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ORIGINAL ARTICLE

On metric semigroups-valued functions of bounded Riesz-Φ-variation in several variables

M. Bracamonte[1] · J. Ereú[1] · J. Giménez[2] · N. Merentes[3]

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Abstract Using classical techniques related to the so-called Hardy–Vitali variation, we present the class of X-valued functions of bounded Φ-variation in several variables, where $(X, d, +)$ is a metric semigroup. We exhibit some of the main properties of this class; among them, we show that this class can be made into a normed space and present a counterpart of the renowned Riesz's Lemma for the case in which $X = \mathbb{R}$ with its usual metric.

Keywords Metric semigroup · Φ-Variation · φ-function · Riesz's Lemma

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✉ M. Bracamonte
mireyabracamonte@ucla.edu.ve

J. Ereú
jereu@ucla.edu.ve

J. Giménez
jgimenez@ula.ve

N. Merentes
nmerucv@gmail.com

1 Departamento de Matemáticas, Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Venezuela

2 Departamento de Matemáticas, Universidad de Los Andes, Mérida, Venezuela

3 Escuela de Matemáticas, Universidad Central de Venezuela, Caracas, Venezuela

1 Introduction

BV functions of a single variable were first introduced by Camille Jordan, in a paper [21] that deals with the convergence of Fourier series. Soon after Jordan's work, many mathematicians began to study notions of bounded variation for functions of several variables. There are various approaches to the notion of variation for functions of several variables. We can mention those belonging to Vitali, Hardy, Krause, Arzela, Frechét, Tonelli, Hahn, Kronrod-Vitushkin, Minlos, and others. Functions of bounded variation in n variables ($n > 1$) belonging to each of these classes have more or less the same properties as the functions of bounded variation of one variable. However, there are some properties in the one-dimensional case that cannot be transferred automatically to the multidimensional one (see [22]). On the other hand, functions of bounded variation in $\mathbb{R}^n$ can be identified with n-dimensional normal currents in $\mathbb{R}^n$. This point of view is due to Federer [18].

In the literature, the many notions of bounded variation are mainly studied for functions defined on a rectangle $J \subset \mathbb{R}^n$. A definition of the variation in the sense of Hardy and Krause is given in [22]. Let $f : [0, 1]^n \to \mathbb{R}$. Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be elements of $[0, 1]^n$ such that $\mathbf{a} < \mathbf{b}$ (see Sect. 2). The n-dimensional difference operator Δ^n, which assigns to the axis-parallel rectangle $[\mathbf{a}, \mathbf{b}]$ a n-dimensional quasi-volume

$$\Delta^{(n)}(f; [\mathbf{a}, \mathbf{b}]) := \sum_{j_1=0}^{1} \cdots \sum_{j_n=0}^{1} (-1)^{j_1+\cdots+j_n} f(b_1 + j_1(a_1 - b_1), \dots, b_n + j_n(a_n - b_n)).$$

For $s = 1, \dots, n$ let $0 = x_0^{(s)} < x_1^{(s)} < \dots x_{m_s}^{(s)} = 1$ be a partition of $[0, 1]$, and $\mathcal{P}$ be the partition of $[0, 1]^n$ which is given by

$$\mathcal{P} = \left\{ \left[x_{l_1}^{(1)}, x_{l_1+1}^{(1)} \right] \times \cdots \times \left[x_{l_n}^{(n)}, x_{l_n+1}^{(n)} \right], l_s = 0, \dots, m_s - 1, s = 1, \dots, n \right\}. \tag{1.1}$$

Then the variation of f on $[0, 1]^n$ in the sense of Vitali is given by

$$V^{(d)}(f; [0, 1]^n) := \sup_{\mathcal{P}} \sum_{A \in \mathcal{P}} |\Delta^{(n)}(f; A)|,$$

where the supremum is extended over all partitions of $[0, 1]^n$ into axis-parallel boxes generated by d one-dimensional partitions of $[0, 1]$, as in (1.1).

If the same functions restricted to the various faces of $[0, 1]^s$ with $s = 1, \dots, n$ are of bounded variation in the sense of Vitali over each of them, then f is said to be of bounded variation on $[0, 1]^s$ in the sense of Hardy and Krause, that is, for $1 \le s \le n$ and $1 \le i_1 < \cdots < i_s \le n$, let $V^{(s)}(f; i_1, \dots, i_s; [0, 1]^n)$ denote the s-dimensional variation in the sense of Vitali of the restriction of f to the face

$$U_d^{(i_1,\ldots,i_s)} = \{(x_1,\ldots,x_n) \in [0,1]^n : x_j = 1 \quad \text{for all} \quad j \neq i_1,\ldots,i_s\}$$

of $[0,1]^n$. Then the variation of f on $[0,1]^n$ in the sense of Hardy and Krause anchored at $\mathbf{1}$, abbreviated by HK-variation, is given by

$$\mathrm{Var}_{HK}(f;[0,1]^n) = \sum_{s=1}^{n} \sum_{1\leq i_1<\cdots<i_s\leq n} V^{(s)}(f;i_1,\ldots,i_s;[0,1]^n). \tag{1.2}$$

Note that for the definition of the HK-variation in (1.2), we add the n-dimensional variation in the sense of Vitali plus the variation in the sense of Vitali on all lower dimensional faces of $[0,1]^n$ which are adjacent to $\mathbf{1}$.

On the other hand, a function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be of bounded Tonelli-variation if a.e. in $(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)$ it is of bounded variation in each variable x_j for all $1 \leq j \leq n$ and if these variations $\mathrm{BV}_j(f(x)) := \mathrm{BV}_{x_j\in\mathbb{R}(f(x))}$ are Lebesgue integrable as functions of the other $n-1$ variables $x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$:

$$\mathrm{VT}(f) := \sum_{j=1}^{n} \int_{\mathbb{R}^{n-1}} \mathrm{BV}_j(f(x)) \prod_{\substack{k=1\\k\neq j}}^{n} \mathrm{d}x_k,$$

which for a smooth enough function f, it is equal to

$$\mathrm{VT}(f) = \int_{\mathbb{R}} \sum_{j=1}^{n} \left|\frac{\partial f(x)}{\partial x_j}\right| \mathrm{d}x.$$

Among the sources dealing with the Tonelli variation, let us mention [2,7,9,16,27].

Until now it seems ([20,22]) that only the approach due to Vitali–Hardy–Krause gives a notion of variation for real-valued functions of several variables such that a complete analogue of the Helly theorem holds with respect to the pointwise convergence of extracted subsequences. However, the point of view which is nowadays accepted in the literature as most efficient generalization of the 1-dimensional theory is due to De Giorgi and Fichera (see [17,19]).

Thus, the unvarying interest generated by the classical notion of *function of bounded variation* has led to some generalizations of the concept, mainly, intended to the search of bigger classes of functions whose elements have pointwise convergent Fourier series or to find applications in geometric measure theory, calculus of variations, and mathematical physics. As in the classical case, these generalizations have found many applications in the study of certain differential and integral equations (see e.g., [8]).

In this paper, we present a detailed study of the space of functions of bounded Riesz-Φ-variation, which was introduced previously in [5,6], for real-valued functions of several variables.

This extends the work done in [3] (resp. in [4]), in which the authors present the notion of real-valued function (resp. vector valued) of bounded Riesz-Φ-variation, that are defined on a rectangle of $\mathbb{R}^2$. In particular, we extend some results due to Chistyakov ([15]) and give a version of Riesz's Lemma for the case of functions of several real variables which take values in a reflexive Banach space.

2 Notation and preliminary

To start, we give some notations and definitions that will be used throughout the rest of this paper (see [5,12,13]).

As usual, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ denotes the set of all positive integers, non-negative integers and real numbers, respectively. A typical point of $\mathbb{X}^n$ ($\mathbb{N}, \mathbb{N}_0$ or $\mathbb{R}$) is denoted as $\mathbf{x} = (x_1, x_2, \ldots, x_n) := (x_i)_{i=1}^n$, but the canonical unit vectors of $\mathbb{R}^n$ are denoted by $\mathbf{e_j}$ ($j = 1, 2, \ldots, n$); that is, $\mathbf{e_j} := (e_1^j, e_2^j, \ldots, e_n^j)$ where, $e_r^j := \begin{cases} 0 & \text{if } r \neq j \\ 1 & \text{if } r = j. \end{cases}$ The zero n-tuple $(0, 0, \ldots, 0)$ will be denoted by $\mathbf{0}$, and by $\mathbf{1}$ we will denote the n-tuple $\mathbf{1} = (1, 1, \ldots, 1)$.

If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a n-tuple of non-negative integers then we call $\boldsymbol{\alpha}$ a *multi-index* ([1]).

