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ORIGINAL ARTICLE

Real canonical forms for the adjoint action of the Lie groups Sp(V, B) and O(V, B) on their Lie algebras

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Abstract We determine the canonical form of a Hamiltonian matrix $X \in \mathfrak{sp}(2n, \mathbb{R})$ under symplectic similarity, and the canonical form of a matrix $Y \in \mathfrak{o}(m)$ in the orthogonal Lie algebra under similarity. This is a well known problem, and it has been solved by means of different techniques. Our contribution is to provide a new solution through elementary linear algebra. As an application, a list of the non-equivalent two- and four-dimensional quadratic Hamiltonians is given.

Keywords Classical Lie groups · Adjoint action · Canonical forms · Cyclic subspaces · Eigenvalue problem · Hamiltonian matrix · Jordan canonical form

Mathematics Subject Classification Primary 15A21; Secondary 17B20

1 Introduction

Let (V, B) be a $2n$-dimensional real symplectic space endowed with a symplectic form $B : V \times V \to \mathbb{R}$. Given two real linear transformations X_1, X_2 in the symplectic Lie

Honoring the 60th birthday of O. A. Sánchez-Valenzuela.

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algebra $\mathfrak{sp}(V)$, a very well known problem in Lie theory is to determine if X_1, X_2 are simplectically similar, that is, to find out if there exists a symplectic transformation $T \in \mathrm{Sp}(V)$ such that $X_2 = T^{-1} \circ X_1 \circ T$. This is equivalent to determine if X_1 and X_2 are congruent under the adjoint action of the Lie group $\mathrm{Sp}(V)$ into its Lie algebra $\mathfrak{sp}(V)$. Then, determining the different real canonical forms of $X \in \mathfrak{sp}(V)$ under symplectic similarity is equivalent to classify the $\mathrm{Sp}(V)$-adjoint orbits of $\mathfrak{sp}(V)$. Letting (V, B) be now a m-dimensional real vector space endowed with a inner product $B : V \times V \to \mathbb{R}$, the same problem arises for the orthogonal group $\mathrm{O}(V)$: determining the different real canonical forms of $X \in \mathfrak{o}(m)$ under *similarity* is equivalent to classify the $\mathrm{O}(m)$-adjoint orbits of $\mathfrak{o}(m)$.

The aim of this work is to provide a new solution to this problem through elementary ideas: with basic concepts corresponding to a first course on linear algebra, we can compute the real canonical form of a given linear transformation X in the symplectic Lie algebra $\mathfrak{sp}(V)$ under symplectic similarity, and the real canonical form of a given linear transformation Y in the orthogonal Lie algebra $\mathfrak{o}(V)$ under similarity. At our knowledge, there is not a reference for that mentioned before.

A linear transformation X in the symplectic Lie algebra $\mathfrak{sp}(V)$ (and its matrix representation $[X]$, in some fixed basis of V) is called Hamiltonian. There is a vast literature concerning the problem of computing the real canonical form of a Hamiltonian matrix $[X]$ under symplectic similarity (see [1,3,4,8] and the references given therein). For example, in [8] Williamson determined when two Hamiltonian matrices are simplectically similar, but a constructive procedure for computing the canonical forms was not provided. On the other hand, to give explicit canonical blocks for Hamiltonian matrices, Laub and Meyer analyzed in [3] the canonical form of a Hamiltonian matrix restricted to its generalized eigenspaces; in particular, the non-trivial cases where the matrix has zero or pure imaginary eigenvalues are treated by using an extension of the symplectic form. In [4], Lin, Mehrmann and Xu studied canonical forms for Hamiltonian and symplectic matrices or pencils under equivalence transformations that keep the class invariant, as close as possible to a triangular structure in the class. More recently, in [1], Duong and Ushirobira reviewed the problem of obtaining a classification for the adjoint orbits of $\mathrm{Sp}(2n, \mathbb{R})$ into $\mathfrak{sp}(2n, \mathbb{R})$ and $\mathrm{O}(m)$ into $\mathfrak{o}(m)$, in terms of parametrizing the invertible Fitting decomposition of a skew-symmetric map. On the other hand, the best general reference for the study of conjugacy classes and centralizers in a connected semisimple group is the book written by Humphreys, Conjugacy clases in Semisimple Algebraic Groups (see [2]). In the introduction of this book the author points out that it is both a monograph and a survey, and he provides a detailed review of the classical progress made so far on the subject. We invite the interested reader to consult [2] for more details.

Our approach consists the following: let $\mathfrak{g}(V, B)$ denote either a symplectic Lie algebra $\mathfrak{sp}(V)$ or an orthogonal Lie algebra $\mathfrak{o}(V)$, and let $G(V, B)$ denote the corresponding Lie group. A linear transformation $X \in \mathfrak{g}(V, B)$ induces a decomposition of the vector space V in terms of indecomposable cyclic subspaces $V_\lambda \subset V$ associated to the spectrum $\sigma(X)$ of X. Moreover, for $\lambda \neq 0$, each subspace V_λ comes with its dual pair $V_{-\lambda}$ in such a way that $V_\lambda \oplus V_{-\lambda}$ is a subspace endowed with the same type of geometry as V; whereas if $\lambda = 0$, the cyclic subspace V_0 (if any) also admits a similar decomposition, that is, V_0 can be decomposed as an orthogonal sum of subspaces that are equipped with the same type of geometry as V. We use this idea to understand the interaction of the cyclic subspaces associated to the spectrum $\sigma(X)$, and to obtain the canonical form of X under the action of $G(V, B)$. No attempt has been made here to develop or improve the theory presented in previous works on this topic. The important point here is to note that, with elementary concepts in linear algebra, we can obtain either the canonical form of $X \in \mathfrak{sp}(V)$ under symplectic similarity or the canonical form of $Y \in \mathfrak{o}(V)$ under similarity.

The paper is structured as follows: in Sect. 2 we set up notation, terminology, and we present some preliminaries results. Section 3 provides an explicit way to compute either the real canonical form of $X \in \mathfrak{sp}(V)$ under symplectic similarity, or the canonical form of $Y \in \mathfrak{o}(V)$ under similarity. Finally, Sect. 4 contains a brief summary of how the classification problem for quadratic Hamiltonians corresponds to the classification problem for the $\mathrm{Sp}(2n, \mathbb{R})$-adjoint orbits of $\mathfrak{sp}(2n, \mathbb{R})$. Hence, we apply the results obtained in previous sections to describe the non-equivalent quadratic Hamiltonians for the standard low dimensional real symplectic spaces $(\mathbb{R}^2, \omega)$ and $(\mathbb{R}^4, \omega)$, respectively. Our motivation comes from the fact that in [7], Ovando described mechanical systems for quadratic Hamiltonians on a standard symplectic space $(\mathbb{R}^{2n}, \omega)$, making use of the coadjoint orbits of the $(2n + 1)$-dimensional Heisenberg Lie algebra.

2 Preliminaries

Throughout this work, and unless otherwise is stated, V is a finite dimensional real vector space. However, since the eigenvalues of a linear transformation can be complex, sometimes we shall make use of the field of complex numbers $\mathbb{C}$ to finish the arguments. We shall suppose that every real vector space is embedded in some complex space of the same dimension. For any linear transformation $X : V \to V$, $[X]$ denotes its matrix representation in some fixed basis of V.

Let $B : V \times V \to \mathbb{R}$ be a non-degenerate skew-symmetric bilinear form on a $2n$-dimensional vector space V. Then we say that B is a symplectic geometry on V and in this case the pair (V, B) is called a symplectic vector space. Letting $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ be a complex structure on $\mathbb{R}^{2n}$, the standard example is $(\mathbb{R}^{2n}, \omega)$ where $\omega(x, y) = x^t J y$ for all $x, y \in \mathbb{R}^{2n}$. Let $\mathrm{Sp}(V)$ be the isometry group of B, that is, the group of transformations $T \in \mathrm{GL}(V)$ that preserve the symplectic geometry B:

$$\mathrm{Sp}(V) = \{T \in \mathrm{GL}(V) \mid B(Tu, Tv) = B(u, v),\ \forall\, u, v \in V\}.$$

It is a well known fact that $\mathrm{Sp}(V)$ is a Lie group and it is called symplectic. For the standard symplectic space $(\mathbb{R}^{2n}, \omega)$, this group is denoted by $\mathrm{Sp}(2n, \mathbb{R})$ and it can be identified with the linear Lie group $\{A \in \mathrm{Mat}(2n \times 2n, \mathbb{R}) \mid A^t J A = J\}$. Then, we simply write $\mathrm{Sp}(2n)$ when no confusion can arise. An element $T \in \mathrm{Sp}(V)$ is called a *symplectic transformation*, and without loss of generality we also say that its matrix representation $[T]$ is a symplectic matrix, that is, $[T] \in \mathrm{Sp}(2n, \mathbb{R})$.

It is straightforward to verify that the Lie algebra $\mathfrak{sp}(V)$ of $\mathrm{Sp}(V)$ is given by

$$\mathfrak{sp}(V) = \{X \in \mathrm{End}(V) \mid B(Xu, v) + B(u, Xv) = 0,\ \forall\, u, v \in V\}.$$

For the standard symplectic space $(\mathbb{R}^{2n}, \omega)$, this Lie algebra it is denoted by $\mathfrak{sp}(2n, \mathbb{R})$, and it can be identified with the linear Lie algebra $\{A \in \mathrm{Mat}(2n \times 2n, \mathbb{R}) \mid A^t J + JA = 0\}$. Again, we simply write $\mathfrak{sp}(2n, \mathbb{R})$ when no confusion can arise. A linear transformation $X \in \mathfrak{sp}(V)$ is called a Hamiltonian operator, and without loss of generality, we say that its matrix representation $[X]$ is a Hamiltonian matrix, that is, $[X] \in \mathfrak{sp}(2n, \mathbb{R})$.

