Boletín de la Sociedad Matemática Mexicana (2019) 25:509–528
https://doi.org/10.1007/s40590-018-0220-z

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

On degenerate Apostol-type polynomials and applications

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Received: 28 May 2018 / Accepted: 23 October 2018 / Published online: 8 November 2018

Abstract

The main object of the current paper is to introduce and investigate a new unified class of the degenerate Apostol-type polynomials. These polynomials are studied by means of the generating function, series definition and are framed within the context of monomiality principle. Several important recurrence relations and explicit representations for the antecedent class of polynomials are derived. As the special cases, the degenerate Apostol–Bernoulli, Euler and Genocchi polynomials are obtained and corresponding results are also proved. A fascinating example is constructed in terms of truncated-exponential polynomials, which gives the applications of these polynomials to produce their hybridized forms.

Keywords Apostol-type polynomials · Degenerate Apostol-type polynomials · Quasi-monomiality · Recurrence relation · Explicit representations

Mathematics Subject Classification 11B83 · 11B68 · 65D20 · 65Q30

1 Introduction and preliminaries

On the subject of the Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials and their various extensions, a remarkably large number of investigations have appeared in the literature, see for example [7,12,13]. Many authors achieve certain enthralling results including various relatives of the Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials.

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Recently, many researchers began to study various kinds of degenerate versions of the familiar polynomials like Bernoulli, Euler, falling factorial and Bell polynomials by using generating functions, umbral calculus, and p-adic integrals, see for example [2,8,9,11]. We recall the following definitions:

Definition 1.1 The degenerate Bernoulli polynomials $\beta_n(x;\lambda)$ are defined by means of the following generating function [2]:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\,(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(x;\lambda)\frac{t^n}{n!}. \tag{1.1}$$

When $x = 0$, $\beta_n(\lambda) := \beta_n(0;\lambda)$ are the corresponding degenerate Bernoulli numbers. It is to be noted from Eq. (1.1) that

$$\lim_{\lambda\to 0}\beta_n(x;\lambda)=B_n(x),\quad n\geq 0,$$

where $B_n(x)$ is the n-th order ordinary Bernoulli polynomials [17].

Definition 1.2 The degenerate Euler polynomials $\mathcal{E}_n(x;\lambda)$ are defined by means of the following generating function [9]:

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\,(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_n(x;\lambda)\frac{t^n}{n!}. \tag{1.2}$$

When $x = 0$, $\mathcal{E}_n(\lambda) := \mathcal{E}_n(0;\lambda)$ are the corresponding degenerate Euler numbers. It is to be noted from Eq. (1.2) that

$$\lim_{\lambda\to 0}\mathcal{E}_n(x;\lambda)=E_n(x),\quad n\geq 0,$$

where $E_n(x)$ is the n-th order ordinary Euler polynomials [17].

Definition 1.3 The degenerate Genocchi polynomials $\mathcal{G}_n(x;\lambda)$ are defined by means of the following generating function [11]:

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\,(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{G}_n(x;\lambda)\frac{t^n}{n!}. \tag{1.3}$$

When $x = 0$, $\mathcal{G}_n(\lambda) := \mathcal{G}_n(0;\lambda)$ are the corresponding degenerate Genocchi numbers. It is to be noted from Eq. (1.3) that

$$\lim_{\lambda\to 0}\mathcal{G}_n(x;\lambda)=G_n(x),\quad n\geq 0,$$

where $G_n(x)$ is the n-th order ordinary Genocchi polynomials [18].

We can also find various types of captivating researches on the subject of the Apostol-type polynomials and their properties and generalizations, see, for example, [5,7,12–15].

Motivated by the above-cited work on Apostol-type polynomials in this paper, a unified class of the degenerate Apostol-type polynomials is introduced and studied by means of the generating function, series definition and monomiality principle. Several important recurrence relations and explicit representations for these polynomials are derived. As the special cases, the degenerate Apostol–Bernoulli, Euler and Genocchi polynomials are obtained and corresponding results are proved. An example is constructed in terms of truncated-exponential polynomials to give the applications of main results.

2 Degenerate Apostol-type polynomials

In this section, we introduce a unified class of the degenerate Apostol-type polynomials. Certain properties and explicit formulae for these polynomials are also derived. We give the following definition:

Definition 2.1 Let $x \in \mathbb{R};\ \gamma, \mu, \nu \in \mathbb{C}$ and $n \in \mathbb{N}_0$. The degenerate Apostol-type polynomials denoted by $\mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu)$ of order α are defined by means of the following generating function:

$$\left(\frac{2^\mu t^\nu}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^\alpha (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $\mathcal{P}_n^{(\alpha)}(\lambda; \gamma, \mu, \nu) := \mathcal{P}_n^{(\alpha)}(0; \lambda; \gamma, \mu, \nu)$ are the corresponding degenerate Apostol-type numbers of order α and defined as:

$$\left(\frac{2^\mu t^\nu}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^\alpha = \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha)}(\lambda; \gamma, \mu, \nu)\frac{t^n}{n!}. \tag{2.2}$$

Remark 2.1 In view of Eq. (2.1), we remark that

$$\lim_{\lambda\to 0} \mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu) = \mathcal{F}_n^{(\alpha)}(x; \gamma, \mu, \nu), \quad n \geq 0, \tag{2.3}$$

where $\mathcal{F}_n^{(\alpha)}(x; \gamma, \mu, \nu)$ are the Apostol-type polynomials of order α [14] (see also [16]).

It should be noted that the degenerate Apostol-type polynomials $\mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu)$ include the following special cases:

Remark 2.2 For the special case $\gamma \to -\gamma;\ \mu = 0$ and $\nu = 1$ and on use of relation

$$(-1)^\alpha \mathcal{P}_n^{(\alpha)}(x; \lambda; -\gamma, 0, 1) = \mathfrak{B}_n^{(\alpha)}(x; \lambda; \gamma),$$

we have the degenerate Apostol–Bernoulli polynomials defined by

$$\left(\frac{t}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)\frac{t^n}{n!}, \tag{2.4}$$

where $\mathfrak{B}_n^{(\alpha)}(0;\lambda;\gamma)=:\mathfrak{B}_n^{(\alpha)}(\lambda;\gamma)$ are the degenerate Apostol–Bernoulli numbers of order α and $\mathfrak{B}_n^{(\alpha)}(x;\lambda;1)=:\beta_n^{(\alpha)}(x;\lambda)$ are the degenerate Bernoulli polynomials of order α.

