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ such that

$$\|a - b\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon/2. \tag{4.6}$$

Then $b - b(\infty) \in C_0(\mathbb{R}) \cap V(\mathbb{R})$. By Lemma 3.2,

$$b - b(\infty) \in \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}} \left( C_c^\infty(\mathbb{R}) \right).$$

Then there exists $c \in C_c^\infty(\mathbb{R})$ such that

$$\|b - b(\infty) - c\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon/2. \tag{4.7}$$

It follows from inequalities (4.6) and (4.7) that

$$\|a - (c + b(\infty))\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon.$$

Since $c + b(\infty) \in C_c^\infty(\mathbb{R}) \dot{+} \mathbb{C} \subset SO^3$, we get $a \in \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}(SO^3) = SO_{X(\mathbb{R},w)}$. $\square$

## 5 Almost periodic Fourier multipliers and their products with continuous Fourier multipliers vanishing at infinity

### 5.1 The algebra $AP_{X(\mathbb{R},w)}$ of almost periodic Fourier multipliers

For $\lambda \in \mathbb{R}$, let $T_\lambda$ denote the translation operator defined by

$$(T_\lambda f)(x) = f(x-\lambda), \quad x \in \mathbb{R}.$$

**Lemma 5.1** *Let $X(\mathbb{R})$ be a rearrangement-invariant Banach function space and $w : \mathbb{R} \to [0,\infty]$ be a weight such that $w \in X_{\rm loc}(\mathbb{R})$ and $1/w \in X'_{\rm loc}(\mathbb{R})$. Suppose that $\lambda \in \mathbb{R}$. Then the translation operator $T_\lambda$ is bounded on the Banach function space $X(\mathbb{R},w)$ if and only if the function*

$$v_\lambda(x) := \frac{w(x+\lambda)}{w(x)}, \quad x \in \mathbb{R},$$

*belongs to the space $L^\infty(\mathbb{R})$. In that case $\|T_\lambda\|_{\mathcal{B}(X(\mathbb{R},w))} = \|v_\lambda\|_{L^\infty(\mathbb{R})}$.*

***Proof*** The operator $T_\lambda$ is bounded on the space $X(\mathbb{R},w)$ if and only if the operator $wT_\lambda w^{-1}I = T_\lambda(v_\lambda I)$ is bounded on the space $X(\mathbb{R})$. Moreover, their norms coincide. It is easy to see that for every $f \in X(\mathbb{R})$, the function $T_\lambda f$ is equimeasurable with $f$, whence $\|T_\lambda f\|_{X(\mathbb{R})} = \|f\|_{X(\mathbb{R})}$. Therefore,

$$\|T_\lambda\|_{\mathcal{B}(X(\mathbb{R},w))} = \|T_\lambda(v_\lambda I)\|_{\mathcal{B}(X(\mathbb{R}))} = \|v_\lambda I\|_{\mathcal{B}(X(\mathbb{R}))}.$$

By [31, Theorem 1], the multiplication operator $v_\lambda I$ is bounded on the space $X(\mathbb{R})$ if and only if $v_\lambda \in L^\infty(\mathbb{R})$ and $\|v_\lambda I\|_{\mathcal{B}(X(\mathbb{R}))} = \|v_\lambda\|_{L^\infty(\mathbb{R})}$. Thus, $\|T_\lambda\|_{\mathcal{B}(X(\mathbb{R},w))} = \|v_\lambda\|_{L^\infty(\mathbb{R})}$. □

As a consequence of the previous result, we show that for all $\lambda \in \mathbb{R}$, the exponential functions $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$, are Fourier multipliers on separable rearrangement-invariant Banach function spaces with weights in the sublclass $A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$ of the class of Muckenhoupt weights.