Given a symplectic transformation $T \in \mathrm{Sp}(V)$ and a Hamiltonian operator $X \in \mathfrak{sp}(V)$, it is easy to see that $T^{-1} \circ X \circ T$ is again a Hamiltonian operator. Then we say that two Hamiltonian operators X_1, X_2 are symplectically similar if there exists a symplectic transformation $T \in \mathrm{Sp}(V)$ such that $X_2 = T^{-1} \circ X_1 \circ T$. Clearly this defines an equivalence relation. Since the adjoint action of $\mathrm{Sp}(V)$ in $\mathfrak{sp}(V)$ is defined by $(T, X) \longmapsto T^{-1} \circ X \circ T$, saying that X_1, X_2 are symplectically similar it is equivalent to say that X_1 and X_2 are congruent under the adjoint action of the Lie group $\mathrm{Sp}(V)$ into its Lie algebra $\mathfrak{sp}(V)$.

Suppose now that $B : V \times V \to \mathbb{R}$ is a non-degenerate positive definite symmetric bilinear form on a m-dimensional vector space V. In this case we say that B is an orthogonal geometry on V, and (V, B) is called an orthogonal vector space. The standard example is $(\mathbb{R}^m, g)$ where $g(x, y) = x^t y$ for all $x, y \in \mathbb{R}^m$. Analogously to the previous setting, we can consider the isometry group of B:

$$\mathrm{O}(V) = \{T \in \mathrm{GL}(V) \mid B(Tu, Tv) = B(u, v),\ \forall\, u, v \in V\}.$$

$\mathrm{O}(V)$ is called an orthogonal Lie group. For the standard orthogonal space $(\mathbb{R}^m, g)$, this group is denoted by $\mathrm{O}(m)$ and it can be identified with the linear Lie group $\{A \in \mathrm{Mat}(m \times m, \mathbb{R}) \mid A^t A = I_m\}$. Then we simply write $\mathrm{O}(m)$ when no confusion can arise. An element $T \in \mathrm{O}(V)$ is called an *orthogonal transformation*. The corresponding orthogonal Lie algebra $\mathfrak{o}(V)$ is given by

$$\mathfrak{o}(V) = \{X \in \mathrm{End}(V) \mid B(Xu, v) + B(u, Xv) = 0,\ \forall\, u, v \in V\},$$

and for the standard orthogonal space $(\mathbb{R}^m, g)$, the corresponding Lie algebra can be identified with the linear Lie algebra of skew-symmetric matrices $\mathfrak{o}(m)$. Hence, we simply write $\mathfrak{o}(m)$ when no confusion can arise. We say that two linear operators $X_1, X_2 \in \mathfrak{o}(V)$ are similar if they are congruent under the action of the adjoint map of the Lie group $\mathrm{O}(V)$ into its Lie algebra $\mathfrak{o}(V)$.

The linear Lie groups $\mathrm{Sp}(2n)$, $\mathrm{O}(m)$ and their corresponding linear Lie algebras are called classical. A very well known and interesting problem in Lie Theory is to determine a classification of the adjoint orbits of classical Lie algebras $\mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{o}(m)$ for non-zero natural numbers m, n. This problem has been solved by means of different techniques (see for example, [1,4] and the references given there). In particular, observe that the problem of providing an explicit classification for the $\mathrm{Sp}(2n, \mathbb{R})$-adjoint orbits of $\mathfrak{sp}(2n, \mathbb{R})$ is equivalent to the problem of determining the real canonical form of a Hamiltonian matrix $X \in \mathfrak{sp}(2n, \mathbb{R})$ under symplectic similarity, and it has been solved in [3,5,8], for example.

Symplectic similarity is more restrictive than ordinary similarity. For example, the Hamiltonian matrices $X_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are both the real Jordan forms for the harmonic oscillator; however, there does not exist a real symplectic transformation $T \in \mathrm{Sp}(2, \mathbb{R})$ such that $X_2 = T^{-1} \circ X_1 \circ T$ (see [5] for more details). Therefore, X_1 and X_2 determine different $\mathrm{Sp}(2, \mathbb{R})$-orbits in the Lie algebra $\mathfrak{sp}(2, \mathbb{R})$. This implies that one should expect more canonical forms than the usual Jordan canonical forms. Analogously to the ordinary similarity case, the eigenvalue structure of a Hamiltonian operator plays an important role to determine its canonical form under symplectic similarity. Then the aim of this work is to provide a new solution to this problem through basic elementary concepts studied in first course on linear algebra, as follows:

Let $B : V \times V \to \mathbb{R}$ be either a symplectic or an orthogonal geometry in V and denote by $\mathfrak{g}(V, B)$ its corresponding Lie algebra. For any endomorphism $X \in \mathfrak{g}(V, B)$, let $\sigma(X)$ be the spectrum of X, that is, $\sigma(X)$ is the set of all the eigenvalues of $X : V \to V$. Suppose that X has r (not necessarily distinct) eigenvalues $\lambda_1, \dots, \lambda_r$, and let us consider the decomposition of V in terms of the $(X - \lambda_i \,\mathrm{Id}_V)|_{V_{\lambda_i}}$-cyclic subspaces V_{λ_i} associated to the spectrum $\sigma(X)$ of X:

$$V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r}. \tag{1}$$

We say that each V_{λ_i} in (1) is a cyclic subspace for short. Observe that each V_{λ_i} is indecomposable in the sense that if V_{λ_i} is written as a direct sum of two $(X - \lambda_i \,\mathrm{Id}_V)|_{V_{\lambda_i}}$-invariant subspaces, then one of them has to be zero.

For each cyclic subspace V_{λ_i}, there exists a cyclic basis $\{e_1^i, \dots, e_{m_i}^i\}$ such that the restriction $X|_{V_{\lambda_i}} : V_{\lambda_i} \to V_{\lambda_i}$ satisfies $X(e_1^i) = \lambda_i e_1^i$ and $X(e_k^i) = \lambda_i e_k^i + e_{k-1}^i$ for $2 \le k \le m_i$. In this basis, the matrix representation of $X|_{V_{\lambda_i}} : V_{\lambda_i} \to V_{\lambda_i}$ is given by

$$[X|_{V_{\lambda_i}}] = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}.$$

Let $m_i = \dim_{\mathbb{R}} V_{\lambda_i}$, then $[X|_{V_{\lambda_i}}] = \lambda_i I_{m_i} + N_{m_i}$, where $I_{m_i} \in \mathrm{Mat}(m_i \times m_i, \mathbb{R})$ denotes the identity matrix, and $N_k \in \mathrm{Mat}(k \times k, \mathbb{R})$ denotes the matrix given by

$$N_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since λ_i is assumed given, $[X|_{V_{\lambda_i}}] = \lambda_i I_{m_i} + N_{m_i}$ is a Jordan block on the Jordan canonical form of $[X]$, and it is completely specified by its size, $m_i = \dim_{\mathbb{R}} V_{\lambda_i}$. Furthermore, it is a well known fact that each V_{λ_i} is an $(X - \lambda_i \operatorname{Id}|_{V_{\lambda_i}})$-invariant subspace of V. Then, the Jordan form Theorem implies the existence of a Jordan matrix associated to $X : V \to V$ with q (not necessarily distinct) Jordan blocks $[X|_{V_{\lambda_i}}]$ written in terms of the cyclic bases of each V_{λ_i}.

Thus, given either a symplectic or an orthogonal geometry $B : V \times V \to \mathbb{R}$ in V, we shall first understand how the different cyclic subspaces V_{λ_i} mutually interact.

For simplicity of the following calculations, in each cyclic subspace V_{λ_i} we fix a cyclic basis $\{e^i_1, \ldots, e^i_{m_i}\}$, and we define $e^i_0 := 0$ for all $1 \le i \le r$.

Lemma 2.1 *Let $X \in \mathfrak{g}(V, B)$ be a linear transformation having $\lambda_i, \lambda_j \in \sigma(X)$ as two (not necessarily distinct) eigenvalues. Let V_{λ_i} and V_{λ_j} be the cyclic subspaces of V associated to λ_i and λ_j, respectively. If $\{e^i_1, \ldots, e^i_{m_i}\}$ and $\{e^j_1, \ldots, e^j_{m_j}\}$ are cyclic bases of V_{λ_i} and V_{λ_j}, respectively, then*

$$(\lambda_i + \lambda_j) B(e^i_k, e^j_l) = -B(e^i_{k-1}, e^j_l) - B(e^i_k, e^j_{l-1})$$

for all $1 \le k \le m_i$ and $1 \le l \le m_j$.

Proof Since $X \in \mathfrak{g}(V, B)$ satisfies that $B(Xu, v) + B(u, Xv) = 0$ for all $u, v \in V$, it is enough to take $u = e^i_k$ for all $1 \le k \le m_i$ and $v = e^j_l$ for all $1 \le l \le m_j$. □

Proposition 2.2 *Let $X \in \mathfrak{g}(V, B)$ be a linear transformation having $\lambda_i, \lambda_j \in \sigma(X)$ as two (not necessarily distinct) eigenvalues. Let V_{λ_i} and V_{λ_j} be cyclic subspaces of V associated to λ_i and λ_j, respectively.*

(1) *If $\lambda_i + \lambda_j \neq 0$, then the restriction $B|_{V_{\lambda_i} \times V_{\lambda_j}} : V_{\lambda_i} \times V_{\lambda_j} \to \mathbb{R}$ is identically zero.*
(2) *If $\lambda_i + \lambda_j = 0$ and the restriction $B|_{(V_{\lambda_i} \oplus V_{\lambda_j}) \times (V_{\lambda_i} \oplus V_{\lambda_j})} : (V_{\lambda_i} \oplus V_{\lambda_j}) \times (V_{\lambda_i} \oplus V_{\lambda_j}) \to \mathbb{R}$ is a non-degenerate bilinear form, then there exist bases for V_{λ_i} and V_{λ_j} for which the matrix of the restriction $B|_{(V_{\lambda_i} \oplus V_{\lambda_j}) \times (V_{\lambda_i} \oplus V_{\lambda_j})}$ is given by*

$$[B|_{(V_{\lambda_i} \oplus V_{\lambda_j}) \times (V_{\lambda_i} \oplus V_{\lambda_j})}] = \begin{bmatrix} 0 & I_{m_i} \\ -I_{m_i} & 0 \end{bmatrix},$$

where $m_i = \dim_{\mathbb{R}} V_{\lambda_i}$.

