



A comprehensive subclass of bi-univalent functions associated with Chebyshev polynomials of the second kind

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Abstract

Our objective in this paper is to introduce and investigate a newly constructed subclass of normalized analytic and bi-univalent functions by means of the Chebyshev polynomials of the second kind. Upper bounds for the second and third Taylor–Maclaurin coefficients, and also Fekete–Szegő inequalities of functions belonging to this subclass are founded. Several connections to some of the earlier known results are also pointed out.

Keywords Analytic functions · Univalent and bi-univalent functions · Taylor–Maclaurin series · Fekete–Szegő problem · Chebyshev polynomials · Coefficient bounds · Subordination

Mathematics Subject Classification Primary 30C45; Secondary 30C50

1 Introduction, definitions and notations

Let \mathcal{A} denote the class of all analytic functions f defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$. Thus each $f \in \mathcal{A}$ has a Taylor–Maclaurin series expansion of the form:

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \quad (1.1)$$

Further, let \mathcal{S} denote the class of all functions $f \in \mathcal{A}$ which are univalent in \mathbb{U} (for details, see [16]; see also some of the recent investigations [1,4–8]).

Given two functions f and g in \mathcal{A} . The function f is said to be subordinate to g in \mathbb{U} , written as $f(z) \prec g(z)$, if there exists a Schwartz function $\omega(z)$, analytic in \mathbb{U} , with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \text{ for all } z \in \mathbb{U},$$

such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$. Furthermore, if the function g is univalent in \mathbb{U} , then we have the following equivalence (see [22,31]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Two of the important and well-investigated subclasses of the analytic and univalent function class \mathcal{S} are the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in \mathbb{U} and the class $\mathcal{K}(\alpha)$ of convex functions of order α in \mathbb{U} . By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}, \quad (1.2)$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}. \quad (1.3)$$

It is clear from the definitions (1.2) and (1.3) that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$. Also we have

$$f(z) \in \mathcal{K}(\alpha) \text{ iff } zf'(z) \in \mathcal{S}^*(\alpha),$$

and

$$f(z) \in \mathcal{S}^*(\alpha) \text{ iff } \int_0^z \frac{f(t)}{t} dt = F(z) \in \mathcal{K}(\alpha).$$

It is well known [16] that every function $f \in \mathcal{S}$ has an inverse map f^{-1} that satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.4)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{U} if both $f(z)$ and $f^{-1}(z)$ are univalent in \mathbb{U} . Let Σ denote the class of bi-univalent functions in \mathbb{U} given by (1.1). For a brief history and some interesting examples of functions and characterization of the class Σ , see Srivastava et al. [27], Frasin and Aouf [17], and Magesh and Yamini [20].

In 1967, Lewin [18] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [10] conjectured that $|a_2| \leq \sqrt{2}$. Later, Netanyahu [23] showed that $\max |a_2| = \frac{4}{3}$ if $f \in \Sigma$. Brannan and Taha [11] introduced certain subclasses of a bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α ($0 \leq \alpha < 1$), respectively (see [9]). Thus, following the works of Brannan and Taha [11], for $0 \leq \alpha < 1$, a function $f \in \Sigma$ is in the class $\mathcal{S}_\Sigma^*(\alpha)$ of bi-starlike functions of order α ; or $\mathcal{K}_\Sigma(\alpha)$ of bi-convex functions of order α if both f and f^{-1} are, respectively, starlike or convex functions of order α . Recently, many researchers have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. In fact, the aforementioned work of Srivastava et al. [27] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [17], Xu et al. [28], Çağlar et al. [14], and others (see, for example, [2, 19, 24–26, 29]). The coefficient estimate problem for each of the following Taylor–Maclaurin coefficients $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$) for each $f \in \Sigma$ given by (1.1) is still an open problem.