If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ we use the notation $\mathbf{a} < \mathbf{b}$ to mean that $a_i < b_i$ for each $i = 1, \ldots, n$ and similarly are defined $\mathbf{a} = \mathbf{b}$, $\mathbf{a} \leq \mathbf{b}$, $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} > \mathbf{b}$. If $\mathbf{a} < \mathbf{b}$, the set $\mathbf{J} := [\mathbf{a}, \mathbf{b}] = \prod_{i=1}^n [a_i, b_i]$ will be called a *n-dimensional closed interval*. The Euclidean volume of an n-dimensional closed interval will be denoted by $\mathrm{Vol}[\mathbf{a}, \mathbf{b}]$; that is, $\mathrm{Vol}[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^n (b_i - a_i)$.

In addition, for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ we will use the notations $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\alpha\mathbf{x} := (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n)$.

In this work, we will consider functions whose domain is a n-dimensional closed interval $[\mathbf{a}, \mathbf{b}]$ and whose range is an invariant metric semigroup $(X, d, +, \cdot)$; i.e., (X, d) is a complete metric space, d is a translation invariant metric on X, $(X, +)$ is an commutative semigroup. In particular, the triangle inequality implies that, for all $u, v, p, q \in X$,

$$\begin{aligned} d(u, v) &\leq d(p, q) + d(u + p, v + q) \quad \text{and} \\ (u + p, v + q) &\leq d(u, v) + d(p, q). \end{aligned} \tag{2.1}$$

Clearly any normed space is a metric semigroup.

The following standard notation (see [14]) will be used: $\mathcal{N}$ will denote the set of all continuous convex functions $\Phi : [0, +\infty) \to [0, +\infty)$ such that $\Phi(t) = 0$ if and only if $t = 0$, and $\mathcal{N}_\infty$ the set of all functions $\Phi \in \mathcal{N}$, for which the Orlicz condition (also called ∞_1 condition) holds: $\lim_{t\to\infty} \dfrac{\Phi(t)}{t} = +\infty$. Following [23], functions from $\mathcal{N}$ are called φ-functions. Any function $\Phi \in \mathcal{N}$ is strictly increasing, and so, its inverse Φ^{-1} is continuous and concave; besides, the functions $t \longmapsto \dfrac{\Phi(t)}{t}$ and $t \longmapsto t\Phi^{-1}\left(\dfrac{1}{t}\right)$ are nondecreasing for $t > 0$.

One says that a function $\Phi \in \mathcal{N}$ satisfies a Δ_2 condition, and writes $\Phi \in \Delta_2$, if there are constants $K > 2$ and $t_0 > 0 \in \mathbb{R}$ such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for all} \quad t \geq t_0. \tag{2.2}$$

For instance, if $\Phi(x) := t^p,\ p > 1$, one may choose the optimal constant $K = 2^p$.

Now we define two sets that will play an important role in this work:

$$\mathcal{E}(n) := \{\theta \in \mathbb{N}_0^n : \theta \leq \mathbf{1} \text{ and } |\theta| \text{ is even}\},$$
$$\mathcal{O}(n) := \{\theta \in \mathbb{N}_0^n : \theta \leq \mathbf{1} \text{ and } |\theta| \text{ is odd}\}.$$

Notice that these sets are related in one-to-one correspondence; indeed, if $\theta = (\theta_1, \ldots, \theta_n) \in \mathcal{E}(n)$ then we can define $\widetilde{\theta} := (\theta_1, \ldots, \theta_{i-1}, 1-\theta_i, \theta_{i+1}, \ldots \theta_n) \in \mathcal{O}(n)$, and this operation is clearly invertible.

Definition 2.1 [10,11,26] Given $f : [\mathbf{a}, \mathbf{b}] \to X$, we define the *n-dimensional Vitali difference* of f over an n-dimensional interval $[\mathbf{x}, \mathbf{y}] \subseteq [\mathbf{a}, \mathbf{b}]$, by

$$\Delta_n(f, [\mathbf{x}, \mathbf{y}]) := d\left(\sum_{\theta \in \mathcal{E}(n)} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}),\ \sum_{\theta \in \mathcal{O}(n)} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y})\right). \tag{2.3}$$

Note that in the case $n = 2$, we have $\mathcal{E}(2) := \{(0,0), (1,1)\}$ and $\mathcal{O}(2) = \{(1,0), (0,1)\}$, thus $\Delta_2(f, [\mathbf{x}, \mathbf{y}]) = d(f(x_1, x_2) + f(y_1, y_2), f(y_1, x_2) + f(x_1, y_2))$.

Even when $\Delta_n(f, [\mathbf{x}, \mathbf{y}])$, in (2.3), is defined for $\mathbf{x} < \mathbf{y}$, note that if $x_i = y_i$ for some i, then the right-hand side of (2.3) is equal to zero for all maps $f : [\mathbf{a}, \mathbf{b}] \to X$. This difference is also called *mixed difference* and it is usually associated to the names of Vitali, Lebesgue, Hardy, Krause, Fréchet and De la Vallée Poussin ([10,11,20]).

Now, we are going to define the Φ-variation of a function $f : [\mathbf{a}, \mathbf{b}] \to X$ (see [5,6]). To that end, we consider *net* partitions of $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^n [a_i, b_i]$; that is, partitions of the kind

$$\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_n \text{ with } \xi_i := \left\{t_j^{(i)}\right\}_{j=0}^{k_i}, \quad i = 1, \ldots, n, \tag{2.4}$$

where $\{k_i\}_{i=1}^n \subset \mathbb{N}$ and for each i, ξ_i is a partition of $[a_i, b_i]$. The set of all net partitions of an interval $[\mathbf{a}, \mathbf{b}]$ will be denoted by $\Lambda([\mathbf{a}, \mathbf{b}])$.

A point in a net partition ξ is called a *node* ([25]) and it is of the form

$$\mathbf{t}_{\boldsymbol{\alpha}} := (t_{\alpha_1}^{(1)}, t_{\alpha_2}^{(2)}, t_{\alpha_3}^{(3)}, \ldots, t_{\alpha_n}^{(n)}),$$

where $\mathbf{0} \leq \boldsymbol{\alpha} = (\alpha_i)_{i=1}^n \leq \kappa$, with $\kappa := (k_i)_{i=1}^n$, as a result, $t_{\alpha_i}^{(j)} \in [a_j, b_j]$.

For the sake of simplicity in notation, we will simply write $\xi = \{\mathbf{t}_{\boldsymbol{\alpha}}\}$, to refer to all nodes determined by a given partition ξ.

A cell of an n-dimensional interval $[\mathbf{a}, \mathbf{b}]$ is an n-dimensional subinterval of the form $[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$, for $\mathbf{0} < \boldsymbol{\alpha} \leq \kappa$.

Note that

$$\mathbf{t_0} = (t_0^{(1)}, t_0^{(2)}, \ldots, t_0^{(n)}) = (a_1, a_2, \ldots, a_n) \text{ and } \mathbf{t}_\kappa = (t_{k_1}^{(1)}, t_{k_2}^{(2)}, \ldots, t_{k_n}^{(n)}) = (b_1, b_2, \ldots, b_n).$$

3 $\mathbf{RV}_\Phi^n([\mathbf{a}, \mathbf{b}]; X)$

Now we introduce the Riesz-Φ-variation of a function f.

Definition 3.1 Let $f : [\mathbf{a}, \mathbf{b}] \to X$ and $\Phi \in \mathcal{N}$. The Φ-*variation, in the sense of Vitali–Riesz* of f is defined as

$$\mathrm{RV}_\Phi^n(f, [\mathbf{a}, \mathbf{b}]) := \sup_{\xi \in \Lambda([\mathbf{a}, \mathbf{b}])} \mathrm{RV}_\Phi^n(f, [\mathbf{a}, \mathbf{b}], \xi), \tag{3.1}$$

where $\xi = \{\mathbf{t}_\alpha\}$, and

$$\mathrm{RV}_\Phi^n(f, [\mathbf{a}, \mathbf{b}], \xi) := \sum_{\mathbf{1} \le \alpha \le \kappa} \Phi\left(\frac{\Delta_n(f, [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha])}{\mathrm{Vol}\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]}\right) \mathrm{Vol}\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha].$$

The main objective of this section is to define the Riesz-Φ-variation of a function f. To do that it will be necessary to define the variation of a function when we consider that certain variables are fixed, thus as it was done in [11], we now define *the truncation* of a point, of an interval and of a function, by a given multi-index $\mathbf{0} \le \eta \le \mathbf{1}$, with $\mathbf{0} \ne \eta$. Notice that in this case, the entries of a such η are either 0 or 1.