Proof Assume that $X \in \mathfrak{g}(V, B)$ satisfies the conditions stated above. Then,

(1) If $\lambda_i + \lambda_j \neq 0$, it follows from Lemma 2.1 that $B(e^i_1, e^j_1) = 0$. Now, applying again Lemma 2.1 to $B(e^i_1, e^j_l)$ for $l \ge 2$, we obtain that

$$\begin{aligned}(\lambda_i + \lambda_j) B(e^i_1, e^j_l) &= -B(e^i_0, e^j_l) - B(e^i_1, e^j_{l-1}) \\ &= -B(e^i_1, e^j_{l-1}),\end{aligned}$$

and then $B(e_1^j, e_l^i) = 0$ for all $1 \leq l \leq m_i$. Analogously, we have that $B(e_2^j, e_1^i) = 0$ and hence $B(e_2^j, e_l^i) = 0$ for all $1 \leq l \leq m_i$. In the same way we conclude that $B(e_k^j, e_l^i) = 0$ for all $1 \leq k \leq m_i$ and $1 \leq l \leq m_j$.

(2) Suppose now that $\lambda_i + \lambda_j = 0$. Since $\lambda_i = -\lambda_j$, the bases for V_{λ_i} and V_{λ_j} have the same cardinality, let us say that it is m_i. From Lemma 2.1 it follows that

$$B(e_k^i, e_{l-1}^j) = -B(e_{k-1}^i, e_l^j) \quad \text{for all } 1 \leq k, l \leq m_i.$$

Consequently, we have that $B(e_1^i, e_p^j) = 0$ for all $1 \leq p \leq m_i - 1$ and $B(e_1^i, e_{m_i}^j) \neq 0$ since B is a non-degenerate bilinear form. In the same manner we can see that $B(e_k^j, e_l^i) = 0$ when $k + l \leq m_i$, whereas $B(e_k^j, e_l^i) = (-1)^{m_i - l} B(e_{k-(m_i-l)}^j, e_{m_i}^i)$ when $k + l > m_i$.

Letting $B_{kl} = (-1)^{m_i - l} B(e_{k-(m_i-l)}^j, e_{m_i}^i)$ when $k + l > m_i$, it follows that the matrix $[B_{kl}]$ representation of $B|_{V_{\lambda_i} \times V_{\lambda_j}}$ is given by

$$[B_{kl}] = \begin{bmatrix} 0 & 0 & \dots & 0 & B_{1,m_i} \\ 0 & 0 & \dots & -B_{1,m_i} & B_{2,m_i} \\ \vdots & \vdots & & \vdots & \vdots \\ (-1)^{m_i-1} B_{1,m_i} & (-1)^{m_i-2} B_{2,m_i} & \dots & -B_{m_i-1,m_i} & B_{m_i,m_i} \end{bmatrix},$$

which it is clearly non-singular.

A straightforward calculation shows that $[B_{kl}]^{-1}$ is given by

$$[B_{kl}]^{-1} = \begin{bmatrix} A_{11} & -A_{12} & \dots & (-1)^{m_i-2} A_{1,m_i-1} & (-1)^{m_i-1} A_{1,m_i} \\ A_{12} & -A_{13} & \dots & (-1)^{m_i-2} A_{1,m_i} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_{1,m_i} & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Observe that taking $f_k^i = \sum_{s=1}^{m_i-k+1} (-1)^{s-1} A_{1,k+s-1} e_s^i$, the matrix $[B_{kl}]^{-1}$ changes the basis $\{e_1^i, \dots, e_{m_i}^i\}$ into a basis $\{f_1^i, \dots, f_{m_i}^i\}$ such that $B(f_k^i, e_l^j) = \delta_k^l$. Thus, the matrix representation of $B|_{(V_{\lambda_i} \oplus V_{\lambda_j}) \times (V_{\lambda_i} \oplus V_{\lambda_j})}$ is given by

$$[B|_{(V_{\lambda_i} \oplus V_{\lambda_j}) \times (V_{\lambda_i} \oplus V_{\lambda_j})}] = \begin{bmatrix} 0 & I_{m_i} \\ -I_{m_i} & 0 \end{bmatrix}.$$

□

Remark 2.3 Since the second statement of Proposition 2.2 above implies that there exist bases of V_{λ_i} and V_{λ_j} such that

$$[B|_{(V_{\lambda_i} \oplus V_{\lambda_j}) \times (V_{\lambda_i} \oplus V_{\lambda_j})}] = \begin{bmatrix} 0 & I_{m_i} \\ -I_{m_i} & 0 \end{bmatrix},$$

we can identify the cyclic subspace V_{λ_j} with its dual space $V_{\lambda_i}^*$. In this case we say that the subspace V_{λ_j} is a dual pair of V_{λ_i}.

Corollary 2.4 *Let $X \in \mathfrak{g}(V, B)$ be a linear transformation as in Proposition 2.2(2) above. Then, the matrix representation of $X|_{V_{\lambda_i} \oplus V_{\lambda_j}} : V_{\lambda_i} \oplus V_{\lambda_j} \to V_{\lambda_i} \oplus V_{\lambda_j}$ is given by*

$$[X|_{V_{\lambda_i} \oplus V_{\lambda_j}}] = \left[\begin{array}{c|c} -\lambda_j I_{m_i} - N_{m_i}^t & 0 \\ \hline 0 & \lambda_j I_{m_i} + N_{m_i} \end{array}\right].$$

Proof Since $\lambda_i + \lambda_j = 0$, there exist bases $\{f_1^i, \ldots, f_{m_i}^i\}$ and $\{e_1^j, \ldots, e_{m_i}^j\}$ of V_{λ_i} and V_{λ_j}, respectively, such that

$$[B|_{(V_{\lambda_i} \oplus V_{\lambda_j}) \times (V_{\lambda_i} \oplus V_{\lambda_j})}] = \begin{bmatrix} 0 & I_{m_i} \\ -I_{m_i} & 0 \end{bmatrix}.$$

Then, it will be enough to understand the action of the restriction $X|_{V_{\lambda_i}} : V_{\lambda_i} \to V_{\lambda_i}$ in the basis $\{f_1^i, \ldots, f_{m_i}^i\}$. Recalling that $f_k^i = \sum_{s=1}^{m_i-k+1} (-1)^{s-1} A_{1,k+s-1} e_s^i$, for $1 \leq k < m_i$ we have that

$$\begin{aligned} X(f_k^i) &= \sum_{s=1}^{m_i-k+1} (-1)^{s-1} A_{1,k+s-1} X(e_s^i) \\ &= \sum_{s=1}^{m_i-k+1} (-1)^{s-1} A_{1,k+s-1} \left(\lambda_i e_s^i + e_{s-1}^i\right) \\ &= \lambda_i \sum_{s=1}^{m_i-k+1} (-1)^{s-1} A_{1,k+s-1} e_s^i + \sum_{r=1}^{m_i-k+1} (-1)^{s-1} A_{1,k+s-1} e_{s-1}^i \\ &= \lambda_i f_k^i - f_{k+1}^i \\ &= -\lambda_j f_k^i - f_{k+1}^i, \end{aligned}$$

whereas for $k = m_i$, it follows that

$$\begin{aligned} X(f_{m_i}^i) &= A_{1,m_i} X(e_1^i) \\ &= \lambda_i A_{1,m_i} e_1^i = \lambda_i f_{m_i}^i \\ &= -\lambda_j f_{m_i}^i. \end{aligned}$$

Then, the matrix representation of $X|_{V_{\lambda_i}} : V_{\lambda_i} \to V_{\lambda_i}$ is given by

$$[X|_{V_{\lambda_i}}] = [-\lambda_j I_{m_i} - N_{m_i}^t],$$

where $N_{m_i}^t$ is the transpose matrix of N_{m_i}. □

Remark 2.5 Without loss of generality, consider the standard symplectic space $(\mathbb{R}^{2n}, \omega)$. Clearly, a Hamiltonian matrix $X \in \mathfrak{sp}(2n, \mathbb{R})$ satisfies that $[X]^t[B] + [B][X] = 0$, and hence

$$[X] = \left[\begin{array}{c|c} A & C \\ \hline D & -A^t \end{array}\right], \tag{2}$$

where A, C, and $D \in \mathrm{Mat}(n \times n, \mathbb{R})$ with C and D symmetric.