Remark 2.3 For the special case $\mu=1$ and $\nu=0$ and on use of

$$\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,1,0)=\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma),$$

we have the degenerate Apostol–Euler polynomials defined by

$$\left(\frac{2}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)\frac{t^n}{n!}, \tag{2.5}$$

where $\mathfrak{E}_n^{(\alpha)}(0;\lambda;\gamma)=:\mathfrak{E}_n^{(\alpha)}(\lambda;\gamma)$ are the degenerate Apostol–Euler numbers of order α and $\mathfrak{E}_n^{(\alpha)}(x;\lambda;1)=:\mathcal{E}_n^{(\alpha)}(x;\lambda)$ are the degenerate Euler polynomials of order α.

Remark 2.4 For the special case $\mu=1$ and $\nu=1$ and on use of

$$\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,1,0)=\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma),$$

we have the degenerate Apostol–Genocchi polynomials defined by

$$\left(\frac{2t}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)\frac{t^n}{n!}, \tag{2.6}$$

where $\mathcal{G}_n^{(\alpha)}(0;\lambda;\gamma)=:\mathcal{G}_n^{(\alpha)}(\lambda;\gamma)$ are the degenerate Apostol–Genocchi numbers of order α and $\mathcal{G}_n^{(\alpha)}(x;\lambda;1)=:\mathcal{G}_n^{(\alpha)}(x;\lambda)$ are the degenerate Genocchi polynomials of order α.

To prove several formulae and identities for the aforementioned polynomials, we recall the following definitions:

Definition 2.2 The Stirling numbers of the first kind $S_1(n,m)$ [20] are defined by

$$\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}=\frac{(\log(1+t))^m}{m!}. \tag{2.7}$$

Definition 2.3 The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k), \tag{2.8}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$, it follows that

$$(x|\lambda)_n = \sum_{k=0}^{n} S_1(n,k)\lambda^{n-k}x^k. \tag{2.9}$$

From Binomial Theorem, we have

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x|\lambda)_n \frac{t^n}{n!}. \tag{2.10}$$

Theorem 2.4 *The degenerate Apostol-type polynomials $\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ are defined by the following series expansion:*

$$\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}^{(\alpha)}(\lambda;\gamma,\mu,\nu)\ (x|\lambda)_k. \tag{2.11}$$

Proof Using Eqs. (2.2) and (2.10) in the left hand side of generating function (2.1) and by applying the Cauchy-product rule in the resultant equation, it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\mathcal{P}_{n-k}^{(\alpha)}(\lambda;\gamma,\mu,\nu)\ (x|\lambda)_k t^n}{(n-k)!\ k!} = \sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}. \tag{2.12}$$

Equating the coefficients of same powers of t in Eq. (2.12), yields assertion (2.11). □

Theorem 2.5 *The following implicit summation formula for the degenerate Apostol-type polynomials $\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ holds true:*

$$\mathcal{P}_n^{(\alpha+\beta)}(x+y;\lambda;\gamma,\mu,\nu) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\mathcal{P}_k^{(\beta)}(y;\lambda;\gamma,\mu,\nu). \tag{2.13}$$

Proof Replacing x by $x+y$ and α by $\alpha+\beta$ in generating relation (2.1), we have

$$\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha+\beta}(1+\lambda t)^{\frac{x+y}{\lambda}} = \sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha+\beta)}(x+y;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}, \tag{2.14}$$

which on using generating function (2.1) becomes

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\mathcal{P}_k^{(\beta)}(y;\lambda;\gamma,\mu,\nu)\frac{t^{n+k}}{n!k!}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha+\beta)}(x+y;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}. \tag{2.15}$$

Using Cauchy product rule in the left hand side and then equating the coefficients of the same powers of t in both sides of the resultant equation yields assertion (2.13). □

The notion of quasi-monomiality was introduced and studied by Dattoli [3], in details. The main motive behind this is to find the multiplicative and derivative operators. Further, to frame the degenerate Apostol-type polynomials $\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ within the context of the monomiality principle, we prove the following result:

Theorem 2.6 *The degenerate Apostol-type polynomials $\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{\mathcal{P}}=\frac{x}{e^{\lambda D_x}}+\frac{\alpha\nu\lambda}{e^{\lambda D_x}-1}-\frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{\gamma(e^{D_x})+1} \tag{2.16}$$

and

$$\hat{P}_{\mathcal{P}}=\frac{e^{\lambda D_x}-1}{\lambda}. \tag{2.17}$$

Proof Consider the following identity:

$$t\left\{e^{x\ln(1+\lambda t)^{\frac{1}{\lambda}}}\right\}=\frac{e^{\lambda D_x}-1}{\lambda}\left\{e^{x\ln(1+\lambda t)^{\frac{1}{\lambda}}}\right\}. \tag{2.18}$$

Differentiating generating function (2.1) partially with respect to t, it follows that

$$\left(\frac{x}{1+\lambda t}+\frac{\alpha\nu}{t}-\frac{\alpha\gamma(1+\lambda t)^{\frac{1}{\lambda}-1}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)\left\{\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}\right\}$$
$$=\sum_{n=0}^{\infty}\mathcal{P}_{n+1}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}, \tag{2.19}$$

which in view of identity (2.18) and then use of generating function (2.1) in the left hand side of resulting equation and after rearranging the summation, we have

$$\sum_{n=0}^{\infty}\left(\frac{x}{e^{\lambda D_x}}+\frac{\alpha\nu\lambda}{e^{\lambda D_x}-1}-\frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{\gamma(e^{D_x})+1}\right)\left\{\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}\right\}$$
$$=\sum_{n=0}^{\infty}\mathcal{P}_{n+1}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}. \tag{2.20}$$

On equating the coefficients of same powers of t in both sides of Eq. (2.20) and in view of monomiality principle equation $\hat{M}\{p_n(x)\} = p_{n+1}(x)$, assertion (2.16) follows.

