Bol. Soc. Mat. Mex. (2017) 23:257–265
DOI 10.1007/s40590-016-0124-8

ORIGINAL ARTICLE

On the free loop spaces of a toric space

A. Bahri[1] · **M. Bendersky**[2] · **F. R. Cohen**[3] · **S. Gitler**[4]

Received: 11 April 2015 / Revised: 2 October 2015 / Accepted: 13 March 2016 /
Published online: 19 April 2016

Abstract In this note, it is shown that the Hilbert–Poincaré series for the rational homology of the free loop space on a moment-angle complex is a rational function if and only if the moment-angle complex is a product of odd spheres and a disk. A partial result is included for the Davis–Januszkiewicz spaces. The opportunity is taken to correct the result (Bahri et al., Proceedings of the Steklov Institute of Mathematics, Russian Academy of Sciences, vol. 286, pp. 219–223. doi:10.1134/S0081543814060121, 2014) which used a theorem from Berglund and Jöllenbeck (J Algebra 315:249–273, 2007).

Keywords Rational homotopy · Free loop space · Rationally elliptic and hyperbolic · Moment-angle complex · Davis–Januszkiewicz space

This paper is dedicated to Samuel Gitler Hammer who brought us much joy and interest in mathematics.

✉ A. Bahri
bahri@rider.edu

M. Bendersky
mbenders@hunter.cuny.edu

F. R. Cohen
cohf@math.rochester.edu

1 Department of Mathematics, Rider University, Lawrenceville, NJ 08648, USA

2 Department of Mathematics, CUNY, East 695 Park Avenue, New York, NY 10065, USA

3 Department of Mathematics, University of Rochester, Rochester, NY 14625, USA

4 Department of Mathematics, Cinvestav, San Pedro Zacatenco, Apartado Postal 14-740, 07360 Mexico, DF, Mexico

Mathematics Subject Classification Primary 55P62 · 55P35; Secondary 52B11 · 55U10

1 Introduction

Let

$$Z_K = Z(K; (D^2, S^1))$$

be a moment-angle complex (a special case of a polyhedral product), where K is a finite simplicial complex with m vertices of dimension $n-1$ [1,4,5]. In the special cases for which K is a polytopal sphere, Z_K is a manifold with orbit space given by a simple convex polytope

$$P^n(K) = Z_K/T^m,$$

where the torus of rank m, T^m, acts naturally on Z_K. The topology/geometry of the free loop space of the Davis–Januszkiewicz space $DJ(K) = ET^m \times_{T^m} Z_K$ and related spaces here is tightly tied to the geometry of $P^n(K)$.

Félix and Halperin showed [11,12] that there is a dichotomy for simply connected finite CW-complexes X. Their theorem is the following.

Theorem 1.1 *Either*

(1) *$\pi_*(X) \otimes \mathbb{Q}$ is a finite $\mathbb{Q}$-vector space, in which case X is called rationally elliptic or,*
(2) *$\pi_*(X) \otimes \mathbb{Q}$ grows exponentially, in which case X is called rationally hyperbolic.*

The purpose of this note is to develop the dichotomy in the next Theorem 1.2 arising from LX the free loop space of a space X together with the connections to $P^n(K)$. For a definition of the term *exponential growth*, see [16, p. 9].

Theorem 1.2 *The Hilbert–Poincaré series for the rational homology of*

$$LZ(K; (D^2, S^1))$$

has exponential growth if and only if $Z(K; (D^2, S^1))$ contains a wedge of two spheres as a rational retract, and so is hyperbolic. Thus the following are equivalent:

(1) *The Hilbert–Poincaré series for the rational homology of $LZ(K; (D^2, S^1))$ has sub-exponential growth.*
(2) *The space $Z(K; (D^2, S^1))$ has totally finite rational homotopy groups, in other words $Z(K; (D^2, S^1))$ is elliptic.*

The previous theorem follows by combining theorems of Lambrechts [15], Neisendorfer and Miller [18] together with Theorem 1.3 of [2], which illustrates this dichotomy in the case of Z_K. (The opportunity is taken here to correct this result in Section 2.) The growth of free loop spaces has also been developed in [10].