- The truncation of a point $\mathbf{x} \in \mathbb{R}^n$ by a multi-index $\mathbf{0} \le \eta \le \mathbf{1}$ with $\mathbf{0} \ne \eta$, which is denoted by $\mathbf{x}\lfloor\eta$, is defined as the $|\eta|$-tuple that is obtained if we suppress from $\mathbf{x}$ the entries for which the corresponding entries of η are equal to 0. That is, $\mathbf{x}\lfloor\eta = (x_i : i \in \{1, 2, \ldots, n\}, \eta_i = 1)$. For instance, if $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ and $\eta = (0, 1, 1, 0, 1)$ then $\mathbf{x}\lfloor\eta = (x_2, x_3, x_5)$.
- The truncation of an n-dimensional interval $[\mathbf{a}, \mathbf{b}]$ by a multi-index $\mathbf{0} \le \eta \le \mathbf{1}$ with $\mathbf{0} \ne \eta$, is defined as $[\mathbf{a}, \mathbf{b}]\lfloor\eta := [\mathbf{a}\lfloor\eta, \mathbf{b}\lfloor\eta]$.
- Given a function $f : [\mathbf{a}, \mathbf{b}] \to X$, a multi-index $\mathbf{0} \le \eta \le \mathbf{1}$ with $\mathbf{0} \ne \eta$ and a point $\mathbf{z} \in [\mathbf{a}, \mathbf{b}]$, we define $f_\eta^{\mathbf{z}} : [\mathbf{a}, \mathbf{b}]\lfloor\eta \to X$, the truncation of f by η, by the formula

$$f_\eta^{\mathbf{z}}(\mathbf{x}\lfloor\eta) := f(\eta\mathbf{x} + (\mathbf{1} - \eta)\mathbf{z}), \quad x \in [\mathbf{a}, \mathbf{b}].$$

Note that the function $f_\eta^{\mathbf{z}}$ depends only on the $|\eta|$ variables x_i for which $\eta_i = 1$.

Remark 3.2 Given a function $f : [\mathbf{a}, \mathbf{b}] \to X$ and a multi-index $\eta \ne \mathbf{0}$, then the $|\eta|$-dimensional Vitali difference for $f_\eta^{\mathbf{a}}$ (cf. (2.3)), is given by

$$\Delta_{|\eta|}(f_\eta^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}]) := d\left(\sum_{\substack{\theta\in\mathcal{E}(n)\\ \theta\le\eta}} f\left(\eta(\theta\mathbf{x} + (\mathbf{1}-\theta)\mathbf{y}) + (\mathbf{1}-\eta)\mathbf{a}\right), \sum_{\substack{\theta\in\mathcal{O}(n)\\ \theta\le\eta}} f(\eta(\theta\mathbf{x} + (\mathbf{1}-\theta)\mathbf{y}) + (\mathbf{1}-\eta)\mathbf{a})\right).$$

The Vitali-type nth variation of $f : [\mathbf{a}, \mathbf{b}] \to X$ is defined by

$$V_n(f, [\mathbf{a}, \mathbf{b}]) := \sup \sum_{\mathbf{1}\le\eta\le\kappa} \Delta_{|\eta|}(f_\eta^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}]),$$

the supremum being taken over all multiindices κ and all net partitions of $[\mathbf{a}, \mathbf{b}]$.

The total variation of $f : [\mathbf{a}, \mathbf{b}] \to X$ in the sense of Hildebrandt and Leonov (see [20,22]) is defined by

$$\mathrm{TV}(f, [\mathbf{a}, \mathbf{b}]) := \sum_{\mathbf{0}\ne\alpha\le\mathbf{1}} V_{|\alpha|}(f_\alpha^{\mathbf{a}}, [\mathbf{a}, \mathbf{b}]\lfloor\alpha),$$

the summations here and throughout the paper being taken over n-dimensional multiindices in the ranges specified under the summation sign.

Definition 3.3 Let $\Phi \in \mathcal{N}$ and let $f : [\mathbf{a}, \mathbf{b}] \to X$ be a function. We call

$$\mathrm{TRV}_\Phi(f, [\mathbf{a}, \mathbf{b}]) := \sum_{\mathbf{0}\ne\eta\le\mathbf{1}} \mathrm{RV}_\Phi^{|\eta|}(f_\eta^{\mathbf{a}}, [\mathbf{a}, \mathbf{b}]\lfloor\eta)$$

the Φ-variation of f in the sense of Vitali–Hardy–Riesz, briefly: Riesz-Φ-variation of f, in $[\mathbf{a}, \mathbf{b}]$. The set of all functions f satisfying the condition $\mathrm{TRV}_\Phi(f, [\mathbf{a}, \mathbf{b}]) < \infty$ will be denoted by $\mathrm{RV}_\Phi^n([\mathbf{a}, \mathbf{b}]; X)$.

It is easy to check that if f is a constant function then $\Delta_n(f, [\mathbf{x}, \mathbf{y}]) = 0$ and consequently $\mathrm{TRV}_\Phi(f, [\mathbf{a}, \mathbf{b}]) = 0$. In fact, $\mathrm{TRV}_\Phi(f, [\mathbf{a}, \mathbf{b}]) = 0$ if and only if f is constant, as we show next.

Theorem 3.4 $\mathrm{TRV}_\Phi(f, [\mathbf{a}, \mathbf{b}]) = 0$ *if and only if f is a constant function.*

Proof We just prove the necessity of the condition. Suppose $\mathrm{TRV}_\Phi(f, [\mathbf{a}, \mathbf{b}]) = 0$ and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a point in $[\mathbf{a}, \mathbf{b}]$. Then, $\mathbf{x}$ determines, for each $1 \le i \le n$, the partitions $\xi_i := \{a_i, x_i, b_i\} := \{t_0^{(i)}, t_1^{(i)}, t_2^{(i)}\}$.

Since $\mathrm{TRV}_\Phi(f, [\mathbf{a}, \mathbf{b}]) = 0$, we must have $\Delta_{|\eta|}(f_\eta^{\mathbf{a}}, [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]) = 0$ for every $\mathbf{1} \le \boldsymbol{\alpha} \le \mathbf{2}$ and every $\mathbf{0} \ne \eta \le \mathbf{1}$. Consequently, if $\eta = \mathbf{e}_i$ and $\alpha = \mathbf{1}$ we obtain

$$0 = d\left(f(\eta\mathbf{t_1} + (\mathbf{1}-\eta)\mathbf{a}), f(\mathbf{a})\right).$$

Hence,

$$f(a_1, \ldots, x_i, a_{i+1}, \ldots, a_n) = f(\eta\mathbf{t_1} + (\mathbf{1}-\eta)\mathbf{a}) = f(\mathbf{a}), \text{ for all } 1 \le i \le n. \tag{3.2}$$

On the other hand, if $\eta = \mathbf{e}_i + \mathbf{e}_j$ with $i < j$ then

$$d(f(\eta\mathbf{t_1} + (\mathbf{1} - \eta)\mathbf{a}) + f(\mathbf{a}), f(\mathbf{e}_j\mathbf{t}_1 + (\mathbf{1} - \mathbf{e}_j)\mathbf{a}) + f(\mathbf{e}_i\mathbf{t}_1 + (\mathbf{1} - \mathbf{e}_i)\mathbf{a})) = 0. \tag{3.3}$$

Thus, using (3.3) and (3.2) we have

$$\begin{aligned} f(\eta\mathbf{t_1} + (\mathbf{1} - \eta)\mathbf{a}) + f(\mathbf{a}) &= f(\mathbf{e}_j\mathbf{t}_1 + (\mathbf{1} - \mathbf{e}_j)a) + f(\mathbf{e}_i\mathbf{t}_1 + (\mathbf{1} - \mathbf{e}_i)a) \\ f(\eta\mathbf{t_1} + (\mathbf{1} - \eta)\mathbf{a}) + f(\mathbf{a}) &= f(\mathbf{a}) + f(\mathbf{a}). \end{aligned}$$

Equivalently

$$f(\eta\mathbf{x} + (\mathbf{1} - \eta)\mathbf{a}) = f(\mathbf{a}). \tag{3.4}$$

Now, suppose that (3.4) holds for any multi-index η, $\mathbf{0} \leq \eta \leq \mathbf{1}$, $\mathbf{0} \neq \eta$, with k non-zero entries.

