Boletín de la Sociedad Matemática Mexicana (2019) 25:589–596
https://doi.org/10.1007/s40590-018-0210-1

Sociedad Matemática Mexicana

ORIGINAL ARTICLE

Normal functions concerning derivatives and shared sets

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Received: 27 February 2018 / Accepted: 12 July 2018 / Published online: 27 July 2018

Abstract
Let $f(z)$ be meromorphic in Δ, $E_1 = \{a_1, a_2, a_3\}$ and $E_2 = \{b_1, b_2, b_3\}$ be two sets in $\mathbb{C}$, $k \in Z^+$. Suppose that $f(z) \in E_1 \Leftrightarrow f^{(k)}(z) \in E_2$ and $\max_{0 \le i \le k-1} |f^{(i)}(z)| = 0$ whenever $f(z) \in E_1$, then $f(z)$ is a normal function.

Keywords Meromorphic functions · Normal functions · Shared sets

Mathematics Subject Classification 30D45 · 30D35

1 Introduction and main results

By the Bloch principle, the criteria of normal functions are studied, which are consistent with the known criteria of normal families. For example, corresponding to the well-known Montel's theorem of normal families, Lehto and Virtanen [4] showed that if $f(z)$ is meromorphic in Δ and $f(z) \neq a, b, c$, then $f(z)$ is a normal function, where a, b, c are three distinct points in $\overline{\mathbb{C}}$. But, they are not always right when the criteria of normal functions are related to derivatives. For instance, corresponding to the well-known Miranda criterion for the family of holomorphic functions, Hayman and Storvick [1] proved that there exists a non-normal function $f(z)$ satisfying $f(z) \neq 0$ and $f'(z) \neq 1$ in Δ. In the following, we focus on the criteria of normal functions concerning derivatives.

This work is partially supported by the National Natural Science Foundation of China (11501367, 61673257), the Natural Science Foundation of Shanghai (17ZR1419900).

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The normality of families of meromorphic functions concerning shared values was proved by Schwick [8] in 1992, which is listed below.

Theorem A *[8, Theorem 2] Let $\mathcal{F}$ be a family of meromorphic in D, and let a_1, a_2, a_3 be three distinct complex numbers in $\mathbb{C}$. Suppose that, $f(z)$ and $f'(z)$ share a_1, a_2, a_3, for every function $f(z) \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In 2007, Liu-Pang [5] improved Schwick's result, by means of substituting sharing the set $\{a_1, a_2, a_3\}$ for sharing three values a_1, a_2, a_3 in Theorem A, as follows:

Theorem B *[5, Theorem 1] Let $\mathcal{F}$ be a family of meromorphic in D, and let $E = \{a_1, a_2, a_3\}$ be a set in $\mathbb{C}$. Suppose that $f(z)$ and $f'(z)$ share E, for every function $f \in \mathcal{F}$; then $\mathcal{F}$ is normal in D.*

Recently, we consider the question about normal functions concerning derivatives and shared sets. And, we get the following results:

Theorem 1 *Let $f(z)$ be meromorphic in Δ, $E_1 = \{a_1, a_2, a_3\}$ and $E_2 = \{b_1, b_2, b_3\}$ be two sets in $\mathbb{C}$, $k \in Z^+$. Suppose that $f(z) \in E_1 \Leftrightarrow f^{(k)}(z) \in E_2$ and $\max\limits_{0\le i\le k-1} |f^{(i)}(z)| = 0$ whenever $f(z) \in E_1$; then $f(z)$ is a normal function.*

And for the case of holomorphic functions, the following result can be obtained.

Theorem 2 *Let $f(z)$ be holomorphic in Δ, a_1, a_2, a_3 be three distinct complex numbers in $\mathbb{C}$, and $A > 0$. Suppose that $|f'(z)| \le A$ whenever $f(z) = a_i (i = 1, 2, 3)$; then $f(z)$ is a normal function.*

Remark 1 The condition "$\max\limits_{0\le i\le k-1} |f^{(i)}(z)| = 0$ whenever $f(z) \in E_1$" in Theorem 1 holds naturally for $k = 1$. Or more accurately, if $k = 1$, then $i = 0$ and the condition "$\max\limits_{0\le i\le k-1} |f^{(i)}(z)| = 0$ whenever $f(z) \in E_1$" is removed. And if $k \ge 2$, then $1 \le i \le k-1$ and the condition is needed. Furthermore, in the latter case, the multiplicities of zeros of $f(z) - a$ are at least k, where $a \in E_1$.