Gurvich in his thesis [14] showed that in the case K is a polytopal sphere, then Z_K is elliptic if and only if $P^n(K)$ is a product of simplices. (This result is generalized for any K in [2].) The next corollary follows from Gurvich's result together with Theorem 1.2.

Corollary 1.3 *Let K be a polytopal sphere. Then following are equivalent:*

(1) *The Hilbert–Poincaré series for the rational homology of $LZ(K; (D^2, S^1))$ has sub-exponential growth.*
(2) *The space $Z(K; (D^2, S^1))$ is elliptic, and so has totally finite rational homotopy groups.*
(3) *The simple polytope $P^n(K)$ is a product of simplices.*

In what follows, a related theorem is stated in which $Z(K; (D^2, S^1))$ is replaced by either $DJ(K)$ the associated Davis–Januszkiewicz space or mildly more general spaces.

Remarks addressing earlier work on irrational Hilbert–Poincaré series follow next. Roos first proved that the Hilbert–Poincaré series for the free loop space of $S^3 \vee S^3$ is irrational [19], following Serre's method for proving that the Hilbert–Poincaré series for $\Omega^2(S^3 \vee S^3)$ is irrational [21]. One common theme here is the application of the Lech–Mahler–Skolem theorem which identifies whether certain infinite series are given by rational functions [19,21]. However, it is unclear whether these methods extend directly to many of the cases in this paper.

A result due to Lambrechts is described next [15]. Lambrechts proves that if X is a coformal, 1-connected CW complex of finite type, and is hyperbolic, then the rational Betti numbers of the free loop space have exponential growth. Examples are wedges of two spheres each of dimension greater than 1. [Aside: let X be a simply connected CW complex with rational cohomology of finite type. Let $\Lambda(V; d)$ denote the Sullivan minimal model for X. Then $\Lambda(V; d)$ is said to be coformal provided $d^2(V) \subset \Lambda^2 V$.]

By Theorem 1.3 in [2] (corrected below), either Z_K is rationally homotopy equivalent to a finite product of odd spheres in which case Z_K is elliptic, or rationally Z_K has a wedge of two spheres both of dimension greater than one as a retract in which case, it is hyperbolic. The structure of the minimal non-faces determines whether the moment-angle complex is elliptic or hyperbolic.

A related result holds for the Davis–Januszkiewicz spaces and mild generalizations.

Theorem 1.4 *Let $X = DJ(K)$ or $X = ET^m \times_{T^q} Z_K$ where $T^q \subset T^m$. Then if the space $Z(K; (D^2, S^1))$ is elliptic (and so has totally finite rational homotopy groups), the Hilbert–Poincaré series for the rational homology of LX has sub-exponential growth.*

Example Let K be the simplicial complex consisting of two disjoint points and Q a simplicial complex with one edge and a disjoint point. Then, $Z(K; (D^2, S^1)) = S^3$ is elliptic, and $Z(Q; (D^2, S^1))$ is a wedge of spheres and so is hyperbolic. Further,

$$DJ(K) \simeq \mathbb{CP}^\infty \vee \mathbb{CP}^\infty.$$

On the other hand, the Hilbert–Poincaré series for the rational homology of

$$LDJ(Q) \simeq L\big((\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \vee \mathbb{CP}^\infty\big)$$

may have exponential growth.

Since the Hochschild homology of the cohomology ring for $DJ(K)$ is the cohomology of the free loop space of $DJ(K)$ as a special case of [13], the next result follows.

