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ORIGINAL ARTICLE

Weak solutions for multiquasilinear elliptic–parabolic systems: application to thermoelectrochemical problems

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Abstract
This paper investigates the existence of weak solutions of biquasilinear boundary value problem for a coupled elliptic–parabolic system of divergence form with discontinuous leading coefficients. The mathematical framework addressed in the article considers the presence of an additional nonlinearity in the model which reflects the radiative thermal boundary effects in some applications of interest. The results are obtained via the Rothe–Galerkin method. Only weak assumptions are made on the data and the boundary conditions are allowed to be on a general form. The major contribution of the current paper is the explicit expressions for the constants appeared in the quantitative estimates that are derived. These detailed and explicit estimates may be useful for the study on nonlinear problems that appear in the real-world applications. In particular, they clarify the smallness conditions. In conclusion, we illustrate how the above results may be applied to the thermoelectrochemical phenomena in an electrolysis cell. This problem has several applications as, for instance, to optimize the cell design and operating conditions.

Keywords Rothe–Galerkin method · Radiative thermal boundary effects · Thermoelectrochemical system

Mathematics Subject Classification 35R05 · 35J62 · 35K59 · 78A57 · 80A20 · 35Q79

List of symbols

c	Molar concentration (molarity) (mol m^{-3})
D	Diffusion coefficient ($\mathrm{m}^2\,\mathrm{s}^{-1}$)
D'	Dufour coefficient ($\mathrm{m}^2\,\mathrm{s}^{-1}\mathrm{K}^{-1}$)
h	Heat transfer coefficient (W $\mathrm{m}^{-2}\,\mathrm{K}^{-1}$)
k	Thermal conductivity (W $\mathrm{m}^{-1}\,\mathrm{K}^{-1}$)

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S	Soret coefficient (thermal diffusion) ($m^2\ s^{-1}\ K^{-1}$)
t	Transference number (dimensionless)
u	Ionic mobility ($m^2\ V^{-1}\ s^{-1}$)
z	Valence (dimensionless)
α_S	Seebeck coefficient ($V\ K^{-1}$)
ϕ	Electric potential (V)
π	Peltier coefficient (V)
σ	Electrical conductivity ($S\ m^{-1}$)
θ	Absolute temperature (K)

1 Introduction

The main gap between theory and practice has been the assumption of some simplifications. Among them, they are the constant coefficients of the time derivative term in parabolic equations, or its independence on the space variable (commonly the density). In the real-world applications, there are three terms that destroy the regularity of the solutions. The first quasilinear term classically stands for the spatial gradient of the solution, the second one stands for the time derivative, and the third one appears from the power-type boundary condition. This power-type boundary condition represents the radiative heat transfer existent on a part of boundary. We mention to Ref. [26] for the transient radiative heat transfer equations in the one-dimensional slab.

Quantitative estimates take the characteristics of the coefficients into account, but usually include constants that hide some intrinsic characteristics of the domain. We seek for the complete explicitness of the constants that are involved on the quantitative estimates, and their effectiveness. We emphasize that their sharpness remains as an open problem. The main purpose is the analysis of a weak formulation of the corresponding boundary- and initial-value elliptic–parabolic problem. To that aim, we approximate the problem via implicit time discretization, by the classical Rothe method.

We point out that, in addition to the fact that Galerkin and Rothe methods are convenient tools for the theoretical analysis of elliptic and evolution problems [3,12,21,28], it is of particular interest from the numerical point of view [17,23,24]. Different versions of the primal discontinuous Galerkin methods to treat the coupling of flow and transport and the coupling of transport and reaction have recently gained popularity because they are easier to implement than most traditional finite element methods, from a computer science point of view (see [29] and the references therein). Lipschitz continuity property is commonly assumed as a data character, which simplifies the Rothe method [15,27].

The paper [10] deals with modeling of quasilinear thermoelectric phenomena, including the Peltier and Seebeck effects. In Ref. [6], the spatial distribution of the variables such as the electrolyte temperature, which is subject to local cell conditions, is studied. To optimize cell operations is the aim for the long-term sustainability of the aluminum smelting industry.

The heat transfer modeling on electrochemical devices is gaining interest in the literature, as, for instance, the thermoelectrochemical cells (TECs) are of low cost and it is an effective approach to harvest waste thermal energy [5,14,20]. Here, no internal interfaces are considered in the model, which amounts to neglecting possible material heterogeneities as done in Refs. [7–9]. These works deal with weak solutions related to thermoelectrochemical devices with radiative effects in a part of the boundary, involving the cross effects. A particular feature is the mixture of some kind of (nonlinear) Neumann and Robin boundary conditions. Also, quantitative estimates are stated for the norm (steady state in Ref. [7] and unsteady state in Ref. [8]) under appropriate assumptions on the data, where the constants are given explicitly. Within this state of mind, we close this paper by applying the theoretical coupled elliptic–parabolic system to the thermoelectrochemical phenomena.

The structure of the paper is as follows. We begin by introducing the functional framework, the data under consideration and the main theorem in Sect. 2. The main ingredient of the proof is the Rothe method presented in Sect. 3. Section 4 deals with the existence proof of the corresponding elliptic problem. The idea of the proof is based on classical Galerkin approximation argument (Sect. 4.1). In Sect. 5, we derive a priori estimates for the approximate problem, getting compactness properties that allow the existence proof of the main theorem via the passage to the limit as the time step vanishes. As a consequence of the main theoretical result, the existence of a weak solution to a thermoelectrochemical problem is stated in Sect. 6.

2 Statement of the problem

Let $[0, T] \subset \mathbb{R}$ be the time interval with $T > 0$ being an arbitrary (but preassigned) time. Let Ω be a bounded domain (that is, connected open set) in $\mathbb{R}^n$ ($n \geq 2$). Its boundary $\partial\Omega$ is constituted by three pairwise disjoint open $(n-1)$-dimensional sets, namely the electrode surface Γ, the wall surface $\Gamma_{\rm w}$, and the remaining outer surface $\Gamma_{\rm o}$, such that $\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma}_{\rm w} \cup \overline{\Gamma}_{\rm o}$. Observe that the electrode surface Γ consists of the anode $\Gamma_{\rm a}$ and the cathode $\Gamma_{\rm c}$. Figure 1 displays two schematic geometrical representations of the domain Ω and of its boundary $\partial\Omega$ to identify the various subsets into which the boundary is decomposed and, as a consequence, to better understand the physical significance of the enforced boundary conditions. Hence, further, we set $Q_T = \Omega\times]0, T[$ and $\Sigma_T = \partial\Omega\times]0, T[$.

We are interested in the following boundary value problem in the sense of distributions. Find the functions $(\mathbf{u}, \phi) : Q_T \to \mathbb{R}^{\rm I+2}$, with I being an integer number, that solve

$$\mathsf{B}(u_{\rm I+1})\partial_t\mathbf{u} - \nabla\cdot(\mathsf{A}(\mathbf{u})\nabla\mathbf{u}) = \nabla\cdot(\mathbf{F}(\mathbf{u})\nabla\phi); \tag{1}$$

$$-\nabla\cdot(\sigma(\mathbf{u})\nabla\phi) = \nabla\cdot(\mathbf{G}(\mathbf{u})\nabla\mathbf{u}) \text{ in } Q_T, \tag{2}$$

with the following meaning of notation, for $j = 1, \ldots, \rm I + 1$,

$$\nabla\cdot(\mathsf{A}(\mathbf{u})\nabla\mathbf{u}) = \sum_{k=1}^{n}\partial_k\left(\sum_{l=1}^{\rm I+1} a_{j,l}(\mathbf{u})\partial_k u_l\right);$$

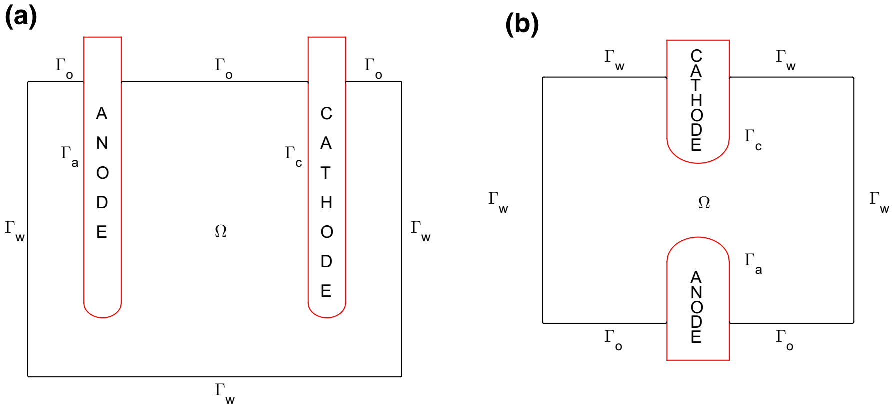

Fig. 1 Schematic 2D representation of two cells of one compartment (not in scale). **a** An electrolytic cell. **b** TEC device design: heating bottom plate and two electrodes symmetrically placed [19]

$$\nabla \cdot (\mathbf{F}(\mathbf{u})\nabla\phi) = \sum_{k=1}^{n} \partial_k (F_j(\mathbf{u})\partial_k\phi);$$
$$\nabla \cdot (\mathbf{G}(\mathbf{u})\nabla\mathbf{u}) = \sum_{k=1}^{n} \partial_k \left(\sum_{l=1}^{\mathrm{I}+1} G_l(\mathbf{u})\partial_k u_l\right).$$

Here A and B are $(\mathrm{I}+1)^2$-matrices such that

(A) the leading matrix A is supposed to be uniformly elliptic, of quadratic-growth, and with real-valued L^∞ components;
(B) B is the diagonal matrix with non-zero components

$$b_{j,j} = \begin{cases} 1 \text{ if } 1 \leq j \leq \mathrm{I} \\ b \text{ if } j = \mathrm{I}+1. \end{cases}$$

Only $(\mathrm{I}+1)$ parabolic equation is in fact known as the doubly nonlinear elliptic–parabolic equation which has been investigated by several authors when Dirichlet conditions are taken into account on the boundary (we refer, for example, to the works [4,27] and the references cited therein for some details).

The Kirchhoff transformation could be applied to the $(\mathrm{I}+1)$ parabolic equation to be useful in the time discretization because

$$b(u)\partial_t u = \partial_t \left(\int^u b(z)\mathrm{d}z\right), \tag{3}$$

although it is not truly useful as change variable because the function b depends on the space variable and $\nabla\left(\int^u b(r)\mathrm{d}r\right)$ may be ill defined.

The boundary conditions are in the concise form:

$$(\mathsf{A}(\mathbf{u})\nabla\mathbf{u} + \mathbf{F}(\mathbf{u})\nabla\phi)\cdot\mathbf{n} + \mathbf{b}(u_{\mathrm{I}+1})^\top\mathbf{u} = \mathbf{h}; \tag{4}$$

$$(\sigma(\mathbf{u})\nabla\phi + \mathbf{G}(\mathbf{u})\nabla\mathbf{u})\cdot\mathbf{n} = g\chi_\Gamma \text{ on } \Sigma_T, \tag{5}$$

with $\mathbf{n}$ denoting the outward unit normal to the boundary $\partial\Omega$, and

$$b_j = \begin{cases} 0 & \text{if } 1 \le j \le \mathrm{I} \\ \gamma & \text{if } j = \mathrm{I}+1. \end{cases}$$

Here, the boundary coefficient γ stands for the Robin-type boundary effects on Γ, and for the power-type boundary effects on Γ_{w}. The functions $\mathbf{h}$ and g stand for the boundary sources.

Finally, let the initial condition be

$$\mathbf{u}(\cdot, 0) = \mathbf{u}^0 \text{ in } \Omega. \tag{6}$$

In the framework of Sobolev and Lebesgue functional spaces, we use the following spaces of test functions:

$$\begin{aligned} V(\Omega) &= \{v \in H^1(\Omega) : \int_\Omega v\mathrm{d}x = 0\}; \\ V(\partial\Omega) &= \{v \in H^1(\Omega) : \int_{\partial\Omega} v\mathrm{d}s = 0\}; \\ V_\ell(\Omega) &= \{v \in H^1(\Omega) : v|_{\Gamma_{\mathrm{w}}} \in L^\ell(\Gamma_{\mathrm{w}})\}; \\ V_\ell(Q_T) &= \{v \in L^2(0,T; H^1(\Omega)) : v|_{\Gamma_{\mathrm{w}}\times]0,T[} \in L^\ell(\Gamma_{\mathrm{w}}\times]0,T[)\} \end{aligned}$$

with their usual norms, $\ell > 1$. Hereafter, we use the notation "ds" for the surface element in the integrals on the boundary as well as any subpart of the boundary $\partial\Omega$. Notice that $V_\ell(\Omega) \equiv H^1(\Omega)$ if $\ell < 2_*$, where 2_* is the critical trace continuity constant, i.e., $2_* = 2(n-1)/(n-2)$ if $n > 2$ and $2_* > 1$ is arbitrary if $n = 2$.