Then, if λ is a multi-index such that $\mathbf{0} \leq \lambda \leq \mathbf{1}$, $\mathbf{0} \neq \lambda$, with $k+1$ non-zero entries and $\Delta_{|\lambda|}(f_\lambda^{\mathbf{a}}, [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]) = 0$ for all $\mathbf{1} \leq \boldsymbol{\alpha} \leq \mathbf{2}$, then

$$\sum_{\substack{\theta\in\mathcal{E}(n)\\ \theta\leq\lambda}} f(\lambda(\theta\mathbf{t_0}+(\mathbf{1}-\theta)\mathbf{t_1})+(\mathbf{1}-\lambda)\mathbf{a}) = \sum_{\substack{\theta\in\mathcal{O}(n)\\ \theta\leq\lambda}} f(\lambda(\theta\mathbf{t_0}+(\mathbf{1}-\theta)\mathbf{t_1}) + (\mathbf{1}-\lambda)\mathbf{a})\,. \tag{3.5}$$

Notice that if $\theta \neq \mathbf{0}$ then $\lambda(\theta\mathbf{t_0}+(\mathbf{1}-\theta)\mathbf{t_1})+(\mathbf{1}-\lambda)\mathbf{a}$ has at most k entries equal to the corresponding entries of $\mathbf{x}$ and the remaining entries equal to the corresponding entries of $\mathbf{a}$. In this case, (3.4) implies that $f(\lambda(\theta\mathbf{t_0} + (\mathbf{1} - \theta)\mathbf{t_1}) + (\mathbf{1} - \lambda)\mathbf{a}) = f(\mathbf{a})$. Hence, since $\mathcal{E}(n)$ has the same number of elements as $\mathcal{O}(n)$, it follows from identity (3.4) that $f(\lambda\mathbf{t_1} + (\mathbf{1} - \lambda)\mathbf{a}) = f(\mathbf{a})$. We conclude that f is a constant function. □

Remark 3.5 Note that if X is a normed space, then (2.3) can be replaced by

$$\Delta_n(f, [\mathbf{x}, \mathbf{y}]) := \left\| \sum_{\theta\leq\mathbf{1}} (-1)^{|\theta|} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) \right\|.$$

Example 3.6 Let c_0 be the space of all null sequences with the ∞-norm, and let $f : [0, 1] \times [0, 1] \times [0, 1] \to c_0$ be defined by

$$f(\mathbf{x}) := \left\{ \sum_{i=1}^{3} (-1)^{i+1} \frac{x_i}{n} \right\}_{n\geq1}.$$

If $\xi = \xi_1 \times \xi_2 \times \xi_3$, where $\xi_i := \{t_1^{(i)}, t_2^{(i)}, \dots, t_{k_i}^{(i)}\}$, $i = 1, 2, 3$. Then,

$$\left\| \sum_{\theta \le \mathbf{e}_i} (-1)^{|\theta|} f(\mathbf{e}_i(\theta \mathbf{t}_{\alpha-\mathbf{1}} + (\mathbf{1}-\theta)\mathbf{t}_\alpha) + (\mathbf{1}-\mathbf{e}_i)\mathbf{0}) \right\|_\infty = \| f(\mathbf{e}_i \mathbf{t}_\alpha) - f(\mathbf{e}_i\, \mathbf{t}_{\alpha-\mathbf{1}})\|_\infty$$
$$= \left\| \left(\frac{t_{\alpha_i}^{(i)} - t_{\alpha_i-1}^{(i)}}{n} \right)_{n\in\mathbb{N}} \right\|_\infty .$$

Hence,

$$\sum_{\mathbf{1}\le\boldsymbol{\alpha}\le\kappa} \Phi\left(\frac{\Delta_1\left(f^0_{\mathbf{e}_i}, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\lfloor \mathbf{e}_i\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\lfloor \mathbf{e}_i} \right) \mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\lfloor \mathbf{e}_i$$
$$= \sum_{\mathbf{1}\le\boldsymbol{\alpha}\le\kappa} \Phi\left(\frac{\left\| \left(\frac{t_{\alpha_i}^{(i)} - t_{\alpha_{i-1}}^{(i)}}{n} \right)_{n\in\mathbb{N}} \right\|_\infty}{t_{\alpha_i}^{(i)} - t_{\alpha_{i-1}}^{(i)}} \right) (t_{\alpha_i}^{(i)} - t_{\alpha_{i-1}}^{(i)})$$
$$= \sum_{\mathbf{1}\le\boldsymbol{\alpha}\le\kappa} \Phi\left(\frac{(t_{\alpha_i}^{(i)} - t_{\alpha_{i-1}}^{(i)}) \left\| \left(\frac{1}{n} \right)_{n\in\mathbb{N}} \right\|_\infty}{t_{\alpha_i}^{(i)} - t_{\alpha_{i-1}}^{(i)}} \right) (t_{\alpha_i}^{(i)} - t_{\alpha_{i-1}}^{(i)})$$
$$= \sum_{\mathbf{1}\le\boldsymbol{\alpha}\le\kappa} \Phi(1)\, (t_{\alpha_i}^{(i)} - t_{\alpha_{i-1}}^{(i)}) = \Phi(1). \tag{3.6}$$

Consequently, for each $i = 1, 2, 3$ we must have $\mathrm{RV}^1_\Phi(f^0_{\mathbf{e}_i}, [\mathbf{a}, \mathbf{b}]) := \Phi(1)$. In addition,

$$\left\| \sum_{\theta \le \mathbf{e}_1+\mathbf{e}_2} (-1)^{|\theta|} f((\mathbf{e}_1+\mathbf{e}_2)(\theta \mathbf{t}_{\alpha-\mathbf{1}} + (\mathbf{1}-\theta)\mathbf{t}_\alpha) + (\mathbf{1}-(\mathbf{e}_1+\mathbf{e}_2))\mathbf{0}) \right\|_\infty$$
$$= \left\| \left(\frac{t_{\alpha_1}^{(1)} - t_{\alpha_2}^{(2)}}{n} \right) - \left(\frac{t_{\alpha_1-1}^{(1)} - t_{\alpha_2}^{(2)}}{n} \right) - \left(\frac{t_{\alpha_1}^{(1)} - t_{\alpha_2-1}^{(2)}}{n} \right) + \left(\frac{t_{\alpha_1-1}^{(1)} - t_{\alpha_2-1}^{(2)}}{n} \right) \right\| = 0,$$

and therefore, $\mathrm{RV}^2_\Phi(f^0_{(1,1,0)}, [\mathbf{a}, \mathbf{b}]) = 0$.

It can be verified similarly that $\mathrm{RV}^2_\Phi(f^{\mathbf{0}}_{\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3}, [\mathbf{a}, \mathbf{b}]) = \mathrm{RV}^2_\Phi(f^{\mathbf{0}}_{\mathbf{e}_i+\mathbf{e}_j}, [\mathbf{a}, \mathbf{b}]) = 0$, where $i, j = 1, 2, 3$ with $i \neq j$.

From (3.2) we conclude that $\mathrm{TRV}_\Phi(f, [\mathbf{a}, \mathbf{b}]) = 3\Phi(1)$.

A proof of the following lemma can be found in [5] or in [6].

Lemma 3.7 [5,6] *If X is a normed space, then the functional* $\mathrm{TRV}_\Phi(\cdot, [\mathbf{a}, \mathbf{b}])$ *is convex.*

4 Further properties

Theorem 4.1 *Let X be a normed space. If $f \in \mathrm{RV}_\Phi([\mathbf{a},\mathbf{b}];X)$, then f is bounded.*

Proof It is well known that if $f:[a,b]\to X$, $[a,b]\subset\mathbb{R}$, is a function of bounded Riesz-Φ-variation, then f is bounded (see [15]). Hence, if $f:[\mathbf{a},\mathbf{b}]\to X$ is a function of bounded Φ-variation on $[\mathbf{a},\mathbf{b}]:=[a_1,b_1]\times[a_2,b_2]$ then there are constants C_1 and C_2 such that

$$\|f^{\mathbf{a}}_{\mathbf{e}_i}(\mathbf{x}\lfloor\mathbf{e}_i)\|_X \le C_i,\ (i=1,2) \text{ for all } \quad \mathbf{x}\in[\mathbf{a},\mathbf{b}].$$

Suppose that f is not bounded. Then, for each $m\in\mathbb{N}$ that satisfies $m>C_1+C_2$, there exists $\mathbf{x}_m=(x_1^m,x_2^m)\in[\mathbf{a},\mathbf{b}]$ such that $\|f(\mathbf{x}_m)+f(\mathbf{a})\|\ge m$. Hence, for all $m>C_1+C_2$,

$$\begin{aligned}&\|f(\mathbf{x})+f(\mathbf{a})-f(x_1^m,a_2)-f(a_1,x_2^m)\|\\&\quad\ge\|f(\mathbf{x})+f(\mathbf{a})\|-\|f(x_1^m,a_2)\|-\|f(a_1,x_2^m)\|\\&\quad\ge\|f(\mathbf{x})+f(\mathbf{a})\|-C_1-C_2\\&\quad\ge m-C_1-C_2.\end{aligned}\tag{4.1}$$

If $\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]\le 1$, then (4.1) implies

$$\begin{aligned}\Phi(m-C_1-C_2)&\le\Phi\left(\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]\frac{\|f(\mathbf{x})+f(\mathbf{a})-f(x_1^m,a_2)-f(a_1,x_2^m)\|}{\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]}\right)\\&\le\Phi\left(\frac{\|f(\mathbf{x})+f(\mathbf{a})-f(x_1^m,a_2)-f(a_1,x_2^m)\|}{\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]}\right)\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]\\&\le\mathrm{RV}^2_\Phi(f,[\mathbf{a},\mathbf{b}]),\end{aligned}$$

a contradiction (since $\lim_{m\to\infty}\Phi(m-C_1-C_2)=\infty$).