Corollary 1.5 *The Hochschild homology of the Stanley–Reisner ring (or face ring of K) has Hilbert–Poincaré series having sub-exponential growth, if the space $Z(K; (D^2, S^1))$ is elliptic. Furthermore, if K is a polytopal sphere, the Hochschild homology of the Stanley–Reisner ring has Hilbert–Poincaré series which is a rational function if the simple polytope $P^n(K)$ is a product of simplices.*

A related question is to work out the precise cohomology of LX. In the paper [9], Fadell and Husseini computed the cohomology ring of LM for M a sphere or a complex projective space. The Chas–Sullivan rings of the homology of these M have been computed by Cohen et al. [7]. Using more elementary means, the calculation has been done also by Seeliger [20]. In the special case for which $Z(K; (D^2, S^1))$ is rationally elliptic, the homology of the free loop space is just that of a product of odd-dimensional spheres with a product of pointed loop spaces of odd-dimensional spheres. To work out the homology of $LDJ(K)$ in the rationally elliptic case, it suffices to work out the differentials in the spectral sequence for $L(Z(K; (D^2, S^1))) \to L(DJ(K)) \to L(\mathbb{CP}(\infty))^m$ where there is a homotopy equivalence

$$L(\mathbb{CP}(\infty))^m \to \mathbb{CP}(\infty)^m \times (S^1)^m.$$

The examples above arise from Ganea's fibration

$$S^3 \to \mathbb{CP}^\infty \vee \mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty.$$

In this case K has two vertices without an edge between the vertices, $Z_K = S^3$ and $DJ(K) = \mathbb{CP}^\infty \vee \mathbb{CP}^\infty$. The upshot is that Hilbert–Poincaré series for $L(\mathbb{CP}^\infty \vee \mathbb{CP}^\infty)$ has sub-exponential growth.

2 The dichotomy for $Z(K; (D^2, S^1))$: a correction to [2, Theorem 1.3]

In the paper [2], a result from [3] is used to prove that the moment-angle complex $Z(K; (D^2, S^1))$ is rationally elliptic if and only if it is the product of odd spheres and a disk. This occurs if and only if K is the iterated join of simplices and boundaries of simplices.

Recently, counterexamples to the relevant result from [3] have appeared in the literature. This necessitates a repair to [2, Theorem 1.3] which is included below. Our goal is to prove that if a simplicial complex K does not have pairwise disjoint non-faces, then rationally, $Z(K; (D^2, S^1))$ has a wedge of odd spheres as a retract, and so it will be rationally hyperbolic. Notice that, by [2, Corollary 2.7], the hypothesis here is equivalent to K not being the iterated join of simplices and boundaries of simplices. The next proposition will reduce the proof to a simple induction.

Definition 2.1 Let $\mathcal{A}_m$ be the collection of all simplicial complexes on m vertices which have a pair of intersecting minimal non-faces, but no proper full subcomplex with that property.

Example Let $m = 4$ and K have minimal non-faces corresponding to relations in the Stanley–Reisner ring: $v_1v_2v_3$, $v_1v_2v_4$ and v_1v_4. Here, K has no proper full subcomplex with intersecting non-faces.

Proposition 2.2 *Let $K \in \mathcal{A}_m$, then $Z(K; (D^2, S^1))$ has a wedge of odd spheres as a retract.*

Proof Suppose that K has minimal intersecting non-faces corresponding to the following relations in the Stanley–Reisner ring

$$v_1 \dots v_k w_1 \dots w_t \quad \text{and} \quad u_1 \dots u_r w_1 \dots w_t.$$

