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ORIGINAL ARTICLE

Topological rigidity of higher graph manifolds

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Abstract In this short note we prove the Borel conjecture for a family of aspherical manifolds that includes higher graph manifolds.

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1 Introduction

The Borel conjecture is a statement about topological rigidity. It states that a homotopy equivalence between two compact aspherical manifolds is homotopic to a homeomorphism.

A lot of work in geometric topology has been done in the last years with the aim to prove the Borel conjecture using methods involving controlled topology and algebraic

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Dedicated to the memory of Prof. Samuel Gitler.

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K-theory. In particular, the Borel conjecture was shown by Frigerio et al. to hold for the class of graph manifolds studied in [11].

On the other hand, relationships between several generalizations of the concept of finite asymptotic dimension in connection with isomorphism conjectures, in algebraic K and L- theory, as well as coarse versions of these have been carried out by Carlsson and Goldfarb in [7,8,12].

The method of proof of the Borel conjecture in this note uses these previous developments.

Consider the following construction of smooth n-manifolds M, for $n \geq 3$:

Definition 1 1. For every $i = 1, \ldots, r$ take a complete finite-volume non-compact pinched negatively curved n_i-manifold V_i, where $2 \leq n_i \leq n$.
2. Denote by M_i the compact smooth manifold with boundary obtained by "truncating the cusps" of V_i, i.e., by removing from V_i a (nonmaximal) horospherical open neighborhood of each cusp.
3. Take fiber bundles $Z_i \to M_i$ with fiber N_i finitely covered by a compact quotient of an aspherical simply connected Lie group $\widetilde{N_i}$ by the action of a uniform lattice Γ_i, of dimension $n - n_i$, i.e., N_i is diffeomorphic to $H_i \backslash (\widetilde{N_i} / \Gamma_i)$, where $\widetilde{N_i}$ is a simply connected Lie group, Γ_i a uniform lattice, and H_i a finite group.
4. Fix a complete pairing of diffeomorphic boundary components between distinct Z_is, provided one exists, and glue the paired boundary components using diffeomorphisms, to obtain a connected manifold of dimension n.

We will call the Z_i's the pieces of M and when $\dim(M_i) = n$, then we say $Z_i = M_i$ is a pure piece, (short for purely negatively curved).

Remark 1 The construction in the previous definition includes:

1. The class of *generalized graph manifolds* of Frigerio et al. [11]. The pieces V_i in item (1) above are required to be hyperbolic with toral boundary cusps, the N_i in item (3) are required to be tori, and the gluing diffeomorphisms in item (4) are required to be affine diffeomorphims. These authors produce examples of manifolds within this class that do not admit any $CAT(0)$ metric.
2. The family of *cusp-decomposable manifolds* of Phan [14], where interesting (non)rigidity properties are explored. These manifolds only have pure pieces.
3. The *affine twisted doubles* of hyperbolic manifolds, for which Aravinda and Farrell study in [1] the existence of nonpositively curved metrics.
4. The *higher graph manifolds* studied in [9] by Connell and the third named author. In that family, item (3) consists of infra-nilmanifold bundles with affine structure group, which are, moreover, trivial near the cusp boundary of the negatively curved pieces in the base. In item (4) the gluing diffeomorphisms are restricted to those which are isotopic to affine diffeomorphisms. These two restrictions are used in [9] to prove statements about collapsing and computations of minimal volume. They turn out not to be needed in the arguments we present for the Borel conjecture to hold true.

The following theorem is our main result:

Theorem 1 *Let M be an n-dimensional manifold constructed as in Definition 1, for $n \geq 6$, then M satisfies the Borel conjecture, that is, given a homotopy equivalence $f : M \to M^{'}$, where $M^{'}$ is an aspherical n-dimensional manifold, then f is homotopic to a homeomorphism.*

The following section explains the notions of asymptotic dimension, weak regular coherence, and finite decomposition complexity. In the last section a proof of Theorem 1 that uses these properties can be found, and also a proof that presents a slight extension of the general strategy proposed by Frigerio–Lafont–Sisto, and we verify it for the higher graph manifolds whose pieces are trivial bundles.