The following example shows that the condition "$\max\limits_{0\le i\le k-1} |f^{(i)}(z)| = 0$ whenever $f(z) \in E_1$" in Theorem 1 is necessary.

Example 1 Let $\mathcal{F} = \{f_n(z) | f_n(z) = n^2(e^{a_1 z} - e^{a_2 z}), z \in \Delta, n = 1, 2, \ldots\}$, $E_1 = E_2 = \{-1, 0, 1\}$, where $a_1 \ne a_2$ satisfying $a_1^k = a_2^k = 1, k \in Z^+$.

By calculating, it yields that $f_n(z) = f_n^{(k)}(z)$, and $f_n(z) \in E_1 \Leftrightarrow f_n^{(k)} \in E_2$. $\max\limits_{0\le i\le k-1} |f_n^{(i)}(z)| \ne 0$ whenever $f_n(z) \in E_1$.

However, $(1 - |z|^2) f_n^{\sharp}(z)|_{z=0} = (1 - |0|^2) \frac{|f_n'(0)|}{1+|f_n(0)|^2} = n^2|a_1 - a_2| \to \infty$, as $n \to \infty$. Thus, as $n \to \infty$, $f_n(z)$ is not a normal function.

Here and in the sequel, $\mathbb{C}$ is the complex plane and $\overline{\mathbb{C}}$ is the extended complex plane. D is a domain in $\mathbb{C}$. $\Delta(z_0, r) = \{z : |z - z_0| < r\}$, $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$, where $z_0 \in \mathbb{C}, r > 0$. The unit disc is marked as $\Delta = \Delta(0, 1)$.

$f_n(z) \overset{\chi}{\Rightarrow} f(z)$ in D shows that the sequence $\{f_n(z)\}$ converges to $f(z)$ in the spherical metric uniformly in compact subsets of D and $f_n(z) \Rightarrow f(z)$ in D if the convergence is in the Euclidean metric. $f^\sharp(z) = |f'(z)|/(1 + |f(z)|^2)$ is the spherical derivative of $f(z)$.

Let $f(z)$, $L(z)$ be meromorphic in D, a, b be two complex numbers in $\mathbb{C}$, and E_1, E_2 be two sets in $\mathbb{C}$. $f(z) = a \Rightarrow L(z) = b$ if $L(z) = b$ whenever $f(z) = a$, and $f(z) = a \Leftrightarrow L(z) = b$ if $f(z) = a \Rightarrow L(z) = b$ and $L(z) = b \Rightarrow f(z) = a$. When $f(z) = a \Leftrightarrow L(z) = a$ in D, we say that $f(z)$ and $L(z)$ share the value a. $D(f, E_1) := \cup_{a\in E_1}\{z \in D : f(z) = a\}$, $D(L, E_2) := \cup_{a\in E_2}\{z \in D : L(z) = a\}$. If $D(f, E_1) \subset D(L, E_2)$, we write $f(z) \in E_1 \Rightarrow L(z) \in E_2$ in D. Furthermore, if $D(f, E_1) \subset D(L, E_2)$ and $D(L, E_2) \subset D(f, E_1)$, that is, $D(f, E_1) = D(L, E_2)$, we write $f(z) \in E_1 \Leftrightarrow L(z) \in E_2$ in D. If $f(z) \in E \Leftrightarrow L(z) \in E$ in D, we say that $f(z)$ and $L(z)$ share the set E in D.

Recall that a family $\mathcal{F}$ of meromorphic in D is said to be a normal family in D, if each sequence $\{f_n(z)\} \subset \mathcal{F}$ contains a subsequence which converges spherically locally in D. The subtracted set may depend on the subsequence. See [2,3]. A function $f(z)$ meromorphic in Δ, it is said to be a normal function if and only if the family $\{f(L(z))\}$ is normal (see [4]), where $L(z)$ shows an arbitrary one-one conformal mapping of Δ onto itself.

2 Preliminary results

First, we introduce some lemmas which will be used in the proofs of main results.