**Corollary 5.2** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0<\alpha_X \le \beta_X < 1$. Suppose that a weight $w$ belongs to $A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$. Then for every $\lambda \in \mathbb{R}$, the function $e_\lambda$ belongs to $\mathcal{M}_{X(\mathbb{R},w)}$ and $\|e_\lambda\|_{\mathcal{M}_{X(\mathbb{R},w)}} = \|v_\lambda\|_{L^\infty(\mathbb{R})}$.*

***Proof*** It follows from the definition of the classes $A^0_{1/\alpha_X}(\mathbb{R})$ and $A^0_{1/\beta_X}(\mathbb{R})$ that the function $v_\lambda(x) = \frac{w(x+\lambda)}{w(x)}$, $x \in \mathbb{R}$, is bounded for every $\lambda \in \mathbb{R}$. By Lemma 2.3(a), $w \in X_{\rm loc}(\mathbb{R})$ and $1/w \in X'_{\rm loc}(\mathbb{R})$. Then, by Lemma 5.1, the operator $T_\lambda$ is bounded on the Banach function space $X(\mathbb{R},w)$ and

$$\|T_\lambda\|_{\mathcal{B}(X(\mathbb{R},w))} = \|v_\lambda\|_{L^\infty(\mathbb{R})}, \quad \lambda \in \mathbb{R}.$$

It remains to observe that $T_\lambda = W^0(e_\lambda)$. Thus $e_\lambda \in \mathcal{M}_{X(\mathbb{R},w)}$ and

$$\|e_\lambda\|_{\mathcal{M}_{X(\mathbb{R},w)}} = \|W^0(e_\lambda)\|_{\mathcal{B}(X(\mathbb{R},w))} = \|v_\lambda\|_{L^\infty(\mathbb{R})}, \quad \lambda \in \mathbb{R},$$

which completes the proof. □

Corollary 5.2 implies that if $X(\mathbb{R})$ is a separable rearrangement-invariant Banach function spaces and $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$, then $APP \subset \mathcal{M}_{X(\mathbb{R},w)}$. We define the algebra $AP_{X(\mathbb{R},w)}$ of almost periodic Fourier multipliers by

$$AP_{X(\mathbb{R},w)} := \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}\left(APP\right).$$

It is natural to refer to the weights in $A^0_{1/\alpha_X} \cap A^0_{1/\beta_X}$ as suitable Muckenhoupt weights. The class of suitable Muckenhoput weights contains many nontrivial weights as the following example shows.

For $\delta, v, \eta \in \mathbb{R}$, consider the weight

$$w(x) := \begin{cases} \exp\left(\delta + v\sin(\eta\log(\log|x|))\right) & \text{if} \quad |x| \geq e, \\ \exp(\delta) & \text{if} \quad |x| < e. \end{cases}$$

Let $r \in (1, \infty)$. It was shown in [27, Example 4.2] that if

$$-1/r < \delta - |v|\sqrt{\eta^2+1} \leq \delta + |v|\sqrt{\eta^2+1} < 1 - 1/r,$$

then $w \in A^0_r(\mathbb{R})$. Hence if $0 < \alpha_X \leq \beta_X < 1$ and

$$-\alpha_X < \delta - |v|\sqrt{\eta^2+1} \leq \delta + |v|\sqrt{\eta^2+1} < 1 - \beta_X,$$

then $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$.

### 5.2 Products of almost periodic Fourier multipliers and continuous Fourier multipliers vanishing at infinity

The next lemma generalizes [29, Lemma 3.1(iii)] from the setting of Lebesgue spaces to the setting of rearrangement-invariant Banach function spaces with suitable Muckenhoupt weights.

**Lemma 5.3** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X$, $\beta_X < 1$. Suppose that $w$ belongs to $A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$ and $C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$ is defined by (3.17). If $a \in AP_{X(\mathbb{R},w)}$ and $\psi \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$, then $a\psi \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$.*

***Proof*** By Theorem 3.3, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|\psi_n - \psi\|_{\mathcal{M}_{X(\mathbb{R},w)}} = 0. \tag{5.1}$$