On the other hand, if $\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]>1$ then from (4.1) we obtain

$$\begin{aligned}\Phi\left(\frac{m-C_1-C_2}{\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]}\right)&\le\Phi\left(\frac{\|f(\mathbf{x})+f(\mathbf{a})-f(x_1^m,a_2)-f(a_1,x_2^m)\|}{\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]}\right)\\&\le\Phi\left(\frac{\|f(\mathbf{x})+f(\mathbf{a})-f(x_1^m,a_2)-f(a_1,x_2^m)\|}{\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]}\right)\mathrm{Vol}\,[\mathbf{a},\mathbf{x}]\\&\le\mathrm{RV}^2_\Phi(f,[\mathbf{a},\mathbf{b}]),\end{aligned}$$

which again leads to a contradiction; thus, f is a bounded function.

Suppose that the result holds for all the cases in which $[\mathbf{a},\mathbf{b}]$ is a k-dimensional interval with $k<n$.

Consider now the case in which $f:[\mathbf{a},\mathbf{b}]\to X$ where $[\mathbf{a},\mathbf{b}]$ is an n-dimensional interval. Then, for all multi-index $\mathbf{0}<\eta\le\mathbf{1}(\mathbf{0}<\eta)$ that satisfies $|\eta|<n$ there exists a constant M_η such that

$$\|f^{\mathbf{a}}_\eta(\mathbf{x}\lfloor\eta)\|\le M_\eta.$$

In this case, if we suppose that f is not bounded, then for all $m > \sum_{\mathbf{0}<\eta<\mathbf{1}} M_\eta$ there is a point $\mathbf{x}_m = (x_1^m, x_2^m, \ldots, x_n^m)$ such that

$$\|f(\mathbf{x}_m) + (-1)^n f(\mathbf{a})\| > m,$$

and hence

$$\begin{aligned}
\Delta_n(f, [\mathbf{a}, \mathbf{x}_m]) &= \left\| \sum_{\theta \le \mathbf{1}} (-1)^{|\theta|} f(\theta\, \mathbf{a} + (\mathbf{1} - \theta)\mathbf{x}_m) \right\| \\
&\ge \left\| f(\mathbf{x}_m) + (-1)^n f(\mathbf{a}) \right\| - \sum_{\substack{\mathbf{0} \ne \theta \le \mathbf{1} \\ \theta \ne \mathbf{1}}} \|f(\theta\, \mathbf{a} + (\mathbf{1} - \theta)\mathbf{x}_m)\| \\
&= \left\| f(\mathbf{x}_m) + (-1)^n f(\mathbf{a}) \right\| - \sum_{\substack{\mathbf{0} \ne \theta \le \mathbf{1} \\ \theta \ne \mathbf{1}}} \left\| f^{\mathbf{a}}_{\mathbf{1}-\theta}(\mathbf{x}_m \lfloor (\mathbf{1} - \theta)) \right\| \\
&\ge m - \sum_{\substack{\mathbf{0} \ne \theta \le \mathbf{1} \\ \theta \ne \mathbf{1}}} M_{\mathbf{1}-\theta}.
\end{aligned}$$

The result now follows, as in the $n = 2$ case, from the fact that

$$\lim_{m \to \infty} \Phi \left(m - \sum_{\substack{\mathbf{0} \ne \theta \le \mathbf{1} \\ \theta \ne \mathbf{1}}} M_{\mathbf{1}-\theta} \right) = \infty,$$

again, by considering the two cases $\mathrm{Vol}([a, x_m]) \le 1$ and $\mathrm{Vol}([a, x_m]) > 1$. □

Theorem 4.2 *Let X be a normed space. Let $\Phi_1, \Phi_2 \in \mathcal{N}$ such that $\Phi_1(x) \le K\Phi_2(x)$ for all x and some constant K, then* $\mathrm{RV}_{\Phi_2}([\mathbf{a}, \mathbf{b}]; X) \subseteq \mathrm{RV}_{\Phi_1}([\mathbf{a}, \mathbf{b}]; X)$.

Proof Let $f \in \mathrm{RV}_{\Phi_2}([\mathbf{a}, \mathbf{b}]; X)$, then for all net partition $\xi = \{\mathbf{t}_\alpha\} \in \Lambda([\mathbf{a}, \mathbf{b}])$ we have

$$\Phi_1 \left(\frac{\Delta_n(f, [\mathbf{t}_{\alpha-1}, \mathbf{t}_\alpha])}{\mathrm{Vol}[\mathbf{t}_{\alpha-1}, \mathbf{t}_\alpha]} \right) \mathrm{Vol}[\mathbf{t}_{\alpha-1}, \mathbf{t}_\alpha] \le K\Phi_2 \left(\frac{\Delta_n(f, [\mathbf{t}_{\alpha-1}, \mathbf{t}_\alpha])}{\mathrm{Vol}[\mathbf{t}_{\alpha-1}, \mathbf{t}_\alpha]} \right) \mathrm{Vol}[\mathbf{t}_{\alpha-1}, \mathbf{t}_\alpha],$$

from which the proposition follows. □

Theorem 4.3 *Let $\Phi \in \mathcal{N}$ satisfying condition Δ_2 and let X be a normed space. Then* $\mathrm{RV}_\Phi([\mathbf{a}, \mathbf{b}]; X)$ *is a linear space.*

Proof Let K and $x_0 \in \mathbb{R}$ be as in (2.2) and suppose $f, g \in \mathrm{RV}_\Phi([\mathbf{a}, \mathbf{b}]; X)$. Then, for any net partition $\xi = \{\mathbf{t}_\alpha\} \in \Lambda([\mathbf{a}, \mathbf{b}])$

$$\begin{aligned}\Delta_n\left(f+g,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right) &= \left\|\sum_{\theta\leq\mathbf{1}}(-1)^{|\theta|}\,(f+g)(\theta\,\mathbf{x}+(\mathbf{1}-\theta)\mathbf{y})\right\| \\ &\leq \left\|\sum_{\theta\leq\mathbf{1}}(-1)^{|\theta|}\,f(\theta\,\mathbf{x}+(\mathbf{1}-\theta)\mathbf{y})\right\| \\ &\quad+\left\|\sum_{\theta\leq\mathbf{1}}(-1)^{|\theta|}\,g(\theta\,\mathbf{x}+(\mathbf{1}-\theta)\mathbf{y})\right\| \\ &= \Delta_n\left(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)+\Delta_n\left(g,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right).\end{aligned}$$

Put $A_n = \dfrac{\Delta_n\left(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)+\Delta_n\left(g,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}$, then

$$\begin{aligned}&\mathrm{RV}^n_\Phi(f+g,[\mathbf{a},\mathbf{b}],\xi)\\ &= \sum_{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa}\Phi\left(\frac{\Delta_n\left(f+g,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &\leq \sum_{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa}\Phi\left(A_n\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &= \sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa\\ A_n<x_0}}\Phi\left(A_n\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]+\sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa\\ A_n\geq x_0}}\Phi\left(A_n\right)\mathrm{Vol}\,[\mathbf{t}_{(\boldsymbol{\alpha}-\mathbf{1})},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &\leq \sum_{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa}\Phi\left(x_0\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &\quad+\sum_{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa}\Phi\left(\frac{1}{2}\frac{2\Delta_n\left(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}+\frac{1}{2}\frac{2\Delta_n\left(g,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &\leq \Phi\left(x_0\right)\mathrm{Vol}[\mathbf{a},\mathbf{b}]\\ &\quad+\sum_{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa}\frac{1}{2}\Phi\left(\frac{2\Delta_n\left(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right)+\frac{1}{2}\Phi\left(\frac{2\Delta_n\left(g,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &\leq \Phi\left(x_0\right)\mathrm{Vol}[\mathbf{a},\mathbf{b}]\\ &\quad+\sum_{1\leq\boldsymbol{\alpha}\leq\kappa}\frac{K}{2}\Phi\left(\frac{\Delta_n\left(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right)+\frac{K}{2}\Phi\left(\frac{\Delta_n\left(g,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &= \Phi\left(x_0\right)\mathrm{Vol}[\mathbf{a},\mathbf{b}]+\frac{K}{2}\left(\mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}],\xi)+\mathrm{RV}^n_\Phi(g,[\mathbf{a},\mathbf{b}],\xi)\right).\end{aligned}$$

Since this holds for all n, it follows that

$$\mathrm{TRV}_\Phi(f+g,[\mathbf{a},\mathbf{b}])\leq\Phi\left(x_0\right)\mathrm{Vol}[\mathbf{a},\mathbf{b}]+\frac{K}{2}\left(\mathrm{TRV}_\Phi(f,[\mathbf{a},\mathbf{b}])+\mathrm{TRV}_\Phi(g,[\mathbf{a},\mathbf{b}])\right),$$

from which we conclude that $f+g\in\mathrm{RV}_\Phi([\mathbf{a},\mathbf{b}];X)$. On the other hand, if γ is any scalar, then

$$\Delta_n(\gamma f, [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]) = \left\| \sum_{\theta \le \mathbf{1}} (-1)^{|\theta|} \gamma f(\theta\,\mathbf{x} + (\mathbf{1}-\theta)\mathbf{y}) \right\|$$
$$= |\gamma| \left\| \sum_{\theta \le \mathbf{1}} (-1)^{|\theta|} f(\theta\,\mathbf{x} + (\mathbf{1}-\theta)\mathbf{y}) \right\| = |\gamma| \Delta_n(f, [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]) .$$