(Notice that minimality dictates that k, t and r are all ≥ 1.) It follows that the vertex set of K must be

$$\{v_1, \dots, v_k, u_1, \dots, u_r, w_1, \dots, w_t\} \tag{1}$$

for otherwise, removing a vertex from K, which is not among these, will produce a proper full subcomplex contradicting $K \in \mathcal{A}_m$. Next, setting

$$I = \{v_1, \dots, v_k, w_1, \dots, w_t\} \quad \text{and} \quad J = \{u_1, \dots, u_r, w_1, \dots, w_t\}$$

gives retractions off $Z(K; (D^2, S^1))$:

$$Z_{K_I} = S^{2(k+t)-1} \quad \text{and} \quad Z_{K_J} = S^{2(r+t)-1}$$

corresponding to the full subcomplexes K_I and K_J, [8, Theorem 2.2.3]. The stable splitting theorem of [1] distinguishes these two spheres. This gives a map

$$S^{2(k+t)-1} \vee S^{2(r+t)-1} \longrightarrow Z(K; (D^2, S^1)).$$

It remains to show that rationally no cells are attached to this wedge of spheres inside $Z(K; (D^2, S^1))$. Now, the results of [1] imply that all non-trivial attaching maps to this wedge of spheres must be *stably trivial*. The Hilton–Milnor theorem [17, Theorem 4.3.2], gives

$$\begin{aligned} \pi_n(S^{2(k+t)-1} \vee S^{2(r+t)-1}) \cong \pi_n(S^{2(k+t)-1}) &\oplus \pi_n(S^{2(r+t)-1}) \\ &\oplus \pi_n\big(\Sigma(S^{2(k+t)-2} \wedge S^{2(r+t)-2})\big) \\ &\oplus_{j\geq 2} \pi_n\big(\Sigma(S^{2j(k+t)-j} \wedge S^{2(r+t)-1})\big). \end{aligned}$$

The rational homotopy groups of spheres is well known. The only stably trivial non-trivial classes occur in the groups $\pi_{4q-1}(S^{2q})$. In the decomposition above, this requires

$$n \geq 4(2k + 3t + r - 1) - 1.$$

The vertex set of K is given by (1) and so the largest cell possible in $Z(K; (D^2, S^1))$ has dimension $2(k + r + t) - 1$. Now

$$2(k + r + t) - 1 < 4(2k + 3t + r - 1) - 1$$

because k, t and r are all ≥ 1. So rationally, no non-trivial attaching map is possible. □

An induction argument now gives the result.

Theorem 2.3 *Let K be a simplicial complex which contains a pair of minimal intersecting non-faces, then $Z(K; (D^2, S^1))$ is rationally hyperbolic.*

Proof It is straightforward to check that all simplicial complexes on three vertices, which have pairwise intersecting non-faces have a wedge of spheres as a retract and so are rationally hyperbolic.

Suppose by way of induction, that all simplicial complexes with fewer than m vertices, which have pairwise intersecting non-faces, have a wedge of spheres as a rational retract. Let K be a simplicial complex on m vertices, which has pairwise intersecting non-faces. If $K \in \mathcal{A}_m$, the result is true for K by Proposition 2.2. If $K \notin \mathcal{A}_m$, then K has a proper full subcomplex L which has a pair of intersecting non-faces. The induction hypothesis and [8, Theorem 2.2.3] now imply the result. □

3 Proof of Theorem 1.2

Assume that $Z(K; (D^2, S^1))$ is rationally hyperbolic. Thus $Z(K; (D^2, S^1))$ has a rational wedge of two simply connected spheres as a retract. A wedge of two spheres is coformal by a result of Neisendorfer and Miller [18, p. 573]. Appealing to Lambrecht's theorem [15], the Hilbert–Poincaré series for the rational homology of the free loop space of $Z(K; (D^2, S^1))$ has exponential growth as the Hilbert–Poincaré series for the free loop space of a wedge of two simply connected spheres has exponential growth. Thus the rational homology of $LZ(K; (D^2, S^1))$ has exponential growth.

Conversely, note that $Z(K; (D^2, S^1))$ is rationally elliptic if and only if it is rationally homotopy equivalent to a product of odd spheres. The free loop space of a product of odd spheres is rationally (or indeed after inverting 2) homotopy equivalent to the product of odd spheres with the pointed loop space of the finite product of odd spheres. *This product has a cohomology algebra which has sub-exponential growth*.