The problem (1), (2) is in fact a system of $\mathrm{I}+2$ partial differential equations and it may be decomposed into one system of I parabolic equations, one parabolic equation with a quasilinear time derivative, and one third elliptic equation.

Definition 1 We say that a function $(\mathbf{u}, \phi)$ is a weak solution to the problem (1), (2), (4)–(6), if it satisfies (6) and the variational formulation, with $u = u_{\mathrm{I}+1}$,

$$\begin{aligned} &\int_0^T \langle \partial_t u_i, v_i\rangle \mathrm{d}t + \sum_{j=1}^{\mathrm{I}+1} \int_{Q_T} a_{i,j}(\mathbf{u})\nabla u_j \cdot \nabla v_i \mathrm{d}x\mathrm{d}t \\ &\quad = -\int_{Q_T} F_i(\mathbf{u})\nabla\phi\cdot\nabla v_i \mathrm{d}x\mathrm{d}t + \int_{\Sigma_T} h_i v_i \mathrm{d}s\mathrm{d}t, \quad i = 1,\ldots,\mathrm{I}; \end{aligned} \tag{7}$$

$$\int_0^T \langle b(u)\partial_t u, v\rangle \mathrm{d}t + \sum_{j=1}^{\mathrm{I}+1}\int_{Q_T} a_{\mathrm{I}+1,j}(\mathbf{u})\nabla u_j\cdot\nabla v \mathrm{d}x\mathrm{d}t + \int_{\Sigma_T}\gamma(u)uv\mathrm{d}s\mathrm{d}t$$
$$= -\int_{Q_T} F_{\mathrm{I}+1}(\mathbf{u})\nabla\phi\cdot\nabla v\mathrm{d}x\mathrm{d}t + \int_{\Sigma_T} h_{\mathrm{I}+1}v\mathrm{d}s\mathrm{d}t; \qquad (8)$$
$$\int_\Omega \sigma(\mathbf{u})\nabla\phi\cdot\nabla w\mathrm{d}x = -\sum_{j=1}^{\mathrm{I}+1}\int_\Omega G_j(\mathbf{u})\nabla u_j\cdot\nabla w\mathrm{d}x + \int_\Gamma gw\mathrm{d}s, \text{ a.e. in }]0,T[\qquad (9)$$

for all $v_i \in L^2(0,T;V(\Omega))$, $v \in V_\ell(Q_T)$, and $w \in V(\partial\Omega)$.

The symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing $\langle\cdot,\cdot\rangle_{X'\times X}$, with X being a Banach space. The notation X' denotes the dual space of X, and X' is equipped with the usual induced norm $\|f\|_{X'} = \sup\{\langle f,u\rangle,\ u\in X : \|u\|_X \le 1\}$.

The set of hypothesis is as follows.

(H1) The vector-valued functions $\mathbf{F}$ and $\mathbf{G}$, from $\Omega\times\mathbb{R}^{\mathrm{I}+1}$ into $\mathbb{R}^{\mathrm{I}+1}$, are assumed to be Carathéodory, i.e., measurable with respect to $x\in\Omega$ and continuous with respect to other variables, such that they verify

$$\exists F_j^\# > 0 : |F_j(x,\mathbf{e})| \le F_j^\#, \qquad (10)$$
$$\exists G_j^\# > 0 : |G_j(x,\mathbf{e})| \le G_j^\#, \qquad (11)$$

for all $j = 1,\dots,\mathrm{I}+1$, for a.e. $x\in\Omega$, and for all $\mathbf{e}\in\mathbb{R}^{\mathrm{I}+1}$.

(H2) The coefficient b is assumed to be a Carathéodory function from $\Omega\times\mathbb{R}$ to $\mathbb{R}$. Moreover, there exist $b_\#, b^\# > 0$ such that

$$b_\# \le b(x,e) \le b^\#, \qquad (12)$$

for a.e. $x\in\Omega$, and for all $e\in\mathbb{R}$.

(H3) The leading coefficient A has its components $a_{i,j} : \Omega\times\mathbb{R}^{\mathrm{I}+1}\to\mathbb{R}$ being Carathéodory functions. Moreover, they satisfy

$$(a_i)_\# := \min_{(x,\mathbf{e})\in\Omega\times\mathbb{R}^{\mathrm{I}+1}} a_{i,i}(x,\mathbf{e}) > 0; \qquad (13)$$
$$\exists a_{i,j}^\# > 0 : \quad |a_{i,j}(\cdot,\mathbf{e})| \le a_{i,j}^\#, \quad \text{a.e. in } \Omega,\ \forall\mathbf{e}\in\mathbb{R}^{\mathrm{I}+1}, \qquad (14)$$

for all $i,j\in\{1,\dots,\mathrm{I}+1\}$.

(H4) The leading coefficient σ is assumed to be a Carathéodory function from $\Omega\times\mathbb{R}^{\mathrm{I}+1}$ into $\mathbb{R}$. Moreover, there exist $\sigma_\#, \sigma^\# > 0$ such that

$$\sigma_\# \le \sigma(x,\mathbf{e}) \le \sigma^\#, \qquad (15)$$

for a.e. $x\in\Omega$, and for all $\mathbf{e}\in\mathbb{R}^{\mathrm{I}+1}$.

(H5) The boundary coefficient γ is assumed to be a Carathéodory function from $\partial\Omega \times \mathbb{R}$ into $\mathbb{R}$. Moreover, there exist $\gamma_\#, \gamma^\# > 0$ and $\gamma_1 \geq 0$ such that

$$\gamma_\# |e|^{\ell-2} \leq \gamma(\cdot, e) \leq \gamma^\# |e|^{\ell-2} + \gamma_1, \tag{16}$$

a.e. on $\partial\Omega$, and for all $e \in \mathbb{R}$, where the exponent $\ell \geq 2$ stands for the Robin-type boundary condition ($\ell = 2$) on Γ, and for the power-type boundary condition ($\ell > 2$) on Γ_{w}.

Remark 1 The boundary condition (16) may be generalized for a function $\gamma_1 : \partial\Omega \to \mathbb{R}$ belonging to $L^{\ell/(\ell-2)}(\partial\Omega)$ for $\ell \geq 2$. Indeed, Theorem 1 remains valid if (16) is replaced by

$$\begin{aligned} |\gamma(\cdot, e)| &\leq \gamma_1 \text{ a.e. on } \Gamma; \\ \gamma_\# |e|^{\ell-2} \leq \gamma(\cdot, e) &\leq \gamma^\# \ |e|^{\ell-2} + \gamma_1 \text{ a.e. on } \Gamma_{\mathrm{w}}, \end{aligned}$$

for all $e \in \mathbb{R}$, which infer in Sect. 4.1 that the Brouwer fixed point theorem is applied for a different $r > 0$ taking Definition 2 into account.

Hereafter, we will use the Kirchhoff transformation (3) to the time derivative term, i.e., the characterization $\partial_t B(u)$, denoting by B the operator defined by

$$v \in L^2(Q_T) \mapsto B(v) = \int_0^v b(\cdot, z)\mathrm{d}z. \tag{17}$$

Let us state the existence results.

Theorem 1 *Suppose that Assumptions (H1)–(H5), $h_i \in L^2(\Sigma_T)$, $i = 1, \ldots, \mathrm{I}$, $h_{\mathrm{I}+1} \in L^{\ell/(\ell-1)}(\Sigma_T)$, and $g \in L^2(\Gamma)$ be fulfilled. Under the smallness conditions, for $i \in \{1, \ldots, \mathrm{I}+1\}$,*

$$(a_i)_\# > \frac{1}{2}\left(\sum_{\substack{l=1 \\ l \neq i}}^{\mathrm{I}+1} (a_{i,l}^\# + a_{l,i}^\#) + F_i^\# + G_i^\# \right), \tag{18}$$

$$\sigma_\# > \frac{1}{2} \sum_{j=1}^{\mathrm{I}+1} \left(F_j^\# + G_j^\# \right), \tag{19}$$

there exists at least one weak solution

$$(\mathbf{u}, \phi) \in [L^\infty(0, T; L^2(\Omega))]^{\mathrm{I}+1} \times L^2(0, T; V(\partial\Omega))$$

in accordance with Definition 1, with $v \in L^\ell(0, T; V_\ell(\Omega))$, such that

$$\begin{gathered} u_i - u_i^0 \in L^2(0, T; V(\Omega)) \quad \text{and} \quad \partial_t u_i \in L^2(0, T; (V(\Omega))'); \\ u \in V_\ell(Q_T) \quad \text{and} \quad b(u)\partial_t u \in L^{\ell'}(0, T; (V_\ell(\Omega))'), \end{gathered}$$

for $i = 1, \ldots, \mathrm{I}$. *In particular,* $B(u) \in L^\infty(0, T; L^1(\Omega))$.

Here, we consider the Banach spaces that are of direct application for the thermo-electrochemical problem under study. Clearly, Theorem 1 remains valid for any closed subspace V such that $H^1_0(\Omega) \hookrightarrow V \hookrightarrow H^1(\Omega)$ is considered instead of $V(\Omega)$ or $V(\partial\Omega)$ if the Poincaré inequality is verified.

Remark 2 In (7) and (8), the meaning of the time derivative should be understood as in the following weak sense [4]:

$$\int_0^T \langle \partial_t u_i, v_i \rangle \mathrm{d}t = -\int_0^T \int_\Omega u_i \partial_t v_i \mathrm{d}x\mathrm{d}t - \int_\Omega u_i^0 v_i(0)\mathrm{d}x; \tag{20}$$

$$\int_0^T \langle b(u)\partial_t u, v \rangle \mathrm{d}t = -\int_0^T \int_\Omega B(u)\partial_t v \mathrm{d}x\mathrm{d}t - \int_\Omega B(u^0)v(0)\mathrm{d}x, \tag{21}$$

for every test functions $v_i \in L^2(0, T; V(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega))$, for $i \in \{1, \ldots, \mathrm{I}\}$, and $v \in L^\ell(0, T; V_\ell(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega))$ such that $v_i(T) = v(T) = 0$ a.e. in Ω.

3 Time discretization technique

We adopt the weak solvability of $\mathrm{I}+1$ time-dependent partial differential equation with a nonlinear Neumann boundary condition as investigated in Refs. [4,22], while the j parabolic equations ($j = 1, \ldots, \mathrm{I}$) are studied via the classical time discretization technique [21]. We introduce a recurrent system of boundary value problems to be successively solved for $m = 1, \ldots, M \in \mathbb{N}$, starting from the initial function (6).

We decompose the time interval $I = [0, T]$ into M subintervals $I_{m,M}$ of size τ (commonly called time step) such that $M = T/\tau \in \mathbb{N}$, i.e., $I_{m,M} = [(m-1)T/M, mT/M]$ for $m \in \{1, \cdots, M\}$. We set $t_{m,M} = mT/M$.

For any time integrable function $h : \Sigma_T \to \mathbb{R}$, we introduce the (piecewise constant in time) function $h^M \in L^\infty(0, T; L^1(\partial\Omega))$ being given by $h^M(t) = \bar{h}^m$ for $t \in](m-1)\tau, m\tau]$, with

$$\bar{h}^m = \frac{1}{\tau} \int_{(m-1)\tau}^{m\tau} h(\cdot, z)\mathrm{d}z.$$

Then the problem (7)–(9) is approximated by the following recurrent sequence of time discretized problems:

$$\frac{1}{\tau}\int_\Omega u_i^m v_i \mathrm{d}x + \sum_{j=1}^{\mathrm{I}+1} \int_\Omega a_{i,j}(\mathbf{u}^m)\nabla u_j^m \cdot \nabla v_i \mathrm{d}x + \int_\Omega F_i(\mathbf{u}^m)\nabla\phi^m \cdot \nabla v_i \mathrm{d}x$$
$$= \frac{1}{\tau}\int_\Omega u_i^{m-1} v_i \mathrm{d}x + \int_{\partial\Omega} \bar{h}_i^m v_i \mathrm{d}s, \quad i = 1, \ldots, \mathrm{I}, \tag{22}$$

$$\frac{1}{\tau}\int_\Omega B(u^m)v\mathrm{d}x + \sum_{j=1}^{\mathrm{I}+1}\int_\Omega a_{\mathrm{I}+1,j}(\mathbf{u}^m)\nabla u_j^m\cdot\nabla v\mathrm{d}x + \int_\Omega F_{\mathrm{I}+1}(\mathbf{u}^m)\nabla\phi^m\cdot\nabla v\mathrm{d}x$$

$$+\int_{\partial\Omega}\gamma(u^m)u^m v\mathrm{d}s = \frac{1}{\tau}\int_\Omega B(u^{m-1})v\mathrm{d}x + \int_{\partial\Omega}\bar{h}^m_{\mathrm{I}+1}v\mathrm{d}s, \tag{23}$$

$$\int_\Omega \sigma(\mathbf{u}^m)\nabla\phi^m\cdot\nabla w\mathrm{d}x + \sum_{j=1}^{\mathrm{I}+1}\int_\Omega G_j(\mathbf{u}^m)\nabla u_j^m\cdot\nabla w\mathrm{d}x = \int_\Gamma gw\mathrm{d}s, \tag{24}$$

where $\mathbf{u} = (u_1,\dots,u_\mathrm{I},u)$, for all $v_i \in V(\Omega)$, $i = 1,\dots,\mathrm{I}$, $v \in V_\ell(\Omega)$ and $w \in V(\partial\Omega)$. Since $\mathbf{u}^0 \in L^2(\Omega)$ is known, we determine $\mathbf{u}^1$ as the unique solution of Proposition 1, and we inductively proceed.

