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ORIGINAL ARTICLE

On the existence of a local solution for an integro-differential equation with an integral boundary condition

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Abstract
In this paper, we consider a nonlinear hyperbolic equation with a nonlocal boundary condition. We apply the Faedo–Galerkin's method to establish the local existence and uniqueness of a weak solution.

Keywords Nonlinear hyperbolic equation · Faedo–Galerkin's method · Integro-differential equation · Integral boundary condition · Local existence

Mathematics Subject Classification 35L70 · 35R09 · 35A01

1 Introduction

Boundary value problems with integral conditions are an interesting and important class of problems; this is due to the importance of nonlocal conditions appearing in the mathematical modeling of various phenomena of physics, ecology, biology, etc. The starting work on the use of nonlocal boundary conditions has been done by Cannon [4]; the presence of an integral term in boundary conditions can complicate the application of classical methods; therefore, several methods have been proposed for overcoming the difficulties arising from nonlocal conditions as functional methods, approximation methods (see [1,6,7,11]). Pulkina [17] has dealt with a hyperbolic problem with two integral conditions and has established the existence and uniqueness of generalized solutions using the fixed point arguments. The importance of approximation methods

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is that they do not only prove the existence and uniqueness of the solution but they also allow the construction of algorithms for numerical solutions. Rothe's method and Faedo–Galerkin's method are very effective tools in the study of the approximate solution and its convergence to the exact solution. The objective of this work is to apply Faedo–Galerkin's method to the study of a multidimensional nonlinear hyperbolic integro-differential equation with integral conditions.

$$u_{tt} + u_t - \Delta u = |u|^{p-2}u + \int_0^t a(t-\tau)u(x,\tau)\mathrm{d}\tau \quad (x,t) \in Q_T \tag{1.1}$$

$$u(x,t) = \int_\Omega k(x,y)u(y,t)\mathrm{d}y \quad x \in \partial\Omega \tag{1.2}$$

$$u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain and $0 < t < T$. Let $u(t),\; u'(t) = u_t(t),\; u''(t) = u_{tt}(t),\; Q_T = \Omega \times (0, T],\; T > 0,\; p > 2$.

Equation (1.1) represents the second-order telegraph equation and models mixture between diffusion and wave propagation or mass transport equation. It is also used in signal analysis for transmission and propagation of electrical signals [14].

Equation (1.1) has been studied for initial and Dirichlet conditions by several different methods (see [1,3,15,16,19]), but without a Volterra operator

$$\int_0^t a(t-\tau)u(x,t)\mathrm{d}\tau.$$

Many mathematical models contain integro-differential equations; these equations arise in many fields like biological models and fluid dynamics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Let us mention that different methods are used to solve linear and nonlinear integro-differential equations. Balachandran and Park [2] investigated an integro-differential equation of Sobolev type with nonlocal condition and proved the existence of mild and strong solutions using semigroup theory and Schauder fixed point theorem. Merad et al. [12] studied the solvability of the integro-differential hyperbolic equation with purely nonlocal conditions using a priori estimates and Laplace transform method and obtained the solution using a numerical technique.

This paper is organized as follows: In the next section, we specify notations, state some assumptions and prove the existence of a solution using Faedo–Galerkin's method in Sect. 2.1. Finally, Sect. 2.2 is devoted to establish the uniqueness of solution.

2 Preliminaries and main results

In this section, we shall introduce some notations that will be considered. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$. Much of our arguments are based on the functions spaces $C^m(\Omega)$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots.$ are used. Let $\langle ., . \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. Denote by $\|\cdot\|_X$ the norm in the Banach space X. We denote by $L^p(0, T; X)$, $1 \le p \le \infty$, the Banach space of the real functions $u : (0, T) \longrightarrow X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, \mathrm{d}t\right)^{1/p} < \infty \text{ for } 1 \le p \le \infty$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess} \sup_{0<t<T} \|u(t)\|_X \text{ for } p = \infty.$$

On H^1, we shall use the following norm:

$$\|u\|_{H^1} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{1/2}$$

and the compact embedding

$$\|v\|_q \le C_q \|v\|_{H^1}, \quad \forall\, v \in H^1, 1 \le q \le \frac{2N}{N-2}, N \ge 3. \tag{2.1}$$

Define a space V:

$$V = \left\{ v \in H^2(\Omega) : v(x) = \int_\Omega k(x, y)u(y)\mathrm{d}y \;\; x \in \partial\Omega \right\}.$$