By the definition of the algebra $AP_{X(\mathbb{R},w)}$, there exists a sequence $a_n \in APP$ such that

$$\lim_{n\to\infty} \|a_n - a\|_{\mathcal{M}_{X(\mathbb{R},w)}} = 0. \tag{5.2}$$

Then $a_n\psi_n \in C_c^\infty(\mathbb{R}) \subset C(\dot{\mathbb{R}}) \cap V(\mathbb{R})$ for every $n \in \mathbb{N}$. Moreover, (5.1)–(5.2) imply that

$$\lim_{n\to\infty} \|a_n\psi_n - a\psi\|_{\mathcal{M}_{X(\mathbb{R},w)}} = 0.$$

Hence $a\psi \in C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$. In view of the continuous embedding of $\mathcal{M}_{X(\mathbb{R},w)}$ into $L^\infty(\mathbb{R})$ (see Theorem 2.4) and the above equality, we obtain

$$\begin{aligned} |(a\psi)(\infty)| &= \lim_{n\to\infty} |(a_n\psi_n)(\infty) - (a\psi)(\infty)| \le \lim_{n\to\infty} \|a_n\psi_n - a\psi\|_{L^\infty(\mathbb{R})} \\ &\le \lim_{n\to\infty} \|a_n\psi_n - a\psi\|_{\mathcal{M}_{X(R,w)}} = 0. \end{aligned}$$

Thus $(a\psi)(\infty) = 0$ and $a\psi \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$. □

## 6 Proof of the main result

### 6.1 The algebra $\mathcal{A}_u$

For a real-valued monotonically increasing function $u \in C(\overline{\mathbb{R}})$ such that

$$u(-\infty) = 0 \quad u(+\infty) = 1, \tag{6.1}$$

consider the set

$$\mathcal{A}_u := \left\{a = (1-u)a_l + ua_r + a_0 \ : \ a_l, a_r \in AP_{X(\mathbb{R},w)},\ a_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})\right\}.$$

**Lemma 6.1** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X$, $\beta_X < 1$. Suppose that $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$. If $u \in C(\overline{\mathbb{R}})$ is a real-valued monotonically increasing function such that $u(-\infty) = 0$ and $u(+\infty) = 1$, then the set $\mathcal{A}_u$ is an algebra and the mappings $a \mapsto a_l$ and $a \mapsto a_r$ are algebraic homomorphisms of $\mathcal{A}_u$ onto $AP_{X(\mathbb{R},w)}$.*

***Proof*** If $a, b \in \mathcal{A}_u$, then

$$a = (1-u)a_l + ua_r + a_0, \quad b = (1-u)b_l + ub_r + b_0$$

with some $a_l, a_r, b_l, b_r \in AP_{X(\mathbb{R},w)}$ and $a_0, b_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$. Therefore

$$a + b = (1-u)(a_l + b_l) + u(a_r + b_r) + (a_0 + b_0) \tag{6.2}$$

and

$$\begin{aligned} ab &= (1-u)^2 a_l b_l + u^2 a_r b_r + (1-u)u(a_l b_r + a_r b_l) \\ &\quad + \big((1-u)a_l + u a_r\big) b_0 + \big((1-u)b_l + u b_r\big) a_0 + a_0 b_0 \\ &= (1-u)a_l b_l + u a_r b_r + c_0, \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} c_0 &= (u-u^2)\big[(a_l b_r + a_r b_l) - (a_l b_l + a_r b_r)\big] \\ &\quad + \big((1-u)a_l + u a_r\big) b_0 + \big((1-u)b_l + u b_r\big) a_0 + a_0 b_0. \end{aligned} \tag{6.4}$$

Since $1-u, u \in C(\overline{\mathbb{R}}) \cap V(\mathbb{R}) \subset C_{X(\mathbb{R},w)}(\overline{\mathbb{R}})$ and $a_0, b_0 \in C_{0,X(\mathbb{R},w)}(\mathbb{R})$, it follows from Lemma 4.1 that

$$(1-u)a_0, u a_0, (1-u)b_0, u b_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}).$$

Then, by Lemma 5.3,

$$(1-u)a_l b_0, u a_r b_0, (1-u)b_l a_0, u b_r a_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}). \tag{6.5}$$

Since $u - u^2 \in C(\overline{\mathbb{R}}) \cap V(\mathbb{R}) \subset C_{X(\mathbb{R},w)}(\overline{\mathbb{R}})$ and $u(\pm\infty) - u^2(\pm\infty) = 0$, by Lemma 4.1, $u - u^2 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$. Then, in view of Lemma 5.3, we also conclude that

$$(u-u^2)\big[(a_l b_r + a_r b_l) - (a_l b_l + a_r b_r)\big] \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}). \tag{6.6}$$