The Chebyshev polynomials are a sequence of orthogonal polynomials that are related to De Moivre’s formula and which can be defined recursively. They have abundant properties, which make them useful in many areas in applied mathematics, numerical analysis and approximation theory. There are four kinds of Chebyshev polynomials, see for details Doha [15] and Mason [21]. The Chebyshev polynomials of degree n of the second kind, which are denoted $U_n(t)$, are defined for $t \in [-1, 1]$ by the following three-terms recurrence relation:

$$\begin{aligned} U_0(t) &= 1, \\ U_1(t) &= 2t, \\ U_{n+1}(t) &:= 2tU_n(t) - U_{n-1}(t). \end{aligned}$$

The first few of the Chebyshev polynomials of the second kind are

$$U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, \dots \quad (1.5)$$

The generating function for the Chebyshev polynomials of the second kind, $U_n(t)$, is given by:

$$H(z, t) = \frac{1}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} U_n(t)z^n \quad (z \in \mathbb{U}).$$

Yousef et al. [30] introduced the following class $\mathcal{B}_{\Sigma}^{\mu}(\beta, \lambda, \delta)$ of analytic and bi-univalent functions defined as follows.

Definition 1.1 For $\lambda \geq 1, \mu \geq 0, \delta \geq 0$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{\mu}(\beta, \lambda, \delta)$ if the following conditions hold for all $z, w \in \mathbb{U}$:

$$\operatorname{Re} \left((1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) \right) > \beta \quad (1.6)$$

and

$$\operatorname{Re} \left((1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) \right) > \beta, \quad (1.7)$$

where the function $g(w) = f^{-1}(w)$ is defined by (1.4) and $\xi = \frac{2\lambda + \mu}{2\lambda + 1}$.

This work is concerned with the coefficient bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and the Fekete–Szegő inequality for functions belonging to the class $\mathcal{B}_{\Sigma}^{\mu}(\lambda, \delta, t)$ defined as follows.

Definition 1.2 For $\lambda \geq 1, \mu \geq 0, \delta \geq 0$ and $t \in (\frac{1}{2}, 1)$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{\mu}(\lambda, \delta, t)$ if the following subordinations hold for all $z, w \in \mathbb{U}$:

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) \prec H(z, t) := \frac{1}{1 - 2tz + z^2} \quad (1.8)$$

and

$$(1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) \prec H(w, t) := \frac{1}{1 - 2tw + w^2}, \quad (1.9)$$

where the function $g(w) = f^{-1}(w)$ is defined by (1.4) and $\xi = \frac{2\lambda + \mu}{2\lambda + 1}$.

The following special cases of Definition 1.2 are worthy of note.

Remark 1.3 Note that for $\lambda = 1, \mu = 1$ and $\delta = 0$, the class of functions $\mathcal{B}_{\Sigma}^1(1, 0, t) := \mathcal{B}_{\Sigma}(t)$ have been introduced and studied by Altinkaya and Yalçın [3], for $\mu = 1$ and $\delta = 0$, the class of functions $\mathcal{B}_{\Sigma}^1(\lambda, 0, t) := \mathcal{B}_{\Sigma}(\lambda, t)$ have been introduced and studied by Bulut et al. [12], for $\delta = 0$, the class of functions

$\mathcal{B}_\Sigma^\mu(\lambda, 0, t) := \mathcal{B}_\Sigma^\mu(\lambda, t)$ have been introduced and studied by Bulut et al. [13], and for $\mu = 1$, the class of functions $\mathcal{B}_\Sigma^1(\lambda, \delta, t) := \mathcal{B}_\Sigma(\lambda, \delta, t)$ have been introduced and studied by Yousef et al. [32].

2 Coefficient bounds for the function class $\mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$

In this section, we establish coefficient bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ of the function $f \in \mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$. Several corollaries of the main result are also considered.