Lemma 1 *[7, Lemma 2] Let $\mathcal{F}$ be a family of meromorphic in D, all of whose zeros have multiplicities at least $k(\in Z^+)$, and suppose that there exists $M \geq 1$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$ and $f(z) \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each α, $0 \leq \alpha \leq k$, there exist a sequence of complex numbers $z_n \in D$, $z_n \to z_0$, positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$L_n(\xi) = \frac{f_n(z_n + \rho_n\xi)}{\rho_n^\alpha} \overset{\chi}{\Rightarrow} L(\xi),$$

where $L(\xi)$ is nonconstant and meromorphic in $\mathbb{C}$, all of whose zeros have multiplicities at least k, such that $L^\sharp(\xi) \leq L^\sharp(0) = kM + 1$. Moreover, $L(\xi)$ has order at most 2.

Lemma 2 *[6] Let $f(z)$ be meromorphic in Δ. Suppose that $f(z)$ is not a normal function, then there exists a sequence of points $z_n \in \Delta$ and positive numbers $\rho_n \to 0$ such that*

$$L_n(z) = f(z_n + \rho_n z) \overset{\chi}{\Rightarrow} L(z)$$

in $\mathbb{C}$, where $L(z)$ is nonconstant and meromorphic in $\mathbb{C}$.

Lemma 3 *[3, Theorem 1.5] Let $f(z)$ be nonconstant and meromorphic, $a_1, a_2, \cdots, a_q$ be $q(> 2)$ distinct complex numbers in $\mathbb{C}$. Then*

$$m(r, f) + \sum_{i=1}^{q} m\left(r, \frac{1}{f - a_i}\right) \leq 2T(r, f) - N_1(r) + S(r, f),$$

where $N_1(r) = 2N(r, f) - N(r, f') + N(r, \frac{1}{f'})$ and $S(r, f) = o(T(r, f)$, as $r \to \infty$ possibly outside a set of finite measure.

Lemma 4 *[2, Corollary to Theorem 3.5] Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$ and $k \in Z^+$. Then $f(z)$ or $f^{(k)}(z) - 1$ has infinitely many zeros.*

3 Proof of theorems

Proof of Theorem 1 Suppose, to the contrary, that $f(z)$ is not a normal function in Δ. Then, based on Lemma 2, there exist points $z_n \in \Delta$, $\rho_n \to 0^+$ such that

$$L_n(\eta) = f(z_n + \rho_n \eta) \overset{\chi}{\Rightarrow} L(\eta), \tag{1}$$

where $L(\eta)$ is nonconstant and meromorphic in $\mathbb{C}$.

And we claim that the multiplicities of all zeros of $L(\eta) - a_i$ (for each $a_i \in E_1$) are no less than $k + 1$. In fact, assume that $L(\eta_0) - a_1 = 0$, it gets that there exist points $\eta_n \to \eta_0$ such that $f(z_n + \rho_n \eta_n) = a_1$ by Hurwitz's theorem and $L(\eta)$ is nonconstant. According to $f(z) \in E_1 \Leftrightarrow f^{(k)}(z) \in E_2$ and $\max\limits_{0 \leq i \leq k-1} |f^{(i)}(z)| = 0$ whenever $f(z) \in E_1$, it obtains $f^{(k)}(z_n + \rho_n \eta_n) \in E_2$ and $f^{(i)}(z_n + \rho_n \eta_n) = 0 (0 \leq i \leq k - 1)$. Then $L_n^{(i)}(\eta_n) = \rho_n^i f^{(i)}(z_n + \rho_n \eta_n) = \rho_n^i \cdot 0 = 0$, $(0 \leq i \leq k - 1)$, and $L_n^{(k)}(\eta_n) = \rho_n^k f^{(k)}(z_n + \rho_n \eta_n) = \rho_n^k b_i$. Clearly, $L^{(i)}(\eta_0) = \lim\limits_{n \to \infty} L_n^{(i)}(\eta_n) = 0 (0 \leq i \leq k)$. Thus, the multiplicities of all zeros of $L(\eta) - a_1$ are no less than $k + 1$. Similar results that the multiplicities of all zeros of $L(\eta) - a_i (i = 2, 3)$ are no less than $k + 1$ can be obtained. So, the claim is proved.