It follows from (6.4) to (6.6) that $c_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$. In view of this observation and equalities (6.2)–(6.3), we see that $a+b, ab \in \mathcal{A}_u$. Therefore, $\mathcal{A}_u$ is an algebra. It is clear that the mappings $a \mapsto a_l$ and $a \mapsto a_r$ are algebraic homomorphisms of $\mathcal{A}_u$ onto $AP_{X(\mathbb{R},w)}$. □

### 6.2 The multiplier norm of $a = (1-u)a_r + u a_r + a_0 \in \mathcal{A}_u$ dominates the multiplier norms of $a_r$ and $a_l$

In this section we will prepare the proof of the fact that the algebraic homomorphisms $\mathcal{A}_u \to AP_{X(\mathbb{R},w)}$ given by $a \mapsto a_l$ and $a \mapsto a_r$ are actually Banach algebra homomorphisms of norm 1. To this end, we will show that for $a \in \mathcal{A}_u$,

$$\|a_r\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}, \quad \|a_l\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}. \tag{6.7}$$

For $a \in L^\infty(\mathbb{R})$ and $h \in \mathbb{R}$, we define

$$a^h(x) := a(x+h), \quad x \in \mathbb{R}.$$

The following consequence of Kronecker's theorem (see, e.g., [10, Theorem 1.12]) plays a crucial role in the proof of inequalities (6.7).

**Lemma 6.2** *If $a_1, \ldots, a_k \in APP$ is a finite collection of almost periodic polynomials, then there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ of real numbers such that $h_n \to +\infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \|a_m^{\pm h_n} - a_m\|_{L^\infty(\mathbb{R})} = 0$$

*for each* $m \in \{1, \ldots, k\}$.

For the sign "+", the proof of the above lemma is given in [10, Lemma 10.2], for the sign "–", the proof is analogous.

We start the proof of inequalities (6.7) for $a = (1-v)a_l + va_r + a_0$ with a nice function $v$ in place of $u$ and nice functions $a_l, a_r$ and $a_0$.

**Lemma 6.3** *Let* $X(\mathbb{R})$ *be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying* $0<\alpha_X,\ \beta_X<1$. *Suppose that* $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$. *Let* $v \in C(\overline{\mathbb{R}})$ *be any real-valued monotonically increasing function such that there exists a point* $x_0 > 0$ *such that* $v(x) = 0$ *for* $x < -x_0$ *and* $v(x) = 1$ *for* $x > x_0$. *If* $a_l, a_r \in APP$, $a_0 \in C_c^\infty(\mathbb{R})$, *and*

$$a = (1-v)a_l + va_r + a_0, \tag{6.8}$$

*then inequalities* (6.7) *hold.*

***Proof*** The idea of the proof is borrowed from the proof of [27, Theorem 3.1]. By Lemma 6.2, there is a sequence $\{h_n\}_{n\in\mathbb{N}}$ of real numbers such that $h_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \|a_r^{h_n} - a_r\|_{L^\infty(\mathbb{R})} = 0, \quad \lim_{n\to\infty} \|(a_r')^{h_n} - a_r'\|_{L^\infty(\mathbb{R})} = 0, \tag{6.9}$$

$$\lim_{n\to\infty} \|a_l^{-h_n} - a_l\|_{L^\infty(\mathbb{R})} = 0, \quad \lim_{n\to\infty} \|(a_l')^{-h_n} - a_l'\|_{L^\infty(\mathbb{R})} = 0. \tag{6.10}$$

Let us show that

$$\operatorname*{s-lim}_{n\to\infty} e_{h_n} W^0(a) e_{-h_n} I = W^0(a_r), \quad \operatorname*{s-lim}_{n\to\infty} e_{-h_n} W^0(a) e_{h_n} I = W^0(a_l) \tag{6.11}$$

on the space $X(\mathbb{R}, w)$. As

$$e_{\pm h_n} W^0(a) e_{\mp h_n} I = W^0(a^{\pm h_n}),$$

we have to prove that for every $f \in X(\mathbb{R}, w)$,

$$\lim_{n\to\infty} \left\| W^0(a^{h_n} - a_r) f \right\|_{X(\mathbb{R},w)} = 0, \tag{6.12}$$

$$\lim_{n\to\infty} \left\| W^0(a^{-h_n} - a_l) f \right\|_{X(\mathbb{R},w)} = 0. \tag{6.13}$$

Since the operators $W^0(a^{h_n} - a_r)$ and $W^0(a^{-h_n} - a_l)$ are uniformly bounded in $n \in \mathbb{N}$ and the set $\mathcal{S}_0(\mathbb{R})$ is dense in the space $X(\mathbb{R}, w)$ in view of Lemma 2.3, applying [34, Lemma 1.4.1], we conclude that it is enough to prove equalities (6.12)–(6.13) for all $f \in \mathcal{S}_0(\mathbb{R})$.