2 Finite asymptotic dimension and weak regular coherence

2.1 Asphericity

Consider the next definition, following [11], which will be used later on:

Definition 2 The boundaries of the pieces Z_i that are identified together in Definition 1 will be called the internal walls of M.

Now we will prove, via an adaptation of the arguments of Frigerio et al. that the manifolds we are interested in are in fact aspherical.

Lemma 2 *If M is a manifold (possibly with boundary) constructed as in Definition 1, then M is aspherical.*

Proof This proof is by induction on the number of internal walls c of M. If $c = 0$ then $M = Z$ for some bundle Z over a closed, negatively curved base. It follows from the homotopy exact sequence for the bundle Z that M is aspherical in this case, establishing the base case for our inductive argument.

Assume $c > 0$, and that the result holds for manifolds constructed as in Definition 1, with strictly less than c internal walls. Cut open M along an arbitrary internal wall W. Our inductive hypothesis implies that now M is obtained by gluing one or two (depending on whether W separates M or not) aspherical spaces. Since the inclusion of W in the piece(s) in M it belongs to is π_1-injective, it follows from a classical result of Whitehead [16] that M is aspherical. □

2.2 Finite asymptotic dimension

Let G be a finitely presented group. Fix a finite generator set S and consider the word metric d_S induced by the generating set. With this metric, the group G is a proper metric space.

Definition 3 A family $\{U\}$ of subsets in a metric space X is D-disjoint if $d(U, U^{'}) > D$ for all subsets in the family. The asymptotic dimension asdim X of X is the smallest number n such that for any $D > 0$ there is a uniformly bounded cover of X by $n+1$-families of D-disjoint families of subsets.

An example of spaces (and groups) for which their asymptotic dimension can be explicitly computed are precisely quotients of simply connected Lie groups:

Theorem 3 (Carlsson and Goldfarb, Corollary 3.6 in [6]) *Let* Γ *be a cocompact lattice in a connected Lie group* G *with maximal compact subgroup* K*. Then* $\operatorname{asdim}\Gamma = dim(G/K)$.

For spaces that are built up using smaller subsets, there is a theorem that allows us to bound the asymptotic dimension of the total space. Let X be a metric space. The family $\{X_\alpha\}$ of subsets of X is said to satisfy the inequality $\operatorname{asdim} X_\alpha \le n$ uniformly if for every $r < \infty$ a constant R can be found so that for every α there exists R-disjoint families $U_\alpha^0, U_\alpha^1, U_\alpha^2, \ldots, U_\alpha^n$ of R-bounded subsets of X_α covering X_α.

Theorem 4 (Union theorem, Bell and Dranishnikov, Theorem 25 in [3]) *Let* $X = \bigcup_\alpha X_\alpha$ *be a metric space where the family* $\{X_\alpha\}$ *satisfies the inequality* $\operatorname{asdim} X_\alpha \le n$ *uniformly. Suppose further that for every* r *there is a* $Y_r \subset X$ *with* $\operatorname{asdim} Y_r \le n$ *so that* $d(X_\alpha - Y_r, X_{\alpha'} - Y_r) \ge r$ *whenever* $X_\alpha \ne X_{\alpha'}$*. Then* $\operatorname{asdim} X \le n$.

Lemma 5 *The fundamental group* $\pi_1(M)$ *of a manifold* M *of dimension* n *constructed as in Definition 1 has finite asymptotic dimension.*

Proof The fundamental groups of the pieces $\pi_1(Z_i)$ fit in an exact sequence

$$1 \to \pi_1(N_i) \to \pi_1(Z_i) \to \pi_1(M_i) \to 1.$$

The asymptotic dimension of $\pi_1(M_i)$ equals $n - n_i$ by the Cartan–Hadamard Theorem. On the other hand, the asymptotic dimension of the fibers, which are finitely covered by quotients of Lie groups G under the action of a uniform lattice, is finite itself because $\dim(G/K) < \infty$ by Theorem 3.