We use the following notation:
$k_1(x)$: norm of $\nabla k(x, y)$ in $L^2(\Omega)$ with respect to y,
i.e., $k_1(x) = \left(\int_\Omega |\nabla k(x, y)|^2 \mathrm{d}y\right)^{1/2}$;
$k_2(x)$: norm of $k(x, y)$ in $L^2(\Omega)$ with respect to y,
i.e., $k_2(x) = \left(\int_\Omega |k(x, y)|^2 \mathrm{d}y\right)^{1/2}$.
Next, we make the following assumptions:

(H_1): $2 < p \le \frac{2N-2}{N-2}$, $N \ge 3$,
(H_2): $|a(t - \tau)| \le a_2$
(H_3): For any $x \in \partial\Omega$, $k_1(x) < \infty$, $k_2(x) < \infty$,
(H_4): $\int_{\partial\Omega} k_1(x)k_2(x)\mathrm{d}x \le C$.

Theorem 2.1 *Suppose that* $(H_1) - -(H_4)$ *hold and initial data* $(u_0, u_1) \in H^2 \times H^1$ *satisfy the compatibility condition*

$$u_0(x) = \int_\Omega k(x, y)u_0(y)dy$$

The problem (1.1)–(1.3) *has a unique local solution*

$$u \in L^\infty(0, T_*; H^2), \quad u_t \in L^\infty(0, T_*; H^1), \quad u_{tt} \in L^\infty(0, T_*; L^2).$$

for $T_* > 0$ small enough.

2.1 Existence of solutions

Proof Our main tool to prove the existence in time is the Faedo–Galerkin's method, which consists of constructing approximations of the solutions, then we obtain a priori estimates necessary to guarantee the convergence of approximations. Our proof is organized as follows. In the first step, we define an approximate problem in bounded dimension space which has a unique solution. In the second step, we derive the various a priori estimates. In the third step, we will pass to the limit of the approximations using the compactness of some embedding in the Sobolev spaces.

Step 1. Approximate solutions: Since V is a subspace of $H^2(\Omega)$ which is separable Hilbert space. Then, there exists a family of subspaces $\{V_n\}$ such that

(i) $V_n \subset V \;(\dim V_n < \infty), \quad \forall n \in \mathbb{N}$.
(ii) $V_n \to V$, such that there exists a dense subspace ϑ in V and for all $v \in \vartheta$, we can get sequence $\{v_n\} \subseteq V_n$, and $v_n \to v$ in V.
(iii) $V_n \subset V_{n+1}$ and $\overline{\cup_{n\in\mathbb{N}} V_n} = V$.

We can choose a countable basis of elements $\{w_j(x),\ j = 1, 2, \ldots\}$, which generate V and are orthogonal in $L^2(\Omega)$. Let V_m be the subspace of V generated by the first m elements $\{w_1, w_2, \ldots, w_m\}$, $m \in \mathbb{N}$ we will try to find an approximate solution of the problem (1.1)-(1.3) in the form:

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j(x), \tag{2.2}$$

where the coefficient functions $\left(c_{mj(t)}\right)_{j=1}^{m}$ remain to be determined.

The approximations of the functions $u_0(x)$ and $u_1(x)$ are denoted, respectively, by

$$\begin{aligned} u_{m0}(x) &= \sum_{j=1}^{m} u_{0j} w_j \to u_0 \text{ in } H^2(\Omega), \\ u_{m1}(x) &= \sum_{j=1}^{m} u_{1j} w_j \to u_1 \text{ in } H^1(\Omega), \\ c_{mj}(0) &= u_{0j}, \quad c'_{mj}(0) = u_{1j}, \end{aligned}$$

where

$$\begin{aligned} u_{0j} &= \int_\Omega u_0 w_j(x) \mathrm{d}x \\ u_{1j} &= \int_\Omega u_1 w_j(x) \mathrm{d}x \end{aligned}$$

Multiplying both sides of equation (1.1) by w_l, then by integrating over Ω, we get

$$\begin{cases} \langle u''_m(t), w_l\rangle + \langle u'_m(t), w_l\rangle + \langle \nabla u_m(t), \nabla w_l\rangle \\ = \left\langle |u_m(t)|^{p-2}u_m(t), w_l\right\rangle + \int_{\partial\Omega} \langle \nabla k(x,y), u_m(t)\rangle w_l \mathrm{d}s \\ \quad + \int_0^t a(t-\tau)\langle u_m(\tau), w_l\rangle \mathrm{d}\tau \\ u_m(0) = u_0, \quad u'_m(0) = u_1. \end{cases} \tag{2.3}$$