Fix $f \in \mathcal{S}_0(\mathbb{R})$. Then, by a smooth version of Urysohn's lemma (see, e.g., [17, Proposition 6.5]), there is a function $\psi \in C_c^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $\operatorname{supp}\mathcal{F}f \subset \operatorname{supp}\psi$ and $\psi|_{\operatorname{supp}\mathcal{F}f} = 1$. Therefore, for all $n \in \mathbb{N}$,

$$W^0(a^{h_n}-a_r)f=\mathcal{F}^{-1}(a^{h_n}-a_r)\psi\mathcal{F}f,\quad W^0(a^{-h_n}-a_l)f=\mathcal{F}^{-1}(a^{-h_n}-a_l)\psi\mathcal{F}f$$

and

$$\left\|W^0(a^{h_n}-a_r)f\right\|_{X(\mathbb{R},w)}\le\|(a^{h_n}-a_r)\psi\|_{\mathcal{M}_{X(\mathbb{R},w)}}\|f\|_{X(\mathbb{R},w)},\tag{6.14}$$

$$\left\|W^0(a^{-h_n}-a_l)f\right\|_{X(\mathbb{R},w)}\le\|(a^{-h_n}-a_l)\psi\|_{\mathcal{M}_{X(\mathbb{R},w)}}\|f\|_{X(\mathbb{R},w)}.\tag{6.15}$$

Since $v(x)=1$ for $x>x_0$ and $v(x)=0$ for $x<-x_0$ and $a_0\in C_c^\infty(\mathbb{R})$, there exists $N\in\mathbb{N}$ such that for all $x\in\operatorname{supp}\psi$ and $n>N$,

$$v(x+h_n)=1,\quad v(x-h_n)=0,\quad a_0(x\pm h_n)=0.$$

Hence, for all $n>N$ and $x\in\mathbb{R}$,

$$\left(a^{h_n}(x)-a_r(x)\right)\psi(x)=\left(a_r^{h_n}(x)-a_r(x)\right)\psi(x),\tag{6.16}$$

$$\left(a^{-h_n}(x)-a_l(x)\right)\psi(x)=\left(a_l^{-h_n}(x)-a_l(x)\right)\psi(x).\tag{6.17}$$

It is clear that the functions on the right-hand sides of (6.16)–(6.17) belong to $C_c^\infty(\mathbb{R})$. Therefore, by the Stechkin-type inequality (1.4), for all $n>N$,

$$\begin{aligned}\left\|\left(a^{h_n}-a_r\right)\psi\right\|_{\mathcal{M}_{X(\mathbb{R},w)}}&=\left\|\left(a_r^{h_n}-a_r\right)\psi\right\|_{\mathcal{M}_{X(\mathbb{R},w)}}\\&\le c_{X(\mathbb{R},w)}\left\|\left(a_r^{h_n}-a_r\right)\psi\right\|_{V(\mathbb{R})}\\&=c_{X(\mathbb{R},w)}\left\|\left(a_r^{h_n}-a_r\right)\psi\right\|_{L^\infty(\mathbb{R})}\\&\quad+c_{X(\mathbb{R},w)}\int_{\mathbb{R}}\left|(a_r^{h_n})'(x)-a_r'(x)\right|\,|\psi(x)|\,dx\\&\quad+c_{X(\mathbb{R},w)}\int_{\mathbb{R}}\left|a_r^{h_n}(x)-a_r(x)\right|\,|\psi'(x)|\,dx\\&\le c_{X(\mathbb{R},w)}\left(\|\psi\|_{L^\infty(\mathbb{R})}+\|\psi'\|_{L^1(\mathbb{R})}\right)\left\|a_r^{h_n}-a_r\right\|_{L^\infty(\mathbb{R})}\\&\quad+c_{X(\mathbb{R},w)}\|\psi\|_{L^1(\mathbb{R})}\left\|(a_r^{h_n})'-a_r'\right\|_{L^\infty(\mathbb{R})}\end{aligned}\tag{6.18}$$