These remarks imply Theorem 1.2 since any space of the homotopy type of a finite, simply connected CW-complex is either elliptic, or hyperbolic.

Remark 3.1 The calculations of the Chas–Sullivan string topology rings of $H_*(LM)$ for $M = S^n$ and $\mathbb{CP}^n$, mentioned above [7,20], yield a quotient of a finitely generated free associative algebra by an ideal. In particular, using the Chas–Sullivan product one sees that the homology of these free loops are rationally elliptic. Now $Z(K; (D^2, S^1))$ is a manifold if K is a triangulation of a sphere. So, the string topology rings of $LZ(K; (D^2, S^1))$ are defined for such K. It follows from Theorem 1.2 that the Chas–Sullivan string topology of the free loops on a moment angle manifold $Z(K; (D^2, S^1))$ cannot be a quotient of a finitely generated free associative algebra unless $Z(K; (D^2, S^1))$ is a product of odd spheres.

4 Proof of Theorem 1.4

Suppose condition (1) holds, namely that the rational cohomology $LZ(K; (D^2, S^1)$ has sub-exponential growth. In this case, $Z(K; (D^2, S^1)$ is rationally elliptic and so, by the results of [2], must be rationally homotopy equivalent to a product of odd spheres. Recall next ([6, p. 339], for example), that there is a homotopy equivalence

$$\Omega(DJ(K)) \longrightarrow \Omega(Z(K; (D^2, S^1))) \times T^m. \tag{2}$$

This implies that the rational cohomology of $\Omega(DJ(K))$ is a tensor product of a polynomial algebra and an exterior algebra. Next, the Serre spectral sequence of the fibration

$$\Omega(DJ(K)) \longrightarrow L(DJ(K)) \longrightarrow DJ(K) \tag{3}$$

has an E_2 term which is a tensor product of a polynomial algebra, an exterior algebra and the Stanley–Reisner ring. So, the rational cohomology $L(DJ(K))$ must have sub-exponential growth. This completes the proof of the theorem for the case of $DJ(K)$. The proof of the theorem for the space $ET^m \times_{T^q} Z_K$ is entirely analogous.

5 Free loop spaces in the elliptic case

Assume that $Z_K = Z(K; (D^2, S^1))$ is rationally elliptic, then it is a finite product of odd-dimensional spheres by [2]. The free loop space $L(S^{2n+1})$ is homotopy equivalent to

$$S^{2n+1} \times \Omega(S^{2n+1})$$

as long as the prime 2 has been inverted. In this case of $L(Z_K)$, the free loop space is a product of free loop spaces of odd-dimensional spheres.

One remark is that the natural spectral sequence for

$$L(DJ(K)) \to L(\mathbb{CP}^\infty)^m$$

frequently supports a non-trivial differential as in the case of the free loops of Ganea's fibration

$$L(S^3) \to L(\mathbb{CP}^\infty \vee \mathbb{CP}^\infty) \to L(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$$

for which K is two points, and $DJ(K) = \mathbb{CP}^\infty \vee \mathbb{CP}^\infty$. This differential propagates to several related cases.

It is natural to conjecture that if Z_K is rationally elliptic, then the Hilbert–Poincaré series for the free loop space of $L(ET^m \times_{T^q} Z_K)$ is a rational function.

Acknowledgements The authors are grateful to Ran Levi and Kathryn Hess for useful suggestions and also to Jason McCullough. The comments of the referee have improved the exposition. A. Bahri was supported in part by Grant No. 210386 from the Simons Foundation.