Finally, we invoke Theorem 4, from which we conclude that the asymptotic dimension of $\pi_1(X)$ is finite. □

Alternatively, another result of Bell and Dranishnikov, Theorem 77 in [3], shows that the asymptotic dimension of a graph of groups is finite provided each vertex group has finite asymptotic dimension.

2.3 Finite decomposition complexity

We will briefly define the notion of straight finite decomposition complexity, since we use it as a key property in the proof of the main result presented below.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two families of metric spaces, and $R > 0$. The family $\mathcal{X}$ is called R-decomposable over $\mathcal{Y}$ if, for any space X in $\mathcal{X}$ there are collections of subsets $\{U_{1,\alpha}\}$ and $\{U_{2,\beta}\}$ such that

$$X = \bigcup_{i=1,2,\gamma=\alpha,\beta} U_{i,\gamma}.$$

Each $U_{i,\gamma}$ is a member of the family $\mathcal{Y}$, and each of the collections $\{U_{1,\alpha}\}$ and $\{U_{2,\beta}\}$ is R-disjoint. A family of metric spaces is called bounded if there is a uniform bound on the diameters of the spaces in the family.

Definition 4 A metric space X has straight finite decomposition complexity if, for any sequence $R_1 \leq R_2 \leq \ldots$ of positive numbers, there exists a finite sequence of metric families $V_1, V_2, \ldots, V_n$ such that X is R_1-decomposable over V_1, V_1 is R_2-decomposable over V_2, etc., and the family V_n is bounded.

The following well-known lemma ties this notion with that of asymptotic dimension, for completeness we include a proof.

Lemma 6 *If a group has finite asymptotic dimension, then it has straight finite decomposition complexity.*

Proof It was shown by Guentner et al. that a countable metric space of finite asymptotic dimension has finite decomposition complexity in [13]. As part of their study of straight finite decomposition complexity, Dranishnikov and Zarichnyi showed in [10] that groups with finite decomposition complexity have straight finite decomposition complexity. □

3 Two proofs

3.1 The Carlsson–Goldfarb approach to the Borel conjecture

Let Γ be the fundamental group of a manifold constructed as in Definition 1. The strategy for proving Theorem 1 for manifolds with fundamental group Γ consists of showing that Γ satisfies the following properties:

1. Γ has finite asymptotic dimension.
2. Γ has a finite model for the classifying space $B\Gamma$.

A group satisfying these two conditions has been proven to also satisfy the integral isomorphism conjecture in algebraic K-theory, according to Theorem 3.11 of Goldfarb in [12].

Proof (of Theorem 1)
Item (1) was shown in Lemma 5 above.
Item (2) follows from the fact that these are fundamental groups of compact aspherical manifolds (possibly with boundary). Therefore, the Borel conjecture holds for the manifolds in Definition 1. □

This simple strategy provides an alternative to the one laid out by Frigerio et al. in [11]. We also present in the following a modified version of their strategy, and verify that it can be carried out for certain manifolds within those of Definition 1.

In a series of articles, Goldfarb and Carlsson have investigated several notions which generalize that of regular coherence for the group ring of infinite groups. The main geometric interest on this situation relies on the fact that these conditions are strong enough to allow the vanishing of the Whitehead group and negative algebraic K-theory groups of group rings, and weak enough to be handled with methods dealing with coarse versions of the isomorphism conjecture in algebraic K-theory [6,12].

We will recall some definitions and fundamental results related to finite asymptotic dimension and the coarse assembly map in the boundedly controlled setting. See, for example [5,12], for further reference.

Let $\mathcal{P}(G)$ be the power set viewed as a category where morphisms are inclusions of subsets. Let R be a noetherian ring and consider a finitely generated $R[G]$-module $\mathcal{M}$. A G-filtration of $\mathcal{M}$ is a functor $f : \mathcal{P}(G) \to R - Sub(\mathcal{M})$ to the category of R-submodules of $\mathcal{M}$ such that $f(G) = \mathcal{M}$, and each bounded set in the word metric d_S, $T \subset G$ is mapped to a finitely generated R-submodule. Such a functor f is equivariant if $f(gS) = gf(S)$.