Observe that the Jordan canonical form of such $X \in \mathfrak{sp}(2n, \mathbb{R})$ is not necessarily a Hamiltonian matrix. However, if $X \in \mathfrak{sp}(2n, \mathbb{R})$ has eigenvalues $\lambda_i, \lambda_j \in \sigma(X)$ satisfying the conditions given in Proposition 2.2 (2), then $V_{\lambda_i} \oplus V_{\lambda_j} \subset V$ is a symplectic subspace and, as a consequence, $X|_{V_{\lambda_i} \oplus V_{\lambda_j}}$ is also a Hamiltonian transformation. Thus, from Corollary 2.4 we conclude that the canonical form of $X|_{V_{\lambda_i} \oplus V_{\lambda_j}} : V_{\lambda_i} \oplus V_{\lambda_j} \to V_{\lambda_i} \oplus V_{\lambda_j}$ is a Hamiltonian matrix:

$$[X|_{V_{\lambda_i} \oplus V_{\lambda_j}}] = \left[\begin{array}{c|c} -\lambda_j I_{m_i} - N_{m_i}^t & 0 \\ \hline 0 & \lambda_j I_{m_i} + N_{m_i} \end{array}\right]. \tag{3}$$

From Proposition 2.2 we also have the following corollaries:

Corollary 2.6 *Let $X \in \mathfrak{g}(V, B)$ be a linear transformation as in Proposition 2.2 above. If $\mu \neq 0$ and 0 are eigenvalues of X, then the restriction $B|_{V_0 \times V_\mu} : V_0 \times V_\mu \to \mathbb{R}$ is identically zero.*

Remark 2.7 If $X \in \mathfrak{g}(V, B)$ is a linear transformation satisfying the conditions stated in Corollary 2.6 above, it follows that the restriction $B|_{V_0 \times V_0} : V_0 \times V_0 \to \mathbb{R}$ is non-degenerate. Furthermore, Proposition 2.2 (2) implies that there exists a basis of V_0 such that

$$[B|_{V_0 \times V_0}] = \begin{bmatrix} 0 & I_{m_i} \\ -I_{m_i} & 0 \end{bmatrix}.$$

Hence, V_0 can be decomposed as follows:

$$V_0 = U_1 \oplus \cdots \oplus U_r \oplus (U_{s_1} \oplus U_{s_1}^*) \oplus \cdots \oplus (U_{s_k} \oplus U_{s_k}^*),$$

where for each $1 \leq i \leq r$, U_i is a subspace endowed with the same type of geometry as V; whereas for each $1 \leq j \leq k$, $U_{s_j}^*$ is the dual pair of the subspace U_{s_j}. Then it follows that $(U_{s_j} \oplus U_{s_j}^*)$ is endowed with the same type of geometry as V.

Corollary 2.8 *Let $X \in \mathfrak{g}(V, B)$ be a linear transformation as in Proposition 2.2 above. If $\lambda \neq 0$ is an eigenvalue of X, then the restriction $B|_{V_\lambda \times V_\lambda} : V_\lambda \times V_\lambda \to \mathbb{R}$ is identically zero.*

Corollary 2.9 *Let $X \in \mathfrak{g}(V, B)$ be a linear transformation as in Proposition 2.2* (2) *above. If $\lambda \neq 0$ is an eigenvalue of X, then $-\lambda$ is also an eigenvalue of X.*

Corollary 2.10 *Let $X \in \mathfrak{g}(V, B)$ be a linear transformation as in Proposition 2.2 above. If $\lambda = a + bi$ is a complex eigenvalue of X with $a \neq 0$ and $b \in \mathbb{R}$, then $-\lambda, \overline{\lambda}$ and $-\overline{\lambda}$ are also eigenvalues of X.*

Now, we can state the main Theorem of this section:

Theorem 2.11 *Let $B : V \times V \to \mathbb{R}$ be a geometry on V and consider the Lie algebra $\mathfrak{g}(V, B)$. Then, any linear transformation $X \in \mathfrak{g}(V, B)$ induces a decomposition of V as an orthogonal sum of the cyclic subspace V_0 associated to $\lambda = 0$ (if any), and dual pairs associated to the nonzero eigenvalues $\pm\lambda_1, \dots, \pm\lambda_t$ of the spectrum $\sigma(X)$ of X:*

$$V = V_0 \oplus (V_{\lambda_1} \oplus V_{-\lambda_1}) \oplus \cdots \oplus (V_{\lambda_t} \oplus V_{-\lambda_t}).$$

Moreover, V_0 also admits a similar decomposition, that is,

$$V_0 = U_1 \oplus \cdots \oplus U_r \oplus (U_{s_1} \oplus U_{s_1}^*) \oplus \cdots \oplus (U_{s_k} \oplus U_{s_k}^*),$$

where each one of the subspaces U_i $(1 \leq i \leq r)$, $U_{s_j} \oplus U_{s_j}^*$ $(1 \leq j \leq k)$ and $V_{\lambda_r} \oplus V_{-\lambda_r}$ $(1 \leq r \leq t)$ are endowed with the same type of geometry as V.

Proof It is a direct consequence of Proposition 2.2. □

Remark 2.12 Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite dimensional representation of a Lie algebra $\mathfrak{g}$ and let $B : V \times V \to \mathbb{R}$ be a bilinear form in V. We say that B is *ρ-invariant* if

$$B(\rho(x)u, v) = B(u, \rho(x)v) \quad \forall x \in \mathfrak{g}, \quad \forall u, v \in V.$$

Suppose that (V, B) is a vector space endowed with either a symplectic or an orthogonal geometry $B : V \times V \to \mathbb{R}$, and let $\rho : \mathfrak{g}(V, B) \to \mathfrak{gl}(V)$ be a completely reducible finite dimensional representation of $\mathfrak{g}(V, B)$. If B is ρ-invariant, then V can be decomposed as follows:

$$V = U_1 \oplus \cdots \oplus U_k \oplus (W_1 \oplus W_1^*) \oplus \cdots \oplus (W_l \oplus W_l^*),$$

where each one of the subspaces U_i $(1 \leq i \leq k)$ and $W_j \oplus W_j^* (1 \leq j \leq l)$ are endowed with the same type of geometry as V (see [6]).

3 Real canonical forms of a linear transformation $X \in \mathfrak{g}(V, B)$ for symplectic and orthogonal geometries

Let V be a real vector space endowed with either a symplectic or an orthogonal geometry $B : V \times V \to \mathbb{R}$, and consider its corresponding Lie algebra $\mathfrak{g}(V, B)$. Using the results obtained in Sect. 2, in this section we shall to determine the real canonical form of a Hamiltonian operator $X \in \mathfrak{sp}(V)$ under symplectic similarity, and the canonical form of a linear operator $X \in \mathfrak{o}(V)$ under similarity.

3.1 Real canonical forms of Hamiltonian operators

From Theorem 2.11 we know that $X \in \mathfrak{sp}(V)$ induces a decomposition of V as an orthogonal sum of the cyclic subspaces associated to the spectrum $\sigma(X)$ of X; hence, we shall first compute the real canonical forms of the restrictions $X|_{V_\lambda} : V_\lambda \to V_\lambda$ for any $\lambda \in \sigma(X)$ and secondly, we shall determine how these real canonical forms mutually interact.

To do this, recall that the complex eigenvalues of a Hamiltonian transformation X appear in foursomes $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ where $\lambda = a + bi$ with $a, b \in \mathbb{R}$, or in real pairs $\lambda, -\lambda$, in each case with equal algebraic multiplicities. First suppose that $X \in \mathfrak{sp}(V)$ has a pair of nonzero real eigenvalues $\lambda, -\lambda$ satisfying the conditions stated in Proposition 2.2 (2). Then $V_\lambda \oplus V_{-\lambda}$ is a symplectic space and as consequence, $X|_{V_\lambda \oplus V_{-\lambda}} \in \mathfrak{sp}(V_\lambda \oplus V_{-\lambda})$. If $X|_{V_\lambda \oplus V_{-\lambda}}$ is diagonalizable, we are done. On the other case, from Corollary 2.4 follows that the matrix representation of $X|_{V_\lambda \oplus V_{-\lambda}} : V_\lambda \oplus V_{-\lambda} \to V_\lambda \oplus V_{-\lambda}$ is a Hamiltonian matrix:

$$[X|_{V_\lambda \oplus V_{-\lambda}}] = \left[\begin{array}{c|c} -\lambda I_m - N_m^t & 0 \\ \hline 0 & \lambda I_m + N_m \end{array}\right].$$

Suppose now that $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ are nonzero complex eigenvalues of $X \in \mathfrak{sp}(V)$, where $\lambda = a + bi$ with $a, b \in \mathbb{R}$. Then real canonical form of $X|_{V_\lambda \oplus V_{-\lambda}}$ is not necessarily given by a Hamiltonian matrix, but we can determine a suitable real basis of the symplectic space $W = (V_\lambda \oplus V_{\lambda}) \oplus (V_{-\lambda} \oplus V_{-\bar{\lambda}})$ for which the restriction $X|_W \in \mathfrak{sp}(V)$ is represented by a Hamiltonian matrix.

Lemma 3.1 *Let $X \in \mathfrak{sp}(V)$ be a linear transformation having a nonzero complex eigenvalue $\lambda = a + ib$ with $a, b \in \mathbb{R}$. Suppose that V_λ is an m-dimensional complex cyclic subspace of V. Then, there exists a basis of the symplectic subspace $W = (V_\lambda \oplus V_{\bar{\lambda}}) \oplus (V_{-\lambda} \oplus V_{-\bar{\lambda}})$ such that the real canonical form of the restriction $X|_W : W \to W$ is given by*

$$[X|_W] = \left[\begin{array}{ccccc|ccccc} A & I_2 & 0 & \dots & 0 & & & & & \\ 0 & A & I_2 & \dots & 0 & & & & & \\ \vdots & 0 & A & \dots & 0 & & & & & \\ & & & \ddots & I_2 & & & & & \\ 0 & \dots & & 0 & A & & & & & \\ \hline & & & & & -A^T & 0 & 0 & \dots & 0 \\ & & & & & -I_2 & -A^T & 0 & \dots & 0 \\ & & & & & 0 & -I_2 & -A^T & \dots & 0 \\ & & & & & & & & \ddots & 0 \\ & & & & & 0 & \dots & 0 & -I_2 & -A^T \end{array}\right],$$

where $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ *and* $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Proof Let $X \in \mathfrak{sp}(V)$ be a Hamiltonian operator having $\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$ as nonzero complex eigenvalues where $\lambda = a + bi$ with $a, b \in \mathbb{R}$. Since $V_{\bar{\lambda}}$, $V_{-\lambda}$ and $V_{-\bar{\lambda}}$ are also m-dimensional complex subspaces of V, there exist real bases $\{e_1 \dots e_{2m}\}$ of $V_\lambda \oplus V_{\bar{\lambda}}$, and $\{f_1 \dots f_{2m}\}$ of $V_{-\lambda} \oplus V_{-\bar{\lambda}}$, such that the canonical form of the restrictions $X|_{V_\lambda \oplus V_{\bar{\lambda}}} : V_\lambda \oplus V_{\bar{\lambda}} \to V_\lambda \oplus V_{\bar{\lambda}}$ and $X|_{V_{-\lambda} \oplus V_{-\bar{\lambda}}} : V_{-\lambda} \oplus V_{-\bar{\lambda}} \to V_{-\lambda} \oplus V_{-\bar{\lambda}}$ are given by