Among $L(\eta) - a_i (i = 1, 2, 3)$, there exists at least one term which has zeros, according to Lemma 3 and $L(\eta)$ is nonconstant. Suppose that η_0 is a zero of $L(\eta) - a_1$ with multiplicities l. It can find $\delta > 0$, for large enough n, such that $L_n(\eta)$ is holomorphic in $\Delta(\eta_0, \delta)$.

Let

$$\varphi_n(\eta) = \frac{L_n(\eta) - a_1}{\rho_n^k}. \tag{2}$$

Then $\{\varphi_n(\eta)\}$ is holomorphic in $\Delta(\eta_0, \delta)$.

It is asserted that (i) $\{\varphi_n(\eta)\}$ is not normal at η_0, and (ii) $|\varphi_n^{(k)}(\eta)| \leq |b_1| + |b_2| + |b_3|$ whenever $\varphi_n(\eta) = 0$.

First of all , we prove the claim (i). Suppose, to the contrary, that $\{\varphi_n(\eta)\}$ is normal at η_0. According to the definition of normal family, there exist $0 < \delta_1 < \delta$ and a subsequence of $\{\varphi_n(\eta)\}$ (still denoted by $\{\varphi_n(\eta)\}$), such that

$$\varphi_n(\eta) \Rightarrow \varphi(\eta)$$

in $\Delta(\eta_0, \delta_1)$, where $\varphi(\eta)$ is holomorphic or identical to infinity in $\Delta(\eta_0, \delta_1)$. Since $L(\eta_0) = a_1$ and $L(\eta)$ is nonconstant, there exist points $\eta_n \to \eta_0$, for large enough n, such that $L_n(\eta_n) - a_1 = 0$ by Hurwitz's theorem. And hence

$$\varphi(\eta_0) = \lim_{n\to\infty} \varphi_n(\eta_n) = \lim_{n\to\infty} \frac{L_n(\eta_n) - a_1}{\rho_n^k} = 0. \tag{3}$$

On the other side, it can find that $\eta_0' \in \Delta(\eta_0, \delta_1)$ such that $\eta_0' \neq \eta_0$ and $L(\eta_0') \neq a_1$ according to the isolated of zeros. Thus, it gets, for large enough n, $|L_n(\eta_0') - a_1| > |L(\eta_0') - a_1|/2 > 0$. Hence

$$|\varphi_n(\eta_0')| = \frac{|L_n(\eta_0') - a_1|}{\rho_n^k} > \frac{|L(\eta_0') - a_1|}{2\rho_n^k} \to \infty.$$

Then, $\varphi_n(\eta) \Rightarrow \infty$ in $\Delta(\eta_0, \delta_1)$, which contradicts Eq. (3). So the claim (i) is proved.

Next, we prove the claim (ii). Indeed, assume that $\varphi_n(\eta) = 0$, it gets that $f(z_n + \rho_n\eta) = a_1 \in E_1$ by Eqs. (1) and (2). By the condition $f(z) \in E_1 \Rightarrow f^{(k)}(z) \in E_2$, it gets $f^{(k)}(z_n + \rho_n\eta) \in E_2$. Hence, $|\varphi_n^{(k)}(\eta)| = |f^{(k)}(z_n + \rho_n\eta)| \leq |b_1| + |b_2| + |b_3|$. Thus the claim (ii) holds immediately.

Then, according to the claim (i) and Lemma 1, there exist a subsequence of $\{\varphi_n(\eta)\}$ (still marked as $\{\varphi_n(\eta)\}$), points $\eta_n \to \eta_0$, and $\zeta_n \to 0^+$ such that

$$\Phi_n(\xi) = \frac{\varphi_n(\eta_n + \zeta_n\xi)}{\zeta_n^k} = \frac{L_n(\eta_n + \zeta_n\xi) - a_1}{\zeta_n^k\rho_n^k} \overset{\chi}{\Rightarrow} \Phi(\xi), \tag{4}$$

where $\Phi(\xi)$ is meromorphic and nonconstant in $\mathbb{C}$. What is more, based on the claim (ii), it has $\Phi^\sharp(\xi) \leq \Phi^\sharp(0) = k(|b_1| + |b_2| + |b_3| + 1) + 1$.