Substituting the approximate solution in Eq. (2.3) yields

$$\begin{cases} c''_{mj}(t)\langle w_j, w_l\rangle + c'_{mj}(t)\langle w_j, w_l\rangle + c_{mj}(t)\langle \nabla w_j, \nabla w_l\rangle \\ = \psi_l(t) + c_{mj}(t)\int_{\partial\Omega} w_l(s)\int_\Omega \nabla k(x,y) w_j \mathrm{d}y\mathrm{d}s \\ \quad + \int_0^t a(t-\tau)c_{mj}(\tau)\langle w_j, w_l\rangle \mathrm{d}\tau, \ \ 1 \le j \le m, \\ c_{mj}(0) = u_{0j}, \quad c'_{mj}(0) = u_{1j}. \end{cases}$$

where

$$\psi_l(t) = \int_\Omega \Big|\sum_{j=1}^m c_{mj}(t) w_j\Big|^{p-2} \sum_{j=1}^m c_{mj}(t) w_j w_l \mathrm{d}x.$$

we obtain a system of differential equations of second order respect to the variable t, by the theory of ordinary differential equations [5] we see that there exists a unique global solution $c_{mj} \in H^3[0,T]$, and using the embedding $H^m[0,T] \hookrightarrow C^{m-1}[0,T]$, we deduce that the solution $c_{mj} \in C^2[0,T]$. In turn, this gives a unique u_m of the problem (2.3) on some interval $[0, T_m] \subset [0,T]$. For proving the convergence of solutions, we need a priori estimates of solutions $\{u_m\}$ independent of m and T.

Step 2. A priori estimates: The next estimates prove that the energy of our problem is bounded to conclude that the maximal time T_m of existence can be extended to T.

The first estimate: multiplying the System (2.3) by $\big(c_{mj}(t)\big)'$ and summing up with respect to j we conclude that

$$\begin{aligned} &\|u'_m(t)\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla u_m(t)\|_2^2 + \|u'_m(t)\|_2^2\Big) \\ &\quad = \left\langle |u_m(t)|^{p-2}u_m(t), u'_m(t)\right\rangle \\ &\qquad + \int_{\partial\Omega} \langle \nabla k(x,y), u_m(t)\rangle\, u'_m(t)\mathrm{d}s + \int_0^t a(t-\tau)\langle u_m(\tau), u'_m(t)\rangle \mathrm{d}\tau \end{aligned} \tag{2.4}$$

Integrating by parts with respect to the time variable from 0 to t after some rearrangements, we get

$$\begin{aligned}
&\|u_m'(t)\|_2^2 + 2\int_0^t \|u_m'(\tau)\|_2^2 \mathrm{d}\tau + \|\nabla u_m(t)\|_2^2 \\
&= \|\nabla u_m(0)\|_2^2 + \|u_m'(0)\|_2^2 + 2\int_0^t \left\langle |u_m(\tau)|^{p-2} u_m(\tau), u_m'(\tau)\right\rangle \mathrm{d}\tau \\
&\quad +2\int_0^t \int_{\partial\Omega} \langle \nabla k(x,y), u_m(\tau)\rangle\, u_m'(\tau)\mathrm{d}s\mathrm{d}\tau + 2\int_0^t \int_0^s a(s-\tau)\langle u_m(\tau), u_m'(s)\rangle \mathrm{d}\tau\mathrm{d}s \\
&= \|\nabla u_m(0)\|_2^2 + \|u_m'(0)\|_2^2 + 2\sum_{j=i}^{3} I_i
\end{aligned}$$

We choose

$$\varphi_m(t) = \|u_m'(t)\|_2^2 + \|\nabla u_m(t)\|_2^2 + 2\int_0^t \|u_m'(\tau)\|_2^2 \mathrm{d}\tau$$

we obtain

$$\varphi_m(t) = 2\sum_{j=i}^{3} I_i + \varphi_m(0). \tag{2.5}$$

The first term on the right-hand side of (2.5) can be estimated as follows:

$$\begin{aligned}
2I_1 &= 2\int_0^t \left\langle |u_m(\tau)|^{p-2} u_m(\tau), u_m'(\tau)\right\rangle \mathrm{d}\tau \le 2\int_0^t \left\| |u_m(\tau)|^{p-1} \right\| \|u_m'(\tau)\| \mathrm{d}\tau \\
&\le \int_0^t \left\| |u_m(\tau)|^{p-1} \right\|_2^2 \mathrm{d}\tau + \int_0^t \|u_m'(\tau)\|_2^2 \mathrm{d}\tau \\
&\le \int_0^t \left\| |u_m(\tau)|^{p-1} \right\|_2^2 d\tau + \int_0^t \varphi_m(\tau) \mathrm{d}\tau \\
&= \int_0^t \|u_m(\tau)\|_{L^{2p-2}}^{2p-2} \mathrm{d}\tau + \int_0^t \varphi_m(\tau)\mathrm{d}\tau \\
&\le C_{2p-2}^{2p-2} \int_0^t \|u_m(\tau)\|_{H^1}^{2p-2} d\tau + \int_0^t \varphi_m(\tau)\mathrm{d}\tau,
\end{aligned} \tag{2.6}$$