$$[X|_{V_\lambda \oplus V_{\bar{\lambda}}}] = \begin{bmatrix} A & I_2 & 0 & \dots & 0 \\ 0 & A & I_2 & \dots & 0 \\ \vdots & & A & \dots & 0 \\ & & \ddots & & I_2 \\ 0 & \dots & & 0 & A \end{bmatrix}$$

and

$$[X|_{V_{-\lambda} \oplus V_{-\bar{\lambda}}}] = \begin{bmatrix} -A & I_2 & 0 & \dots & 0 \\ 0 & -A & I_2 & \dots & 0 \\ \vdots & & -A & \dots & 0 \\ & & \ddots & & I_2 \\ 0 & \dots & & 0 & -A \end{bmatrix}, \tag{4}$$

where $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then, to obtain the required Hamiltonian matrix, it will be enough to choose a suitable basis of $V_{-\lambda} \oplus V_{-\bar{\lambda}}$. Observe that from Eq. (4) follows that the action of $X|_{V_{-\lambda} \oplus V_{-\bar{\lambda}}}$ in the basis $\{f_1 \dots f_{2m}\}$ of $V_{-\lambda} \oplus V_{-\bar{\lambda}}$ is given by:

$$\begin{aligned} X(f_1) &= -af_1 + bf_2, \\ X(f_2) &= -bf_1 - af_2, \\ X(f_i) &= f_{i-2} - af_i + bf_{i+1}, \quad \text{for odd } i \text{ such that } 3 \le i \le 2m-1, \\ X(f_i) &= f_{i-2} - bf_i - af_{i+1}, \quad \text{for even } i \text{ such that } 4 \le i \le 2m. \end{aligned}$$

Now, defining $\widetilde{f_k} = (-1)^{[2+\frac{k-1}{2}]} f_{2r-(k-1)}$ for $1 \le k \le 2m$, we have that

$$\begin{aligned} X(\widetilde{f_1}) &= -a\widetilde{f_1} - b\widetilde{f_2} - \widetilde{f_3}, \\ X(\widetilde{f_2}) &= b\widetilde{f_1} - a\widetilde{f_2} + \widetilde{f_4}, \\ X(\widetilde{f_i}) &= -a\widetilde{f_i} - 1\widetilde{f}_{i+2} - b\widetilde{f}_{i+3}, \quad \text{for odd } i \text{ such that } 3 \le i < 2m-1, \\ X(\widetilde{f_i}) &= -a\widetilde{f_i} - b\widetilde{f}_{i+1} - 1\widetilde{f}_{i+2}, \quad \text{for even } i \text{ such that } 4 \le i < 2m, \\ X(\widetilde{f}_{2r-1}) &= -a\widetilde{f}_{2r-1} - b\widetilde{f}_{2r}, \\ X(\widetilde{f}_{2r}) &= b\widetilde{f}_{2r-1} - a\widetilde{f}_{2r}. \end{aligned}$$

Hence $\{f_{2m}, f_{2m-1}, \ldots, (-1)^{[m+1]}f_2, (-1)^{[m+1]}f_1\}$ is a basis of $V_{-\lambda}\oplus V_{-\bar{\lambda}}$ for which the canonical form of the restriction $X|_{V_{-\lambda}\oplus V_{-\bar{\lambda}}}$ is given by:

$$[X|_{V_{-\lambda}\oplus V_{-\bar{\lambda}}}] = \begin{bmatrix} -A^t & 0 & 0 & \ldots & 0 \\ -I_2 & -A^t & 0 & \ldots & 0 \\ & -I_2 & -A^t & \ldots & 0 \\ & & \ddots & 0 & \\ 0 & \ldots & & -I_2 & -A^t \end{bmatrix}.$$

As a consequence, letting $W = V_\lambda \oplus V_{\bar{\lambda}} \oplus V_{-\lambda} \oplus V_{-\bar{\lambda}}$, the restriction $X|_W : W \to W$ has a canonical form given by a Hamiltonian matrix. □

Remark 3.2 If $\alpha = bi$ with $b \neq 0$ is an eigenvalue of $X \in \mathfrak{sp}(2n, \mathbb{R})$, we can apply 3.1 to obtain the real canonical form of $X|_{V_\lambda\oplus V_{\bar{\lambda}}}$. For example, if $X \in \mathfrak{sp}(4, \mathbb{R})$ has only one eigenvalue $\alpha = bi$ with $b \neq 0$, its real canonical forms under symplectic similarity are:

$$\begin{bmatrix} 0 & \pm b & 0 & 0 \\ \mp b & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp b \\ 0 & 0 & \pm b & 0 \end{bmatrix}.$$

Now, if $X \in \mathfrak{sp}(V)$ is a linear transformation having $\lambda = 0$ as an eigenvalue, from Proposition 2.2 (2) follows the well known result:

Corollary 3.3 *Let $X \in \mathfrak{sp}(V)$ and suppose that $0 \in \sigma(X)$. Then the algebraic multiplicity of 0 is even.*

Remark 3.4 If the linear transformation $X \in \mathfrak{sp}(V)$ is such that $0 \in \sigma(X)$, recall that Theorem 2.11 implies that the corresponding cyclic subspace V_0 can be decomposed as follows:

$$V_0 = U_1 \oplus \cdots \oplus U_r \oplus (U_{s_1} \oplus U_{s_1}^*) \oplus \cdots \oplus (U_{s_k} \oplus U_{s_k}^*),$$

where for each $1 \leq i \leq r$ (if any) U_i is a symplectic subspace endowed with the same type of geometry as V; whereas for each $1 \leq j \leq k$, $U_{s_j}^*$ denotes the dual pair of the subspace U_{s_j} and moreover, $U_{s_j} \oplus U_{s_j}^*$ is also a symplectic subspace. Since U_i and $U_{s_j} \oplus U_{s_j}^*$ are symplectic subspaces of V, we can apply the ideas described before to determine the canonical form of the restriction $X|_{V_0} : V_0 \to V_0$.

Without loss of generality let us suppose that $\lambda, \mu \in \sigma(X)$ are two not necessarily distinct eigenvalues of $X \in \mathfrak{sp}(V)$. Now, the following Lemma indicates how to obtain the canonical form of the restriction $X|_{(V_\lambda\oplus V_{-\lambda})\oplus(V_\mu\oplus V_{-\mu})} : (V_\lambda \oplus V_{-\lambda}) \oplus (V_\mu \oplus V_{-\mu}) \to (V_\lambda \oplus V_{-\lambda}) \oplus (V_\mu \oplus V_{-\mu})$, in such a way that its matrix representation $[X|_{(V_\lambda\oplus V_{-\lambda})\oplus(V_\mu\oplus V_{-\mu})}]$ satisfies Eq. (2).

Lemma 3.5 *Let $X \in \mathfrak{sp}(V)$ be a Hamiltonian operator and consider the decomposition of V induced by X, as an orthogonal direct sum of symplectic subspaces [see Theorem 2.11)]. If $V_\lambda \oplus V_{-\lambda}$ and $V_\mu \oplus V_{-\mu}$ are two symplectic subspaces in this decomposition, then the canonical form of the restriction $X|_{(V_\lambda \oplus V_{-\lambda})\oplus(V_\mu \oplus V_{-\mu})}$: $(V_\lambda \oplus V_{-\lambda}) \oplus (V_\mu \oplus V_{-\mu}) \to (V_\lambda \oplus V_{-\lambda}) \oplus (V_\mu \oplus V_{-\mu})$ is given by*

$$[X\ |_{(V_\lambda \oplus V_{-\lambda})\oplus(V_\mu \oplus V_{-\mu})}] = \left[\begin{array}{cc|cc} -\lambda I_m - N_m^t & 0 & & \\ 0 & -\mu I_n - N_n^t & & \\ \hline & & \lambda I_m + N_m & 0 \\ & & 0 & \mu I_n + N_n \end{array}\right].$$

Proof Let $V_\lambda \oplus V_{-\lambda}$ and $V_\mu \oplus V_{-\mu}$ be two symplectic subspaces in the decomposition of V induced by X. Then there exist bases $\{e_1, \dots, e_r, f_1, \dots, f_r\}$ of $V_\lambda \oplus V_{-\lambda}$ and $\{g_1, \dots g_s, h_1, \dots, h_s\}$ of $V_\mu \oplus V_{-\mu}$, respectively, such that the restrictions $X|_{V_\lambda \oplus V_{-\lambda}}$ and $X|_{V_\mu \oplus V_{-\mu}}$ have the following matrix representations:

$$[X|_{V_\lambda \oplus V_{-\lambda}}] = \left[\begin{array}{c|c} -\lambda I_m - N_m^t & \\ \hline & \lambda I_m + N_m \end{array}\right],$$

$$[X|_{V_\mu \oplus V_{-\mu}}] = \left[\begin{array}{c|c} -\mu I_n - N_n^t & \\ \hline & \mu I_n + N_n \end{array}\right].$$

Now, it is enough to observe that $\{e_i, g_j, f_i, h_j \mid 1 \le i \le m, 1 \le j \le n\}$ is a basis of the symplectic subspace $(V_\lambda \oplus V_{-\lambda}) \oplus (V_\mu \oplus V_{-\mu})$, and for this basis we have that:

$$[X|_{(V_\lambda \oplus V_{-\lambda})\oplus(V_\mu \oplus V_{-\mu})}] = \left[\begin{array}{cc|cc} -\lambda I_m - N_m^t & 0 & & \\ 0 & -\mu I_n - N_n^t & & \\ \hline & & \lambda I_m + N_m & 0 \\ & & 0 & \mu I_n + N_n \end{array}\right].$$