Using generating function (2.1) in identity (2.18) after simplification, we have

$$\sum_{n=0}^{\infty} \frac{e^{\lambda D_x} - 1}{\lambda} \left\{ \mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu) \frac{t^n}{n!} \right\} = \sum_{n=1}^{\infty} \mathcal{P}_{n-1}^{(\alpha)}(x; \lambda; \gamma, \mu, \nu) \frac{t^n}{(n-1)!}. \quad (2.21)$$

On equating the coefficients of the same powers of t on both sides of the Eq. (2.21) and in view of monomiality principle equation $\hat{P}\{p_n(x)\} = n\ p_{n-1}(x)$, assertion (2.17) follows. □

Using expressions (2.16) and (2.17) in monomiality principle equation $\hat{M}\hat{P}\{p_n(x)\} = n\ p_n(x)$, we get the following result:

Corollary 2.1 *The degenerate Apostol-type polynomials* $\mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu)$ *satisfies the following differential equation:*

$$\left(\alpha\nu + \frac{x}{e^{\lambda D_x}} \frac{e^{\lambda D_x} - 1}{\lambda} - \frac{\alpha\gamma (e^{D_x})^{1-\lambda}}{\gamma(e^{D_x}) + 1} \left(\frac{e^{\lambda D_x} - 1}{\lambda} \right) - n \right) \mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu) = 0. \quad (2.22)$$

In view of Remarks 2.2–2.4, we can find the analogous results for the degenerate Apostol–Bernoulli, Euler and Genocchi polynomials, $\mathfrak{B}_n^{(\alpha)}(x; \lambda; \gamma)$, $\mathfrak{E}_n^{(\alpha)}(x; \lambda; \gamma)$ and $\mathcal{G}_n^{(\alpha)}(x; \lambda; \gamma)$, respectively. We present these results in Table 1.

In the next section, recurrence relation and explicit representations for the degenerate Apostol-type polynomials are established.

3 Recurrence relation and explicit representations

In this section, we derive the several recurrence relations and explicit formulas for the degenerate Apostol-type polynomials $\mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu)$. We prove the following theorems:

Theorem 3.1 *For any integral* $n \geq 1$, $\gamma \in \mathbb{C}$ *and* $\alpha \in \mathbb{N}$, *the following recurrence relation for the degenerate Apostol-type polynomials* $\mathcal{P}_n^{(\alpha)}(x; \lambda; \gamma, \mu, \nu)$ *holds true:*

$$\left(\frac{\alpha\nu}{n+1} - 1 \right) \mathcal{P}_{n+1}^{(\alpha)}(x; \lambda; \gamma, \mu, \nu) = \frac{\alpha\gamma}{2^\mu} \frac{n!}{(n+\nu)!} \mathcal{P}_{n+\nu}^{(\alpha+1)}(x + 1 - \lambda; \lambda; \gamma, \mu, \nu) - x\mathcal{P}_n^{(\alpha)}(x - \lambda; \lambda; \gamma, \mu, \nu). \quad (3.1)$$

Table 1 Results for the $\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)$, $\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)$ and $\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)$

S. No.	Special polynomials	Differential operators	Differential equations
I.	$\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)$	$\hat{M}_{\mathfrak{B}} := \left(\frac{x}{e^{\lambda D_x}} + \frac{\alpha\lambda}{e^{\lambda D_x}-1} + \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{1-\gamma(e^{D_x})}\right)$	$\left(\alpha + \left(\frac{x}{e^{\lambda D_x}} + \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{1-\gamma(e^{D_x})}\right)\frac{e^{\lambda D_x}-1}{\lambda} - n\right)\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma) = 0$
		$\hat{P}_{\mathfrak{B}} := \frac{e^{\lambda D_x}-1}{\lambda}$	
II.	$\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)$	$\hat{M}_{\mathfrak{E}} := \left(\frac{x}{e^{\lambda D_x}} - \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{1+\gamma(e^{D_x})}\right)$	$\left(\left(\frac{x}{e^{\lambda D_x}} - \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{1+\gamma(e^{D_x})}\right)\frac{e^{\lambda D_x}-1}{\lambda} - n\right)\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma) = 0$
		$\hat{P}_{\mathfrak{E}} := \frac{e^{\lambda D_x}-1}{\lambda}$	
III.	$\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)$	$\hat{M}_{\mathcal{G}} := \left(\frac{x}{e^{\lambda D_x}} + \frac{\alpha\lambda}{e^{\lambda D_x}-1} - \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{1+\gamma(e^{D_x})}\right)$	$\left(\alpha + \left(\frac{x}{e^{\lambda D_x}} - \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{1+\gamma(e^{D_x})}\right)\frac{e^{\lambda D_x}-1}{\lambda} - n\right)\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma) = 0$
		$\hat{P}_{\mathcal{G}} := \frac{e^{\lambda D_x}-1}{\lambda}$	

Proof Differentiating both sides of Eq. (2.1) with respect to t, it follows that

$$\frac{\alpha\nu}{t}\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}-\frac{\alpha\gamma}{2^{\mu}t^{\nu}}\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha+1}(1+\lambda t)^{\frac{x+1}{\lambda}-1}$$
$$+x\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}-1}=\sum_{n=0}^{\infty}n\mathcal{P}_{n}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^{n-1}}{n!}, \tag{3.2}$$

which on using generating function (2.1) yields

$$\frac{\alpha\nu}{t}\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^{n}}{n!}-\frac{\alpha\gamma}{2^{\mu}t^{\nu}}\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha+1)}(x+1-\lambda;\lambda;\gamma,\mu,\nu)\frac{t^{n}}{n!}$$
$$+x\sum_{n=0}^{\infty}\mathcal{P}_{n}^{(\alpha)}(x-\lambda;\lambda;\gamma,\mu,\nu)\frac{t^{n}}{n!}=\sum_{n=1}^{\infty}\mathcal{P}_{n}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^{n}}{n!}. \tag{3.3}$$

On comparing the coefficients of the same powers of t on both sides of Eq. (3.3), assertion (3.1) follows. □

Next, we derive the explicit representations for the degenerate Apostol-type polynomials $\mathcal{P}_{n}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$. For this, we recall the following definition:

Definition 3.2 The generalized Hurwitz–Lerch Zeta function $\Phi_{\mu}(z,s,a)$ [6] is defined by

$$\Phi_{\mu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{n}}{n!}\,\frac{z^{n}}{(n+a)^{s}}, \tag{3.4}$$

which for $\mu=1$ becomes the Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ [19] (see also [1,10]).