since $1 \le 2 \le 2p-2 \le \frac{2N}{N-2}$, $H^1(\Omega) \hookrightarrow L^{2p-2}(\Omega)$.
Now, we will estimate the first term in the last inequality of (2.6). By the definition of $\varphi_m(t)$, it is easy to check that

$$\begin{aligned}
\|u_m(t)\|_{H^1}^2 &= \|u_m(t)\|_2^2 + \|\nabla u_m(t)\|_2^2 \\
&= \left[\|u_0\|_2 + \int_0^t \|u_m'(s)\|_2 \mathrm{d}s\right]^2 + \|\nabla u_m(t)\|_2^2 \\
&\le 2\|u_0\|_2^2 + \varphi_m(t) + 2t\int_0^t \varphi_m(s) ds.
\end{aligned} \tag{2.7}$$

Using the following inequality:

$$(a+b+c)^q \le 3^{q-1}(a^q+b^q+c^q) \text{ for all } q \ge 1,\ a,b,c \ge 0.$$

We have

$$\begin{aligned}\|u_m(t)\|_{H^1}^{2p-2} &\le \left[2\|u_0\|_2^2+\varphi_m(t)+2t\int_0^t \varphi_m(s)\mathrm{d}s\right]^{p-1}\\ &\le 3^{p-2}2^{p-1}\|u_0\|_2^{2p-2}+3^{p-1}(\varphi_m(t))^{p-1}+3^{p-2}2^{p-1}t^{2p-3}\int_0^t (\varphi_m(s))^{p-1}\mathrm{d}s.\end{aligned} \tag{2.8}$$

Combining (2.6) with (2.8), we get

$$2I_1 \le C_T + C_T\int_0^t (\varphi_m(\tau))^{p-1}\,\mathrm{d}\tau + \int_0^t \varphi_m(\tau)\mathrm{d}\tau. \tag{2.9}$$

Next, we estimate the second and the third terms in the right-hand side of (2.5) as follows:

$$2I_2 = 2\int_0^t \int_{\partial\Omega} \langle \nabla k(x,y), u_m(\tau)\rangle\, u_m'(\tau)\mathrm{d}s\mathrm{d}\tau$$

For $x \in \partial\Omega$, we have

$$\begin{aligned}|\langle \nabla k(x,y), u_m(\tau)\rangle| &= \left|\int_\Omega \nabla k(x,y)u_m(y,\tau)\mathrm{d}y\right| \le \int_\Omega |\nabla k(x,y)u_m(y,\tau)|\mathrm{d}y\\ &\le \|\nabla k(x,y)\|_2\|u_m(\tau)\|_2 = k_1(x)\|u_m(\tau)\|_2\end{aligned}$$

and

$$|u_m'(x,t)| \le \int_\Omega |k(x,y)u_m'(y,t)|dy \le \|k(x,y)\|_2\|u_m'(t)\|_2 = k_2(x)\|u_m'(t)\|_2$$

Then, using Holder's inequality, (H_3)–(H_4) and (2.7), we have

$$\begin{aligned}2I_2 &\le 2\int_0^t \int_{\partial\Omega} k_1(x)k_2(x)\|u_m(\tau)\|_2\|u_m'(\tau)\|_2\mathrm{d}s\mathrm{d}\tau\\ &\le 2\int_0^t \left(\int_{\partial\Omega} k_1(x)k_2(x)\mathrm{d}s\right)\|u_m(\tau)\|_2\|u_m'(\tau)\|_2\mathrm{d}\tau\\ &\le C\int_0^t \|u_m(\tau)\|_2^2\mathrm{d}\tau + C\int_0^t \|u_m'(\tau)\|_2^2\mathrm{d}\tau\\ &\le C_2^2C\int_0^t \|u_m(\tau)\|_{H^1}^2 d\tau + C\int_0^t \varphi_m(\tau)\mathrm{d}\tau\\ &\le C_T + C_T\int_0^t \varphi_m(\tau)\mathrm{d}\tau.\end{aligned} \tag{2.10}$$