The existence of the above system of elliptic problems is established in the following proposition.

Proposition 1 *Let $m \in \{1,\cdots,M\}$ be fixed, and $\mathbf{u}^{m-1}$ be given. Then there exists a unique solution $(\mathbf{u}^m,\phi^m) \in [V(\Omega)]^\mathrm{I}\times V_\ell(\Omega)\times V(\partial\Omega)$ to the variational system (22)–(24).*

This existence of solution is proved in Sect. 4 via the Galerkin method (cf. Sect. 4.1).

Let us recall the technical result [4,22].

Lemma 1 *Denoting by*

$$\Psi(s) := B(s)s - \int_0^s B(r)\mathrm{d}r = \int_0^s (B(s)-B(r))\mathrm{d}r,$$

there holds

$$\int_\Omega (B(u)-B(v))u\mathrm{d}x \geq \int_\Omega \Psi(u)\mathrm{d}x - \int_\Omega\Psi(v)\mathrm{d}x. \tag{25}$$

In particular, if Assumption (12) is fulfilled then there holds

$$\int_\Omega \Psi(u)\mathrm{d}x \leq \int_\Omega B(u)u\mathrm{d}x \leq b^\#\|u\|^2_{2,\Omega}.$$

Under Assumption (12), the operator B verifies

$$(B(u)-B(v),u-v) \geq b_\#\|u-v\|^2_{2,\Omega}. \tag{26}$$

To control the time dependence, we begin by recalling the following remarkable lemma [4, Lemma 1.9].

Lemma 2 *Suppose u_m weakly converge to u in $L^p(0,T;W^{1,p}(\Omega))$, $p>1$, with the estimates*

$$\int_\Omega \Psi(u_m(t))\mathrm{d}x \leq C \quad \textit{for } 0<t<T,$$

and for $z > 0$

$$\int_0^{T-z}\int_\Omega (B(u_m(t+z)) - B(u_m(t)))(u_m(t+z) - u_m(t))\mathrm{d}x\mathrm{d}t \leq Cz, \tag{27}$$

with C *being positive constants. Then* $B(u_m) \to B(u)$ *in* $L^1(Q_T)$ *and* $\Psi(u_m) \to \Psi(u)$ *almost everywhere in* Q_T.

In the sequel, we will also need both the discrete Gronwall inequality and the Aubin–Lions theorem. Let us recall the following discrete version of the Gronwall inequality [22].

Lemma 3 *(Discrete Gronwall inequality) Let* $\{a_m\}_{m\in\mathbb{N}}$ *and* $\{A_m\}_{m\in\mathbb{N}}$ *be sequences of nonnegative real numbers such that* A_m *is nondecreasing and*

$$a_m \leq A_m + \tau L \sum_{j=1}^{m} a_j,$$

for each $m \in \mathbb{N}$ *and for some* $0 < \tau L < 1$. *Then there holds*

$$a_m \leq \frac{A_m}{1-\tau L}\exp[(m-1)\tau].$$

Let us recall the following version of the Aubin–Lions theorem for piecewise constant functions [13].

Theorem 2 *(Aubin–Lions) Let* X, B, *and* Y *be Banach spaces such that the embeddings* $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$ *hold, and let* $T > 0$ *and* $1 \leq p < \infty$. *Let* $\{u_M\}_{M\in\mathbb{N}}$ *be a sequence of functions, which are constant on each time subinterval* $](k-1)\tau, k\tau]$ *with uniform time step* $\tau = T/M$, *satisfying*

$$\tau^{-1}\|u_M - u_{M-1}\|_{L^1(\tau,T;Y)} + \|u_M\|_{L^p(0,T;X)} \leq C_0, \quad \forall\tau > 0,$$

where C_0 *is a positive constant independent on* τ. *Then there exists a subsequence of* $\{u_M\}_{M\in\mathbb{N}}$ *strongly converging in* $L^p(0,T;B)$.

4 Proof of Proposition 1

Let $m \in \{1,\cdots,M\}$ be fixed, and $\mathbf{u}^{m-1}$ be given. Set $\mathbf{f} = \mathbf{u}^{m-1}$, and $\mathbf{g}$ be such that

$$g_j = \begin{cases} \bar{h}_j^m & \text{if } 1 \leq j \leq \mathrm{I}+1 \\ g\chi_\Gamma & \text{if } j = \mathrm{I}+2. \end{cases} \tag{28}$$

Set the $(\mathrm{I}+2)^2$-matrix

$$\mathsf{L}(\mathbf{u}) = \begin{bmatrix} \mathsf{A}(\mathbf{u}) & \mathbf{F}(\mathbf{u}) \\ \mathbf{G}^\top(\mathbf{u}) & \sigma(\mathbf{u}) \end{bmatrix}. \tag{29}$$

Using Assumptions (10), (11) and (13)–(15), we find

$$\sum_{j,l=1}^{\mathrm{I}+2}\sum_{\iota=1}^{n}\left(L_{j,l}(\mathbf{u})\xi_{l,\iota}\right)\xi_{j,\iota}\geq\sum_{j=1}^{\mathrm{I}+2}\sum_{\iota=1}^{n}(L_j)_\#|\xi_{j,\iota}|^2, \tag{30}$$

where for $j=1,\dots,\mathrm{I}+1$,

$$(L_j)_\# = (a_j)_\# - \frac{1}{2}\left(\sum_{\substack{l=1\\ l\neq j}}^{\mathrm{I}+1}(a^\#_{l,j}+a^\#_{j,l}) + F^\#_j + G^\#_j\right);$$
$$(L_{\mathrm{I}+2})_\# = \sigma_\# - \frac{1}{2}\sum_{j=1}^{\mathrm{I}+1}\left(F^\#_j + G^\#_j\right).$$

Remark 3 Although the positive definiteness implies invertibility, there are invertible matrices that are not positive definite. The existence of the inverse matrix L^{-1} may be consequence of $\det(\mathsf{L})\neq 0$. An alternative sufficient condition is that $\mathrm{rank}(\mathsf{L})=\mathrm{I}+2$.

Definition 2 We call by $K_2(P_2+1)$ the constant that verifies

$$\|v\|_{2,\Gamma}\leq K_2\left(\|v\|_{2,\Omega}+\|\nabla v\|_{2,\Omega}\right)\leq K_2(P_2+1)\|\nabla v\|_{2,\Omega},\quad \forall v\in H^1(\Omega). \tag{31}$$

Here, K_2 stands to the continuity constant of the trace embedding $H^1(\Omega)\hookrightarrow L^2(\Gamma)$, and P_2 stands to the Poincaré constant correspondent to the space exponent 2.

4.1 Galerkin approximation technique

The Banach space $\mathbf{V} := [V(\Omega)]^{\mathrm{I}}\times V_\ell(\Omega)\times V(\partial\Omega)$ admits linearly independent functions $\mathbf{w}^\nu$, $\nu=1,\dots,N$, such that the finite-dimensional subspace $\mathbf{V}_N=\mathrm{span}\{\mathbf{w}^1,\dots,\mathbf{w}^N\}$ is dense in $\mathbf{V}$, for every $N\in\mathbb{N}$.

Introduce the continuous function $P:\mathbb{M}_{(\mathrm{I}+2)\times N}\to\mathbb{M}_{(\mathrm{I}+2)\times N}$ that maps $\left[\lambda_{j,\nu}\right]$ into $\left[\beta_{j,\nu}\right]$, defined by for each $\nu=1,\dots,N$

$$\begin{aligned}
\beta_{j,\nu} &= \frac{1}{\tau}\int_\Omega U^N_j w^\nu_j \mathrm{d}x + \sum_{l=1}^{\mathrm{I}+2}\int_\Omega\left(L_{j,l}(\mathbf{u}^N)\nabla U^N_l\right)\cdot\nabla w^\nu_j\mathrm{d}x\\
&\quad -\frac{1}{\tau}\int_\Omega f_j w^\nu_j\mathrm{d}x - \int_{\partial\Omega} g_j w^\nu_j\mathrm{d}s,\quad \forall j=1,\dots,\mathrm{I};\\
\beta_{j,\nu} &= \frac{1}{\tau}\int_\Omega b(U^N_{\mathrm{I}+1})U^N_j w^\nu_j\mathrm{d}x + \sum_{l=1}^{\mathrm{I}+2}\int_\Omega\left(L_{j,l}(\mathbf{u}^N)\nabla U^N_l\right)\cdot\nabla w^\nu_j\mathrm{d}x\\
&\quad +\int_{\partial\Omega}\gamma(U^N_{\mathrm{I}+1})U^N_j w^\nu_j\mathrm{d}s - \frac{1}{\tau}\int_\Omega b(U^N_{\mathrm{I}+1})f_j w^\nu_j\mathrm{d}x - \int_{\partial\Omega} g_j w^\nu_j\mathrm{d}s,\quad j=\mathrm{I}+1;\\
\beta_{j,\nu} &= \sum_{l=1}^{\mathrm{I}+2}\int_\Omega\left(L_{j,l}(\mathbf{u}^N)\nabla U^N_l\right)\cdot\nabla w^\nu_j\mathrm{d}x - \int_{\partial\Omega} g_j w^\nu_j\mathrm{d}s,\quad j=\mathrm{I}+2,
\end{aligned}$$

with the function $\mathbf{U}^N \in \mathbf{V}_N$ being in the form

$$U_j^N(x) = \sum_{\nu=1}^{N} \lambda_{j,\nu}^N w_j^\nu(x), \quad j = 1, \dots, \mathrm{I}+2.$$

Here, we set $\mathbf{u}^N = (U_1^N, \dots, U_{\mathrm{I}+1}^N)$, for the sake of simplicity.

To apply the Brouwer fixed point theorem [25], we must prove that P satisfies $(P\lambda, \lambda) > 0$ for all $\lambda \in \mathbb{M}_{(\mathrm{I}+2)\times N}$ such that

$$|\lambda| = \left(\sum_{j=1}^{\mathrm{I}+1} \sum_{\nu=1}^{N} \lambda_{j,\nu}^2 \right)^{1/2} = r,$$

and (β, λ) stands for the inner product in $\mathbb{M}_{(\mathrm{I}+2)\times N}$. To this aim, we compute

$$\begin{aligned}(P\lambda, \lambda) &= \sum_{j=1}^{\mathrm{I}+2} \sum_{\nu=1}^{N} \beta_{j,\nu} \lambda_{j,\nu} \\ &= \frac{1}{\tau} \sum_{j=1}^{\mathrm{I}+1} \int_\Omega b_{j,j}(U_{\mathrm{I}+1}^N) |U_j^N|^2 \mathrm{d}x + \sum_{j=1}^{\mathrm{I}+2} \sum_{l=1}^{\mathrm{I}+2} \int_\Omega \left(L_{j,l}(\mathbf{u}^N) \nabla U_l^N \right) \cdot \nabla U_j^N \mathrm{d}x \\ &\quad + \int_{\partial\Omega} \gamma(U_{\mathrm{I}+1}^N) |U_{\mathrm{I}+1}^N|^2 \mathrm{d}s - \frac{1}{\tau} \sum_{j=1}^{\mathrm{I}+1} \int_\Omega b_{j,j}(U_{\mathrm{I}+1}^N) f_j U_j^N \mathrm{d}x - \sum_{j=1}^{\mathrm{I}+2} \int_{\partial\Omega} g_j U_j^N \mathrm{d}s.\end{aligned}$$