To derive the explicit representations for the degenerate Apostol-type polynomials $\mathcal{P}_{n}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$, we prove the following results:

Theorem 3.3 *The following explicit formula for the degenerate Apostol type polynomials $\mathcal{P}_{n}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ in terms of the Stirling number of the first kind $S_1(n,m)$ holds true:*

$$\begin{aligned}\mathcal{P}_{k+1}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)&=\sum_{n=0}^{k}\sum_{m=0}^{n}\Bigg\{x^{m}\frac{\alpha\nu}{n}\binom{k}{n-1}\mathcal{P}_{k-(n-1)}^{(\alpha)}(\lambda;\gamma,\mu,\nu)\\&\quad-\frac{\alpha\gamma}{2^{\mu}}(x-\lambda+1)^{m}\frac{(n-\nu)!}{n!}\\&\quad\times\binom{k}{n-\nu}\mathcal{P}_{k-(n-\nu)}^{(\alpha+1)}(\lambda;\gamma,\mu,\nu)\\&\quad+x(x-\lambda)^{m}\binom{k}{n}\mathcal{P}_{k-n}^{(\alpha)}(\lambda;\gamma,\mu,\nu)\Bigg\}\lambda^{n-m}S_{1}(n,m).\end{aligned} \tag{3.5}$$

Proof Rewriting Eq. (3.2) in the following form:

$$\begin{aligned}
&\frac{\alpha\nu}{t}\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha} e^{\frac{x}{\lambda}\ln(1+\lambda t)} - \frac{\alpha\gamma}{2^{\mu}t^{\nu}}\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha+1} e^{\frac{x+1-\lambda}{\lambda}\ln(1+\lambda t)} \\
&\quad + x\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha} e^{\frac{x-\lambda}{\lambda}\ln(1+\lambda t)} = \sum_{n=0}^{\infty} n\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^{n-1}}{n!}.
\end{aligned} \tag{3.6}$$

Expanding the exponential and then using Eqs. (2.1) and (2.7) in the resultant equation and after simplification, it follows that

$$\begin{aligned}
&\frac{\alpha\nu}{t}\sum_{k=0}^{\infty}\mathcal{P}_k^{(\alpha)}(\lambda;\gamma,\mu,\nu)\frac{t^k}{k!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\frac{x}{\lambda}\right)^{m} S_1(n,m)\lambda^{n}\right)\frac{t^n}{n!} \\
&\quad - \frac{\alpha\gamma}{2^{\mu}t^{\nu}}\sum_{k=0}^{\infty}\mathcal{P}_k^{(\alpha+1)}(\lambda;\gamma,\mu,\nu)\frac{t^k}{k!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\frac{x-\lambda+1}{\lambda}\right)^{m} S_1(n,m)\lambda^{n}\right)\frac{t^n}{n!} \\
&\quad + x\sum_{k=0}^{\infty}\mathcal{P}_k^{(\alpha)}(\lambda;\gamma,\mu,\nu)\frac{t^k}{k!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\frac{x-\lambda}{\lambda}\right)^{m} S_1(n,m)\lambda^{n}\right)\frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty}\mathcal{P}_{k+1}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^k}{k!}.
\end{aligned} \tag{3.7}$$

On comparing the coefficients of the same powers of t on both sides of Eq. (3.7) and interchanging the sides of the resultant equation, assertion (3.5) follows. □

Theorem 3.4 *The following explicit formula for the degenerate Apostol-type polynomials $\mathcal{P}_n(\alpha)(x;\lambda;\gamma,\mu,\nu)$ in terms of the generalized Hurwitz–Lerch Zeta function $\Phi_{\mu}(z,s,a)$ holds true:*

$$\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu) = \sum_{m=0}^{n-\nu\alpha} 2^{\mu\alpha}\,(\nu\alpha)!\binom{n}{\nu\alpha}\Phi_{\alpha}(-\gamma,-m,x)\,\lambda^{n-m-\nu\alpha}S_1(n-\nu\alpha,m). \tag{3.8}$$

Proof The generating relation (2.1) can be simplified in the following form:

$$\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!} = (2^{\mu}t^{\nu})^{\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}(-\gamma)^k\,(1+\lambda t)^{\frac{k+x}{\lambda}},$$

which gives

$$\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!} = (2^{\mu}t^{\nu})^{\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}(-\gamma)^k\,\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(k+x)^m S_1(n,m)\,\lambda^{n-m}\frac{t^n}{n!}, \tag{3.9}$$

Use of Eq. (2.7) in above equation and after simplification, we find

$$\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n-\nu\alpha}\left\{2^{\mu\alpha}\,(\nu\alpha)!\binom{n}{\nu\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\frac{(-\gamma)^k}{(k+x)^{-m}}\right.$$
$$\left.\times\,\lambda^{n-m-\nu\alpha}\,S_1(n-\nu\alpha,m)\right\}\frac{t^n}{n!},\tag{3.10}$$

which on using Eq. (3.4) and then comparing the coefficients of same powers of t on both sides of the resultant equation yields assertion (3.8). □

Theorem 3.5 *The following explicit formula for the degenerate Apostol-type polynomials $\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ in terms of the degenerate Apostol–Bernoulli polynomials $\mathfrak{B}_n(x;\lambda;\gamma)$ holds true:*

$$\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\Big(\gamma\,\mathfrak{B}_k(x+1;\lambda;\gamma)$$
$$-\,\mathfrak{B}_k(x;\lambda;\gamma)\Big)\mathcal{P}_{n+1-k}^{(\alpha)}(\lambda;\gamma,\mu,\nu).\tag{3.11}$$

Proof From generating function (2.1), we have

$$\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}$$
$$=\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}\frac{1}{t}\left(\frac{\gamma\,t}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)(1+\lambda t)^{\frac{x+1}{\lambda}}$$
$$-\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}\frac{1}{t}\left(\frac{t}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)(1+\lambda t)^{\frac{x}{\lambda}}.\tag{3.12}$$