It can be asserted that (iii) $\Phi(\xi)$ is an entire function in $\mathbb{C}$, (iv) $\Phi(\xi)$ has no more than l distinct zeros, and (v) $\Phi(\xi) = 0$ if and only if $\Phi^{(k)}(\xi) \in E_2$.

We first prove the claim (iii). Indeed, based on the fact that $\{\Phi_n(\xi)\}$ is holomorphic in $\Delta(\eta_0, \delta)$, and $\eta_n + \zeta_n\xi \to \eta_0$ for each $\xi \in \mathbb{C}$, it obtains that $\Phi(\xi)$ is an entire function in $\mathbb{C}$ by Eq. (4). Thus, the claim (iii) is proved.

In the following, we prove the claim (iv). Suppose, to the contrary, that $\Phi(\xi)$ has (no less than) $l+1$ distinct zeros: $\xi_1, \xi_2, \ldots, \xi_{l+1}$. Then there exist $l+1$ distinct sequences $\{\xi_{nj}\}$ such that $\xi_{nj} \to \xi_j$ and $\Phi_n(\xi_{nj}) = 0 (j = 1, 2, \ldots, l+1)$ by Hurwitz's theorem and Eq. (4). Hence $L_n(\eta_n + \zeta_n\xi_{nj}) - a_1 = 0$. Obviously, $\eta_n + \zeta_n\xi_{nj} \to \eta_0$ and $\eta_n + \zeta_n\xi_{ni} \neq \eta_n + \zeta_n\xi_{nj}$ for $1 \leq i < j \leq l+1$. It gets that η_0 is a zero of $L(\eta) - a_1$ with multiplicities at least $l+1$ by Eq. (1). But, this contradicts the fact that η_0 is a zero of $L(\eta) - a_1$ with multiplicities l. So, the claim (iv) is true.

Last, we prove the claim (v). On the one hand, set $\Phi(\xi_0) = 0$. Combining $\Phi(\xi) \not\equiv 0$ and Eq. (4) with Hurwitz's theorem, it follows that there exist points $\xi_n \to \xi_0$ such that $\Phi_n(\xi_n) = 0$. Then $L_n(\eta_n + \zeta_n\xi_n) - a_1 = 0$. It gets that

$$f(z_n + \rho_n\eta_n + \rho_n\zeta_n\xi_n) = a_1 \in E_1.$$

According to the condition that $f(z) \in E_1 \Rightarrow f^{(k)}(z) \in E_2$, and $\max\limits_{0 \le i \le k-1} |f^{(i)}(z)| = 0$ whenever $f(z) \in E_1$, it yields

$$\Phi_n^{(i)}(\xi_n) = \frac{f^{(i)}(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n)}{\zeta_n^{k-i} \rho_n^{k-i}} = 0 \ (0 \le i \le k-1)$$

and

$$\Phi_n^{(k)}(\xi_n) = f^{(k)}(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n) \in E_2.$$

It gets that $\Phi^{(i)}(\xi_0) = 0 \ (0 \le i \le k-1)$, $\Phi^{(k)}(\xi_0) \in E_2$ according to $\Phi_n^{(i)}(\xi_n) \to \Phi^{(i)}(\xi_0)$. Further, we obtain that all zeros of $\Phi(\xi)$ have multiplicities at least k.

On the other hand, assume that $\Phi^{(k)}(\xi_0) \in E_2$, and $\Phi^{(k)}(\xi_0) = b_1$. It can be concluded that $\Phi^{(k)}(\xi) \not\equiv b_1$. Otherwise, $\Phi(\xi) = b_1 \frac{(\xi - \xi_1)^k}{k!}$ since all zeros of $\Phi(\xi)$ have multiplicities at least k. By simple calculation,

$$\Phi^{\sharp}(0) \le \begin{cases} \frac{k}{2}, & \text{if } |\xi_1| \ge 1, \\ |b_1|, & \text{if } |\xi_1| < 1. \end{cases}$$

That is, $\Phi^{\sharp}(0) < k(|b_1| + |b_2| + |b_3| + 1) + 1$ which contradicts the fact that $\Phi^{\sharp}(0) = k(|b_1| + |b_2| + |b_3| + 1) + 1$. Thus, $\Phi^{(k)}(\xi) \not\equiv b_1$. Then, there exist $\xi_n \to \xi_0$ such that $\Phi_n^{(k)}(\xi_n) = b_1$ by Hurwitz's theorem. Clearly,

$$f^{(k)}(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n) = b_1 \in E_2.$$