Applying Assumptions (12) and (16), the Hölder inequality, and (31), we obtain

$$\begin{aligned}(P\lambda, \lambda) \geq& \frac{1}{\tau} \sum_{j=1}^{\mathrm{I}} \left(\|U_j^N\|_{2,\Omega} - \|f_j\|_{2,\Omega} - K_2 \|g_j\|_{2,\partial\Omega} \right) \|U_j^N\|_{2,\Omega} \\ &+ \frac{1}{\tau} \left(b_\# \|U_{\mathrm{I}+1}^N\|_{2,\Omega} - b^\# \|f_{\mathrm{I}+1}\|_{2,\Omega} \right) \|U_{\mathrm{I}+1}^N\|_{2,\Omega} \\ &+ \sum_{j=1}^{\mathrm{I}} \left((L_j)_\# \|\nabla U_j^N\|_{2,\Omega} - K_2 \|g_j\|_{2,\partial\Omega} \right) \|\nabla U_j^N\|_{2,\Omega} \\ &+ (L_{\mathrm{I}+1})_\# \|\nabla U_{\mathrm{I}+1}^N\|_{2,\Omega}^2 \\ &+ \left(\gamma_\# \|U_{\mathrm{I}+1}^N\|_{\ell,\partial\Omega}^{\ell-1} + \gamma_1 \|U_{\mathrm{I}+1}^N\|_{\ell',\partial\Omega} - \|g_{\mathrm{I}+1}\|_{\ell',\partial\Omega} \right) \|U_{\mathrm{I}+1}^N\|_{\ell,\partial\Omega} \\ &+ \left((L_{\mathrm{I}+2})_\# \|\nabla U_{\mathrm{I}+2}^N\|_{2,\Omega} - K_2(P_2+1) \|g_{\mathrm{I}+2}\|_{2,\partial\Omega} \right) \|\nabla U_{\mathrm{I}+2}^N\|_{2,\Omega}.\end{aligned}$$

Then there exists $r > 0$ such that fulfills $(P\lambda, \lambda) > 0$. We are in the position of applying the Brouwer fixed point theorem. Consequently, there exists $\lambda \in \mathbb{M}_{(\mathrm{I}+2)\times N}$ such that $|\lambda| \leq r$ and $P([\lambda_{j,\nu}]) = 0$, i.e., taking the density of $\mathbf{V}_N$ into $\mathbf{V}$:

$$\frac{1}{\tau}\sum_{j=1}^{\mathrm{I}}\int_{\Omega} U_j^N v_j \mathrm{d}x + \frac{1}{\tau}\int_{\Omega} b(U_{\mathrm{I}+1}^N) U_{\mathrm{I}+1}^N v_{\mathrm{I}+1} \mathrm{d}x$$
$$+\sum_{j=1}^{\mathrm{I}+2}\sum_{l=1}^{\mathrm{I}+2}\int_{\Omega}\left(L_{j,l}(\mathbf{u}^N)\nabla U_l^N\right)\cdot\nabla v_j \mathrm{d}x + \int_{\partial\Omega}\gamma(U_{\mathrm{I}+1}^N)U_{\mathrm{I}+1}^N v_{\mathrm{I}+1}\mathrm{d}s$$
$$= \frac{1}{\tau}\sum_{j=1}^{\mathrm{I}}\int_{\Omega} f_j v_j \mathrm{d}x + \frac{1}{\tau}\int_{\Omega} b(U_{\mathrm{I}+1}^N) f_{\mathrm{I}+1} v_{\mathrm{I}+1}\mathrm{d}x + \sum_{j=1}^{\mathrm{I}+2}\int_{\partial\Omega} g_j v_j \mathrm{d}s. \quad (32)$$

To pass to the limit in the variational equality (32) with N, when N tends to infinity, we can extract a subsequence, still denoted by $\mathbf{U}^N$, convergent to $\mathbf{U}$ weakly in $\mathbf{V}$ and strongly in $\mathbf{L}^2(\Omega)$ and in $\mathbf{L}^2(\partial\Omega)$. In particular, $\mathbf{U}^N$ converges to $\mathbf{U}$ a.e. in Ω and on $\partial\Omega$. Applying the Krasnoselski theorem to the Nemytskii operators b and L, we have

$$b(U_{\mathrm{I}+1}^N)v \overset{N\to\infty}{\longrightarrow} b(U_{\mathrm{I}+1})v \text{ in } L^2(\Omega); \quad (33)$$

$$\sum_{j=1}^{\mathrm{I}+2} L_{j,l}(\mathbf{u}^N)\nabla v_j \overset{N\to\infty}{\longrightarrow} \sum_{j=1}^{\mathrm{I}+2} L_{j,l}(\mathbf{u})\nabla v_j \quad \text{in } \mathbf{L}^2(\Omega), \quad (34)$$

for $l = 1, \dots, \mathrm{I}+1$, and for all $v, v_j \in H^1(\Omega)$, making use of the Lebesgue dominated convergence theorem with Assumptions (10)–(15). Similarly, the boundary term $\gamma(U_{\mathrm{I}+1}^N)v$ converges to $\gamma(U_{\mathrm{I}+1})v$ in $L^{\ell'}(\partial\Omega)$, for all $v \in L^{\ell'}(\partial\Omega)$, due to (16). Thus, we are in the condition of passing to the limit in the variational equality (32) as N tends to infinity to conclude that $\mathbf{U}$ is the required limit solution.

5 Passage to the limit as time goes to zero ($M \to +\infty$)

Set $\mathbf{X}_\ell = [V(\Omega)]^{\mathrm{I}} \times V_\ell(\Omega)$. Let $\widetilde{\mathbf{u}}^M :]0, T[\to \mathbf{X}_\ell$ and $\widetilde{\phi}^M :]0, T[\to V(\partial\Omega)$ be the step functions defined by

$$\widetilde{\mathbf{u}}^M(t) = \begin{cases} \mathbf{u}^0 & \text{if } t = 0 \\ \mathbf{u}^m & \text{if } t \in]t_{m-1,M}, t_{m,M}] \end{cases} \quad (35)$$
$$\widetilde{\phi}^M(t) = \phi^m \quad \text{if } t \in]t_{m-1,M}, t_{m,M}]. \quad (36)$$

We begin by establishing the estimates and the weak convergences of the Rothe function

$$(\widetilde{\mathbf{u}}^M, \widetilde{\phi}^M) = (\widetilde{u}_1^M, \dots, \widetilde{u}_{\mathrm{I}}^M, \widetilde{u}^M, \widetilde{\phi}^M)$$

obtained from the discretized solution $(\mathbf{u}^m, \phi^m)$, of variational system (22)–(24), by piecewise constant interpolation with respect to time t.

Proposition 2 *Denoting by* $\{(\widetilde{\mathbf{u}}^M, \widetilde{\phi}^M)\}_{M\in\mathbb{N}}$ *the Rothe sequence, then the following estimate hold, for* $M > T$,

$$\max_{1\leq m\leq M}\left(\sum_{i=1}^{\mathrm{I}}\|u_i^m\|_{2,\Omega}^2 + 2\int_\Omega \Psi(u^m)\mathrm{d}x\right)$$
$$+\sum_{i=1}^{\mathrm{I}}(L_i)_\#\|\nabla\widetilde{u}_i^M\|_{2,Q_T}^2 + (L_{\mathrm{I}+1})_\#\|\nabla\widetilde{u}^M\|_{2,Q_T}^2 + (L_{\mathrm{I}+2})_\#\|\nabla\widetilde{\phi}^M\|_{2,Q_T}^2$$
$$+2\frac{\gamma_\#}{\ell'}\|\widetilde{u}^M\|_{\ell,\Sigma_T}^\ell \leq \left(1+\frac{M}{M-T}\exp[T]\right)\mathcal{R}, \tag{37}$$

where

$$\mathcal{R} = \sum_{i=1}^{\mathrm{I}}\|u_i^0\|_{2,\Omega}^2 + 2b^\#\|u^0\|_{2,\Omega}^2 + T\frac{K_2^2(P_2+1)^2}{(L_{\mathrm{I}+2})_\#}\|g\|_{2,\Gamma_{\mathrm{N}}}^2$$
$$+K_2^2\sum_{i=1}^{\mathrm{I}}\left(1+\frac{1}{(L_i)_\#}\right)\|h_i\|_{2,\Sigma_T}^2 + \frac{1}{\ell'\gamma_\#^{1/(\ell-1)}}\|h\|_{\ell',\Sigma_T}^{\ell'}.$$

Moreover, there exists $(\mathbf{u},\phi) \in [L^2(0,T;V(\Omega))]^{\mathrm{I}} \times V_\ell(Q_T) \times L^2(0,T;V(\partial\Omega))$ *such that*

$$\widetilde{\mathbf{u}}^M \rightharpoonup \mathbf{u} \text{ in } [L^2(0,T;V(\Omega))]^{\mathrm{I}} \times V_\ell(Q_T) \hookrightarrow \mathbf{L}^2(0,T;\mathbf{X}_\ell);$$
$$\widetilde{\phi}^M \rightharpoonup \phi \text{ in } L^2(0,T;V(\partial\Omega))$$

as M tends to infinity (up to subsequences).

Proof Choosing $(\mathbf{v}, v) = \mathbf{u}^m$ and $w = \phi^m$ as test functions in (22)–(24), we sum the obtained relations, and we successively apply the Hölder inequality, to deduce

$$\frac{1}{\tau}\left(\sum_{i=1}^{\mathrm{I}}\int_\Omega\left(u_i^m - u_i^{m-1}\right)u_i^m\mathrm{d}x + \int_\Omega(B(u^m)-B(u^{m-1}))u^m\mathrm{d}x\right)$$
$$+\sum_{j=1}^{\mathrm{I}+1}(L_j)_\#\|\nabla u_j^m\|_{2,\Omega}^2 + (L_{\mathrm{I}+2})_\#\|\nabla\phi^m\|_{2,\Omega}^2 + \gamma_\#\|u^m\|_{\ell,\partial\Omega}^\ell$$
$$\leq \sum_{i=1}^{\mathrm{I}}\|\bar{h}_i^m\|_{2,\partial\Omega}\|u_i^m\|_{2,\partial\Omega} + \|\bar{h}_{\mathrm{I}+1}^m\|_{\ell',\partial\Omega}\|u^m\|_{\ell,\partial\Omega} + \|g\|_{2,\Gamma}\|\phi^m\|_{2,\Gamma}$$
$$:= \sum_{i=1}^{\mathrm{I}}\mathcal{I}_i + \mathcal{I}_{\mathrm{I}+1} + \mathcal{I}_{\mathrm{I}+2}, \tag{38}$$

for all $m \in \{1, \ldots, M\}$. We successively apply (31), and the Young inequality, to obtain

$$\mathcal{I}_i \leq \frac{K_2^2}{2}\left(1+\frac{1}{(L_i)_\#}\right)\|\bar{h}_i^m\|_{2,\partial\Omega}^2 + \frac{1}{2}\|u_i^m\|_{2,\Omega}^2 + \frac{(L_i)_\#}{2}\|\nabla u_i^m\|_{2,\Omega}^2;$$

$$\mathcal{I}_{\mathrm{I}+1} \leq \frac{1}{\ell' \gamma_\#^{1/(\ell-1)}}\|\bar{h}_{\mathrm{I}+1}^m\|_{\ell',\partial\Omega}^{\ell'} + \frac{\gamma_\#}{\ell}\|u^m\|_{\ell,\partial\Omega}^{\ell};$$

$$\mathcal{I}_{\mathrm{I}+2} \leq \frac{K_2^2(P_2+1)^2}{2(L_{\mathrm{I}+2})_\#}\|g\|_{2,\Gamma}^2 + \frac{(L_{\mathrm{I}+2})_\#}{2}\|\nabla\phi^m\|_{2,\Omega}^2.$$

Making recourse to the elementary identity $2(a-b)a = a^2 - b^2 + (a-b)^2$ for all $a, b \in \mathbb{R}$ to the first term on the left-hand side in (38), summing over $k = 1, \ldots, m$, we obtain

$$\sum_{k=1}^{m}\sum_{i=1}^{\mathrm{I}}\int_\Omega \left(u_i^k - u_i^{k-1}\right)u_i^k \mathrm{d}x \geq \frac{1}{2}\sum_{i=1}^{\mathrm{I}}\left(\|u_i^m|_{2,\Omega}^2 - \|u_i^0\|_{2,\Omega}^2\right).$$

Next, applying Lemma 1, we deduce for the second term on the left-hand side in (38)

$$\int_\Omega (B(u^m) - B(u^{m-1}))u^m \mathrm{d}x \geq \int_\Omega (\Psi(u^m) - \Psi(u^{m-1}))\mathrm{d}x.$$

Therefore, summing over $k = 1, \ldots, m$ into (38), inserting the above equalities, and multiplying by 2τ, we obtain

$$\sum_{i=1}^{\mathrm{I}}\|u_i^m\|_{2,\Omega}^2 + 2\int_\Omega \Psi(u^m)\mathrm{d}x + \tau\sum_{k=1}^{m}\left(\sum_{i=1}^{\mathrm{I}}(L_i)_\#\|\nabla u_i^k\|_{2,\Omega}^2\right.$$
$$\left. +2(L_{\mathrm{I}+1})_\#\|\nabla u^k\|_{2,\Omega}^2 + (L_{\mathrm{I}+2})_\#\|\nabla\phi^k\|_{2,\Omega}^2 + \frac{2\gamma_\#}{\ell'}\|u^k\|_{\ell,\partial\Omega}^{\ell}\right)$$
$$\leq \sum_{i=1}^{\mathrm{I}}\|u_i^0\|_{2,\Omega}^2 + 2\int_\Omega \Psi(u^0)\mathrm{d}x + \tau\sum_{k=1}^{m}\sum_{i=1}^{\mathrm{I}}\|u_i^k\|_{2,\Omega}^2$$
$$+\tau\sum_{k=1}^{m}\left(K_2^2\sum_{i=1}^{\mathrm{I}}\left(1+\frac{1}{(L_i)_\#}\right)\|\bar{h}_i^k\|_{2,\partial\Omega}^2 + \frac{2}{\ell'\gamma_\#^{1/(\ell-1)}}\|\bar{h}_{\mathrm{I}+1}^k\|_{\ell',\partial\Omega}^{\ell'}\right)$$
$$+\tau m\frac{K_2^2(P_2+1)^2}{(L_{\mathrm{I}+2})_\#}\|g\|_{2,\Gamma}^2.$$

In particular, the discrete Gronwall inequality (cf. Lemma 3), with $L = 1$ and $\tau = T/M < 1$, implies that

$$\sum_{i=1}^{\mathrm{I}}\|u_i^m\|_{2,\Omega}^2 \leq \frac{M\mathcal{R}}{M-T}\exp[T].$$

Taking the maximum over $m \in \{1, \ldots, M\}$, estimate (37) holds.