Using generating functions (2.1), (2.2) and (2.4) in Eq. (3.12), it follows that

$$\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}=\frac{\gamma}{t}\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathfrak{B}_k(x+1;\lambda;\gamma)\frac{t^k}{k!}$$
$$-\frac{1}{t}\sum_{n=0}^{\infty}\mathcal{P}_n^{(\alpha)}(\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}\sum_{k=0}^{\infty}\mathfrak{B}_k(x;\lambda;\gamma)\frac{t^k}{k!},\tag{3.13}$$

which on using the Cauchy product rule in the right hand side of the above equation and then equating the coefficients of identical powers of t in both sides of resultant equation yields assertion (3.10). □

Theorem 3.6 *The following explicit formula for the degenerate Apostol-type polynomials $\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ in terms of the degenerate Apostol–Euler polynomials*

$\mathfrak{E}_n(x;\lambda;\gamma)$ *holds true:*

$$\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)=\frac{1}{2}\sum_{k=0}^{n}\binom{n+1}{k}\Big(\gamma\ \mathfrak{E}_k(x+1;\lambda;\gamma)+\mathfrak{E}_k(x;\lambda;\gamma)\Big)\mathcal{P}_{n+1-k}^{(\alpha)}(\lambda;\gamma,\mu,\nu). \tag{3.14}$$

Proof Following on the same line of proof as in Theorem 3.5 with use of Eqs. (2.1), (2.2) and (2.5) yields assertion (3.14). Thus we omit it. □

Theorem 3.7 *The following explicit formula for the degenerate Apostol-type polynomials* $\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ *in terms of the degenerate Apostol–Genocchi polynomials* $\mathcal{G}_n(x;\lambda;\gamma)$ *holds true:*

$$\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)=\frac{1}{2(n+1)}\sum_{k=0}^{n}\binom{n+1}{k}\Big(\gamma\ \mathcal{G}_k(x+1;\lambda;\gamma)+\mathcal{G}_k(x;\lambda;\gamma)\Big)\mathcal{P}_{n+1-k}^{(\alpha)}(\lambda;\gamma,\mu,\nu). \tag{3.15}$$

Proof Following on the same line of proof as in Theorem 3.5 with use of Eqs. (2.1), (2.2) and (2.6) yields assertion (3.15). Thus we omit it. □

In view of Remarks 2.2–2.4, we can find the analogous results for the degenerate Apostol–Bernoulli, Euler and Genocchi polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)$, $\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)$ and $\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)$, respectively. We present these results in Table 2.

In the next section, we introduce and study a hybrid form of degenerate Apostol-type polynomials.

4 Example

To introduce the hybridized forms of polynomials and to characterize their properties via generating functions is a recent new approach. To achieve this, we recall the following definition:

Definition 4.1 The generating function for the truncated-exponential polynomials $e_n(x)$ is defined as [4, p.596 (4)]:

$$\frac{e^{xt}}{(1-t)}=\sum_{n=0}^{\infty}e_n(x)\frac{t^n}{n!}. \tag{4.1}$$

The following example can well satisfied the definition of hybridized polynomials:

Table 2 Recurrence relation and explicit representations for the $\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)$, $\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)$ and $\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)$

S. no.	Special polynomials	Recurrence relation and explicit representations
I.	$\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)$	$(\alpha-(n+1))\mathfrak{B}_{n+1}^{(\alpha)}(x;\lambda;\gamma)=-\alpha\gamma\mathfrak{B}_{n+1}^{(\alpha+1)}(x+1-\lambda;\lambda;\gamma)-(n+1)x\mathfrak{B}_n^{(\alpha)}(x-\lambda;\lambda;\gamma)$
		$\mathfrak{B}_{k+1}^{(\alpha)}(x;\lambda;\gamma)=\sum_{n=0}^{k}\sum_{m=0}^{n}\Big\{x^m\frac{\alpha}{n}\mathfrak{B}_{k-(n-1)}^{(\alpha)}(\lambda;\gamma)$
		$+(x-\lambda+1)^m\frac{\alpha\gamma}{n}\binom{k}{n-1}\mathfrak{B}_{k-(n-1)}^{(\alpha+1)}(\lambda;\gamma)+(x-\lambda)^m x\binom{k}{n}\mathfrak{B}_{k-n}^{(\alpha)}(\lambda;\gamma)\Big\}\lambda^{n-m}S_1(n,m)$
		$\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)=\sum_{m=0}^{n-\alpha}(\alpha)!\binom{n}{\alpha}\Phi_\alpha(\gamma,-m,x)\lambda^{n-m-\alpha}S_1(n-\alpha,m)$
II.	$\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)$	$2\mathfrak{E}_{n+1}^{(\alpha)}(x;\lambda;\gamma)=-\alpha\gamma\mathfrak{E}_n^{(\alpha+1)}(x+1-\lambda;\lambda;\gamma)+2x\mathfrak{E}_n^{(\alpha)}(x-\lambda;\lambda;\gamma)$
		$\mathfrak{E}_{k+1}^{(\alpha)}(x;\lambda;\gamma)=\sum_{n=0}^{k}\sum_{m=0}^{n}\Big\{(x-\lambda+1)^m\frac{\alpha\gamma}{2}\binom{k}{n}\mathfrak{E}_{k-n}^{(\alpha+1)}(\lambda;\gamma)$
		$+(x-\lambda)^m x\binom{k}{n}\mathfrak{E}_{k-n}^{(\alpha)}(\lambda;\gamma)\Big\}\lambda^{n-m}S_1(n,m)$
		$\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)=\sum_{m=0}^{n}2^\alpha\,\Phi_\alpha(-\gamma,-m,x)\lambda^{n-m}S_1(n,m)$
III.	$\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)$	$2(\alpha-(n+1))\mathcal{G}_{n+1}^{(\alpha)}(x;\lambda;\gamma)=\alpha\gamma\,\mathcal{G}_{n+1}^{(\alpha+1)}(x+1-\lambda;\lambda;\gamma)-2(n+1)x\,\mathcal{G}_n^{(\alpha)}(x-\lambda;\lambda;\gamma)$
		$\mathcal{G}_{k+1}^{(\alpha)}(x;\lambda;\gamma)=\sum_{n=0}^{k}\sum_{m=0}^{n}\Big\{x^m\frac{\alpha}{n}\binom{k}{n-1}\mathcal{G}_{k-(n-1)}^{(\alpha)}(\lambda;\gamma)$
		$-(x-\lambda+1)^m\frac{\alpha\gamma}{2n}\binom{k}{n-1}\mathcal{G}_{k-(n-1)}^{(\alpha+1)}(\lambda;\gamma)+x(x-\lambda)^m\binom{k}{n}\mathcal{G}_{k-n}^{(\alpha)}(\lambda;\gamma)\Big\}\lambda^{n-m}S_1(n,m)$
		$\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)=\sum_{m=0}^{n-\alpha}2^\alpha\,(\alpha)!\binom{n}{\alpha}\Phi_\alpha(-\gamma,-m,x)\lambda^{n-m-\alpha}S_1(n-\alpha,m)$