Theorem 2.1 *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(\lambda + \mu + 2\xi\delta)^2 - 2[2(\lambda + \mu + 2\xi\delta)^2 - (2\lambda + \mu)(\mu + 1) - 12\xi\delta]t^2|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{4t^2}{(\lambda + \mu + 2\xi\delta)^2} + \frac{2t}{2\lambda + \mu + 6\xi\delta}. \tag{2.2}$$

Proof Let $f \in \mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$. From (1.8) and (1.9), we have

$$\begin{aligned} (1 - \lambda) \left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1} + \xi \delta z f''(z) \\ = 1 + U_1(t)w(z) + U_2(t)w^2(z) + \dots \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} (1 - \lambda) \left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w}\right)^{\mu-1} + \xi \delta w g''(w) \\ = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \dots, \end{aligned} \tag{2.4}$$

for some analytic functions

$$w(z) = c_1z + c_2z^2 + c_3z^3 + \dots \quad (z \in \mathbb{U}),$$

and

$$v(w) = d_1w + d_2w^2 + d_3w^3 + \dots \quad (w \in \mathbb{U}),$$

such that $w(0) = v(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$) and $|v(w)| < 1$ ($w \in \mathbb{U}$).

It follows from (2.3) and (2.4) that

$$\begin{aligned} (1 - \lambda) \left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z}\right)^{\mu-1} + \xi \delta z f''(z) \\ = 1 + U_1(t)c_1z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \dots \end{aligned}$$

and

$$\begin{aligned} & (1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) \\ & = 1 + U_1(t) d_1 w + \left[U_1(t) d_2 + U_2(t) d_1^2 \right] w^2 + \dots \end{aligned}$$

Equating the coefficients yields

$$(\lambda + \mu + 2\xi\delta) a_2 = U_1(t) c_1, \quad (2.5)$$

$$(2\lambda + \mu) \left[\left(\frac{\mu - 1}{2} \right) a_2^2 + \left(1 + \frac{6\delta}{2\lambda + 1} \right) a_3 \right] = U_1(t) c_2 + U_2(t) c_1^2, \quad (2.6)$$

and

$$-(\lambda + \mu + 2\xi\delta) a_2 = U_1(t) d_1, \quad (2.7)$$

$$(2\lambda + \mu) \left[\left(\frac{\mu + 3}{2} + \frac{12\delta}{2\lambda + 1} \right) a_2^2 - \left(1 + \frac{6\delta}{2\lambda + 1} \right) a_3 \right] = U_1(t) d_2 + U_2(t) d_1^2. \quad (2.8)$$

From (2.5) and (2.7), we obtain

$$c_1 = -d_1, \quad (2.9)$$

and

$$2(\lambda + \mu + 2\xi\delta)^2 a_2^2 = U_1^2(t) (c_1^2 + d_1^2). \quad (2.10)$$

By adding (2.6) to (2.8), we get

$$(2\lambda + \mu) \left[1 + \mu + \frac{12\delta}{2\lambda + 1} \right] a_2^2 = U_1(t) (c_2 + d_2) + U_2(t) (c_1^2 + d_1^2). \quad (2.11)$$

By using (2.10) in (2.11), we obtain

$$\left[(2\lambda + \mu)(\mu + 1) + 12\xi\delta - \frac{2U_2(t) (\lambda + \mu + 2\xi\delta)^2}{U_1^2(t)} \right] a_2^2 = U_1(t) (c_2 + d_2). \quad (2.12)$$

It is fairly well known [16] that if $|w(z)| < 1$ and $|v(w)| < 1$, then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \quad (2.13)$$

By considering (1.5) and (2.13), we get from (2.12) the desired inequality (2.1).

Next, by subtracting (2.8) from (2.6), we have

$$\begin{aligned}
 &2(2\lambda + \mu) \left(1 + \frac{6\delta}{2\lambda + 1}\right) a_3 - 2(2\lambda + \mu) \left(1 + \frac{6\delta}{2\lambda + 1}\right) a_2^2 \\
 &= U_1(t) (c_2 - d_2) + U_2(t) (c_1^2 - d_1^2).
 \end{aligned}
 \tag{2.14}$$

Further, in view of (2.9), it follows from (2.14) that

$$a_3 = a_2^2 + \frac{U_1(t)}{2(2\lambda + \mu + 6\xi\delta)} (c_2 - d_2).
 \tag{2.15}$$

By considering (2.10) and (2.13), we get from (2.15) the desired inequality (2.2). This completes the proof of Theorem 2.1. \square

Taking $\lambda = 1, \mu = 1$ and $\delta = 0$ in Theorem 2.1, we get the following consequence.