Thus, we can extract a subsequence, still denoted by $(\widetilde{\mathbf{u}}^M, \widetilde{\phi}^M)$, weakly convergent to $(\mathbf{u}, \phi) \in [L^2(0, T; V(\Omega))]^{\mathrm{I}} \times V_\ell(Q_T) \times L^2(0, T; V(\partial\Omega))$. □

Let us introduce some Rothe functions obtained by piecewise linear interpolation with respect to time t.

Definition 3 We say that $\{(\mathbf{U}^M, B^M)\}_{M\in\mathbb{N}}$ is the Rothe sequence if

$$U_i^M(x,t) = u_i^{m-1}(x) + \frac{t - t_{m-1,M}}{\tau}\left(u_i^m(x) - u_i^{m-1}(x)\right), \quad i = 1, \ldots, \mathrm{I};$$
$$B^M(x,t) = B(x, u^{m-1}(x)) + \frac{t - t_{m-1,M}}{\tau}\left(B(x, u^m(x)) - B(x, u^{m-1}(x))\right),$$

for all $(x, t) \in \Omega \times I_{m,M}$, $m \in \{1, \ldots, M\}$.

The discrete derivative with respect to the time has the following characterization.

Proposition 3 *Let* $\widetilde{\mathbf{Z}}^M : [0, T[\to [L^2(\Omega)]^{\mathrm{I}+1}$ *be defined by*

$$\widetilde{\mathbf{Z}}^M(t) = \begin{cases} \mathbf{Z}^0 & \text{if } t = 0 \\ \mathbf{Z}^m & \text{if } t \in]t_{m-1,M}, t_{m,M}] \end{cases} \quad \text{in } \Omega$$

with $\mathbf{Z}^0 = (u_1^0, \ldots, u_{\mathrm{I}}^0, B(u^0))$*, and the discrete derivative with respect to* t *at the time* $t = t_{m,M}$ *being such that*

$$Z_i^m := \frac{u_i^m - u_i^{m-1}}{\tau}, \quad i = 1, \ldots, \mathrm{I}; \tag{39}$$

$$Z_{\mathrm{I}+1}^m := \frac{B(u^m) - B(u^{m-1})}{\tau}. \tag{40}$$

Then there exists $\mathbf{Z} \in [L^2(0, T; (V(\Omega))')]^{\mathrm{I}} \times L^{\ell'}(0, T; (V_\ell(\Omega))')$ *such that*

$$\widetilde{\mathbf{Z}}^M \rightharpoonup \mathbf{Z} \quad \text{in } [L^2(0, T; (V(\Omega))')]^{\mathrm{I}} \times L^{\ell'}(0, T; (V_\ell(\Omega))'). \tag{41}$$

Proof Let $\{(\mathbf{U}^M, B^M)\}_{M\in\mathbb{N}}$ be the Rothe sequence in accordance with Definition 3. For $i = 1, \ldots, \mathrm{I}$, by definition of norm, we have

$$\|\partial_t U_i^M\|_{L^2(0,T;(V(\Omega))')} = \sup_{\substack{v \in L^2(0,T;V(\Omega)) \\ \|v\| \le 1}} \sum_{m=1}^M \int_{(m-1)\tau}^{m\tau} \langle Z_i^m, v \rangle \mathrm{d}t.$$

Applying Proposition 2 to equality (22) being rewritten as

$$\int_\Omega Z_i^m v \mathrm{d}x = \int_{\partial\Omega} \bar{h}_i^m v \mathrm{d}s - \sum_{j=1}^{\mathrm{I}+1} \int_\Omega a_{i,j}(\mathbf{u}^m) \nabla u_j^m \cdot \nabla v \mathrm{d}x - \int_\Omega F_i(\mathbf{u}^m) \nabla \phi^m \cdot \nabla v \mathrm{d}x,$$

we conclude

$$\|\partial_t U_i^M\|_{L^2(0,T;(V(\Omega))')} \leq C,$$

with $C > 0$ being a constant independent on M. Analogously, applying Proposition 2 to equality (23) we find

$$\|\partial_t B^M\|_{L^{\ell'}(0,T;(V_\ell(\Omega))')} = \sup_{\substack{v\in L^\ell(0,T;V_\ell(\Omega))\\ \|v\|\leq 1}} \sum_{m=1}^{M} \int_{(m-1)\tau}^{m\tau} \langle Z^m_{\mathrm{I}+1}, v\rangle \mathrm{d}t \leq C,$$

with $C > 0$ being a constant independent on M.

Hence, we can extract a subsequence, still denoted by $\widetilde{\mathbf{Z}}^M$, weakly convergent to $\mathbf{Z} \in [L^2(0,T;(V(\Omega))')]^{\mathrm{I}} \times L^{\ell'}(0,T;(V_\ell(\Omega))')$. □

In the following proposition, we state some strong convergences that allow, up to a subsequence, a.e. pointwise convergences.

Proposition 4 *Let $(\widetilde{\mathbf{u}}^M, \widetilde{\phi}^M)$ be according to Proposition 2. Under (10)–(14) and (16), for a subsequence, there hold*

$$\widetilde{\mathbf{u}}^M \to \mathbf{u} \ \ in \ \ \mathbf{L}^2(Q_T), \tag{42}$$

$$B(\widetilde{u}^M) \to B(u) \ \ in \ \ L^1(Q_T), \tag{43}$$

as M tends to infinity. Also, $\widetilde{u}^M$ strongly converges to u in $\mathbf{L}^2(\Sigma_T)$.

Proof To prove (42), we make recourse to the discrete version of the Aubin–Lions theorem 2. Thanks to Proposition 2, we have

$$\|\widetilde{\mathbf{u}}^M\|^2_{L^2(0,T;\mathbf{X}_\ell(\Omega))} \leq T \sup_{t\in]0,T[} \sum_{i=1}^{\mathrm{I}} \|\widetilde{u}_i^M\|^2_{2,\Omega} + \|\widetilde{u}^M\|^\ell_{\ell,\Sigma_T} + \|\nabla\widetilde{\mathbf{u}}^M\|^2_{2,Q_T} \leq C,$$

with $C > 0$ being a constant independent on M.

For a fixed $t \in]0,T[$, there exists $m \in \{1,\ldots,M\}$ such that $t \in]t_{m-1,M}, t_{m,M}]$. For $i = 1,\ldots,\mathrm{I}$, by applying (10) and (14) to (22)–(23), we deduce

$$\|u_i^m - u_i^{m-1}\|_{V'(\Omega)} \leq \tau \sup_{v\in V(\Omega):\ \|v\|=1} \Big(\|\bar{h}_i^m\|_{2,\partial\Omega}\|v\|_{2,\partial\Omega} + \left(\max(a^\#_{ij})\|\nabla\mathbf{u}^m\|_{2,\Omega} + \max(F^\#_j)\|\nabla\phi^m\|_{2,\Omega}\right)\|\nabla v\|_{2,\Omega}\Big).$$

While for $i = \mathrm{I}+1$, by applying (12), (26), (10), (14) and (16), we deduce

$$\|u^m - u^{m-1}\|_{V'_\ell(\Omega)} \leq \frac{\tau}{b_\#} \sup_{v\in V_\ell(\Omega):\ \|v\|=1} \Big(\|\bar{h}^m_{\mathrm{I}+1}\|_{2,\partial\Omega}\|v\|_{2,\partial\Omega} + \left(\max(a^\#_{ij})\|\nabla\mathbf{u}^m\|_{2,\Omega} + \max(F^\#_j)\|\nabla\phi^m\|_{2,\Omega}\right)\|\nabla v\|_{2,\Omega} + \|(\gamma^\#|u^m|^{\ell-2} + \gamma_1)u^m\|_{\ell',\partial\Omega}\|v\|_{\ell,\partial\Omega}\Big).$$

Applying Proposition 2, we find

$$\tau^{-1}\int_\tau^T \|\widetilde{u}_i^M - \widetilde{u}_i^{M-1}\|_{V'(\Omega)}\mathrm{d}t = \sum_{k=1}^M \|\widetilde{u}_i^k - \widetilde{u}_i^{k-1}\|_{V'(\Omega)} \leq C;$$
$$\tau^{-1}\int_\tau^T \|\widetilde{u}^M - \widetilde{u}^{M-1}\|_{V'_\ell(\Omega)}\mathrm{d}t = \sum_{k=1}^M \|\widetilde{u}^k - \widetilde{u}^{k-1}\|_{V'_\ell(\Omega)} \leq C,$$

with $C > 0$ being constants independent of M. Taking the Kondrachov–Sobolev embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ and $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\partial\Omega)$, we conclude the proof of strong convergences of $\widetilde{\mathbf{u}}^M$ due to the Aubin–Lions theorem 2.

To prove the convergence (43), we will apply Lemma 2. Considering the weak convergence of $\widetilde{u}^M$ established in Proposition 2 and estimate (37) to apply Lemma 2 it remains to prove that the condition (27) is fulfilled. Let $0 < z < T$ be arbitrary. Since the objective is to find convergences, it suffices to take $M > T/z$, which means $\tau < z$. Thus, there exists $k \in \mathbb{N}$ such that $k\tau < z \leq (k+1)\tau$. Moreover, we may choose $M > k+1$ deducing

$$\int_0^{T-z}\int_\Omega (B(\widetilde{u}^M(t+z)) - B(\widetilde{u}^M(t)))(\widetilde{u}^M(t+z) - \widetilde{u}^M(t))\mathrm{d}x\mathrm{d}t \leq$$
$$\leq \sum_{l=1}^{M-k}\int_{(l-1)\tau}^{(l+k)\tau}\int_\Omega (B(u^{l+k}) - B(u^l))(u^{l+k} - u^l)\mathrm{d}x. \tag{44}$$

Let us sum up (23) for $m = l+1,\ldots,l+k$ and multiply by τ, obtaining

$$\int_\Omega (B(u^{l+k}) - B(u^l))v\mathrm{d}x \leq \mathcal{I}^l_{\partial\Omega} + \mathcal{I}^l_\Omega, \tag{45}$$

where

$$\mathcal{I}^l_{\partial\Omega} := \tau\sum_{m=l+1}^{l+k}\int_{\partial\Omega} |(\gamma(u^m)u^m - \bar{h}^m_{\mathrm{I}+1})v|\mathrm{d}s;$$
$$\mathcal{I}^l_\Omega := \tau\sum_{m=l+1}^{l+k}\int_\Omega \left|\left(\sum_{j=1}^{\mathrm{I}+1} a_{\mathrm{I}+1,j}(\mathbf{u}^m)\nabla u_j^m + F_{\mathrm{I}+1}(\mathbf{u}^m)\nabla\phi^m\right)\cdot\nabla v\right|\mathrm{d}x.$$

Applying the Hölder inequality and using Assumptions (16), (14) and (10), we deduce

$$\mathcal{I}^l_{\partial\Omega} \leq \int_{l\tau}^{(l+k)\tau}\left(\gamma^\#\|\widetilde{u}^M\|^{\ell-1}_{\ell,\partial\Omega} + \gamma_1\|\widetilde{u}^M\|_{\ell',\partial\Omega} + \|h_{\mathrm{I}+1}\|_{\ell',\partial\Omega}\right)\|v\|_{\ell,\partial\Omega}\mathrm{d}t;$$
$$\mathcal{I}^l_\Omega \leq \int_{l\tau}^{(l+k)\tau}\left(\sum_{j=1}^{\mathrm{I}+1} a^\#_{\mathrm{I}+1,j}\|\nabla\widetilde{u}_j^M\|_{2,\Omega} + F^\#_{\mathrm{I}+1}\|\nabla\phi^M\|_{2,\Omega}\right)\|\nabla v\|_{2,\Omega}\mathrm{d}t.$$

Making use of the Hölder inequality and estimate (37) in the above inequalities, we conclude from (45)

$$\int_\Omega (B(u^{l+k}) - B(u^l))v\mathrm{d}x \leq \|v\|_{\ell,\partial\Omega} C(k\tau)^{1/\ell} + \|\nabla v\|_{2,\Omega} C\sqrt{k\tau}.$$

Taking $v = u^{l+k} - u^l$ in the above inequality, first gathering with (44), second, applying the Hölder inequality and after estimate (37), we obtain

$$\begin{aligned}
&\int_0^{T-z}\int_\Omega (B(\widetilde{u}^M(t+z)) - B(\widetilde{u}^M(t)))(\widetilde{u}^M(t+z) - \widetilde{u}^M(t))\mathrm{d}x\mathrm{d}t \\
&\leq \sum_{l=1}^{M-k}\int_{(l-1)\tau}^{(l+k)\tau}\left(\|u^{l+k} - u^l\|_{\ell,\partial\Omega} C(k\tau)^{1/\ell} + \|\nabla(u^{l+k} - u^l)\|_{2,\Omega} C\sqrt{k\tau}\right) \\
&\leq C\left((k\tau)^{1/\ell}(k\tau+\tau)^{1/\ell'} + (k\tau)^{1/2}(k\tau+\tau)^{1/2}\right) = C\left(2^{1/\ell'} + 2^{1/2}\right) z,
\end{aligned}$$

which implies (27).