and, analogously,

$$\begin{aligned}\left\|\left(a^{-h_n}-a_l\right)\psi\right\|_{\mathcal{M}_{X(\mathbb{R},w)}}&\le c_{X(\mathbb{R},w)}\left(\|\psi\|_{L^\infty(\mathbb{R})}+\|\psi'\|_{L^1(\mathbb{R})}\right)\left\|a_l^{-h_n}-a_l\right\|_{L^\infty(\mathbb{R})}\\&\quad+c_{X(\mathbb{R},w)}\|\psi\|_{L^1(\mathbb{R})}\left\|(a_l^{-h_n})'-a_l'\right\|_{L^\infty(\mathbb{R})}.\end{aligned}\tag{6.19}$$

Combining (6.14)–(6.15) and (6.18)–(6.19) with (6.9)–(6.10), we see that equalities (6.12)–(6.13) hold for every $f\in\mathcal{S}_0(\mathbb{R})$. Therefore, (6.11) are fulfilled for every $f\in X(\mathbb{R},w)$. Hence, by the Banach-Steinhaus theorem (see, e.g., [34, Theorem 1.4.2]),

$$\begin{aligned}\|a_r\|_{\mathcal{M}_{X(\mathbb{R},w)}} &= \|W^0(a_r)\|_{\mathcal{B}(X(\mathbb{R},w))} \le \liminf_{n\to\infty} \|e_{h_n} W^0(a) e_{-h_n} I\|_{\mathcal{B}(X(\mathbb{R},w))} \\ &\le \|W^0(a)\|_{\mathcal{B}(X(\mathbb{R},w))} = \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}\end{aligned}$$

and, analogously,

$$\begin{aligned}\|a_l\|_{\mathcal{M}_{X(\mathbb{R},w)}} &= \|W^0(a_l)\|_{\mathcal{B}(X(\mathbb{R},w))} \le \liminf_{n\to\infty} \|e_{-h_n} W^0(a) e_{h_n} I\|_{\mathcal{B}(X(\mathbb{R},w))} \\ &\le \|W^0(a)\|_{\mathcal{B}(X(\mathbb{R},w))} = \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}},\end{aligned}$$

which completes the proof of (6.7). □

Now we extend the previous result for functions $a$ of the form (6.8) with general $a_l, a_r \in AP_{X(\mathbb{R},w)}$ and $a_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$, keeping the nice function $v$ as above.

**Lemma 6.4** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0<\alpha_X,\ \beta_X<1$. Suppose that $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$. Let $v \in C(\overline{\mathbb{R}})$ be any real-valued monotonically increasing function such that there exists a point $x_0 > 0$ such that $v(x) = 0$ for $x < -x_0$ and $v(x) = 1$ for $x > x_0$. If $a_l, a_r \in AP_{X(\mathbb{R},w)}$, $a_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$, where $C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$ is defined by (3.17), and $a$ is given by equality (6.8), then inequalities (6.7) hold.*

***Proof*** By the definition of $AP_{X(\mathbb{R},w)}$, there are sequences $\{a_l^{(n)}\}_{n\in\mathbb{N}}, \{a_r^{(n)}\}_{n\in\mathbb{N}}$ in $APP$ such that

$$\lim_{n\to\infty} \|a_l^{(n)} - a_l\|_{\mathcal{M}_{X(\mathbb{R},w)}} = 0, \quad \lim_{n\to\infty} \|a_r^{(n)} - a_r\|_{\mathcal{M}_{X(\mathbb{R},w)}} = 0. \tag{6.20}$$

On the other hand, by Theorem 3.3, there is a sequence $\{a_0^{(n)}\}_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|a_0^{(n)} - a_0\|_{\mathcal{M}_{X(\mathbb{R},w)}} = 0. \tag{6.21}$$