It follows from $f(z) \in E_1 \Leftarrow f^{(k)}(z) \in E_2$ that $f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n) \in E_1$. And we claim that there exist a subsequence of $\{f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n)\}$ (still marked as $\{f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n)\}$) such that

$$f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n) = a_1.$$

Otherwise, there exists $f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n) = a_2$ for large enough n. Then, Eq. (4) deduces

$$\Phi(\xi_0) = \lim_{n \to \infty} \Phi_n(\xi_n) = \lim_{n \to \infty} \frac{f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n) - a_1}{\zeta_n^k \rho_n^k} = \lim_{n \to \infty} \frac{a_2 - a_1}{\zeta_n^k \rho_n^k} = \infty,$$

which contradicts the fact that $\Phi^{(k)}(\xi_0) = b_1$. Thus, there exist a subsequence of $\{f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n)\}$ (still marked as $\{f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n)\}$) such that $f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n) = a_1$. Obviously,

$$\Phi(\xi_0) = \lim_{n \to \infty} \Phi_n(\xi_n) = \lim_{n \to \infty} \frac{f(z_n + \rho_n \eta_n + \rho_n \zeta_n \xi_n) - a_1}{\zeta_n^k \rho_n^k} = 0.$$

Hence, the claim (v) is proved.

The claims (iv) and (v) imply that $\Phi^{(k)}(\xi) - b_1$, $\Phi^{(k)}(\xi) - b_2$ and $\Phi^{(k)}(\xi) - b_3$ have only finitely many zeros. Thus, $\Phi(\xi)$ is a polynomial according to the claim (iii) and Lemma 4.

Set

$$\Phi(\xi) = c_p\xi^p + c_{p-1}\xi^{p-1} + \cdots + c_0,$$

where $p \in Z^+$, $c_0, c_1, \ldots, c_p(\neq 0)$ are complex constants. And Lemma 3 yields

$$2T(r, \Phi^{(k)}) \leq \sum_{i=1}^{3} N\left(r, \frac{1}{\Phi^{(k)} - b_i}\right) + S(r, \Phi^{(k)}). \tag{5}$$

Furthermore, the claim (v) deduces

$$\sum_{i=1}^{3} N\left(r, \frac{1}{\Phi^{(k)} - b_i}\right) \leq N\left(r, \frac{1}{\Phi}\right) = p\log r. \tag{6}$$

Obviously, $T(r, \Phi^{(k)}) = (p - k)\log r$ and $S(r, \Phi^{(k)}) = O(1)$. Combining this fact and Eq. (6) with Eq. (5), it yields

$$(p - 2k)\log r \leq O(1), \text{ as } r \to \infty.$$

So, $p \leq 2k$. Moreover, we know that all zeros of $\Phi(\xi)$ have multiplicity at least k. Thus, four cases are divided as follows:

1. When $1 \leq p \leq k - 1$. It obtains that $\Phi(\xi)$ is a polynomial with degree at most $k - 1$. This contradicts the fact that all zeros of $\Phi(\xi)$ have multiplicities at least k.

2. When $p = k$. It gets that $\Phi(\xi) = c_k\frac{(\xi-\xi_1)^k}{k!}$ and $\Phi^{(k)}(\xi) \equiv c_k$, where $c_k \neq 0$. It follows from the claim v that $c_k \in E_2$. Thus, $\Phi^{(k)}(\xi) \in E_2$ for each $\xi \in \mathbb{C}$. However, $\Phi(\xi)$ has only one distinct zero. This contradicts the claim v.

3. When $k + 1 \leq p \leq 2k - 1$. It yields that $\Phi(\xi) = c_p\frac{(\xi-\xi_1)^p}{p!}$. And $\Phi^{(k)}(\xi)$ is a polynomial with degree $p - k$ (≥ 1). Thus, $\Phi^{(k)}(\xi) = a_i (i = 1, 2, 3)$ has at least three distinct zeros. But, $\Phi(\xi)$ has only one distinct zero. This contradicts the claim v.