Thus, all hypotheses of Lemma 2 are fulfilled. Therefore, Lemma 2 assures that $B(u^M)$ strongly converges to $B(u)$ in $L^1(Q_T)$, which concludes the proof of (43). □

Proposition 5 *If* $\mathbf{Z}$ *satisfies Proposition 3, then*

$$\mathbf{Z} = \partial_t\,(\mathbf{u}, B(u)) \ \text{ in } [L^2(0,T;(V(\Omega))')]^{\mathrm{I}} \times L^{\ell'}(0,T;(V_\ell(\Omega))')$$

in the weak sense (cf. Remark 2).

Proof Let $t \in]0, T[$ be arbitrary, but a fixed number. Thus, there exists $m \in \{1, \dots, M\}$ such that $t \in]t_{m-1,M}, t_{m,M}]$. For $j = 1, \dots, \mathrm{I}+1$, we have

$$\begin{aligned}
\int_0^t \widetilde{Z}_j^M(z)\mathrm{d}z &= \sum_{k=1}^{m-1}\int_{(k-1)\tau}^{k\tau} Z_j^k \mathrm{d}z + \int_{(m-1)\tau}^{t} Z_j^m \mathrm{d}z \\
&= \tau\sum_{k=1}^{m-1} Z_j^k + (t-(m-1)\tau)Z_j^m \quad \text{in } \Omega.
\end{aligned}$$

From Definitions (39)–(40), we have

$$\int_0^t \widetilde{\mathbf{Z}}^M(z)\mathrm{d}z = \begin{cases} U_j^M(t) - u_j^0 & \text{for } j = 1,\dots,\mathrm{I} \\ B^M(t) - B(u^0) & \text{for } j = \mathrm{I}+1 \end{cases}.$$

The bounded linear functional $\mathbf{v} \in \mathbf{L}^2(\Omega) \mapsto \int_0^t (\widetilde{\mathbf{Z}}^M(z), \mathbf{v})\mathrm{d}z$ is (uniquely) representable by the element $\left(\mathbf{U}^M - \mathbf{u}^0, B^M - B(u^0)\right)$ from $\mathbf{L}^2(\Omega)$ due to the Riesz theorem.

Observing that by the application of the change of variables, we have

$$\int_0^T \int_\Omega u(x,t-\tau)v(x,t)\mathrm{d}x\mathrm{d}t = \int_{-\tau}^{T-\tau} \int_\Omega u(x,t)v(x,t+\tau)\mathrm{d}x\mathrm{d}t,$$

for every $u, v \in L^2(Q_T)$, we find, for $i = 1, \dots, \mathrm{I}$,

$$\begin{aligned}
\mathcal{J}_i^M &:= \int_0^T \int_\Omega \widetilde{Z}_i^M v\mathrm{d}x\mathrm{d}t = \frac{1}{\tau}\left(\int_{T-\tau}^T \int_\Omega u_i^M(x)v(x,t)\mathrm{d}x\mathrm{d}t\right. \\
&\left. - \int_0^{T-\tau} \int_\Omega \widetilde{u}_i^M(x,t)\triangle_\tau v(x,t)\mathrm{d}x\mathrm{d}t - \int_{-\tau}^0 \int_\Omega u_i^0(x)v(x,t+\tau)\mathrm{d}x\mathrm{d}t\right); \\
\mathcal{J}_{\mathrm{I}+1}^M &:= \int_0^T \int_\Omega \widetilde{Z}_{\mathrm{I}+1}^M v\mathrm{d}x\mathrm{d}t = \frac{1}{\tau}\left(\int_{T-\tau}^T \int_\Omega B(u^M)(x)v(x,t)\mathrm{d}x\mathrm{d}t\right. \\
&\left. - \int_0^{T-\tau} \int_\Omega B(\widetilde{u}^M)(x,t)\triangle_\tau v(x,t)\mathrm{d}x\mathrm{d}t - \int_{-\tau}^0 \int_\Omega B(u^0)(x)v(x,t+\tau)\mathrm{d}x\mathrm{d}t\right),
\end{aligned}$$

where $\triangle_\tau v(x,t) = v(x,t+\tau) - v(x,t)$ for a.e. $(x,t) \in Q_T$.

The objective is to pass to the limit $\mathcal{J}_j^M$, for $j = 1, \dots, \mathrm{I}+1$, as M tends to infinity. To this end, each term is separately evaluated.

First, the weak convergence (41) assures that

$$\mathcal{J}^M \overset{M\to\infty}{\longrightarrow} \langle \mathbf{Z}, \mathbf{v}\rangle,$$

for all $\mathbf{v} \in [L^2(0,T;V(\Omega))]^{\mathrm{I}} \times L^\ell(0,T;V_\ell(\Omega))$.

Considering that $\|v(T)\|_{2,\Omega} = 0$, we evaluate the following term as follows:

$$\frac{1}{\tau}\left|\int_{T-\tau}^T \int_\Omega u_i^M(x)v(x,t)\mathrm{d}x\mathrm{d}t\right| \le \|u_i^M\|_{2,\Omega}\frac{1}{\tau}\int_{T-\tau}^T \|v\|_{2,\Omega}\mathrm{d}t \overset{M\to\infty}{\longrightarrow} 0$$

with Proposition 2 ensuring the uniform boundedness of u_i^M in $L^2(\Omega)$. Considering that $\|v(T)\|_{\infty,\Omega} = 0$ and that Proposition 4 ensures the uniform boundedness of $B(u^M)$ in $L^1(\Omega)$, the similar following term is evaluated as follows:

$$\frac{1}{\tau}\left|\int_{T-\tau}^T \int_\Omega B(u^M)(x)v(x,t)\mathrm{d}x\mathrm{d}t\right| \le \|B(u^M)\|_{1,\Omega}\frac{1}{\tau}\int_{T-\tau}^T \|v\|_{\infty,\Omega}\mathrm{d}t \overset{\tau\to 0}{\longrightarrow} 0.$$

The difference quotient $\triangle_\tau/\tau$ approximates the time derivative ∂_t, that is, $\triangle_\tau v/\tau \to \partial_t v$ a.e. in Q_T as τ tends to zero. Moreover, it verifies

$$\|\triangle_\tau v\|_{L^1(\tau,T;X)} \le \|\partial_t v\|_{L^1(0,T;X)},$$

with X being a Banach space, whenever $\partial_t v \in L^1(0,T;X)$. Thanks to Proposition 4, up to a subsequence $\widetilde{\mathbf{u}}^M \to \mathbf{u}$ and $B(u^M) \to B(u)$ a.e. in Q_T. Hence, there hold

$$\frac{1}{\tau}\int_0^{T-\tau}\int_\Omega \widetilde{u}_i^M(x,t)\triangle_\tau v_i(x,t)\mathrm{d}x\mathrm{d}t \overset{M\to\infty}{\longrightarrow} \int_0^T\int_\Omega u_i\partial_t v_i\mathrm{d}x\mathrm{d}t,$$
$$\frac{1}{\tau}\int_0^{T-\tau}\int_\Omega B(\widetilde{u}^M)(x,t)\triangle_\tau v(x,t)\mathrm{d}x\mathrm{d}t \overset{M\to\infty}{\longrightarrow} \int_0^T\int_\Omega B(u)\partial_t v\mathrm{d}x\mathrm{d}t,$$

for all $v_i, v \in L^2(0,T;H^1(\Omega))$ such that $\partial_t v_i \in L^2(Q_T)$ and $\partial_t v \in L^\infty(Q_T)$.

For $v \in W^{1,1}(0,T;L^2(\Omega)) \hookrightarrow C([0,T];L^2(\Omega))$, we have

$$\frac{1}{\tau}\int_{-\tau}^0 v(t+\tau)\mathrm{d}t = \frac{1}{\tau}\int_0^\tau v(t)\mathrm{d}t \overset{\tau\to 0}{\longrightarrow} v(0) \quad \text{in } L^2(\Omega).$$

Therefore, we find (20) and (21). □

Finally, we are in a condition to establish the passage to the limit as time goes to zero ($M \to +\infty$) in the Neumann–Robin elliptic problems (22)–(24).

Proposition 6 *Let $(\mathbf{u},\phi)$ be in accordance with Proposition 2, then the pair solves (7)–(9), i.e., it is the required solution to Theorem 1.*

Proof Let $(\widetilde{\mathbf{u}}^M, \widetilde{\phi}^M)$ the corresponding Rothe sequence of the steady-state solutions to the variational system (22)–(24). For each $M \in \mathbb{N}$, it satisfies

$$\int_0^T \langle Z_i^M, v_i\rangle\mathrm{d}t + \sum_{j=1}^{\mathrm{I}+1}\int_{Q_T} a_{i,j}(\widetilde{\mathbf{u}}^M)\nabla\widetilde{u}_j^M\cdot\nabla v_i\mathrm{d}x\mathrm{d}t$$
$$= -\int_{Q_T} F_i(\widetilde{\mathbf{u}}^M)\nabla\widetilde{\phi}^M\cdot\nabla v_i\mathrm{d}x\mathrm{d}t + \int_{\Sigma_T} h_i^M v_i\mathrm{d}s\mathrm{d}t, \quad i=1,\dots,\mathrm{I}, \tag{46}$$

$$\int_0^T \langle Z^M, v\rangle\mathrm{d}t + \sum_{j=1}^{\mathrm{I}+1}\int_{Q_T} a_{\mathrm{I}+1,j}(\widetilde{\mathbf{u}}^M)\nabla\widetilde{u}_j^M\cdot\nabla v\mathrm{d}x\mathrm{d}t + \int_{\Sigma_T}\gamma(\widetilde{u}^M)\widetilde{u}^M v\mathrm{d}s\mathrm{d}t$$
$$= -\int_{Q_T} F_{\mathrm{I}+1}(\widetilde{\mathbf{u}}^M)\nabla\widetilde{\phi}^M\cdot\nabla v\mathrm{d}x\mathrm{d}t + \int_{\Sigma_T} h_{\mathrm{I}+1}^M v\mathrm{d}s\mathrm{d}t, \tag{47}$$

$$\int_{Q_T}\sigma(\widetilde{\mathbf{u}}^M)\nabla\widetilde{\phi}^M\cdot\nabla w\mathrm{d}x = -\sum_{j=1}^{\mathrm{I}+1}\int_{Q_T} G_j(\widetilde{\mathbf{u}}^M)\nabla\widetilde{u}_j^M\cdot\nabla w\mathrm{d}x + \int_0^T\int_\Gamma gw\mathrm{d}s, \tag{48}$$

for all $v_i \in L^2(0,T;V(\Omega))$, $v \in V_\ell(Q_T)$, and $w \in V(\partial\Omega)$.