□

Summarizing, we can compute the real canonical form of a given Hamiltonian operator $X \in \mathfrak{sp}(V)$ under symplectic similarity as follows: first we compute the spectrum $\sigma(X)$ of X and consider the indecomposable cyclic subspaces V_{λ_i}, $i = 1, \dots, r$, associated to the spectrum $\sigma(X)$ of X. Then following Theorem 2.11, we decompose the vector space V as an orthogonal direct sum of symplectic subspaces, $V = (V_{\lambda_1} \oplus V_{-\lambda_1}) \oplus \cdots \oplus (V_{\lambda_t} \oplus V_{-\lambda_t})$, and we may reorder this decomposition in terms of the increasing dimension of each V_{λ_i}. Now, for those pairs of real eigenvalues $\lambda_\ell, -\lambda_\ell$, Corollary 2.4 indicates how to compute the canonical form of the restriction $X|_{V_{\lambda_\ell} \oplus V_{-\lambda_\ell}}$ under symplectic similarity. Whereas for those foursome complex eigenvalues $\lambda_j, \overline{\lambda}_j, -\lambda_j, -\overline{\lambda}_j$, from Lemma 3.1 we can obtain the real canonical form of the restriction $X|_W$ under symplectic similarity, where $W = V_{\lambda_j} \oplus V_{\bar{\lambda}_j} \oplus V_{-\lambda_j} \oplus V_{-\bar{\lambda}_j}$. The next step is to apply Lemma 3.5 to $V_{\lambda_1} \oplus V_{-\lambda_1}$ and $V_{\lambda_2} \oplus V_{-\lambda_2}$ to obtain the real canonical form of the restriction $X|_{(V_{\lambda_1} \oplus V_{-\lambda_1})\oplus(V_{\lambda_2} \oplus V_{-\lambda_2})}$ under symplectic similarity. Finally, to obtain the canonical form of a Hamiltonian operator $X \in \mathfrak{sp}(V)$, we shall continue with this process a finite number of times up to $V_{\lambda_t} \oplus V_{-\lambda_t}$. Observe that in this setting, we can apply the ideas presented above to deal with the cases of zero and pure imaginary complex eigenvalues.

3.2 Real canonical forms for linear transformations in the orthogonal Lie algebra $\mathfrak{o}(V)$

Given a linear transformation $X \in \mathfrak{o}(V)$, we are interested in obtaining its real canonical form under similarity. Since Theorem 2.11 also holds for this case, we can get a decomposition of the vector space V in terms of indecomposable cyclic subspaces V_λ associated to the spectrum $\sigma(X)$ of X. Again, we may reorder the decomposition of V in terms of the increasing dimension of the subspaces V_λ, and we can repeat the procedure described in Sect. 3.1, just noticing that in this case it is not necessary to use Lemma 3.5 because (V, B) is a vector space endowed with an orthogonal geometry $B : V \times V \to \mathbb{R}$.

4 Classification of low-dimensional quadratic Hamiltonians

In this section we describe the non-equivalent quadratic Hamiltonians for a standard real symplectic space of dimension 2 and 4. Our motivation comes from [7], since in this work the coadjoint orbits of the $(2n + 1)$-dimensional Heisenberg Lie algebra are used to describe mechanical systems for quadratic Hamiltonians on a standard symplectic space $(\mathbb{R}^n, \omega)$, where $[\omega] = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. In particular, the motion of n uncoupled harmonic oscillators is described with this setting. For this reason, in this section we shall apply the results obtained in Sects. 2 and 3 before, to provide a list of the nonequivalent quadratic Hamiltonians defined in the standard real symplectic spaces $(\mathbb{R}^2, \omega)$ and $(\mathbb{R}^4, \omega)$.

Consider the standard symplectic space $(\mathbb{R}^{2n}, \omega)$. A *quadratic Hamiltonian* is a quadratic form $H : \mathbb{R}^{2n} \to \mathbb{R}$ such that

$$H(u) = \frac{1}{2} g(Su, u), \tag{5}$$

where the vector $u = (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ is written in terms of a symplectic basis and $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symmetric linear transformation with respect to the canonical inner product g of $\mathbb{R}^{2n}$, that is, S satisfies that $g(Su, v) = g(u, Sv)$ for all $u, v \in \mathbb{R}^{2n}$. It is usual to say that such linear transformation S is a *Hamiltonian operator*, and we denote by M the space of Hamiltonian operators, that is,

$$M := \{S \in \mathfrak{gl}(\mathbb{R}^{2n}) \mid g(Su, v) = g(u, Sv) \quad \text{for all } u, v \in \mathbb{R}^{2n}\}.$$

It is a well known fact that the Hamiltonian equation can be written as follows:

$$u' = JSu, \tag{6}$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is a complex structure in $\mathbb{R}^{2n}$.

Observe that the symplectic space $(\mathbb{R}^{2n}, \omega)$ is also an abelian Lie algebra equipped with two ad-invariant non-degenerate bilinear forms: the symplectic form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ and the canonical inner product $g : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ (moreover, ω is an integrable form). Then, there exists a linear transformation $R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that for all $u, v \in \mathbb{R}^{2n}$,

$$\omega(u, v) = g(u, Rv). \tag{7}$$

Hence, fixing a symplectic basis for $\mathbb{R}^{2n}$, it is easy to verify that $R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ corresponds to the complex structure J given in Eq. (6).

Now consider a linear transformation $X \in \mathfrak{sp}(2n, \mathbb{R})$, then by Eq. (7) we have that

$$g(Xu, Jv) + g(u, JXv) = 0.$$

Since for all $u, v \in \mathbb{R}^{2n}$, $g(Ju, v) + g(u, Jv) = 0$ and $J^2 = -I_{2n}$, we conclude that

$$g(JXu, v) = g(u, JXv).$$

That is, JX is a symmetric linear transformation with respect to the canonical inner product g of $\mathbb{R}^{2n}$.

On the other hand, consider now a Hamiltonian operator $S \in M$. Since $(\mathbb{R}^{2n}, \omega)$ is an abelian Lie algebra and the canonical inner product $g : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is ad-invariant, it follows that $g(v, Su) = g(Su, v) = g(u, Sv)$ for all $u, v \in \mathbb{R}^{2n}$. Thus, by Eq. (7), this is equivalent to:

$$\omega(v, JSu) = \omega(u, JSv);$$

that is,

$$-\omega(JSu, v) - \omega(u, JSv) = 0;$$

and therefore $-JS \in \mathfrak{sp}(2n, \mathbb{R})$. Summarizing, we have proved:

Proposition 4.1 *There is a bijection between the space of Hamiltonian operators and the symplectic Lie algebra* $\mathfrak{sp}(2n, \mathbb{R})$.

Now we are interested in determining the conditions under which two quadratic Hamiltonians are equivalent.

We say that two quadratic Hamiltonians H_1 and H_2 are *equivalent* if and only if there exists a map $T \in \mathrm{Sp}(2n, \mathbb{R})$ such that $H_1(u) = H_2(Tu)$ for all $u \in \mathbb{R}^{2n}$. By definition we know that $H_1(u) = \frac{1}{2}g(S_1u, u)$ and $H_2(T(u)) = \frac{1}{2}g(S_2Tu, Tu) = g(Tu, S_2Tu)$ for all $u \in \mathbb{R}^{2n}$, where S_1 and $S_2 \in M$ are Hamiltonian operators. Thus fixing a symplectic basis for $(\mathbb{R}^{2n}, \omega)$, condition $H_1(u) = H_2(Tu)$ holds if and only if the corresponding matrix representations satisfy

$$[S_1]^t = [T]^t[S_2][T].$$

Table 1 Nonequivalent quadratic Hamiltonians for $\mathbb{R}^2$

Eigenvalues $\lambda \in \sigma(X)$	Decomposition of $\mathbb{R}^2$ given by case	Canonical forms for $X \in \mathfrak{sp}(2, \mathbb{R})$	Quadratic Hamiltonian $H(x, y)$
$\lambda = 0$	4.1.1.1	0	0
$\lambda = 0$	4.1.1.2	$\begin{bmatrix} 0 & \pm 1 \\ 0 & 0 \end{bmatrix}$	$\pm y^2/2$
$\lambda \in \mathbb{R} - \{0\}$	4.1.2.1	$\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$	$\lambda x y$
$\lambda = bi, b \in \mathbb{R} - \{0\}$	4.1.2.2	$\begin{bmatrix} 0 & \pm b \\ \mp b & 0 \end{bmatrix}$	$\mp b(x^2 + y^2)/2$

Since $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear transformation in the symplectic Lie group $\mathrm{Sp}(2n, \mathbb{R})$, it is easy to verify that $[T]^t = -J[T]^{-1}J$ and hence,

$$[T]^t[S_2][T] = -J[T]^{-1}J[S_2][T].$$

Now from Proposition 4.1 above it follows that $JX_2 \in \mathfrak{sp}(2n, \mathbb{R})$. Thus, the classification problem for quadratic Hamiltonians corresponds to the classification problem for the orbits of the adjoint action of the symplectic Lie group $\mathrm{Sp}(2n, \mathbb{R})$ into its symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$:

$$\mathrm{Sp}(2n, \mathbb{R}) \times \mathfrak{sp}(2n, \mathbb{R}) \to \mathfrak{sp}(2n, \mathbb{R}),$$
$$(T, X) \mapsto T^{-1}XT.$$

Hence, the results established in Sect. 2 can be used to determine the canonical form of a Hamiltonian transformation $X : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ under symplectic similarity. We shall exemplify this in the following setting: first, consider the real symplectic space $(\mathbb{R}^2, \omega)$. To determine when two quadratic Hamiltonians H_1 and H_2 are equivalent, we shall compute the canonical forms of the corresponding Hamiltonian transformations $X_1, X_2 \in \mathfrak{sp}(2n, \mathbb{R})$ under symplectic similarity. Then we consider the real symplectic space $(\mathbb{R}^4, \omega)$ and we proceed in the same way. Hence, following the notation introduced in Sects. 4.1 and 4.2 below, the next Theorem summarizes the main results of this section:

Theorem 4.2 *For $n = 1, 2$ consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega)$. Given a Hamiltonian transformation $X \in \mathfrak{sp}(2n, \mathbb{R})$ with spectrum $\sigma(X)$, let $[X]$ denote its canonical form under symplectic similarity.*

(i) *For $n = 1$, consider $X \in \mathfrak{sp}(2, \mathbb{R})$, then the list of non-equivalent quadratic Hamiltonians $H : \mathbb{R}^2 \to \mathbb{R}$ is given by Table 1 as above.*

(ii) *For $n = 2$, consider $X \in \mathfrak{sp}(4, \mathbb{R})$. Then the list of non-equivalent quadratic Hamiltonians $H : \mathbb{R}^4 \to \mathbb{R}$ is given by Table 2 as follows:*

Table 2 Nonequivalent quadratic Hamiltonians for $\mathbb{R}^4$

Eigenvalues $\lambda \in \sigma(X)$	Decomposition of $\mathbb{R}^4$ given by case	Canonical form for $X \in \mathfrak{sp}(4, \mathbb{R})$	Quadratic Hamiltonian $H(u_1, u_2, v_1, v_2)$
$\lambda = 0$	4.2.1.1	0	0
$\lambda = 0$	4.2.1.2	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$u_2 v_1 \pm {v_2}^2/2$
$\lambda = 0$	4.2.1.3	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$u_2 v_1$
$\lambda = 0$ $\mu \in \mathbb{R} - \{0\}$	4.2.1.4	$\begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\mu u_1 v_1$
$\lambda = 0$ $\mu = bi, b \in \mathbb{R} - \{0\}$	4.2.1.5	$\begin{bmatrix} 0 & 0 & \pm b & 0 \\ 0 & 0 & 0 & 0 \\ \mp b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\pm b(u_1^2 + v_1^2)/2$
$\lambda = 0$ $\mu \in \mathbb{R} - \{0\}$	4.2.1.6	$\begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\mu u_1 v_1 \pm v_2^2/2$
$\lambda = 0$ $\mu = bi, b \in \mathbb{R} - \{0\}$	4.2.1.7	$\begin{bmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & \pm 1 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & -b & 0 \\ 0 & 0 & 0 & \pm 1 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$(b(u_1^2 + v_1^2) \pm v_2^2)/2$ $(-b(u_1^2 + v_1^2) \pm v_2^2)/2$
$\mu = a + bi$ $a, b \in \mathbb{R} - \{0\}$	4.2.2.1	$\begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & -a & b \\ 0 & 0 & -b & -a \end{bmatrix}$	$a(u_1 v_1 + u_2 v_2) + b(u_2 v_1 - u_1 v_2)$
$\mu = bi$ $b \in \mathbb{R} - \{0\}$	4.2.2.2	$\begin{bmatrix} 0 & 0 & 0 & \mp b \\ 0 & 0 & \mp b & 0 \\ 0 & \pm b & 0 & 0 \\ \pm b & 0 & 0 & 0 \end{bmatrix}$	$\mp b(u_1 u_2 + v_1 v_2)$
$\mu = bi$ $b \in \mathbb{R} - \{0\}$	4.2.2.3	$\begin{bmatrix} 0 & \pm b & 1 & 0 \\ \mp b & 0 & 0 & 1 \\ 0 & 0 & 0 & \pm b \\ 0 & \mp b & 0 & 0 \end{bmatrix}$	$(v_1^2 + v_2^2)/2 \pm b(u_2 v_1 - u_1 v_2)$

Table 2 continued

Eigenvalues $\lambda \in \sigma(X)$	Decomposition of $\mathbb{R}^4$ given by case	Canonical form for $X \in \mathfrak{sp}(4, \mathbb{R})$	Quadratic Hamiltonian $H(u_1, u_2, v_1, v_2)$
$\mu = bi$ $\nu = ci$ $b, c \in \mathbb{R} - \{0\}$	4.2.2.4	$\begin{bmatrix} 0 & 0 & \mp c & 0 \\ 0 & 0 & 0 & -b \\ \pm c & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & \mp c & 0 \\ 0 & 0 & 0 & b \\ \pm c & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{bmatrix}$	$\mp(c(u_1^2 + v_1^2)/2 \pm (u_2^2 + v_2^2))/2$ $\mp(c(u_1^2 + v_1^2)/2 \mp (u_2^2 + v_2^2))/2$
$\mu = bi, b \in \mathbb{R} - \{0\}$ $\nu \in \mathbb{R} - \{0\}$	4.2.2.5	$\begin{bmatrix} -\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp b \\ 0 & 0 & \nu & 0 \\ 0 & \pm b & 0 & 0 \end{bmatrix}$	$-\nu u_1 v_1 \mp b(u_1^2 + v_2^2)/2$
$\mu \in \mathbb{R} - \{0\}$	4.2.2.6	$\begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix}$	$\mu(u_1 v_1 + u_2 v_2)$
$\mu \in \mathbb{R} - \{0\}$	4.2.2.7	$\begin{bmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & -1 & -\mu \end{bmatrix}$	$\mu(u_1 v_1 + u_2 v_2) + u_2 v_1$
$\mu, \nu \in \mathbb{R} - \{0\}$ $\mu \neq \nu$	4.2.2.8	$\begin{bmatrix} -\nu & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix}$	$-\mu u_2 v_2 - \nu u_1 v_1$

Proof Given $X \in \mathfrak{sp}(2n, \mathbb{R})$ for $n = 1, 2$, we shall analyze the spectrum $\sigma(X)$ to provide its possible canonical forms under symplectic similarity, and thus, to establish the corresponding quadratic Hamiltonian. For this we shall proceed by cases, as follows.

4.1 Case 1

$(\mathbb{R}^2, \omega)$. Let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ be a complex structure on $(\mathbb{R}^2, \omega)$ and let $X \in \mathfrak{sp}(2, \mathbb{R})$ be a Hamiltonian operator with spectrum $\sigma(X)$. Since $\mathbb{R}^2$ is an irreducible symplectic space, we have to consider the following cases:
4.1.1. Suppose that $\mathbb{R}^2 = V_0$, that is, $\lambda = 0$ is the unique eigenvalue of $X \in \mathfrak{sp}(2, \mathbb{R})$. Thus, we have the following:
4.1.1.1 When $X \in \mathfrak{sp}(2, \mathbb{R})$ is diagonalizable (that is, X is identically zero), we have that $H(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$.
4.1.1.2 If $X \in \mathfrak{sp}(2, \mathbb{R})$ is not diagonalizable, then their canonical forms under symplectic similarity are given by

$$[X] = \begin{bmatrix} 0 & \pm 1 \\ 0 & 0 \end{bmatrix},$$

and the corresponding quadratic Hamiltonians are $H(x, y) = \pm y^2/2$.
4.1.2. Suppose that $\mathbb{R}^2 = V_\lambda \oplus V_{-\lambda}$ with $\lambda \neq 0$, that is, $\sigma(X) = \{\lambda, -\lambda | \lambda \neq 0\}$, and let $(x, y) \in (\mathbb{R}^2, \omega)$. We shall consider two cases: either $\lambda \in \mathbb{R} - \{0\}$ or $\lambda \in \mathbb{C} - \{0\}$.
4.1.2.1 In the first case, X is diagonalizable, its canonical form under simplectic similarity is given by

$$[X] = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix},$$

and the corresponding quadratic Hamiltonian is $H(x, y) = \lambda xy$.
4.1.2.2 In the second case we necessarily have $Re(\lambda) = 0$, that is, $\lambda = bi$ with $b \neq 0$. Then the non-equivalent real canonical forms of X under symplectic similarity and their corresponding quadratic Hamiltonians are given by

$$[X] = \begin{bmatrix} 0 & \pm b \\ \mp b & 0 \end{bmatrix}, \qquad H(x, y) = \mp b(x^2 + y^2)/2.$$

4.2 Case 2

$(\mathbb{R}^4, \omega)$. Let $X : \mathbb{R}^4 \to \mathbb{R}^4$ be a Hamiltonian operator with spectrum $\sigma(X)$. Analogously to Case 1 above, we shall decompose $\mathbb{R}^4$ as a direct sum of cyclic subspaces, according to $\lambda = 0 \in \sigma(X)$ or $\lambda = 0 \notin \sigma(X)$.
4.2.1. If $\lambda = 0 \in \sigma(X)$, then its algebraic multiplicity must be either 2 or 4.

Suppose that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 4 and let $(u_1, u_2, v_1, v_2) \in (\mathbb{R}^4, \omega)$. Then $\mathbb{R}^4 = V_0$ or it can be decomposed as $\mathbb{R}^4 = U \oplus U^*$, where $U \subset \mathbb{R}^4$ is a two-dimensional subspace and U^* is its dual pair.
4.2.1.1 Suppose that $\mathbb{R}^4 = V_0$. The trivial case is obtained when $X : V_0 \to V_0$ is diagonalizable (that is, X is identically zero) and thus, $H(u_1, u_2, v_1, v_2) = 0$.
4.2.1.2 If $X : V_0 \to V_0$ is not diagonalizable, we compute its Jordan form on the basis of eigenvectors. From this basis we can obtain suitable symplectic bases to obtain two non-equivalent Hamiltonian matrix representations for X:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to [X] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

and hence $H(u_1, u_2, v_1, v_2) = u_2 v_1 \pm {v_2}^2/2$.
4.2.1.3 Suppose now that $\mathbb{R}^4 = U \oplus U^*$, where $U \subset V$ is a two-dimensional subspace and U^* is its dual space. Then the Jordan form of X can be turned into a Hamiltonian matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to [X] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

and the corresponding quadratic Hamiltonian is $H(u_1, u_2, v_1, v_2) = u_2 v_1$.