By applying Hölder's inequality, Young's inequality, (H_2), (2.1) and (2.7), the third term can be estimated as follows:

$$\begin{aligned}
2I_3 &= 2\int_0^t \int_0^s a(s-\tau)\langle u_m(\tau), u_m'(s)\rangle \mathrm{d}\tau \mathrm{d}s \\
&= 2\int_0^t \int_\Omega u_m'(s) \int_0^s a(s-\tau) u_m(\tau) d\tau \mathrm{d}x\mathrm{d}s \\
&\le \int_0^t \|u_m'(\tau)\|_2^2 \mathrm{d}\tau + \int_0^t \int_\Omega \left(\int_0^s a(s-\tau) u_m(\tau) d\tau \right)^2 \mathrm{d}x\mathrm{d}s \\
&\le \int_0^t \varphi_m(\tau)\mathrm{d}\tau + \int_0^t \int_\Omega \left(\int_0^s (a(s-\tau))^2 \mathrm{d}\tau \right) \left(\int_0^s (u_m(\tau))^2 d\tau \right) \mathrm{d}x\mathrm{d}s \\
&\le \int_0^t \varphi_m(\tau)\mathrm{d}\tau + a_2^2 T^2 \int_0^t \|u_m(\tau)\|_2^2 \mathrm{d}\tau \\
&\le \int_0^t \varphi_m(\tau) d\tau + a_2^2 T^2 C_2^2 \int_0^t \|u_m(\tau)\|_{H^1}^2 \mathrm{d}\tau \\
&\le C_T + C_T \int_0^t \varphi_m(\tau)\mathrm{d}\tau.
\end{aligned} \tag{2.11}$$

Combining estimations of all terms, we obtain after some rearrangements

$$\varphi_m(t) \le C_T \left(1 + \int_0^t \varphi_m(\tau)\mathrm{d}\tau + \int_0^t (\varphi_m(\tau))^{p-1} \mathrm{d}\tau \right), \quad 0 \le t \le T_m, \tag{2.12}$$

where C_T always indicates a constant depending on T. Then, by solving a nonlinear Volterra integral inequality (2.12), because we cannot applied the Granwall's Lemma (nonlinear integral inequality), we need the following lemma:

Lemma 2.2 ([9,16]) *There exists a constant T_* depending on T (independent of m) such that*

$$\varphi_m(t) \le D_T \quad \forall\, m \in \mathbb{N}, \ \ \forall\, t \in [0, T_*]. \tag{2.13}$$

The second estimate: Now, we are going to estimate $u_m''(0)$

Letting $t \to 0_+$ in equation (2.3) multiplying the result by $c_{mj}''(0)$, we get

$$\begin{cases} \langle u_m''(0), u_m''(0)\rangle + \langle u_m'(0), u_m''(0)\rangle + \langle \nabla u_m(0), \nabla u_m''(0)\rangle \\ = \left\langle |u_m(0)|^{p-2} u_m(0), u_m''(0)\right\rangle + \int_{\partial\Omega} \langle \nabla k(x,y), u_m(0)\rangle u_m''(0) \mathrm{d}s \end{cases} \tag{2.14}$$

Then

$$\begin{cases} \|u_m''(0)\|_2^2 = \langle u_1, u_m''(0)\rangle - \langle \Delta u_0, u_m''(0)\rangle \\ + \left\langle |u_0|^{p-2} u_0, u_m''(0)\right\rangle + |\partial\Omega| k_1 \langle u_0, u_m''(0)\rangle \end{cases} \tag{2.15}$$

This implies that

$$\|u_m''(0)\| \le \|u_1\| + \|\Delta u_0\| + \left\| |u_0|^{p-1} \right\| + K\|u_0\| = M_0 \quad \text{for all } m \tag{2.16}$$

where M_0 is a constant depending only on p, u_0, u_1. Now, by differentiating (2.3) with respect to t and substituting $w_j = u_m''(t)$, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m''(t)\|_2^2 + \|\nabla u_m'(t)\|_2^2\right) + \|u_m''(t)\|_2^2\\&= (p-1)\left\langle |u_m(t)|^{p-2}u_m'(t), u_m''(t)\right\rangle\\&\quad + \int_{\partial\Omega}\langle \nabla k(x,y), u_m'(t)\rangle u_m''(t)\mathrm{d}s + a(0)\langle u_m''(t), u_m'(t)\rangle\\&\quad - a(t)\langle u_m''(t), u_m(0)\rangle\end{aligned}$$

Integrating with respect to the time variable from 0 to t, we get

$$\begin{aligned}&\|u_m''(t)\|_2^2 + \|\nabla u_m'(t)\|_2^2 + 2\int_0^t \|u_m''(\tau)\|_2^2\mathrm{d}\tau\\&= \|u_m''(0)\|_2^2 + \|\nabla u_m'(0)\|_2^2 + 2(p-1)\int_0^t \left\langle |u_m(\tau)|^{p-2}u_m'(\tau), u_m''(\tau)\right\rangle \mathrm{d}\tau\\&\quad +2\int_0^t\int_{\partial\Omega}\langle \nabla k(x,y), u_m'(\tau)\rangle u_m''(\tau)\mathrm{d}s\mathrm{d}\tau\\&\quad +2a(0)\int_0^t \langle u_m''(\tau), u_m'(\tau)\rangle \mathrm{d}\tau - 2\int_0^t a(\tau)\langle u_m''(\tau), u_m(0)\rangle\mathrm{d}\tau\end{aligned} \tag{2.17}$$