Applying Proposition 4, and the Krasnoselski theorem to the Nemytskii operators A, $\mathbf{F}$, $\mathbf{G}$, and σ, we have

$$\begin{aligned}
&a_{i,j}(\widetilde{\mathbf{u}}^M)\nabla v \longrightarrow a_{i,j}(\mathbf{u})\nabla v \text{ in } \mathbf{L}^2(Q_T);\\
&F_j(\widetilde{\mathbf{u}}^M)\nabla v \longrightarrow F_j(\mathbf{u})\nabla v \text{ in } \mathbf{L}^2(Q_T);\\
&G_j(\widetilde{\mathbf{u}}^M)\nabla v \longrightarrow G_j(\mathbf{u})\nabla v \text{ in } \mathbf{L}^2(Q_T);\\
&\sigma(\widetilde{\mathbf{u}}^M)\nabla v \longrightarrow \sigma(\mathbf{u})\nabla v \text{ in } \mathbf{L}^2(Q_T) \quad \text{as } M \to +\infty,
\end{aligned}$$

for every $i, j = 1, \dots, \mathrm{I}+1$, and for all $v \in H^1(\Omega)$. Thanks to Propositions 2, 3 and 5, we may pass to the limit in (46) and (48), as M tends to infinity, concluding that u_i and ϕ verify, respectively, (7), for $i = 1, \dots, \mathrm{I}$, and (9).

Similar argument is valid to pass to the limit in (47), considering that

$$\nabla(\widetilde{\mathbf{u}}^M, \widetilde{\phi}^M) \rightharpoonup \nabla(\mathbf{u}, \phi) \quad \text{in} \quad \mathbf{L}^2(Q_T), \tag{49}$$

$$\gamma(\widetilde{u}^M)v \to \gamma(u)v \quad \text{in} \quad L^{\ell/(\ell-1)}(\Gamma_{\mathrm{w}} \times]0,T[), \tag{50}$$

$$\widetilde{u}^M \rightharpoonup u \quad \text{in} \quad L^{\ell}(\Gamma_{\mathrm{w}} \times]0,T[), \tag{51}$$

and that $\gamma(\widetilde{u}^M)v$ strongly converges to $\gamma(u)v$ in $L^2(\Gamma \times]0,T[)$, which corresponds to the Robin-type boundary condition ($\ell = 2$). □

6 Application example

The domain Ω stands for the representation of electrolysis cells (see Fig. 1). Electrolysis of metals are well known for lead bromide, magnesium chloride, potassium chloride, sodium chloride, and zinc chloride, to mention a few.

The phenomenological fluxes $\mathbf{q}$, $\mathbf{J}_i$ and $\mathbf{j}$ are, respectively, the measurable heat flux (in W m^{-2}), the ionic flux of component i (in mol m^{-2} s^{-1}), and the electric current density (in C m^{-2} s^{-1}), and they are explicitly driven by gradients of the temperature θ, the molar concentration vector $\mathbf{c} = (c_1, \dots, c_{\mathrm{I}})$, and the electric potential ϕ, in the form (up to some temperature- and concentration-dependent factors): [1,2,9,16,30,31]

$$\mathbf{q} = -k(\theta)\nabla\theta - R\theta^2 \sum_{i=1}^{\mathrm{I}} D_i'(c_i,\theta)\nabla c_i - \Pi(\theta)\sigma(\mathbf{c},\theta)\nabla\phi, \tag{52}$$

$$\mathbf{J}_i = -c_i S_i(c_i,\theta)\nabla\theta - D_i(\theta)\nabla c_i - u_i c_i \nabla\phi, \quad (i = 1, \dots, \mathrm{I}), \tag{53}$$

$$\mathbf{j} = -\alpha_{\mathrm{S}}(\theta)\sigma(\mathbf{c},\theta)\nabla\theta - F\sum_{i=1}^{\mathrm{I}} z_i D_i(\theta)\nabla c_i - \sigma(\mathbf{c},\theta)\nabla\phi. \tag{54}$$

It includes the Fourier law (with the thermal conductivity k), the Fick's law (with the diffusion coefficient D_i), the Ohm's law (with the electrical conductivity σ), the Peltier–Seebeck cross effect (with the Peltier coefficient Π and the Seebeck coefficient α_{S} being correlated by the first Kelvin relation), and the Dufour–Soret cross effect (with the Dufour coefficient D_i' and the Soret coefficient S_i). Hereafter the subscript i stands

Table 1 Universal constants

F	Faraday constant	9.6485×10^4 C mol^{-1}
R	Gas constant	8.314 J mol^{-1}K^{-1}
σ_{SB}	Stefan–Boltzmann constant (for blackbodies)	5.67×10^{-8} W m^{-2}K^{-4}

for the correspondence to the ionic component i intervened in the reaction process. Table 1 displays the universal constants R and F.

Every ionic mobility $u_i = z_i D_i F/(R\theta)$ satisfies the Nernst–Einstein relation $\sigma_i = F z_i u_i c_i$, with $\sigma_i = t_i \sigma$ representing ionic conductivity, and t_i is the transference number (or transport number) of species i. Indeed, the electrical conductivity is a function of temperature and concentration vector as reported in the Debye and Hückel theory [11]. After several approximation attempts [18], the most accepted approximation is the Debye–Hückel–Onsager equation. The thermal conductivity of the electrodes can significantly vary from sample to sample due to the variability in manufacturing techniques, carbon paper grades and amounts of particular compounds. The thermal conductivity is frequently estimated to be in the range 0.1–1.6 W m^{-1} K^{-1}, based on the material composition. In particular, the thermal conductivity of nonmetallic liquids under normal conditions is much lower than that of metals and ranges from 0.1 to 0.6 W m^{-1} K^{-1}, while the thermal conductivity of liquids may change by a factor of 1.1 to 1.6, in the interval between the melting point and the boiling point.

Let $T > 0$ be an arbitrary (but preassigned) time. From the conservation of energy, the mass balance equations, and the conservation of electric charge, we derive, respectively, in $Q_T = \Omega \times]0, T[$

$$\rho c_{\mathrm{v}} \frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{q} = 0; \tag{55}$$

$$\frac{\partial c_i}{\partial t} + \nabla \cdot \mathbf{J}_i = 0; \tag{56}$$

$$\nabla \cdot \mathbf{j} = 0, \tag{57}$$

where the density ρ and the specific heat capacity c_{v} (at constant volume) are assumed to be dependent on temperature and space variable. The absence of external forces, assumed in (55)–(57), is due to their occurrence at the surface of the electrodes Γ_l ($l = \mathrm{a, c}$), i.e., for a.e. in $]0, T[$,

$$\mathbf{q} \cdot \mathbf{n}_l = h_{\mathrm{C}}(\theta - \theta_l), \quad -F z_i \mathbf{J}_i \cdot \mathbf{n}_l = g_{i,l}, \quad -\mathbf{j} \cdot \mathbf{n} = g, \tag{58}$$

where h_{C} denotes the conductive heat transfer coefficient, θ_l denotes a prescribed surface temperature, $g_{i,l}$ may represent a truncated version of the Butler–Volmer expression (cf. [8,9] and the references therein), and g denotes a prescribed surface electric current assumed to be tangent to the surface for all $t > 0$.

The parabolic–elliptic system (55)–(57) is accomplished by (58) and the remaining boundary conditions. For a.e. in $]0, T[$, we consider

$$\mathbf{q}\cdot\mathbf{n} = h_\mathrm{R}|\theta|^{\ell-2}\theta - h \quad \text{on } \Gamma_\mathrm{w}, \tag{59}$$
$$\mathbf{q}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_\mathrm{o}, \tag{60}$$
$$\mathbf{J}_i\cdot\mathbf{n} = \mathbf{j}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_\mathrm{w}\cup\Gamma_\mathrm{o}, \qquad (i = 1,\dots,I). \tag{61}$$

The radiative condition (59), with a general exponent $\ell \geq 2$ [9] and h_R denoting the radiative heat transfer coefficient that may depend both on the space variable and the temperature function θ, accounts, for instance, for the radiation behavior of the heavy water electrolysis, namely the Stefan–Boltzmann radiation law if $\ell = 5$, i.e., $h_\mathrm{R} = \sigma_\mathrm{SB}\epsilon$, and $h = \sigma_\mathrm{SB}\alpha\theta_\mathrm{w}^4$. The parameters, ϵ and α, represent the emissivity and the absorptivity, respectively, θ_w denotes a prescribed wall surface temperature, and σ_SB stands for Stefan–Boltzmann constant for blackbodies (cf. Table 1).

Definition 4 We call by the thermoelectrochemical (TEC) problem, the finding of the temperature–concentration–potential triplet $(\theta, \mathbf{c}, \phi)$ satisfying (55)–(57), under (52)–(54), accomplished with (58)–(61), and the initial conditions $\theta(0) = \theta_0$ and $\mathbf{c}(0) = \mathbf{c}^0$ in Ω.

We assume

(A1) The coefficients ρ and c_v are assumed to be Carathéodory functions from $\Omega\times\mathbb{R}$ into $\mathbb{R}$. Moreover, there exist $b_\#, b^\# > 0$ such that

$$b_\# \leq \rho(x,e)c_\mathrm{v}(x,e) \leq b^\#,$$

for a.e. $x \in \Omega$, and for all $e \in \mathbb{R}$. Although the specific heat coefficient of most liquid metals for which data are available is negative, it is positive at high temperatures, and often invariant with temperature.

(A2) The electrical and thermal conductivities, Peltier, Seebeck, Soret, Dufour, and diffusion coefficients $\sigma, k, \Pi, \alpha, S_i, D_i', D_i$ $(i = 1,\dots,\mathrm{I})$ are Carathéodory functions such that verify (15),

$$\begin{aligned}
\exists k_\#, k^\# > 0: &\qquad k_\# \leq k(x,e) \leq k^\#,\\
\exists \Pi^\# > 0: &\qquad |\Pi(x,e)| \leq \Pi^\#,\\
\exists \alpha^\# > 0: &\qquad |\alpha_\mathrm{S}(x,e)| \leq \alpha^\#,\\
\exists S_i^\# > 0: &\qquad |dS_i(x,d,e)| \leq S_i^\#,\\
\exists (D_i')^\# > 0: &\qquad Re^2|D_i'(x,d,e)| \leq (D_i')^\#,\\
\exists D_i^\# > 0: &\qquad F|z_i|D_i(x,e) \leq D_i^\#,\\
\exists (D_i)_\# > 0: &\qquad D_i(x,e) \geq (D_i)_\#,
\end{aligned}$$

for a.e. $x \in \Omega$, and for all $d, e \in \mathbb{R}$.

(A3) The transference coefficient $t_i \in L^\infty(\Omega)$ is such that

$$\exists t_i^\# > 0: \quad 0 \leq t_i(x) \leq F|z_i|t_i^\#, \quad \text{for a.e. } x \in \Omega.$$

(A4) The boundary operator h_R is a Carathéodory function from $\Gamma_\mathrm{w} \times \mathbb{R}$ to $\mathbb{R}$ such that it verifies

$$\exists \gamma_\#, \gamma^\# > 0 : \quad \gamma_\# \leq h_\mathrm{R}(x, e) \leq \gamma^\# \quad \text{for a.e. } x \in \Gamma_\mathrm{w}, \quad \forall e \in \mathbb{R}.$$

(A5) The boundary function h_C is measurable from $\Gamma \times]0, T[$ into $\mathbb{R}$ satisfying

$$\exists h_\#, h^\# > 0 : \qquad h_\# \leq h_\mathrm{C}(x) \leq h^\#, \quad \text{for a.e. } x \in \Gamma.$$

(A6) $g \in L^2(\Gamma)$, $h \in L^{\ell/(\ell-1)}(\Gamma_\mathrm{w} \times]0, T[)$, $\theta_\mathrm{a} \in L^2(\Gamma_\mathrm{a} \times]0, T[)$, and $\theta_\mathrm{c} \in L^2(\Gamma_\mathrm{c} \times]0, T[)$.

(A7) For each $i = 1, \ldots, \mathrm{I}$, $g_{i,\mathrm{a}}$ and $g_{i,\mathrm{c}}$ belong to $L^2(\Gamma_\mathrm{a} \times]0, T[)$ and $L^2(\Gamma_\mathrm{c} \times]0, T[)$, respectively.

(A8) $\theta_0, c_i^0 \in L^2(\Omega)$, $i = 1, \ldots, \mathrm{I}$.

The main result of existence to the TEC problem is the following theorem.