4. When $p = 2k$, it obtains that $\Phi(\xi) = c_{2k}\frac{(\xi-\xi_1)^k(\xi-\xi_2)^k}{(2k)!}$ or $\Phi(\xi) = c_{2k}\frac{(\xi-\xi_1)^{2k}}{(2k)!}$, and $\Phi^{(k)}(\xi)$ is a polynomial with degree k. Thus, $\Phi^{(k)}(\xi) = a_i (i = 1, 2, 3)$ has at least three distinct zeros. However, $\Phi(\xi)$ has at most two distinct zeros. This contradicts the claim v. Therefore, Theorem 1 is proved.

Proof of Theorem 2 Suppose, to the contrary, $f(z)$ is not a normal function in Δ. Then, based on Lemma 2, there exist points $z_n \in \Delta$, positive numbers $\rho_n \to 0$ such that

$$L_n(\eta) = f(z_n + \rho_n\eta) \overset{\chi}{\Rightarrow} L(\eta),$$

where $L(\eta)$ is a nonconstant holomorphic function in $\mathbb{C}$.

It is asserted that $L'(\eta) = 0$ whenever $L(\eta) = a_1, a_2, a_3$. In fact, assume that $L(\eta_0) = a_1$; then there exist points $\eta_n \to \eta_0$, for large enough n, such that $a_1 = L_n(\eta_n) = f(z_n + \rho_n \eta_n)$ by Hurwitz's theorem and $L(\eta)$ is nonconstant. Thus $|f'(z_n + \rho_n \eta_n)| \leq A$ according to the condition that $|f'(z)| \leq A$ whenever $f(z) = a_i (i = 1, 2, 3)$. Then $|L_n'(\eta_n)| = |\rho_n f'(z_n + \rho_n \eta_n| \leq \rho_n A$. Clearly, $L'(\eta_0) = \lim_{n \to \infty} L_n'(\eta_n) = 0$. Then $L'(\eta) = 0$ whenever $L(\eta) = a_1$. Similarly, it gets that $L'(\eta) = 0$ whenever $L(\eta) = a_2$ or a_3. Then, the claim is proved.

Referring to the fact that $L(\eta)$ is nonconstant, one may easily get $L'(\eta) \not\equiv 0$. And based on the fact $L(\eta)$ is nonconstant and Lemma 3, it follows that

$$
\begin{aligned}
2T(r, L) &\leq \sum_{i=1}^{3} \overline{N}(r, \frac{1}{L - a_i}) + S(r, L) \\
&\leq N(r, \frac{1}{L'}) + S(r, L) \\
&\leq T(r, \frac{1}{L'}) + S(r, L) \\
&\leq T(r, L') + S(r, L) \\
&\leq T(r, L) + S(r, L).
\end{aligned}
$$

One can then obtain $T(r, L) = S(r, L)$, which is a contradiction. Thus, $f(z)$ is a normal function in Δ. This completes the proof of Theorem 2. □

Acknowledgements We thank the referee for his/her valuable comments and suggestions made to this paper. This work is partially supported by the National Natural Science Foundation of China (11501367, 61673257), the Natural Science Foundation of Shanghai (17ZR1419900)

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Hayman, W., Storvick, D.: On normal functions. Bull. Lond. Math. Soc. **3**(2), 193–194 (1971)
2. Hayman, W.K.: Meromorphic functions, vol. 78. Oxford Clarendon Press, Oxford (1964)
3. Le, Y.: Value distribution theory. Springer, Berlin (1993)
4. Lehto, O., Virtanen, K.I.: Boundary behaviour and normal meromorphic functions. Acta Math. **97**(1), 47–65 (1957)
5. Liu, X., Pang, X.: Shared values and normal families. Acta Math. Sin. Chin. Ser. **50**(2), 409–412 (2007)
6. Lohwater, A., Pommerenke, C.: On normal meromorphic functions. Ann. Acad. Sci. Fenn. Ser. A1-Math. **550**, 1–12 (1973)
7. Pang, X., Zalcman, L.: Normal families and shared values. Bull. Lond. Math. Soc. **32**(3), 325–331 (2000)
8. Schwick, W.: Sharing values and normality. Arch. Math. **59**(1), 50–54 (1992)