Table 3 Results for ${}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$

S. no.	Results	Expressions
I.	Multiplicative and derivative operators	$\hat{M}_{eP} = \frac{x}{e^{\lambda D_x}} + \frac{1}{1-D_x} + \frac{\alpha\nu\lambda}{e^{\lambda D_x}-1} - \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{\gamma e^{D_x}+1}$ $\hat{P}_{eP} = \frac{e^{\lambda D_x}-1}{\lambda}$
II.	Differential equation	$\left(\alpha\nu + \left(\frac{x}{e^{\lambda D_x}} + \frac{1}{1-D_x} - \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{\gamma e^{D_x}+1}\right)\left(\frac{e^{\lambda D_x}-1}{\lambda}\right) - n\right) {}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu) = 0$
III.	Recurrence relation	$\left(\frac{\alpha\nu}{n+1} - 1\right) {}_e\mathcal{P}_{n+1}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu) = \frac{\alpha\gamma}{2^\mu} \frac{n!}{(n+\nu)!} {}_e\mathcal{P}_{n+\nu}^{(\alpha+1)}(x+1-\lambda;\lambda;\gamma,\mu,\nu)$ $-x\,{}_e\mathcal{P}_n^{(\alpha)}(x-\lambda;\lambda;\gamma,\mu,\nu) - \sum_{r=0}^{n}\sum_{m=0}^{r}\binom{n}{r} m!\lambda^r S_1(r,m)\, {}_e\mathcal{P}_{n-r}^{(\alpha)}(x-\lambda;\lambda;\gamma,\mu,\nu)$
IV.	Explicit representations	${}_e\mathcal{P}_{k+1}^{(\alpha)}(x;\lambda;\gamma,\mu,\nu) = \sum_{n=0}^{k}\sum_{m=0}^{n}\Big\{x^m \frac{\alpha\nu}{n}\binom{k}{n-1} {}_e\mathcal{P}_{k-(n-1)}^{(\alpha)}(\lambda;\gamma,\mu,\nu) - \frac{\alpha\gamma}{2^\mu}(x-\lambda+1)^m \frac{(n-\nu)!}{n!}$ $\times\binom{k}{n-\nu} {}_e\mathcal{P}_{k-(n-\nu)}^{(\alpha+1)}(\lambda;\gamma,\mu,\nu) + x(x-\lambda)^m\binom{k}{n} {}_e\mathcal{P}_{k-n}^{(\alpha)}(\lambda;\gamma,\mu,\nu)\Big\}\lambda^{n-m} S_1(n,m)$ $+\sum_{r,n=0}^{r+n\le k}\sum_{m=0}^{n}\Big\{\binom{k}{r,n}\frac{{}_e\mathcal{P}_{k-r-n}^{(\alpha)}(\lambda;\gamma,\mu,\nu)}{(k-r-n)!} e_r(0;\lambda)(x-\lambda)^m\Big\}\lambda^{n-m} S_1(n,m).$ ${}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu) = \sum_{r=0}^{n-q-\nu\alpha}\sum_{q=0}^{n-\nu\alpha}\sum_{p=0}^{q}\frac{2^{\mu\alpha} n!\, r!}{(n-q-\nu\alpha)!\, q!}\Phi_\alpha(-\gamma,-p,x)\,\lambda^{n-p-\nu\alpha}$ $\times S_1(q,p)\, S_1(n-q-\nu\alpha)$

Table 4 Special cases of the ${}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$

S. no.	Values of the parameters	Relation between the ${}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ and its special cases	Name of the resultant special polynomials	Generating function of the resultant special polynomials
I.	$\gamma \to -\gamma,$	$(-1)^{\alpha}\,{}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;-\gamma,0,1)$	Truncated-exponential	$\left(\frac{t}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{\alpha}\frac{(1+\lambda t)^{\frac{x}{\lambda}}}{\left(1-\ln(1+\lambda t)^{\frac{1}{\lambda}}\right)}$
	$\mu=0,\ \mu=1$	$={}_e\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)$	Degenerate Apostol–Bernoulli polynomials of order α	$=\sum_{n=0}^{\infty}{}_e\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)\frac{t^n}{n!}$
II.	$\mu=1,\ \nu=0$	${}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,1,0)$	Truncated-exponential	$\left(\frac{2}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}\frac{(1+\lambda t)^{\frac{x}{\lambda}}}{\left(1-\ln(1+\lambda t)^{\frac{1}{\lambda}}\right)}$
		$={}_e\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)$	Degenerate Apostol–Euler polynomials of order α	$=\sum_{n=0}^{\infty}{}_e\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)\frac{t^n}{n!}$
III.	$\mu=1,\ \nu=1$	${}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,1,1)$	Truncated-exponential	$\left(\frac{2t}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}\frac{(1+\lambda t)^{\frac{x}{\lambda}}}{\left(1-\ln(1+\lambda t)^{\frac{1}{\lambda}}\right)}$
		$={}_e\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)$	Degenerate Apostol–Genocchi polynomials of order α	$=\sum_{n=0}^{\infty}{}_e\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)\frac{t^n}{n!}$

Table 5 Results for ${}_{e}\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)$