Suppose that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 2. Then, V_0 is an irreducible two-dimensional symplectic space and thus, $\mathbb{R}^4 = V_0 \oplus (V_\mu \oplus V_{-\mu})$ with $\mu \in \mathbb{F} - \{0\}$. From Case 1 above follows that:

4.2.1.4 $\mathbb{R}^4 = V_0 \oplus (V_\mu \oplus V_{-\mu})$, where both $X|_{V_0}$ and $X|_{V_\mu \oplus V_{-\mu}}$ are diagonalizable over $\mathbb{R}$. In this case necessarily, $\mu \in \mathbb{R} - \{0\}$, then

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix} \to [X] = \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and the corresponding quadratic Hamiltonian is $H(u_1, u_2, v_1, v_2) = \mu u_1 v_1$.

4.2.1.5 $\mathbb{R}^4 = V_0 \oplus (V_\mu \oplus V_{-\mu})$, where $X|_{V_0}$ is diagonalizable over $\mathbb{R}$ but $X|_{V_\mu \oplus V_{-\mu}}$ is not. In this case, we necessarily have that $\mu = -bi$ with $b \in \mathbb{R} - \{0\}$. Then we obtain two non-equivalent real Hamiltonian matrices and their corresponding quadratic Hamiltonians:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm b \\ 0 & 0 & \mp b & 0 \end{bmatrix} \to [X] = \begin{bmatrix} 0 & 0 & \pm b & 0 \\ 0 & 0 & 0 & 0 \\ \mp b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with $H(u_1, u_2, v_1, v_2) = \pm b(u_1^2 + v_1^2)/2$.

4.2.1.6 $\mathbb{R}^4 = V_0 \oplus (V_\mu \oplus V_{-\mu})$, where $X|_{V_\mu \oplus V_{-\mu}}$ is diagonalizable over $\mathbb{R}$ but $X|_{V_0}$ is not. In this case it follows that $\mu \in \mathbb{R} - \{0\}$. Then we obtain two non-equivalent Hamiltonian matrices and their corresponding quadratic Hamiltonians:

$$\begin{bmatrix} 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix} \to [X] = \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with $H(u_1, u_2, v_1, v_2) = \mu u_1 v_1 \pm v_2^2/2$.

4.2.1.7 $\mathbb{R}^4 = V_0 \oplus (V_\mu \oplus V_{-\mu})$, where neither $X|_{V_0}$ nor $X|_{V_\mu \oplus V_{-\mu}}$ are diagonalizable over $\mathbb{R}$. In this case it follows that $\mu = -bi$ with $b \in \mathbb{R} - \{0\}$, and we obtain four non-equivalent canonical forms under symplectic similarity:

$$\begin{bmatrix} 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{bmatrix} \to [X] = \begin{bmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & \pm 1 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = (b(u_1^2 + v_1^2) \pm v_2^2)/2$, and

$$\begin{bmatrix} 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{bmatrix} \to [X] = \begin{bmatrix} 0 & 0 & -b & 0 \\ 0 & 0 & 0 & \pm 1 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = (-b(u_1^2 + v_1^2) \pm v_2^2)/2$.
4.2.2. If $\lambda = 0 \notin \sigma(X)$, according to Sect. 2 we shall analyze the possibilities for the decomposition of $\mathbb{R}^4$ as a direct sum of cyclic subspaces associated to the spectrum $\sigma(X)$:
4.2.2.1 $\mathbb{R}^4 = V_\mu \oplus V_{-\mu} \oplus V_{\bar{\mu}} \oplus V_{-\bar{\mu}}$ with $\mu = a + bi, a, b \in \mathbb{R} - \{0\}$.

$$\begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & -a & -b \\ 0 & 0 & b & -a \end{bmatrix} \to [X] = \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & -a & b \\ 0 & 0 & -b & -a \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = a(u_1v_1 + u_2v_2) + b(u_2v_1 - u_1v_2)$.
4.2.2.2 $\mathbb{R}^4 = V_\mu \oplus V_{-\mu}$, with $\mu = bi, b \in \mathbb{R} - \{0\}$ an eigenvalue of geometric multiplicity 2. In this case we obtain two non-equivalent real canonical forms under symplectic similarity:

$$\begin{bmatrix} 0 & \pm b & 0 & 0 \\ \mp b & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm b \\ 0 & 0 & \mp b & 0 \end{bmatrix} \to [X] = \begin{bmatrix} 0 & 0 & 0 & \mp b \\ 0 & 0 & \mp b & 0 \\ 0 & \pm b & 0 & 0 \\ \pm b & 0 & 0 & 0 \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = \mp b(u_1u_2 + v_1v_2)$.
4.2.2.3 $\mathbb{R}^4 = V_\mu \oplus V_{-\mu}$, with $\mu = bi, b \in \mathbb{R} - \{0\}$ an eigenvalue of geometric multiplicity 1. In this case we also obtain two non-equivalent real canonical forms under symplectic similarity:

$$\begin{bmatrix} 0 & \pm b & 1 & 0 \\ \mp b & 0 & 0 & 1 \\ 0 & 0 & 0 & \pm b \\ 0 & 0 & \mp b & 0 \end{bmatrix} \to [X] = \begin{bmatrix} 0 & \pm b & 1 & 0 \\ \mp b & 0 & 0 & 1 \\ 0 & 0 & 0 & \pm b \\ 0 & \mp b & 0 & 0 \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = (v_1^2 \pm 2b(u_2v_1 - u_1v_2) + v_2^2)/2$.
4.2.2.4 $\mathbb{R}^4 = (V_\mu \oplus V_{-\mu}) \oplus (V_\nu \oplus V_{-\nu})$, with $\mu \neq \nu$ non-zero purely imaginary eigenvalues such that $\mu = bi$ and $\nu = ci$ with $b, c \in \mathbb{R} - \{0\}$. In this case we have four non-equivalent real canonical forms under symplectic similarity:

$$\begin{bmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm c \\ 0 & 0 & \mp c & 0 \end{bmatrix} \to [X] = \begin{bmatrix} 0 & 0 & \mp c & 0 \\ 0 & 0 & 0 & -b \\ \pm c & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = \mp(c(u_1^2 + v_1^2) - b(u_2^2 + v_2^2))/2$,

$$\begin{bmatrix} 0 & -b & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm c \\ 0 & 0 & \mp c & 0 \end{bmatrix} \rightarrow [X] = \begin{bmatrix} 0 & 0 & \mp c & 0 \\ 0 & 0 & 0 & b \\ \pm c & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = \mp(c(u_1^2 + v_1^2) + b(u_2^2 + v_2^2))/2$.
4.2.2.5 $\mathbb{R}^4 = (V_\mu \oplus V_{-\mu}) \oplus (V_\nu \oplus V_{-\nu})$, with $\mu = bi, b, \nu \in \mathbb{R} - \{0\}$. In this case we have two non-equivalent canonical forms under symplectic similarity:

$$\begin{bmatrix} 0 & \pm b & 0 & 0 \\ \mp b & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & -\nu \end{bmatrix} \rightarrow [X] = \begin{bmatrix} -\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp b \\ 0 & 0 & \nu & 0 \\ 0 & \pm b & 0 & 0 \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = -\nu u_1 v_1 \mp b(u_1^2 + v_2^2)/2$.
4.2.2.6 $\mathbb{R}^4 = V_\mu \oplus V_{-\mu}$, with $\mu \neq 0$ and $X|_{V_\mu}$ is diagonalizable over $\mathbb{R}$:

$$\begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix} \rightarrow [X] = \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = \mu(u_1 v_1 + u_2 v_2)$.
4.2.2.7 $\mathbb{R}^4 = V_\mu \oplus V_{-\mu}$, with $\mu \neq 0$ and $X|_{V_\mu}$ is not diagonalizable over $\mathbb{R}$:

$$\begin{bmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 1 \\ 0 & 0 & 0 & -\mu \end{bmatrix} \rightarrow [X] = \begin{bmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & -1 & -\mu \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = \mu(u_1 v_1 + u_2 v_2) + u_2 v_1$.
4.2.2.8 $\mathbb{R}^4 = (V_\mu \oplus V_{-\mu}) \oplus (V_\nu \oplus V_{-\nu})$, with $\mu \neq \nu$ and $\mu, \nu \in \mathbb{R} - \{0\}$.

$$\begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & -\nu \end{bmatrix} \rightarrow [X] = \begin{bmatrix} -\nu & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix},$$

where $H(u_1, u_2, v_1, v_2) = -\mu u_2 v_2 - \nu u_1 v_1$. □

References

1. Duong, M.T., Ushirobira, R.: Singular quadratic Lie superalgebras. J. Algebra **407**, 372–412 (2014)

2. Humphreys, J.E.: Conjugacy classes in semisimple algebraic groups. In: Mathematical Surveys and Monographs, vol. 43. American Mathematical Society, Providence, RI (1995)
3. Laub, A.J., Meyer, K.: Canonical forms for symplectic and hamiltonian matrices. Celest. Mech. **9**(2), 213–238 (1974)
4. Lin, W., Mehrmann, V., Xu, H.: Canonical Forms for Hamiltonian and symplectic matrices and pencils. Linear Algebra Appl. **302–303**, 469–533 (1999)
5. Meyer, K.R., Hall, G.R., Offin, D.: Introduction to Hamiltonian Dynamical Systems and the N-Body Problem. Applied Mathematical Sciences, vol. 90. Springer (2009)
6. Onischick, A., Vinberg, E.: Lie Groups and Algebraic Groups. Springer Series in Soviet Mathematics (1990)
7. Ovando, G.: Small oscillations and the Heisenberg Lie algebra. J. Phys. A-Math. Gen. **40**, 2407–2424
8. Williamson, J.: On the algebraic problem concerning the normal forms of linear dynamical systems. Am. J. Math. **58**(1), 141–163 (1936)