Thus,

$$\mathrm{RV}^n_\Phi(\gamma f, [\mathbf{a}, \mathbf{b}], \xi) = \sum_{1 \le \boldsymbol{\alpha} \le \kappa} \Phi\left(\frac{\Delta_n(\gamma f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]} \right) \mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$= \sum_{1 \le \boldsymbol{\alpha} \le \kappa} \Phi\left(\frac{|\gamma| \Delta_n(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]} \right) \mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}].$$

If $|\gamma| \le 1$, then

$$\mathrm{RV}^n_\Phi(\gamma f, [\mathbf{a}, \mathbf{b}], \xi) = \sum_{1 \le \boldsymbol{\alpha} \le \kappa} \Phi\left(\frac{|\gamma| \Delta_n(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]} \right) \mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$\le |\gamma| \sum_{1 \le \boldsymbol{\alpha} \le \kappa} \Phi\left(\frac{\Delta_n(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]} \right) \mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$= |\gamma| \mathrm{RV}^n_\Phi(f, [\mathbf{a}, \mathbf{b}], \xi).$$

On the other hand, if $|\gamma| > 1$ then, again by (2.2), there is a constant K' and a point x_0 such that

$$\mathrm{RV}^n_\Phi(\gamma f, [\mathbf{a}, \mathbf{b}], \xi) = \sum_{1 \le \boldsymbol{\alpha} \le \kappa} \Phi\left(\frac{|\gamma| \Delta_n(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]} \right) \mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$\le \Phi(x_0)\mathrm{Vol}[\mathbf{a}, \mathbf{b}] + K' \sum_{1 \le \boldsymbol{\alpha} \le \kappa} \Phi\left(\frac{\Delta_n(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]} \right) \mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$\le \Phi(x_0)\mathrm{Vol}[\mathbf{a}, \mathbf{b}] + K' \,\mathrm{RV}^n_\Phi(f, [\mathbf{a}, \mathbf{b}], \xi).$$

It follows that $\gamma f \in \mathrm{RV}_\Phi([\mathbf{a}, \mathbf{b}]; X)$. We conclude that $\mathrm{RV}_\Phi([\mathbf{a}, \mathbf{b}]; X)$ is a linear space. □

Lemma 4.4 *Let X be a metric semigroup and suppose $f \in \mathrm{RV}_\Phi([\mathbf{a}, \mathbf{b}]; X)$. If $\mathbf{x}, \mathbf{y} \in [\mathbf{a}, \mathbf{b}]$ are such that $x_k = y_k$ for some $0 \le k \le n$, then*

$$\sum_{\theta \in \mathcal{E}(n)} f(\theta\,\mathbf{x} + (\mathbf{1}-\theta)\mathbf{y}) = \sum_{\theta \in \mathcal{O}(n)} f(\theta\,\mathbf{x} + (\mathbf{1}-\theta)\mathbf{y}).$$

Proof Let f, $\mathbf{x}$ and $\mathbf{y}$ be as in the statement. If $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathcal{E}(n)$ then $\widetilde{\theta} := (\theta_1, \theta_2, \ldots, 1-\theta_k, \ldots, \theta_n) \in \mathcal{O}(n)$, thus, $(\theta\,\mathbf{x} + (\mathbf{1}-\theta)\mathbf{y})$ has the same entries as $(\widetilde{\theta}\,\mathbf{x} + (\mathbf{1} - \widetilde{\theta})\mathbf{y})$ since the kth entry $\widetilde{\theta}_k\, x_k + (1 - \widetilde{\theta}_k) y_k = (1 - \theta_k)\, x_k + \theta_k y_k = (1 - \theta_k)\, y_k + \theta_k x_k$. Hence,

$$\sum_{\theta \in \mathcal{E}(n)} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) = \sum_{\theta \in \mathcal{E}(n)} f(\widetilde{\theta}\,\mathbf{x} + (\mathbf{1} - \widetilde{\theta})\mathbf{y}) = \sum_{\theta \in \mathcal{O}(n)} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}).$$

□

Lemma 4.5 *Let X be a metric semigroup and let $[\mathbf{x}, \mathbf{y}]$ be an n-dimensional interval. Suppose that $t^{(j)} \in [x_j, y_j]$ for some $1 \le j \le n$. Then*

$$\Delta_n(f, [\mathbf{x}, \mathbf{y}]) \le \Delta_n(f, [\mathbf{x}, \widetilde{\mathbf{y}}]) + \Delta_n(f, [\widetilde{\mathbf{x}}, \mathbf{y}]),$$

where $\widetilde{\mathbf{x}} := (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ with $\widetilde{x}_i = x_i$ if $i \ne j$ and $\widetilde{x}_j = t^{(j)}$, and $\widetilde{\mathbf{y}} := (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $\widetilde{y}_i = y_i$ if $i \ne j$ and $\widetilde{y}_j = t^{(j)}$.

Proof Note that the interval $[\mathbf{x}, \mathbf{y}]$ can be divided into two intervals, namely $[\mathbf{x}, \widetilde{\mathbf{y}}]$ and $[\widetilde{\mathbf{x}}, \mathbf{y}]$, thus, by virtue of property (2.1) and lemma 4.4 we have

$$\begin{aligned}
&\Delta_n(f, [\mathbf{x}, \mathbf{y}]) \\
&\quad := d\left(\sum_{\theta \in \mathcal{E}(n)} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}), \sum_{\theta \in \mathcal{O}(n)} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y})\right) \\
&\quad = d\left(\sum_{\theta \in \mathcal{E}(n)} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + \sum_{\theta \in \mathcal{E}(n)} f(\theta\,\widetilde{\mathbf{x}} + (\mathbf{1} - \theta)\widetilde{\mathbf{y}}),\right. \\
&\qquad \left.\sum_{\theta \in \mathcal{O}(n)} f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + \sum_{\theta \in \mathcal{E}(n)} f(\theta\,\widetilde{\mathbf{x}} + (\mathbf{1} - \theta)\widetilde{\mathbf{y}})\right) \\
&\quad = d\left(\sum_{\theta \in \mathcal{E}(n)} [f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + f(\theta\,\widetilde{\mathbf{x}} + (\mathbf{1} - \theta)\widetilde{\mathbf{y}})],\right. \\
&\qquad \left.\sum_{\theta \in \mathcal{O}(n)} [f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + f(\theta\,\widetilde{\mathbf{x}} + (\mathbf{1} - \theta)\widetilde{\mathbf{y}})]\right) \\
&\quad = d\left(\sum_{\substack{\theta \in \mathcal{E}(n) \\ \theta_j = 1}} [f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y})\right. \\
&\qquad + f(\theta\,\widetilde{\mathbf{x}} + (\mathbf{1} - \theta)\widetilde{\mathbf{y}})] + \sum_{\substack{\theta \in \mathcal{E}(n) \\ \theta_j = 0}} [f(\theta\,\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + f(\theta\,\widetilde{\mathbf{x}} + (\mathbf{1} - \theta)\widetilde{\mathbf{y}})],
\end{aligned}$$

$$\sum_{\substack{\theta\in\mathcal{O}(n)\\ \theta_j=1}} [f(\theta\,\mathbf{x}+(\mathbf{1}-\theta)\mathbf{y})$$

$$\left. + f(\theta\,\widetilde{\mathbf{x}}+(\mathbf{1}-\theta)\widetilde{\mathbf{y}})] + \sum_{\substack{\theta\in\mathcal{O}(n)\\ \theta_j=0}} [f(\theta\,\mathbf{x}+(\mathbf{1}-\theta)\mathbf{y}) + f(\theta\,\widetilde{\mathbf{x}}+(\mathbf{1}-\theta)\widetilde{\mathbf{y}})]\right)$$

$$= d\left(\sum_{\theta\in\mathcal{E}(n)} f(\theta\,\mathbf{x}+(\mathbf{1}-\theta)\widetilde{\mathbf{y}}) + \sum_{\theta\in\mathcal{E}(n)} f(\theta\,\widetilde{\mathbf{x}}+(\mathbf{1}-\theta)\mathbf{y}),\right.$$

$$\left.\sum_{\theta\in\mathcal{O}(n)} f(\theta\,\mathbf{x}+(\mathbf{1}-\theta)\widetilde{\mathbf{y}}) + \sum_{\theta\in\mathcal{O}(n)} f(\theta\,\widetilde{\mathbf{x}}+(\mathbf{1}-\theta)\mathbf{y})\right)$$

$$\leq \Delta_n(f,[\mathbf{x},\widetilde{\mathbf{y}}]) + \Delta_n(f,[\widetilde{\mathbf{x}},\mathbf{y}]).$$

□

Theorem 4.6 *Let X be a metric semigroup. If $f:[\mathbf{a},\mathbf{b}]\to X$ is a function, then*

$$\mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}],\xi_1\times\cdots\times\{\xi_z\cup\{t^{(z)}\}\}\times\cdots\times\xi_n) \geq \mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}],\xi)$$

for any net partition $\xi=\prod_{i=1}^n \xi_i$ of $[\mathbf{a},\mathbf{b}]$.