S. no.	Results	Expressions
I.	Multiplicative and	$\hat{M}_{eB} = \frac{x}{e^{\lambda D_x}} + \frac{1}{1-D_x} + \frac{\alpha\lambda}{e^{\lambda D_x}-1} + \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{1-\gamma e^{D_x}}$
	derivative operators	$\hat{P}_{eB} = \frac{e^{\lambda D_x}-1}{\lambda}$
II.	Differential equation	$\left(\alpha + \left(\frac{x}{e^{\lambda D_x}} + \frac{1}{1-D_x} + \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{1-\gamma e^{D_x}}\right)\left(\frac{e^{\lambda D_x}-1}{\lambda}\right) - n\right) {}_{e}\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma) = 0$
III.	Recurrence relation	$(\alpha-(n+1))\, {}_{e}\mathfrak{B}_{n+1}^{(\alpha)}(x;\lambda;\gamma) = -\alpha\gamma\, {}_{e}\mathfrak{B}_{n+\nu}^{(\alpha+1)}(x+1-\lambda;\lambda;\gamma)$
		$-x(n+1)\, {}_{e}\mathfrak{B}_n^{(\alpha)}(x-\lambda;\lambda;\gamma) - \sum_{r=0}^{n}\sum_{m=0}^{r}(n+1)\binom{n}{r}m!\ \lambda^r S_1(r,m)\, {}_{e}\mathfrak{B}_{n-r}^{(\alpha)}(x-\lambda;\lambda;\gamma)$
IV.	Explicit representations	${}_{e}\mathfrak{B}_{k+1}^{(\alpha)}(x;\lambda;\gamma) = \sum_{n=0}^{k}\sum_{m=0}^{n}\Big\{x^m \frac{\alpha}{n}\binom{k}{n-1}{}_{e}\mathfrak{B}_{k-(n-1)}^{(\alpha)}(\lambda;\gamma) + \alpha\gamma(x-\lambda+1)^m \frac{(n-1)!}{n!}$
		$\times\binom{k}{n-1}{}_{e}\mathfrak{B}_{k-(n-1)}^{(\alpha+1)}(\lambda;\gamma) + x(x-\lambda)^m\binom{k}{n}{}_{e}\mathfrak{B}_{k-n}^{(\alpha)}(\lambda;\gamma)\Big\}\lambda^{n-m} S_1(n,m)$
		$+\sum_{r,n=0}^{r+n\le k}\sum_{m=0}^{n}\Big\{\binom{k}{r,n}\frac{{}_{e}\mathfrak{B}_{k-r-n}^{(\alpha)}(\lambda;\gamma)}{(k-r-n)!}\, e_r(0;\lambda)(x-\lambda)^m\Big\}\lambda^{n-m} S_1(n,m).$
		${}_{e}\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma) = \sum_{r=0}^{n-q-\alpha}\sum_{q=0}^{n-\alpha}\sum_{p=0}^{q}\frac{n!\,r!}{(n-q-\nu)!\,q!}\,\Phi_\alpha(\gamma,-p,x)\,\lambda^{n-p-\alpha} S_1(q,p)\, S_1(n-q-\alpha,r)$

Table 6 Results for ${}_{e}\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)$

S. no.	Results	Expressions
I.	Multiplicative and derivative operators	$\hat{M}_{eE} = \frac{x}{e^{\lambda D_x}} + \frac{1}{1-D_x} - \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{\gamma e^{D_x}+1}$ $\hat{P}_{eE} = \frac{e^{\lambda D_x}-1}{\lambda}$
II.	Differential equation	$\left(\left(\frac{x}{e^{\lambda D_x}} + \frac{1}{1-D_x} - \frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{\gamma e^{D_x}+1}\right)\left(\frac{e^{\lambda D_x}-1}{\lambda}\right) - n\right) {}_{e}\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma) = 0$
III.	Recurrence relation	${}_{e}\mathfrak{E}_{n+1}^{(\alpha)}(x;\lambda;\gamma) = -\frac{\alpha\gamma}{2}\, {}_{e}\mathfrak{E}_n^{(\alpha+1)}(x+1-\lambda;\lambda;\gamma)$ $-x\, {}_{e}\mathfrak{E}_n^{(\alpha)}(x-\lambda;\lambda;\gamma) - \sum_{r=0}^{n}\sum_{m=0}^{r}\binom{n}{r} m!\ \lambda^r S_1(r,m)\, {}_{e}\mathfrak{E}_{n-r}^{(\alpha)}(x-\lambda;\lambda;\gamma)$
IV.	Explicit representations	${}_{e}\mathfrak{E}_{k+1}^{(\alpha)}(x;\lambda;\gamma) = \left(\sum_{n=0}^{k}\sum_{m=0}^{n}\left\{-\frac{\alpha\gamma}{2}(x-\lambda+1)^m\binom{k}{n}{}_{e}\mathfrak{E}_{k-n}^{(\alpha+1)}(\lambda;\gamma) + x(x-\lambda)^m\binom{k}{n}\right.\right.$ $\left.\left.\times\, {}_{e}\mathfrak{E}_{k-n}^{(\alpha)}(\lambda;\gamma)\right\} + \sum_{r,n=0}^{r+n\le k}\sum_{m=0}^{n}\left\{\binom{k}{r,n}\frac{{}_{e}\mathfrak{E}_{k-r-n}^{(\alpha)}(\lambda;\gamma)}{(k-r-n)!}\ e_r(0;\lambda)(x-\lambda)^m\right\}\right)\lambda^{n-m} S_1(n,m).$ ${}_{e}\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma) = \sum_{r=0}^{n-q}\sum_{q=0}^{n}\sum_{p=0}^{q}\frac{2^\alpha n!\, r!}{(n-q)!\, q!}\ \Phi_\alpha(-\gamma,-p,x)\,\lambda^{n-p}\ S_1(q,p)\ S_1(n-q,r)$

Table 7 Results for ${}_e\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)$