For $n \in \mathbb{N}$, consider the functions

$$a^{(n)} := (1-v)a_l^{(n)} + v a_r^{(n)} + a_0^{(n)}. \tag{6.22}$$

It follows from equalities (6.20)–(6.22) and Lemma 6.3 that

$$\begin{aligned}\|a_l\|_{\mathcal{M}_{X(\mathbb{R},w)}} &= \lim_{n\to\infty} \|a_l^{(n)}\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \lim_{n\to\infty} \|a^{(n)}\|_{\mathcal{M}_{X(\mathbb{R},w)}} = \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}, \\ \|a_r\|_{\mathcal{M}_{X(\mathbb{R},w)}} &= \lim_{n\to\infty} \|a_r^{(n)}\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \lim_{n\to\infty} \|a^{(n)}\|_{\mathcal{M}_{X(\mathbb{R},w)}} = \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}},\end{aligned}$$

which completes the proof of inequalities (6.7). □

Now we observe that the algebra $\mathcal{A}_u$ does not depend on the particular choice of a real-valued monotonically increasing function $u \in C(\overline{\mathbb{R}})$ satisfying conditions (6.1).

**Lemma 6.5** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0<\alpha_X,\ \beta_X<1$. Suppose that*

$w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$. *Let* $u, v \in C(\overline{\mathbb{R}})$ *be two real-valued monotonically increasing functions such that*

$$u(-\infty) = v(-\infty) = 0, \quad u(+\infty) = v(+\infty) = 1.$$

*Then* $\mathcal{A}_u = \mathcal{A}_v$.

***Proof*** If $a \in \mathcal{A}_u$, then $a = (1-u)a_l + ua_r + a_0$ for some $a_l, a_r \in AP_{X(\mathbb{R},w)}$ and $a_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$. On the other hand, $a = (1-v)a_l + va_r + b_0$ with

$$b_0 = (v-u)a_l + (u-v)a_r + a_0 = (u-v)(a_r - a_l) + a_0.$$

Since the functions $u$, $v$ are monotonically increasing, we have $u, v \in V(\mathbb{R})$. Hence $u - v \in V(\mathbb{R}) \cap C(\overline{\mathbb{R}})$ and

$$u(+\infty) - v(+\infty) = u(-\infty) - v(-\infty) = 0.$$

Thus $u - v \in C(\dot{\mathbb{R}}) \cap V(\mathbb{R}) \subset C_{X(\mathbb{R},w)}(\dot{\mathbb{R}})$ and $(u-v)(\infty) = 0$. Since the function $a_r - a_l$ belongs to $AP_{X(\mathbb{R},w)}$, it follows from Lemma 5.3 that

$$(u-v)(a_r - a_l) \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}).$$

Then $b_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$ and $a \in \mathcal{A}_v$. Therefore $\mathcal{A}_u \subset \mathcal{A}_v$. It can be shown analogously that $\mathcal{A}_v \subset \mathcal{A}_u$. Thus $\mathcal{A}_u = \mathcal{A}_v$. □

Combining Lemmas 6.4–6.5, we arrive at the main result of this subsection.

**Theorem 6.6** *Let* $X(\mathbb{R})$ *be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying* $0 < \alpha_X, \ \beta_X < 1$. *Suppose that* $w \in A^0_{1/\alpha_X}(\mathbb{R}) \cap A^0_{1/\beta_X}(\mathbb{R})$. *Let* $u \in C(\overline{\mathbb{R}})$ *be a real-valued monotonically increasing function such that* $u(-\infty) = 0$ *and* $u(+\infty) = 1$. *If* $a \in \mathcal{A}_u$, *that is,*

$$a = (1-u)a_l + ua_r + a_0 \quad \text{with} \quad a_l, a_r \in AP_{X(\mathbb{R},w)}, a_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}}),$$

*where* $C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$ *is defined by* (3.17), *then inequalities* (6.7) *hold.*