Corollary 2.2 [12] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{1-t^2}},$$

and

$$|a_3| \leq t^2 + \frac{2}{3}t.$$

Taking $\mu = 1$ and $\delta = 0$ in Theorem 2.1, we get the following consequence.

Corollary 2.3 [12] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\lambda, t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(\lambda + 1)^2 - 4\lambda^2 t^2|}}$$

and

$$|a_3| \leq \frac{4t^2}{(\lambda + 1)^2} + \frac{2t}{2\lambda + 1}.$$

Taking $\delta = 0$ in Theorem 2.1, we get the following consequence.

Corollary 2.4 [13] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^\mu(\lambda, t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(\lambda + \mu)^2 - 2[2(\lambda + \mu)^2 - (2\lambda + \mu)(\mu + 1)]t^2|}}$$

and

$$|a_3| \leq \frac{4t^2}{(\lambda + \mu)^2} + \frac{2t}{2\lambda + \mu}.$$

Taking $\mu = 1$ in Theorem 2.1, we get the following consequence.

Corollary 2.5 [32] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\lambda, \delta, t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(1 + \lambda + 2\delta)^2 - 4[(\lambda + 2\delta)^2 - 2\delta]|t^2}}$$

and

$$|a_3| \leq \frac{4t^2}{(1 + \lambda + 2\delta)^2} + \frac{2t}{1 + 2\lambda + 6\delta}.$$

3 Fekete–Szegő problem for the function class $\mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$

Now, we are ready to find the sharp bounds of Fekete–Szegő functional $a_3 - \eta a_2^2$ defined for $\mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$ given by (1.1). The results presented in this section improve or generalize the earlier results of Bulut et al. [13], Yousef et al. [32], and other authors in terms of the ranges of the parameter under consideration.

Theorem 3.1 *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$. Then for some $\eta \in \mathbb{R}$,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2t}{2\lambda + \mu + 6\xi\delta}, & |\eta - 1| \leq M \\ \frac{8|\eta - 1|t^3}{|(\lambda + \mu + 2\xi\delta)^2 - 2[2(\lambda + \mu + 2\xi\delta)^2 - ((2\lambda + \mu)(\mu + 1) + 12\xi\delta)]t^2|}, & |\eta - 1| \geq M \end{cases} \tag{3.1}$$

where

$$M := \frac{|(\lambda + \mu + 2\xi\delta)^2 - 2[2(\lambda + \mu + 2\xi\delta)^2 - ((2\lambda + \mu)(\mu + 1) + 12\xi\delta)]t^2|}{4(2\lambda + \mu + 2\xi\delta)t^2}.$$

Proof Let $f \in \mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$. By using (2.12) and (2.15) for some $\eta \in \mathbb{R}$, we get

$$\begin{aligned} a_3 - \eta a_2^2 &= (1 - \eta) \left[\frac{U_1^3(t)(c_2 + d_2)}{((2\lambda + \mu)(\mu + 1) + 12\xi\delta)U_1^2(t) - 2(\lambda + \mu + 2\xi\delta)^2U_2(t)} \right] \\ &\quad + \frac{U_1(t)(c_2 - d_2)}{2(2\lambda + \mu + 6\xi\delta)} \\ &= U_1(t) \left[\left(h(\eta) + \frac{1}{2(2\lambda + \mu + 6\xi\delta)} \right) c_2 + \left(h(\eta) - \frac{1}{2(2\lambda + \mu + 6\xi\delta)} \right) d_2 \right], \end{aligned}$$

where

$$h(\eta) = \frac{U_1^2(t)(1 - \eta)}{((2\lambda + \mu)(\mu + 1) + 12\xi\delta)U_1^2(t) - 2(\lambda + \mu + 2\xi\delta)^2U_2(t)}.$$

Then, in view of (1.5), we easily conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2t}{2\lambda + \mu + 6\xi\delta}, & |h(\eta)| \leq \frac{1}{2(2\lambda + \mu + 6\xi\delta)} \\ 4|h(\eta)|t, & |h(\eta)| \geq \frac{1}{2(2\lambda + \mu + 6\xi\delta)} \end{cases}$$

This proves Theorem 3.1. □

We end this section with some corollaries concerning the sharp bounds of Fekete–Szegő functional $a_3 - \eta a_2^2$ defined for $f \in \mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$ given by (1.1).