We put

$$\psi_m(t) = \|u_m''(t)\|_2^2 + \|\nabla u_m'(t)\|_2^2 + 2\int_0^t \|u_m''(\tau)\|_2^2\mathrm{d}\tau$$

we obtain

$$\begin{aligned}\psi_m(t) &= \psi_m(0) + 2(p-1)\int_0^t \left\langle |u_m(\tau)|^{p-2}u_m'(\tau), u_m''(\tau)\right\rangle \mathrm{d}\tau\\&\quad +2\int_0^t\int_{\partial\Omega}\langle \nabla k(x,y), u_m'(\tau)\rangle u_m''(\tau)\mathrm{d}s\mathrm{d}\tau\\&\quad +2a(0)\int_0^t \langle u_m''(\tau), u_m'(\tau)\rangle \mathrm{d}\tau - 2\int_0^t a(\tau)\langle u_m''(\tau), u_m(0)\rangle\mathrm{d}\tau\\&= \psi_m(0) + \sum_{k=1}^{4} J_k.\end{aligned} \tag{2.18}$$

Now, we estimate the last four term in the right side of (2.18). Firstly, it is easy to check that

$$\begin{aligned}\|u_m'(t)\|_{H^1}^2 &= \|u_m'(t)\|_2^2 + \|\nabla u_m'(t)\|_2^2\\&= \left[\|u_1\|_2 + \int_0^t \|u_m''(s)\|_2 ds\right]^2 + \|\nabla u_m'(t)\|_2^2\\&\le 2\|u_1\|_2^2 + \psi_m(t) + 2t\int_0^t \psi_m(s)\mathrm{d}s.\end{aligned} \tag{2.19}$$

From

$$\begin{aligned}\left\| |u_m(t)|^{p-2}u'_m(t) \right\| &\leq D_p \left[1 + \|u_m(t)\|_{H^1}^{1/N} + \|u_m(t)\|_{H^1}^{p-2} \right] \|u'_m(t)\|_{H^1} \\ &\leq D_p C_T \|u'_m(t)\|_{H^1}\end{aligned}$$

By ([16], Lemmas 2,3 (ii), p.4) and (2.19), we have

$$\begin{aligned}J_1 &= 2(p-1)\int_0^t \left\langle |u_m(\tau)|^{p-2}u'_m(\tau), u''_m(\tau)\right\rangle \mathrm{d}\tau \\ &\leq 2(p-1)\int_0^t \left\| u_m(\tau)|^{p-2}u'_m(\tau)\right\| \|u''_m(\tau)\|\mathrm{d}\tau \\ &\leq 2(p-1)D_pC_T\int_0^t \|u'_m(\tau)\|_{H^1}\|u''_m(\tau)\|\mathrm{d}\tau \\ &\leq (p-1)^2D_p^2C_T^2\int_0^t \|u'_m(\tau)\|_{H^1}^2\mathrm{d}\tau + \int_0^t \|u''_m(\tau)\|_2^2\mathrm{d}\tau \\ &\leq (p-1)^2D_p^2C_T^2\left[\int_0^t \|u'_m(\tau)\|_2^2\mathrm{d}\tau + \int_0^t \|\nabla u'_m(\tau)\|_2^2\mathrm{d}\tau\right] + \int_0^t \|u''_m(\tau)\|_2^2\mathrm{d}\tau \\ &\leq C_T\left(1 + \int_0^t \psi_m(\tau)\mathrm{d}\tau\right) \qquad (2.20)\end{aligned}$$

Using (2.1) and (2.19), we continue to estimate all terms in the right-hand side of (2.18) as below

$$J_2 = 2\int_0^t \int_{\partial\Omega} \langle \nabla k(x,y), u'_m(\tau)\rangle u''_m(\tau)\mathrm{d}s\mathrm{d}\tau \qquad (2.21)$$

For $x \in \partial\Omega$, we have

$$\begin{aligned}\left|\langle \nabla k(x,y), u'_m(\tau)\rangle\right| = \left|\int_\Omega \nabla k(x,y)u'_m(y,t)\mathrm{d}y\right| &\leq \int_\Omega |\nabla k(x,y)u'_m(y,t)|\mathrm{d}y \\ &\leq \|\nabla k(x,y)\|_2\|u'_m(t)\|_2 = k_1(x)\|u'_m(t)\|_2\end{aligned}$$

and

$$|u''_m(x,t)| \leq \int_\Omega |k(x,y)u''_m(y,t)|dy \leq \|k(x,y)\|_2\|u''_m(t)\|_2 = k_2(x)\|u''_m(t)\|_2$$