Proof Suppose $\xi=\prod_{i=1}^n\xi_i$ where $\xi_i=\{a_i=t_0^{(i)},t_1^{(i)},\ldots,t_{k_i}^{(i)}=b_i\}$. Assume $z\in\{1,\ldots,n\}$ and let $t^{(z)}$ be such that $t_0^{(z)}<t_1^{(z)}<\cdots<t_{r-1}^{(z)}<t^{(z)}<t_r^{(z)}<\cdots<t_{k_j}^{(z)}$. Put $\varrho:=\prod_{i=1}^n\varrho_i$ where $\varrho_i=\{a_i=s_0^{(i)},s_2^{(i)},\ldots,s_{\widehat{k}_i}^{(i)}=b_i\}$, with

$$s_j^{(l)}=t_j^{(l)},\ \text{ for } l\neq z \ \text{ and } \begin{cases} s_l^{(z)}=t_l^{(z)} & \text{if } 0\leq l\leq r-1\\ s_r^{(z)}=t^{(z)} & \\ s_l^{(z)}=t_{l-1}^{(z)} & \text{if } l\geq r+1.\end{cases}$$

Then,

$$\sum_{\substack{1\leq\boldsymbol{\alpha}\leq\widehat{\kappa}\\ \alpha_z<r}} \Phi\left(\frac{\Delta_n\,(f,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]}\right) = \sum_{\substack{1\leq\boldsymbol{\alpha}\leq\kappa\\ \alpha_z<r}} \Phi\left(\frac{\Delta_n\,(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right) \text{and}$$

$$\sum_{\substack{1\leq\boldsymbol{\alpha}\leq\widehat{\kappa}\\ \alpha_z>r+1}} \Phi\left(\frac{\Delta_n\,(f,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]}\right) = \sum_{\substack{1\leq\boldsymbol{\alpha}\leq\kappa\\ \alpha_z>r}} \Phi\left(\frac{\Delta_n\,(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}])}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right).$$

Hence,

$$\sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \widehat{\kappa}}} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]}\right) [\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]$$
$$- \sum_{\mathbf{1} \leq \boldsymbol{\alpha} \leq \kappa} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$= \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \widehat{\kappa} \\ \alpha_z = r}} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]$$
$$- \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \kappa \\ \alpha_z = r}} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$+ \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \widehat{\kappa} \\ \alpha_z = r+1}} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}].$$

Now if $\widetilde{t}^{(i)}_{\alpha_i} = t^{(i)}_{\alpha_i}$ for $i \neq z$ and $\widetilde{t}^{(z)}_{\alpha_z} = t^{(z)}$, by Lemma 4.5, we obtain

$$= \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \widehat{\kappa} \\ \alpha_z = r}} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]$$
$$- \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \kappa \\ \alpha_z = r}} \Phi\left(\frac{\Delta_n\left(f, [\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-1}, \mathbf{t}_{\boldsymbol{\alpha}}]} + \frac{\Delta_n\left(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \widetilde{\mathbf{t}}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$+ \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \widehat{\kappa} \\ \alpha_z = r+1}} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]$$
$$= \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \widehat{\kappa} \\ \alpha_z = r}} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]$$
$$- \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \kappa \\ \alpha_z = r}} \Phi\left(\frac{\Delta_n\left(f, [\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]} \frac{\operatorname{Vol}[\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]}{\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]}\right.$$
$$\left. + \frac{\Delta_n\left(f, [\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \widetilde{\mathbf{t}}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \widetilde{\mathbf{t}}_{\boldsymbol{\alpha}}]} \frac{\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \widetilde{\mathbf{t}}_{\boldsymbol{\alpha}}]}{\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]$$
$$+ \sum_{\substack{\mathbf{1} \leq \boldsymbol{\alpha} \leq \widehat{\kappa} \\ \alpha_z = r+1}} \Phi\left(\frac{\Delta_n\left(f, [\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]}\right) \operatorname{Vol}[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{s}_{\boldsymbol{\alpha}}]$$

and, by the convexity of Φ we get:

$$\geq \sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\widehat{\kappa}\\ alpha_z=r}} \Phi\left(\frac{\Delta_n\left(f,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]$$

$$-\sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa\\ \alpha_z=r}} \frac{\mathrm{Vol}\,[\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\Phi\left(\frac{\Delta_n\left(f,[\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]$$

$$-\frac{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}}]}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]}\Phi\left(\frac{\Delta_n\left(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\widetilde{\mathbf{t}}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]$$

$$+\sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\widehat{\kappa}\\ \alpha_z=r+1}} \Phi\left(\frac{\Delta_n\left(f,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]$$

$$=\sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\widehat{\kappa}\\ \alpha_z=r}} \Phi\left(\frac{\Delta_n\left(f,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]$$

$$-\sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa\\ \alpha_z=r+1}} \Phi\left(\frac{\Delta_n\left(f,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]$$

$$-\sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa\\ \alpha_z=r}} \Phi\left(\frac{\Delta_n\left(f,\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]$$

$$+\sum_{\substack{\mathbf{1}\leq\boldsymbol{\alpha}\leq\widehat{\kappa}\\ \alpha_z=r+1}} \Phi\left(\frac{\Delta_n\left(f,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]\right)}{\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]}\right)\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]=0.$$

It follows that

$$\sum_{\mathbf{1}\leq\boldsymbol{\alpha}\leq\widehat{\kappa}} \Phi\left(\Delta_n\left(f,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]\right)\right)\mathrm{Vol}\,[\mathbf{s}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{s}_{\boldsymbol{\alpha}}]$$
$$-\sum_{\mathbf{1}\leq\boldsymbol{\alpha}\leq\kappa} \Phi\left(\Delta_n\left(f,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\right)\right)\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\geq 0,$$

and therefore, $\mathrm{RV}^n_\Phi\left(f,[\mathbf{a},\mathbf{b}],\xi_1\times\cdots\times\xi_z\cup\{t^{(z)}\}\times\cdots\times\xi_n\right)\geq \mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}],$ $\prod_{i=1}^n\xi_i)$. □

Corollary 4.7 *Let X be a metric semigroup. If $f:[\mathbf{a},\mathbf{b}]\to X$, ξ and δ are any net partitions of* $[\mathbf{a},\mathbf{b}]$, *such that $\xi\subseteq\delta$, then $V^n(f,[\mathbf{a},\mathbf{b}],\xi)\leq V^n\left(f,[\mathbf{a},\mathbf{b}],\delta\right)$.*

Proof It suffices to apply Theorem 4.6 finitely many times. □

5 A representation theorem

In our next result, we present a counterpart of the classical Riesz's lemma (cf., [24], [3]) in the n-dimensional case. It will be assumed that $X = \mathbb{R}$ and the following well-known notation will be used: given any multi-index $\beta = (\beta_1, \ldots, \beta_k)$ we define

$$D^{\beta} := \frac{\partial^{\beta_1+\cdots+\beta_k}}{\partial x_1^{\beta_1} \cdots \partial x_k^{\beta_k}}.$$

Theorem 5.1 *Let* $\Phi \in \mathcal{N}$. *If* $f \in \mathrm{RV}_{\Phi}([\mathbf{a}, \mathbf{b}]; \mathbb{R})$ *and is of class* C^n, *then*

$$\sum_{\mathbf{0} \neq \eta \leq \mathbf{1}} (n - |\eta|)! \int_{[\mathbf{a},\mathbf{b}]\lfloor\eta} \Phi\left(\left|D^{\eta} f_{\eta}^{\mathbf{a}}(\mathbf{x}\lfloor\eta)\right|\right) d(\mathbf{x}\lfloor\eta) = \mathrm{TRV}_{\Phi}(f, [\mathbf{a}, \mathbf{b}]). \quad (5.1)$$