Taking $\eta = 1$ in Theorem 3.1, we get the following corollary.

Corollary 3.2 *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^\mu(\lambda, \delta, t)$. Then*

$$|a_3 - a_2^2| \leq \frac{2t}{2\lambda + \mu + 6\xi\delta}.$$

Taking $\lambda = 1, \mu = 1$ and $\delta = 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.3 [13] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(t)$. Then for some $\eta \in \mathbb{R}$,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2}{3}t, & |\eta - 1| \leq \frac{1-t^2}{3t^2} \\ \frac{2|\eta-1|t^3}{1-t^2}, & |\eta - 1| \geq \frac{1-t^2}{3t^2} \end{cases}$$

Taking $\eta = 1$ in Corollary 3.3, we get the following corollary.

Corollary 3.4 [32] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(t)$. Then*

$$|a_3 - a_2^2| \leq \frac{2}{3}t.$$

Taking $\mu = 1$ and $\delta = 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.5 [13] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\lambda, t)$. Then for some $\eta \in \mathbb{R}$,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2t}{1+2\lambda}, & |\eta - 1| \leq \frac{|(1+\lambda)^2 - 4\lambda^2 t^2|}{4(1+2\lambda)t^2} \\ \frac{8|\eta-1|t^3}{|(1+\lambda)^2 - 4\lambda^2 t^2|}, & |\eta - 1| \geq \frac{|(1+\lambda)^2 - 4\lambda^2 t^2|}{4(1+2\lambda)t^2} \end{cases} \tag{3.2}$$

Taking $\eta = 1$ in Corollary 3.5, we get the following corollary.

Corollary 3.6 [32] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\lambda, t)$. Then*

$$|a_3 - a_2^2| \leq \frac{2t}{1 + 2\lambda}.$$

Taking $\delta = 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.7 [13] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^\mu(\lambda, t)$. Then for some $\eta \in \mathbb{R}$,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2t}{2\lambda + \mu}, & |\eta - 1| \leq \frac{[(\lambda + \mu)^2 - 2[2(\lambda + \mu)^2 - (2\lambda + \mu)(\mu + 1)]t^2]}{4(2\lambda + \mu)t^2} \\ \frac{8|\eta - 1|t^3}{|(\lambda + \mu)^2 - 2[2(\lambda + \mu)^2 - (2\lambda + \mu)(\mu + 1)]t^2|}, & |\eta - 1| \geq \frac{[(\lambda + \mu)^2 - 2[2(\lambda + \mu)^2 - (2\lambda + \mu)(\mu + 1)]t^2]}{4(2\lambda + \mu)t^2} \end{cases} \quad (3.3)$$

Taking $\mu = 1$ in Theorem 3.1, we get the following corollary.

Corollary 3.8 [32] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\lambda, \delta, t)$. Then for some $\eta \in \mathbb{R}$,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2t}{1 + 2\lambda + 6\delta}, & |\eta - 1| \leq \frac{[(1 + \lambda + 2\delta)^2 - 4[(\lambda + 2\delta)^2 - 2\delta]t^2]}{4(1 + 2\lambda + 6\delta)t^2} \\ \frac{8|\eta - 1|t^3}{|(1 + \lambda + 2\delta)^2 - 4[(\lambda + 2\delta)^2 - 2\delta]t^2|}, & |\eta - 1| \geq \frac{[(1 + \lambda + 2\delta)^2 - 4[(\lambda + 2\delta)^2 - 2\delta]t^2]}{4(1 + 2\lambda + 6\delta)t^2} \end{cases} \quad (3.4)$$

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