Then

$$\begin{aligned}J_2 &\leq 2\int_0^t \int_{\partial\Omega} k_1(x)k_2(x)\|u'_m(\tau)\|_2\|u''_m(\tau)\|_2\mathrm{d}s\mathrm{d}\tau \\ &\leq 2\int_0^t \left(\int_{\partial\Omega} k_1(x)k_2(x)\mathrm{d}s\right)\|u'_m(\tau)\|_2\|u''_m(\tau)\|_2\mathrm{d}\tau \\ &\leq C_1\int_0^t \|u'_m(\tau)\|_2^2d\tau + C_1\int_0^t \|u''_m(\tau)\|_2^2\mathrm{d}\tau \\ &\leq C_T + C_T\int_0^t \psi_m(\tau)\mathrm{d}\tau,\end{aligned} \qquad (2.22)$$

and

$$\begin{aligned} J_3 &= 2a(0)\int_0^t\int_\Omega u_m''(\tau)u_m'(\tau)\mathrm{d}x\mathrm{d}\tau \\ &\leq a(0)\int_0^t \|u_m''(\tau)\|_2^2 d\tau + a(0)\int_0^t \|u_m'(\tau)\|_2^2\mathrm{d}\tau \\ &\leq C_T + C_T\int_0^t \psi_m(\tau)\mathrm{d}\tau \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} J_4 &= 2\int_0^t a(\tau)\langle u_m''(\tau), u_m(0)\rangle\mathrm{d}\tau = 2\int_0^t\int_\Omega a(\tau)u_m''(\tau)u_m(0)\mathrm{d}x\mathrm{d}\tau \\ &\leq 2a_2\int_0^t\int_\Omega u_m''(\tau)u_m(0)\mathrm{d}x\mathrm{d}\tau \\ &\leq Ta_2\|u_m(0)\|_2^2 + a_2\int_0^t \|u_m''(\tau)\|_2^2 d\tau \\ &\leq C_T + C_T\int_0^t \psi_m(\tau)\mathrm{d}\tau \end{aligned} \tag{2.24}$$

Combining estimations of all terms, we obtain after some rearrangements

$$\psi_m(t) \leq C_T\left(1 + \int_0^t \psi_m(\tau)\mathrm{d}\tau\right) \tag{2.25}$$

where C_T always indicates a constant depending on T. Then, by solving Volterra integral inequality [9], we deduce from (2.25) that

$$\psi_m(t) \leq C_T. \tag{2.26}$$

Step 3. Limiting process: From (2.13) and (2.26), we deduce the existence of a subsequence of $\{u_m\}$ denoted by the same symbol such that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty(0, T_*; H^1), \\ u_m' \overset{*}{\rightharpoonup} u' \ \text{ in } L^\infty(0, T_*; H^1), \\ u_m'' \overset{*}{\rightharpoonup} u'' \ \ in\ L^\infty(0, T_*; L^2). \end{cases} \tag{2.27}$$

By the compactness Lemma of Lions ([10], p.57), we can deduce from (2.27) the existence of a subsequence still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to u \ \text{ strongly in } L^2(Q_{T_*}) \text{ and a.e. in } Q_{T_*}, \\ u_m' \to u' \ \text{ strongly in } L^2(Q_{T_*}) \text{ and a.e. in } Q_{T_*}. \end{cases} \tag{2.28}$$

By means of the continuity of the function $t \to |t|^{p-2}t$, we have

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \text{ and a.e. in } Q_{T_*}. \tag{2.29}$$

On the other hand

$$\begin{aligned}\big\||u_m|^{p-2}u_m\big\|^2_{L^2(Q_{T_*})} &= \int_0^{T_*}\int_\Omega |u_m(x,t)|^{2p-2}\mathrm{d}x\mathrm{d}t \\ &= \int_0^{T_*} \|u_m(x,t)\|^{2p-2}_{L^{2p-2}}\mathrm{d}t \\ &\le \int_0^{T_*}\big(C_{2p-2}\|u_m(x,t)\|_{H^1}\big)^{2p-2}\,\mathrm{d}t \\ &\le C^{2p-2}_{2p-2}T_*\|u_m(x,t)\|^{2p-2}_{L^\infty(0,T_*;H^1)} \le C_T\end{aligned} \tag{2.30}$$