Definition 5 A homomorphism $\phi : F_1 \to F_2$ between finitely generated $R[G]$-modules with fixed filtrations f_1, f_2 is boundedly controlled with respect to the bound $D > 0$ if $\phi(f_1(S)) \subset f_2(B_D(S))$ for each subset $S \subset G$. If ϕ also satisfies $\phi F_1 \cap f_2(S) \subset \phi F_1(B_D(S))$, then F is called boundedly bicontrolled.

Definition 6 Let $\mathcal{M}$ be a finitely presented $R[G]$-module. A finite presentation $F : R[G]^m \to R[G]^n \to \mathcal{M}$ is admissible if the homomorphism F is boundedly bicontrolled.

Definition 7 A group ring $R[G]$ is weakly coherent if every $R[G]$-module with an admissible presentation has a projective resolution of finite type. Similarly, a group ring is weakly regular coherent if every $R[G]$-module with an admissible presentation has finite homological dimension.

Theorem 7 (Carlsson and Goldfarb, Corollary 3.9 in [12]) *Let R be a noetherian ring and let G be a group of finite asymptotic dimension. Then, the group ring $R[G]$ is weakly regular coherent.*

Weak regular coherence has been verified to be enough to guarantee the vanishing of Whitehead groups and negative algebraic K-theory.

Theorem 8 (Goldfarb, Theorem 3.11 in [12]). *Let G be a group of finite asymptotic dimension (or more generally of finite decomposition complexity, as explained in [12]). Assume that there is a finite model for the classifying space $K(G, 1)$. Then, the assembly map in algebraic K-theory is an isomorphism. In particular, the Whitehead group of G vanishes.*

As a consequence of Theorem 8 and Lemma 5, we obtain:

Corollary 9 *The group ring $\mathbb{Z}\pi_1(M)$ of a manifold M constructed as in Definition 1 is weakly regular coherent.*

3.2 An extension of the Frigerio–Lafont–Sisto approach to the Borel conjecture

The proof of the Borel conjecture for the class of manifolds studied by Frigerio et al. in [11] in fact developed a general strategy to be carried out for a given family of manifolds. In their Theorem 3.1 they proved that if a manifold is built up from a geometric decomposition, as are the higher graph manifolds in this paper, and satisfies the following six conditions, then it also satisfies the Borel conjecture:

1. Each of the inclusions $W_{i,j} \to Z_i$ is π_1-injective.
2. Each of the pieces Z_i and each of the walls $W_{i,j}$ are aspherical.
3. Each of the pieces Z_i and each of the walls $W_{i,j}$ satisfy the Borel Conjecture.
4. The rings $\mathbf{Z}\pi_1(W_{i,j})$ are all regular coherent.
5. $Wh_k(\mathbf{Z}\pi_1(W_{i,j})) = 0 = Wh_k(\mathbf{Z}\pi_1(Z_i))$ for $k \leq 1$.
6. Each of the inclusions $\pi_1(W_{i,j}) \to \pi_1(Z_i)$ is square root closed.

We propose a slightly modified version of this strategy, where we replace the last three conditions, (4), (5) and (6), by a couple of new requirements. So that we obtain the following:

Lemma 10 *Let M be a compact manifold of dimension $n \geq 6$ with a topological decomposition* (*as described in* [11]). *Assume the following conditions hold:*

1. *Each of the inclusions $W_{i,j} \to Z_i$ is π_1-injective.*
2. *Each of the pieces Z_i and each of the walls $W_{i,j}$ are aspherical.*
3. *Each of the pieces Z_i and each of the walls $W_{i,j}$ satisfy the Borel Conjecture.*
4. *The group $\Gamma = \pi_1(M)$ has finite decomposition complexity.*
5. *There exists a finite model for the classifying space $K(\Gamma, 1)$.*

Then the manifold M also satisfies the Borel conjecture.