Proof Let $\xi_i := \{t_1^{(i)}, t_2^{(i)}, \ldots, t_{k_i}^{(i)}\}$, $i = 1, \ldots, n$ and let $\xi = \prod_{i=1}^{n} \xi_i = \{\mathbf{t}_{\alpha}\}$ be a net partition of $[\mathbf{a}, \mathbf{b}]$ with $\kappa = (k_1, k_2, \ldots, k_n)$. Then, for all $\boldsymbol{\alpha} \leq \kappa$ we have

$$\sum_{\theta \leq \mathbf{1}} (-1)^{|\theta|} f(\theta \mathbf{t}_{\alpha-\mathbf{1}} + (\mathbf{1} - \theta)\mathbf{t}_{\boldsymbol{\alpha}}) = \sum_{\substack{\theta \leq \mathbf{1} \\ \theta_i = 1}} (-1)^{|\theta|} f(\theta \mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}} + (\mathbf{1} - \theta)\mathbf{t}_{\boldsymbol{\alpha}})$$
$$- \sum_{\substack{1 \leq \boldsymbol{\alpha} \leq \kappa \\ \alpha_z = r}} \sum_{\substack{\theta \leq \mathbf{1} \\ \theta_i = 0}} (-1)^{|\theta|+1} f(\theta \mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}} + (\mathbf{1} - \theta)\mathbf{t}_{\boldsymbol{\alpha}}),$$

where $z \leq k_z$.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{v} = \theta \mathbf{t}_{(\boldsymbol{\alpha}-1)} + (\mathbf{1} - \theta)\mathbf{t}_{\boldsymbol{\alpha}}$ and define the function

$$g_1(x_1) := \sum_{\substack{\theta \leq \mathbf{1} \\ \theta_1 = 1}} (-1)^{|\theta|} f(\mathbf{e}_1 \mathbf{x} + (1 - \mathbf{e}_1)\mathbf{v})$$

($\mathbf{e}_i$ denotes the canonical unit vectors of $\mathbb{R}^n$). Then, since f is differentiable, $g_1 : [t_{\alpha_1-1}^1, t_{\alpha_1}^1] \to \mathbb{R}$, satisfies the conditions of the ordinary *mean value theorem* and thus, there is an $x_{\alpha_1}^1 \in (t_{\alpha_1-1}^1, t_{\alpha_1}^1)$ such that,

$$g_1'(x_{\alpha_1}^1) = \frac{g_1(t_{\alpha_1}^1) - g_1(t_{\alpha_1-1}^1)}{t_{\alpha_1}^1 - t_{\alpha_1-1}^1}.$$

That is,

$$\sum_{\substack{\theta \leq \mathbf{1} \\ \theta_1 = 1}} (-1)^{|\theta|} D^{\mathbf{e}_1} f_{1-\mathbf{e}_1}^{x_{\alpha_1}^1 \mathbf{e}_1}(\mathbf{v}\lfloor(\mathbf{1} - \mathbf{e}_1)) = \frac{\sum_{\substack{\theta \leq \mathbf{1} \\ \theta_1 = 1}} (-1)^{|\theta|} f(\mathbf{v}) - \sum_{\substack{\theta \leq \mathbf{1} \\ \theta_1 = 0}} (-1)^{|\theta|+1} f(\mathbf{v})}{t_{\alpha_1}^1 - t_{\alpha_1-1}^1}.$$

Now define $g_2 : [t^2_{\alpha_2-1}, t^2_{\alpha_2}] \to \mathbb{R}$, by

$$g_2(x_2) := \sum_{\substack{\theta \leq \mathbf{1} \\ \theta_1=\theta_2=1}} (-1)^{|\theta|} D^{\mathbf{e}_1} f_{1-\mathbf{e}_1}^{x^1_{\alpha_1}\mathbf{e}_1+x_2\mathbf{e}_2}(\mathbf{v}\lfloor(1-\mathbf{e}_1)).$$

Then, as before, since g_2 depends only in the second variable, x_2, an application of the mean value theorem implies that there is $x^2_{\alpha_2} \in (t^2_{\alpha_2-1}, t^2_{\alpha_2})$ such that

$$g'_2(x^2_{\alpha_2}) = \frac{g_2(t^2_{\alpha_2}) - g_2(t^2_{\alpha_2-1})}{t^2_{\alpha_2} - t^2_{\alpha_2-1}}.$$

Thus,

$$\begin{aligned}
&\sum_{\substack{\theta \leq \mathbf{1} \\ \theta_1=\theta_2=1}} (-1)^{|\theta|} D^{\mathbf{e}_2} D^{\mathbf{e}_1} f_{1-\mathbf{e}_1-\mathbf{e}_2}^{x^1_{\alpha_1}\mathbf{e}_1+x^2_{\alpha_2}\mathbf{e}_2}(\mathbf{v}\lfloor(1-\mathbf{e}_1-\mathbf{e}_2)) \\
&= \frac{\frac{\sum_{\theta\leq\mathbf{1}}(-1)^{|\theta|} f(\mathbf{v})}{t^1_{\alpha_1} - t^1_{\alpha_1-1}}}{t^2_{\alpha_2} - t^2_{\alpha_2-1}} = \frac{\sum_{\theta\leq\mathbf{1}}(-1)^{|\theta|} f(\mathbf{v})}{(t^1_{\alpha_1} - t^1_{\alpha_1-1})(t^2_{\alpha_2} - t^2_{\alpha_2-1})}.
\end{aligned}$$

By repeating this procedure n times, we obtain that there is $\mathbf{x}_\alpha \in [\mathbf{t}_{\alpha-1}, \mathbf{t}_\alpha]$ such that

$$D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x}_\alpha) = \frac{\sum_{\theta\leq\mathbf{1}}(-1)^{|\theta|} f(\mathbf{v})}{\mathrm{Vol}\,[\mathbf{t}_{(\boldsymbol{\alpha}-1)}, \mathbf{t}_{\boldsymbol{\alpha}}]},$$

and hence

$$\Phi\left(\left|D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x}_\alpha)\right|\right) \mathrm{Vol}\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha] = \Phi\left(\frac{\left|\sum_{\theta\leq\mathbf{1}}(-1)^{|\theta|} f(\mathbf{v})\right|}{\mathrm{Vol}\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]}\right) \mathrm{Vol}\,[\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_\alpha]. \tag{5.2}$$

Since this holds for each $\mathbf{t}_\alpha$, $\boldsymbol{\alpha} \leq \kappa$, of any net partition ξ of $[\mathbf{a}, \mathbf{b}]$, we must have

$$\begin{aligned}
&\sum_{1\leq\boldsymbol{\alpha}\leq\kappa} \Phi\left(\left|D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x}_\alpha)\right|\right) \mathrm{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}] \\
&= \sum_{1\leq\boldsymbol{\alpha}\leq\kappa} \Phi\left(\frac{\left|\sum_{\theta\leq\mathbf{1}}(-1)^{|\theta|} f(\mathbf{v})\right|}{\mathrm{Vol}\,[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]}\right) \mathrm{Vol}[\mathbf{t}_{(\boldsymbol{\alpha}-1)}, \mathbf{t}_{\boldsymbol{\alpha}}].
\end{aligned}$$

Now define,

$$\begin{aligned}
m_\alpha &:= \inf\left\{\Phi\left(\left|D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x})\right|\right) : \mathbf{x} \in [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\right\} \quad \text{and} \\
M_\alpha &:= \sup\left\{\Phi\left(\left|D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x})\right|\right) : \mathbf{x} \in [\mathbf{t}_{\alpha-\mathbf{1}}, \mathbf{t}_{\boldsymbol{\alpha}}]\right\},
\end{aligned}$$

then

$$\begin{aligned}\underline{S}(f,\xi) &:= \sum_{1\le\boldsymbol{\alpha}\le\kappa} m_\alpha \operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &\le \sum_{1\le\boldsymbol{\alpha}\le\kappa} \Phi\left(\left|D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x}_\alpha)\right|\right)\operatorname{Vol}[\mathbf{t}_{(\boldsymbol{\alpha}-1)},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &= \mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}],\xi)\\ &\le \sum_{1\le\boldsymbol{\alpha}\le\kappa} M_\alpha \operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]\\ &=: \overline{S}(f,\xi).\end{aligned}$$

Notice that the lower sum, $\underline{S}$, and $\mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}],\xi)$ are increasing with respect to refinements of the partition ξ while the upper sums, $\overline{S}$, are decreasing. This means that if $k_i\to\infty$ then the upper sums decrease to the limit

$$\overline{\int_{[\mathbf{a},\mathbf{b}]}}\Phi\left(\left|D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x})\right|\right)\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]d\mathbf{x},$$

and the lower sums increase to the limit

$$\underline{\int_{[\mathbf{a},\mathbf{b}]}}\Phi\left(\left|D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x})\right|\right)\operatorname{Vol}[\mathbf{t}_{\boldsymbol{\alpha}-\mathbf{1}},\mathbf{t}_{\boldsymbol{\alpha}}]d\mathbf{x},$$

whereas $\mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}],\xi)$ increases to the limit $\mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}])$, and consequently

$$\mathrm{RV}^n_\Phi(f,[\mathbf{a},\mathbf{b}])=\int_{[\mathbf{a},\mathbf{b}]}\Phi\left(\left|D^{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n} f(\mathbf{x})\right|\right)d\mathbf{x}.$$

Now, since this holds for any function of n variables, with $n\ge 1$, in particular it holds for any truncated function $f^{\mathbf{a}}_\eta$, where $\eta\le\mathbf{1}$, which yields (5.1), since there are $(n-|\eta|)!$ truncations. □

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