Using the Lions Lemma ([10], Lemma 1.3; p.12), it follows from (2.29) and (2.30) that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \ in\ L^2(Q_{T_*}) \text{ weakly.} \tag{2.31}$$

Passing to the limit in (2.3) by (2.27), (2.28) and (2.31) we have u satisfying the problem

$$\begin{cases}\langle u''(t),v\rangle + \langle u'(t),v\rangle + \langle\nabla u(t),\nabla v\rangle \\ = \big\langle |u(t)|^{p-2}u(t),v\big\rangle + \int_{\partial\Omega}\langle\nabla k(x,y),u(t)\rangle v\mathrm{d}s \\ \quad + \int_0^t a(t-\tau)\langle u(\tau),v\rangle\mathrm{d}\tau \\ u(0)=u_0,\quad u'(0)=u_1.\end{cases} \tag{2.32}$$

□

2.2 Uniqueness of the solution

Proof Here, we will prove the uniqueness of solution, for this purpose, let u_1, u_2 be two weak solutions of problem (1.1)–(1.3). Then, we set $u = u_1 - u_2$ to verify

$$\begin{cases}\langle u''(t),v\rangle + \langle\nabla u(t),\nabla v\rangle + \langle u'(t),v\rangle \\ = \big\langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, v\big\rangle \\ \quad + \int_{\partial\Omega}\langle\nabla k(x,y),u(t)\rangle v\mathrm{d}s + \int_0^t a(t-\tau)\langle u(\tau),v\rangle\mathrm{d}\tau \\ u(0)=0,\quad u'(0)=0.\end{cases} \tag{2.33}$$

We take $v = u' = u_1' - u_2'$ and integrating with respect to t, we have

$$\begin{aligned} M(t) &= \|u'(t)\|^2 + \|\nabla u(t)\|^2 = -2\int_0^t\langle u'(\tau),u'(\tau)\rangle\mathrm{d}\tau \\ &+2\Big\langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, u'\Big\rangle \\ &+2\int_0^t\int_{\partial\Omega}\langle\nabla k(x,y),u(\tau)\rangle u'(\tau)\mathrm{d}s\end{aligned}$$

$$+2\int_0^t\int_0^s a(s-\tau)\langle u(\tau),u'(s)\rangle \mathrm{d}s\mathrm{d}\tau$$
$$=\sum_{i=1}^{k} M_i. \tag{2.34}$$

where

$$M_1=-2\int_0^t \langle u'(\tau),u'(\tau)\rangle \mathrm{d}\tau \le 2\int_0^t \|u'(\tau)\|_2^2\mathrm{d}\tau \le 2\int_0^t M(\tau)\mathrm{d}\tau \tag{2.35}$$

By Lemma 2.3 in [16], we get

$$\begin{aligned} M_2 &= 2\left\langle |u_1|^{p-2}u_1-|u_2|^{p-2}u_2, u'(\tau)\right\rangle \\ &\le C_T\int_0^t \|u(\tau)\|_{H^1}^2\mathrm{d}\tau + C_T\int_0^t \|u'(\tau)\|_2^2\mathrm{d}\tau \\ &\le 2(C_T+t^2)\int_0^t M(\tau)\mathrm{d}\tau \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} M_3 &= 2\int_0^t\int_{\partial\Omega} \langle \nabla k(x,y), u(\tau)\rangle u'(\tau)\mathrm{d}s\mathrm{d}\tau \\ &\le C\int_0^t \|u(\tau)\|_2^2\mathrm{d}\tau + C\int_0^t \|u'(\tau)\|_2^2\mathrm{d}\tau \\ &\le 2(C_T+t^2)\int_0^t M(\tau)\mathrm{d}\tau \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} M_4 &= 2\int_0^t\int_0^s a(s-\tau)\langle u(\tau),u'(s)\rangle \mathrm{d}s\mathrm{d}\tau \\ &\le 2\int_0^t\int_0^s |a(s-\tau)|\langle u(\tau),u'(s)\rangle \mathrm{d}s\mathrm{d}\tau \\ &\le 2a_2\int_0^t\int_\Omega u'(s)\int_0^s u(\tau)\mathrm{d}s\mathrm{d}\tau \\ &\le a_2(1+T^2)\int_0^t M(\tau)\mathrm{d}\tau \end{aligned} \tag{2.38}$$

Combining (2.35), (2.36), (2.37) and (2.38), we obtain

$$M(t)\le C_T\int_0^t M(\tau)\mathrm{d}\tau. \tag{2.39}$$

By Gronwall's Lemma, it follows from (2.39) that $M\equiv 0$, i,e. $u_1=u_2$. Then, the second part of Theorem 2.1 is proved. □

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