Proof Conditions (4) and (5) imply that the Whitehead groups $Wh_i(\mathbf{Z}\Gamma) = 0$, for $i \leq 1$, as proved in [12].

Therefore, the rest of the proof presented in Theorem 3.1 [11] goes through, and the result holds. □

Now we will concentrate on certain higher graph manifolds, explained briefly in the introduction (see [9]).

Lemma 11 *Assume M is a higher graph manifold, all of whose pieces are trivial as bundles. Then, each of the pieces $Z_i \cong N_i \times M_i$, and each of the walls $W_{i,j}$, satisfy the fibered isomorphism conjecture (FIC) of Farrell–Jones.*

Proof First notice that the validity of FIC for the walls $W_{i,j}$ follows from the work of Bartels et al. in [2], since these are quotients of Lie groups (see also their Remark 2.13).

As each piece Z_i is a trivial fibre bundle

$$Z_i \cong N_i \times M_i$$

the fundamental group of Z_i is a product

$$\pi_1(Z_i) \cong \pi_1(N_i) \times \pi_1(M_i).$$

Recall that M_i is a manifold that admits a pinched negatively curved metric. So it also admits a $CAT(0)$ metric, and therefore FIC holds for $\pi_1(M_i)$. The fibres satisfy FIC following [2]. Therefore, $\pi_1(Z_i)$ also satisfies FIC, by Theorem 2.9 in [2]. □

As a consequence we obtain that the Borel conjecture holds for each of the pieces Z_i, with trivial fibration structure, and each of the walls $W_{i,j}$, and so condition (3) is verified.

Lemma 12 *Let M be a higher graph manifold, all of whose pieces Z_i are trivial as bundles, and let $W_{i,j}$ denote its internal walls. Then, the rings $\mathbf{Z}\pi_1(W_{i,j})$ are weakly regular coherent.*

Proof From the proof of Lemma 5, we conclude that these groups have finite asymptotic dimension. Now the result follows from Theorem 7. □

Lemma 13 *Let M be a higher graph manifold, all of whose pieces Z_i are trivial as bundles, and let $W_{i,j}$ denote its internal walls. Then, $Wh_k(\mathbf{Z}\pi_1(W_{i,j})) = 0 = Wh_k(\mathbf{Z}\pi_1(Z_i))$ for $k \leq 1$.*

Proof Since each of the walls and pieces are aspherical, their fundamental groups are torsion free. By the previous Lemma 11, the result holds for each of the pieces and walls. Alternatively, the result follows from Theorem 8. □

Lemma 14 *Let M be a manifold constructed as in Definition 1. For every $1 \leq i \leq r$, the map $N_i \to X$ is π_1-injective. Moreover, the image of $\pi_1(N_i)$ is a square root closed subgroup in the group $\pi_1(X)$.*

Proof Consider the long exact sequence of homotopy groups of a fibration:

$$\cdots \to \pi_n(N_i) \to \pi_n(Z_i) \to \pi_n(M_i) \to \cdots .$$

The connectedness of N_i implies the π_1-injectivity condition.

Using proposition VII.2 in page 168 of [4], it suffices to verify the square root closed condition in the fundamental groups of the edges $\pi_1(W_i, j) \to \pi_1(Z_i)$. Using the long exact sequence of the fibration again, this is equivalent to showing that there are no 2-torsion elements in $\pi_1(M_i)$. This is certainly the case, since M_i is an aspherical manifold. □

Lemma 15 *Let M be a manifold constructed as in Definition 1 and $\Gamma = \pi_1(M)$. Then there exists a finite model for $K(\Gamma, 1)$.*

Proof Notice that the manifold M is aspherical and hence it is itself a finite model for $K(\Gamma, 1)$. □

Now we collect all of these auxiliary results to present:

Theorem 16 *Let M be a higher graph manifold of dimension ≥ 6. Assume that all of the pieces of M are trivial bundles. Then M satisfies the Borel conjecture.*

Proof Notice that these higher graph manifolds satisfy all the hypothesis of Lemma 10, as has been shown in Lemmas 6, 11, 13, 14, and 15. □

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