References

1. Bahri, A., Bendersky, M., Cohen, F.R., Gitler, S.: The polyhedral product functor: a method of computation for moment–angle complexes, arrangements and related spaces. Adv. Math. **225**, 1634–1668 (2010)
2. Bahri, A., Bendersky, M., Cohen, F.R., Gitler, S.: On the rational type of moment–angle complexes. In: Proceedings of the Steklov Institute of Mathematics, Russian Academy of Sciences, vol. 286, pp. 219–223 (2014). doi:10.1134/S0081543814060121
3. Berglund, A., Jöllenbeck, M.: On the Golod property of Stanly–Reisner rings. J. Algebra **315**, 249–273 (2007)
4. Buchstaber, V., Panov, T.: Torus actions, combinatorial topology and homological algebra. Uspekhi Mat. Nauk **55**(5, 3) (106) (2000) (Russian). [Russ. Math. Surv. **55**(5), 825–921 (2000) (English translation)]. arXiv:math.AT/0010073
5. Buchstaber, V., Panov, T.: Torus actions and their applications in topology and combinatorics. In: AMS University Lecture Series, vol. 24 (2002)
6. Buchstaber, V., Panov, T.: Toric topology. In: Mathematical Surveys and Monographs, vol. 204. American Mathematical Society, Providence, RI (2015). arXiv:1210.2368
7. Cohen, R.L., Jones, J.D.S., Yan, J.: The loop homology algebra of spheres and projective spaces. In: Categorical Decomposition Techniques in Algebraic Topology, Isle of Skye, 2001. Progr. Math., vol. 215, pp, 77–92. Birkhauser, Basel (2004)
8. Denham, G., Suciu, A.: Moment–angle complexes, monomial ideals and Massey products. Pure Appl. Math. Q **3**(1), 25–60 (2007)
9. Fadell, E., Husseini, S.: Infinite cup length in free loop spaces with an application to a problem of the N-body type. Ann. Inst. H. Poincare Anal. Non Lineaire **9**(3), 305–319 (1992)
10. Félix, Y., Halperin, S., Thomas, J.-C.: On the growth of the homology of a free loop space (English summary). Pure Appl. Math. Q. **9**(1), 167–187 (2013)
11. Felix, Y., Halperin, S.: Formal spaces with finite-dimensional rational homotopy. Trans. Am. Math. Soc. **270**(2), 575–588 (1982)
12. Felix, Y., Halperin, S., Thomas, J.-C.: Rational homotopy theory. In: Graduate Texts in Mathematics. Springer, New York (2002)
13. Goodwillie, T.G.: Cyclic homology, derivations, and the free loopspace. Topology **24**(2), 187–215 (1985)
14. Gurvich, M.: Some results on the topology of quasitoric manifolds and their equivariant mapping spaces. Ph.D. thesis, Series: UC San Diego Electronic Theses and Dissertations (2008)
15. Lambrechts, P.: On the Betti numbers of the free loop space of a coformal space. J. Pure Appl. Algebra **161**(1–2), 177–192 (2001)
16. McCullough, J., Peeva, I.: Infinite graded free resolutions. In: Eisenbud, Iyengar, Singh, Stafford, Van den Bergh (eds.) Commutative Algebra and Noncommutative Algebraic Geometry. Math. Sci. Res. Inst. Publ., Cambridge University Press (2016)

17. Neisendorfer, J.: Algebraic Methods in Unstable Homotopy Theory. Cambridge University Press, Cambridge (2010)
18. Neisendorfer, J., Miller, T.: Formal and coformal spaces. Ill. J. Math. **22**(2), 565–580 (1978)
19. Roos, J.-E.: Homology of free loop spaces, cyclic homology and nonrational Poincaré-Betti series in commutative algebra. In: Algebra Some Current Trends (Varna, 1986). Lecture Notes in Math., vol. 1352, pp. 173–189. Springer, Berlin (1988)
20. Seeliger, N.: Loop homology algebra of spheres and complex projective spaces. Forum Math. **26**(4), 967–981 (2011). doi:10.1515/FORM.2011.161. (ISSN (Online): 1435-5337, ISSN (Print): 0933-7741)
21. Serre, J.-P.: Un exemple de série de Poincaré non rationnelle, (French) Nederl. Akad. Wetensch. Indag. Math. **41**(4), 469–471 (1979)