S. no.	Results	Expressions
I.	Multiplicative and derivative operators	$\hat{M}_{eG}=\frac{x}{e^{\lambda D_x}}+\frac{1}{1-D_x}+\frac{\alpha\lambda}{e^{\lambda D_x}-1}-\frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{\gamma e^{D_x}+1}$
		$\hat{P}_{eG}=\frac{e^{\lambda D_x}-1}{\lambda}$
II.	Differential equation	$\left(\alpha+\left(\frac{x}{e^{\lambda D_x}}+\frac{1}{1-D_x}-\frac{\alpha\gamma(e^{D_x})^{1-\lambda}}{\gamma e^{D_x}+1}\right)\left(\frac{e^{\lambda D_x}-1}{\lambda}\right)-n\right){}_e\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)=0$
III.	Recurrence relation	$\left(\frac{\alpha}{n+1}-1\right){}_e\mathcal{G}_{n+1}^{(\alpha)}(x;\lambda;\gamma)=\frac{\alpha\gamma}{2}\frac{n!}{(n+1)!}\,{}_e\mathcal{G}_{n+1}^{(\alpha+1)}(x+1-\lambda;\lambda;\gamma)$ $-x\,{}_e\mathcal{G}_n^{(\alpha)}(x-\lambda;\lambda;\gamma)-\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{n}{r}m!\ \lambda^r S_1(r,m)\,{}_e\mathcal{G}_{n-r}^{(\alpha)}(x-\lambda;\lambda;\gamma)$
IV.	Explicit representations	${}_e\mathcal{G}_{k+1}^{(\alpha)}(x;\lambda;\gamma)=\sum_{n=0}^{k}\sum_{m=0}^{n}\Big\{x^m\frac{\alpha}{n}\binom{k}{n-1}{}_e\mathcal{G}_{k-(n-1)}^{(\alpha)}(\lambda;\gamma)-\frac{\alpha\gamma}{2}(x-\lambda+1)^m\frac{(n-1)!}{n!}$ $\times\binom{k}{n-v}{}_e\mathcal{G}_{k-(n-1)}^{(\alpha+1)}(\lambda;\gamma)+x(x-\lambda)^m\binom{k}{n}{}_e\mathcal{G}_{k-n}^{(\alpha)}(\lambda;\gamma)\Big\}\lambda^{n-m}S_1(n,m)$ $+\sum_{r,n=0}^{r+n\leq k}\sum_{m=0}^{n}\Big\{\binom{k}{r,n}\frac{{}_e\mathcal{G}_{k-r-n}^{(\alpha)}(\lambda;\gamma)}{(k-r-n)!}\,e_r(0;\lambda)(x-\lambda)^m\Big\}\lambda^{n-m}S_1(n,m).$
		${}_e\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)=\sum_{r=0}^{n-q-v\alpha}\sum_{q=0}^{n-v\alpha}\sum_{p=0}^{q}\frac{2^\alpha n!\,r!}{(n-q-\alpha)!\,q!}\,\Phi_\alpha(-\gamma,-p,x)\,\lambda^{n-p-\alpha}S_1(q,p)\,S_1(n-q-\alpha,r)$

It should be noted that for $\gamma=1$, the results derived above for the truncated-exponential degenerate Apostol–Bernoulli polynomials ${}_e\mathfrak{B}_n^{(\alpha)}(x;\lambda;\gamma)$, truncated-exponential degenerate Apostol–Euler polynomials ${}_e\mathfrak{E}_n^{(\alpha)}(x;\lambda;\gamma)$ and truncated-exponential degenerate Apostol–Genocchi polynomials ${}_e\mathcal{G}_n^{(\alpha)}(x;\lambda;\gamma)$ give the analogous results for the truncated-exponential degenerate Bernoulli polynomials (of order α) ${}_e\beta_n^{(\alpha)}(x;\lambda)$, truncated-exponential degenerate Euler polynomials (of order α) ${}_e\mathcal{E}_n^{(\alpha)}(x;\lambda)$ and truncated-exponential degenerate Genocchi polynomials (of order α) ${}_e\mathcal{G}n^{(\alpha)}(x;\lambda)$, respectively. Also, for $\alpha=1$, the analogous results for the truncated-exponential degenerate Bernoulli polynomials ${}_e\beta_n(x;\lambda)$, truncated-exponential degenerate Euler polynomials ${}_e\mathcal{E}_n(x;\lambda)$ and truncated-exponential degenerate Genocchi polynomials ${}_e\mathcal{G}_n(x;\lambda)$ are obtained

Example 4.1 The truncated-exponential degenerate Apostol-type polynomials ${}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$ by defined by means of the following generating function:

$$\left(\frac{2^{\mu}t^{\nu}}{\gamma(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{\alpha}\frac{(1+\lambda t)^{\frac{x}{\lambda}}}{\left(1-\ln(1+\lambda t)^{\frac{1}{\lambda}}\right)}=\sum_{n=0}^{\infty}{}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)\frac{t^n}{n!}. \tag{4.2}$$

When $x = 0$, ${}_e\mathcal{P}_n^{(\alpha)}(\lambda;\gamma,\mu,\nu) := {}_e\mathcal{P}_n^{(\alpha)}(0;\lambda;\gamma,\mu,\nu)$ are the corresponding truncated-exponential degenerate Apostol-type numbers of order α.

The other results for the truncated-exponential degenerate Apostol-type numbers of order α are given in Table 3.

In view of Remarks 2.2–2.3, we can find the special cases of ${}_e\mathcal{P}_n^{(\alpha)}(x;\lambda;\gamma,\mu,\nu)$. These are given in Table 4.

Now, we obtain the results for the truncated-exponential degenerate Apostol–Bernoulli polynomials. These are given in Table 5 below.

Also, the corresponding results for the truncated-exponential degenerate Apostol–Euler polynomials are obtained. We give these results in Table 6 below. Further, the corresponding results for the truncated-exponential degenerate Apostol–Genocchi polynomials are obtained and these are given in Table 7 below. These hybrid special polynomials are important as they possess essential properties such as recurrence and explicit relations and functional and differential equations, summation formulae, symmetric and convolution identities, etc. These polynomials are useful and possess potential for applications in numerous problems of number theory, combinatorics, classical and numerical analysis, theoretical physics, approximation theory and other fields of pure and applied mathematics. The technique used here could be used to establish further quite a wide variety of formulas for certain other special polynomials and can be extended to derive new relations for conventional and generalized polynomials.

Acknowledgements This work is jointly supported by Senior Research Fellowship [Award letter no. F./2014-15/NFO-2014-15-OBC-UTT-24168/(SA-III/Website)] awarded to Ms. Tabinda Nahid by the University Grants Commission, Government of India, New Delhi and by Post-Doctoral Fellowship (Office Memo no. 2/40(38)/2016/R&D-II/1063) awarded to Dr. Mumtaz Riyasat by the National Board of Higher Mathematics, Department of Atomic Energy, Government of India, Mumbai.

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