### 6.3 Proof of Theorem 1.2

The idea of the proof is borrowed from the proof of [10, Theorem 1.21]. If $a \in AP_{X(\mathbb{R},w)}$, then $a = (1-u)a + ua + 0$, whence $a \in \mathcal{A}_u$. If $f \in C_{X(\mathbb{R},w)}(\overline{\mathbb{R}})$, then the function $f_0 = f - (1-u)f(-\infty) - uf(+\infty)$ belongs to $C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$. Therefore $f = (1-u)f(-\infty) + uf(+\infty) + f_0 \in \mathcal{A}_u$. These observations imply that

$$SAP_{X(\mathbb{R},w)} \subset \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}(\mathcal{A}_u). \tag{6.23}$$

On the other hand, it is obvious that

$$\operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}(\mathcal{A}_u) \subset SAP_{X(\mathbb{R},w)}. \tag{6.24}$$

Combining (6.23)–(6.24), we arrive at the equality

$$SAP_{X(\mathbb{R},w)} = \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}(\mathcal{A}_u). \tag{6.25}$$

By Theorem 6.6, for every $a = (1-u)a_r + ua_r + a_0 \in \mathcal{A}_u$ with $a_l, a_r \in AP_{X(\mathbb{R},w)}$ and $a_0 \in C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$, one has

$$\|a_r\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}, \quad \|a_l\|_{\mathcal{M}_{X(\mathbb{R},w)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R},w)}}. \tag{6.26}$$

Consequently, if $\{(1-u)a_l^{(n)} + ua_r^{(n)} + a_0^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}_u$, where $\{a_l^{(n)}\}_{n\in\mathbb{N}}, \{a_r^{(n)}\}_{n\in\mathbb{N}}$ are sequences in $AP_{X(\mathbb{R},w)}$ and $\{a_0^{(n)}\}_{n\in\mathbb{N}}$ is a sequence in $C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$, then $\{a_l^{(n)}\}_{n\in\mathbb{N}}$ and $\{a_r^{(n)}\}_{n\in\mathbb{N}}$ are Cauchy sequences in $AP_{X(\mathbb{R},w)}$. Consequently, $\{a_0^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$. Since $AP_{X(\mathbb{R},w)}$ is closed by its definition and $C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$ is closed in view of Theorem 3.3, we conclude that the limits

$$a_l := \lim_{n\to\infty} a_l^{(n)}, \quad a_r = \lim_{n\to\infty} a_r^{(n)}$$

belong to $AP_{X(\mathbb{R},w)}$ and that the limit

$$a_0 := \lim_{n\to\infty} a_0^{(n)}$$

belongs to $C_{0,X(\mathbb{R},w)}(\dot{\mathbb{R}})$. Therefore, the limit

$$\lim_{n\to\infty} \left((1-u)a_l^{(n)} + ua_r^{(n)} + a_0^{(n)}\right)$$

belongs to $\mathcal{A}_u$. Thus

$$\operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}(\mathcal{A}_u) = \mathcal{A}_u. \tag{6.27}$$

It follows from (6.25) and (6.27) that $\mathcal{A}_u = SAP_{X(\mathbb{R},w)}$. In particular, every function $a \in SAP_{X(\mathbb{R},w)}$ is of the form

$$a = (1-u)a_l + ua_r + a_0 \tag{6.28}$$

with $a_l, a_r \in AP_{X(\mathbb{R},w)}$ and $a_0 \in C_{0,X(\mathbb{R},w)}$. We infer from (6.26) that the representation (6.28) is unique for the function $u$. Moreover, the proof of Lemma 6.5 shows that $a_l, a_r \in AP_{X(\mathbb{R},w)}$ are independent of the particular choice of the function $u$. By Lemma 6.1, the mappings $a \mapsto a_l$ and $a \mapsto a_r$ are algebraic homomorphisms of $\mathcal{A}_u = SAP_{X(\mathbb{R},w)}$ onto $AP_{X(\mathbb{R},w)}$. In view of (6.26), these homomorphisms are Banach algebra homomorphisms of the Banach algebra $SAP_{X(\mathbb{R},w)}$ onto the Banach algebra $AP_{X(\mathbb{R},w)}$ and the norms of these homomorphisms are not greater than one. For every function $a \in AP_{X(\mathbb{R},w)}$, we have equalities in (6.26) because

$$a = (1 - u)a + ua + 0 = a_l = a_r.$$

Thus, the norms of the homomorphisms $a \mapsto a_l$ and $a \mapsto a_l$ are equal to one. □

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