Theorem 3 *Let Assumptions (A1)–(A8) be fulfilled. In addition, suppose that the smallness conditions*

$$(D_i)_\# > \frac{1}{2}\left(S_i^\# + (D_i')^\# + t_i^\# \sigma^\# + D_i^\#\right), \quad i = 1, \ldots, \mathrm{I}, \tag{62}$$

$$k_\# > \frac{1}{2}\left(\sum_{j=1}^{\mathrm{I}} \left[S_j^\# + (D_j')^\#\right] + \Pi^\# \sigma^\# + \alpha^\# \sigma^\#\right), \tag{63}$$

$$\sigma_\# > \frac{1}{2}\left(\sum_{j=1}^{\mathrm{I}} (t_j^\# \sigma^\# + D_j^\#) + (\Pi^\# + \alpha^\#)\sigma^\#\right) \tag{64}$$

hold. Then there exists at least one weak solution to the TEC problem in the following sense:

$$\int_0^T \langle \partial_t c_i, v_i \rangle \mathrm{d}t + \int_{Q_T} D_i(c_i, \theta) \nabla c_i \cdot \nabla v_i \mathrm{d}x\mathrm{d}t = \sum_{l=\mathrm{a,c}} \int_0^T \int_{\Gamma_l} g_{i,l} v_i \mathrm{d}s\mathrm{d}t$$
$$- \int_{Q_T} \left(c_i S_i(c_i, \theta) \nabla \theta + t_i (F z_i)^{-1} \sigma(\mathbf{c}, \theta) \nabla \phi\right) \cdot \nabla v_i \mathrm{d}x\mathrm{d}t, \quad i = 1, \ldots, \mathrm{I},$$
$$\int_0^T \langle \rho(\theta) c_\mathrm{v}(\theta) \partial_t \theta, v \rangle \mathrm{d}t + \int_{Q_T} k(\theta) \nabla \theta \cdot \nabla v \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Gamma h_\mathrm{C} \theta v \mathrm{d}s\mathrm{d}t$$
$$+ \int_0^T \int_{\Gamma_\mathrm{w}} h_\mathrm{R}(\theta) |\theta|^{\ell-2} \theta v \mathrm{d}s\mathrm{d}t = \sum_{l=\mathrm{a,c}} \int_0^T \int_{\Gamma_l} h_\mathrm{C} \theta_l v \mathrm{d}s\mathrm{d}t + \int_0^T \int_{\Gamma_\mathrm{w}} h v \mathrm{d}s\mathrm{d}t$$
$$- \int_{Q_T} \left(R\theta^2 \sum_{j=1}^{\mathrm{I}} D_j'(c_j, \theta) \nabla c_j + \Pi(\theta) \sigma(\mathbf{c}, \theta) \nabla \phi\right) \cdot \nabla v \mathrm{d}x\mathrm{d}t,$$

$$\int_\Omega \sigma(\mathbf{c},\theta)\nabla\phi\cdot\nabla w \mathrm{d}x = \int_\Gamma g w \mathrm{d}s$$
$$-\int_\Omega \left(\alpha_\mathrm{S}(\theta)\sigma(\mathbf{c},\theta)\nabla\theta + F\sum_{j=1}^{\mathrm{I}} z_j D_j(c_j,\theta)\nabla c_j\right)\cdot\nabla w \mathrm{d}x,\ \textit{a.e. in }]0,T[,$$

for all $v_i \in L^2(0,T;V(\Omega))$*,* $v \in V_\ell(Q_T)$*, and* $w \in V(\partial\Omega)$ *where the time derivative is understood in accordance with Remark 2.*

Proof The existence of weak solutions to the TEC problem is a consequence of Theorem 1, under $u_i = c_i$, $i = 1,\dots,\mathrm{I}$ and $u_{\mathrm{I}+1} = \theta$. The explicit forms of the transport coefficients are $b = \rho c_\mathrm{v}$,

$$a_{i,j}(\mathbf{c},\theta) = \begin{cases} D_i(c_i,\theta)\delta_{i,j} & \text{if } 1 \le i, j \le \mathrm{I} \\ c_i S_i(c_i,\theta) & \text{if } 1 \le i \le \mathrm{I}, \quad j = \mathrm{I}+1 \\ R\theta^2 D'_j(c_j,\theta) & \text{if } i = \mathrm{I}+1, \quad 1 \le j \le \mathrm{I} \\ k(\theta) & \text{if } i = \mathrm{I}+1, \quad j = \mathrm{I}+1, \end{cases}$$
$$F_j(\mathbf{c},\theta) = \begin{cases} t_j(Fz_j)^{-1}\sigma(\mathbf{c},\theta) & \text{if } 1 \le j \le \mathrm{I} \\ \Pi(\theta)\sigma(\mathbf{c},\theta) & \text{if } j = \mathrm{I}+1, \end{cases}$$
$$G_j(\mathbf{c},\theta) = \begin{cases} Fz_j D_j(c_j,\theta) & \text{if } 1 \le j \le \mathrm{I} \\ \alpha_\mathrm{S}(\theta)\sigma(\mathbf{c},\theta) & \text{if } j = \mathrm{I}+1. \end{cases}$$

Assumption (A1) is exactly (H2). Assumptions (A2) and (A3) imply (H1) with

$$F_j^\# = \begin{cases} t_j^\#\sigma^\# & \text{if } 1 \le j \le \mathrm{I} \\ \Pi^\#\sigma^\# & \text{if } j = \mathrm{I}+1 \end{cases}$$
$$G_j^\# = \begin{cases} D_j^\# & \text{if } 1 \le j \le \mathrm{I} \\ \alpha^\#\sigma^\# & \text{if } j = \mathrm{I}+1. \end{cases}$$

Assumption (A2) implies (H4) and (H3) with

$$(a_i)_\# = \begin{cases} (D_i)_\# & \text{if } 1 \le i \le \mathrm{I} \\ k_\# & \text{if } i = \mathrm{I}+1 \end{cases}$$
$$a_{i,j}^\# = \begin{cases} D_i^\#/(F|z_i|)\delta_{i,j} & \text{if } 1 \le i, j \le \mathrm{I} \\ S_i^\# & \text{if } 1 \le i \le \mathrm{I}, \quad j = \mathrm{I}+1 \\ (D'_j)^\# & \text{if } i = \mathrm{I}+1, \quad 1 \le j \le \mathrm{I} \\ k^\# & \text{if } i = \mathrm{I}+1, \quad j = \mathrm{I}+1. \end{cases}$$

Moreover, considering in Sect. 4

$$(L_i)_\# = (D_i)_\# - \frac{1}{2}\left(S_i^\# + (D_i')^\# + F_i^\# + G_i^\#\right), \quad i = 1, \dots, \mathrm{I},$$

$$(L_{\mathrm{I}+1})_\# = k_\# - \frac{1}{2}\left(\sum_{j=1}^{\mathrm{I}}\left[S_j^\# + (D_j')^\#\right] + F_{\mathrm{I}+1}^\# + G_{\mathrm{I}+1}^\#\right),$$

$$(L_{\mathrm{I}+2})_\# = \sigma_\# - \frac{1}{2}\sum_{j=1}^{\mathrm{I}+1}\left(F_j^\# + G_j^\#\right),$$

the smallness conditions (18)–(19) read (62)–(63).

Finally, Assumptions (A4) and (A5) fulfill (H5) with

$$\gamma(x, e) = \begin{cases} h_{\mathrm{C}}(x) & \text{if } x \in \Gamma \\ h_{\mathrm{R}}(x, e)|e|^{\ell-2} & \text{if } x \in \Gamma_{\mathrm{w}} \\ 0 & \text{otherwise} \end{cases},$$

for all $e \in \mathbb{R}$, and Assumptions (A5)–(A8) fulfill the remaining hypothesis of Theorem 1. □

References

1. Adams, E.M., McDonald, I.R., Singer, K.: Collective dynamical properties of molten salts: molecular dynamics calculations on sodium chloride. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **357**(1688), 37–57 (1977)
2. Agar, J.N., Turner, J.C.R.: Thermal diffusion in solutions of electrolytes. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **255**(1282), 307–330 (1960)
3. Alphonse, A., Elliott, C.M., Stinner, B.: An abstract framework for parabolic PDEs on evolving spaces. Port. Math. (N.S.) **72**(1), 1–46 (2015)
4. Alt, H.W., Luckhaus, S.: Quasilinear elliptic-parabolic differential equations. Math. Z. **183**, 311–341 (1983)
5. Buckingham, M., Marken, F., Aldous, L.: The thermoelectrochemistry of the aqueous iron(II)/iron(III) redox couple: significance of the anion and pH in thermogalvanic thermal-to-electrical energy conversion. Sustain. Energy Fuels (2018). https://doi.org/10.1039/C8SE00416A
6. Cheung, C.-Y., Menictas, C., Bao, J., Skyllas-Kazacos, M., Welch, B.J.: Spatial thermal condition in aluminum reduction cells under influences of electrolyte flow. Chem. Eng. Res. Des. **100**, 1–14 (2015)
7. Consiglieri, L.: On the posedness of thermoelectrochemical coupled systems. Eur. Phys. J. Plus **128**(5), 47 (2013)
8. Consiglieri, L.: Sufficient conditions to the existence for solutions of a thermoelectrochemical problem. J. Fixed Point Theory Appl. **17**(4), 669–692 (2015)
9. Consiglieri, L.: Quantitative Estimates on Boundary Value Problems. Smallness Conditions to Thermoelectric and Thermoelectrochemical Problems. LAP LAMBERT Academic Publishing, Saarbrücken (2017)
10. Consiglieri, L.: Weak solutions for a thermoelectric problem with power-type boundary effects. Boll. Unione Mat. Ital. **11**(4), 595–619 (2018)
11. Debye, Von P., Hückel, E.: Zur theorie der elektrolyte. I. Gefrierpunktserniedrigung und verwandte erscheinungen (The theory of electrolytes. I. Lowering of freezing point and related phenomena). Physikalische Zeitschrift **24**(9), 185–206 (1923)
12. Douglas Jr., J., Dupont, T.: Galerkin methods for parabolic equations. SIAM J. Numer. Anal. **7**(4), 575–626 (1970)

13. Dreher, M., Jüngel, A.: Compact families of piecewise constant functions in $L^p(0, T; B)$. Nonlinear Anal. **75**, 3072–3077 (2012)
14. Duan, J., Feng, G., Yu, B., Li, J., Chen, M., Yang, P., Feng, J., Kang, L., Zhou, J.: Aqueous thermogalvanic cells with a high Seebeck coefficient for low-grade heat harvest. Nat. Commun. (2018). https://doi.org/10.1038/s41467-018-07625-9
15. El-Azab, M.S., Ashour, A.A.: Rothe's method to nonlinear parabolic problems with a nonlinear boundary condition. Int. J. Differ. Equ. Appl. **9**(3), 193–212 (2004)
16. Eslahian, K.A., Majee, A., Maskos, M., Würger, A.: Specific salt effects on thermophoresis of charged colloids. Soft Matter **10**, 1931–1936 (2014)
17. Gudi, T., Pani, A.K.: Discontinuous Galerkin methods for quasi-linear elliptic problems of nonmonotone type. SIAM J. Numer. Anal. **45**(1), 163–192 (2007)
18. Harned, H.S., Owen, B.B.: The Physical Chemistry of Electrolytic Solutions. Reinhold Publishing Corporation, New York (1943)
19. Hasan, S.W., Said, S.M., Sabri, M.F.M., Bakar, A.S.A., Hashim, N.A., Hasnan, M.M.I.M., Pringle, J.M., MacFarlane, D.R.: High thermal gradient in thermo-electrochemical cells by insertion of a poly(vinylidene fluoride) membrane. Sci. Rep. **6**, 29328 (2016). https://doi.org/10.1038/srep29328
20. Jin, L., Greene, G., MacFarlane, D.R., Pringle, J.: Redox-active quasi-solid-state electrolytes for thermal energy harvesting. ACS Energy Lett. **1**, 654–658 (2016)
21. Kačur, J.: Application of Rothe's method to nonlinear evolution equations. Mat. Čas. **25**(1), 63–81 (1975)
22. Kačur, J.: Solution to strongly nonlinear parabolic problems by a linear approximation scheme. IMA J. Numer. Anal. **19**, 119–145 (1999)
23. Kačur, J., Makmood, M.S.: Galerkin characteristics method for convection–diffusion problems with memory terms. Int. J. Numer. Anal. Model. **6**(1), 89–109 (2009)
24. Kuusi, T., Monsaingeon, L., Videman, J.H.: Systems of partial differential equations in porous medium. Nonlinear Anal. **133**, 79–101 (2016)
25. Leray, J., Lions, J.L.: Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. Bull. Soc. Math. Fr. **93**, 97–107 (1965)
26. López-Pouso, Ó., Muñoz-Sola, R.: About the solution of the even parity formulation of the transient radiative heat transfer equations. RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A. Mat. **104**(1), 129–152 (2010)
27. Pluschke, V.: Rothe's method for parabolic problems with nonlinear degenerating coefficient. Martin-Luther-University Halle, Department of Mathematics. Report No. 14 (1996)
28. Roubíček, T.: Nonlinear partial differential equations with applications. Birkhäuser Verlag, Basel (2005)
29. Sun, S., Wheeler, M.F.: Discontinuous Galerkin methods for coupled flow and reactive transport problems. Appl. Numer. Math. **52**, 273–298 (2005)
30. Wilson, J.R.: The structure of liquid metals and alloys. Metall. Rev. **10**(40), 381–590 (1965)
31. Wu, J., Black, J.J., Aldous, L.: Thermoelectrochemistry using conventional and novel gelled electrolytes in heat-to-current thermocells. Electrochem. Acta **225**, 482–492 (2017)