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Homotopy decomposition of a suspended real toric space

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Abstract We give p-local homotopy decompositions of the suspensions of real toric spaces for odd primes p. Our decomposition is compatible with the one given by Bahri, Bendersky, Cohen, and Gitler for the suspension of the corresponding real moment-angle complex, or more generally, the polyhedral product. As an application, we obtain a stable rigidity property for real toric spaces.

Keywords Homotopy decomposition · Real toric manifold · Real toric spaces

Mathematics Subject Classification Primary 55P15; Secondary 57S17

In memory of Professor Samuel Gitler.

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1 Introduction

For a simplicial complex K on m-vertices $[m] = \{1, \ldots, m\}$ the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$ (or the polyhedral product $(\underline{D^1}, \underline{S^0})^K$) of K is defined as follows:

$$\begin{aligned}\mathbb{R}\mathcal{Z}_K &= (\underline{D^1}, \underline{S^0})^K \\ &:= \bigcup_{\sigma \in K} \left\{(x_1, \ldots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ when } i \notin \sigma\right\},\end{aligned}$$

where $D^1 = [0, 1]$ is the unit interval and $S^0 = \{0, 1\}$ is its boundary. It should be noted that $\mathbb{R}\mathcal{Z}_K$ is a topological manifold if K is a simplicial sphere [2, Lemma 6.13], and that there is a canonical $\mathbb{F}_2^m$-action on $\mathbb{R}\mathcal{Z}_K$ which comes from the $\mathbb{F}_2$-action on the pair (D^1, S^0).

Let $n \leq m$. A map $\lambda \colon V = [m] \to \mathbb{F}_2^n$ is called a (mod 2) characteristic function of K if it has the property that

$$\lambda(i_1), \ldots, \lambda(i_\ell) \text{ are linearly independent in } \mathbb{F}_2^n \quad \text{if} \quad \{i_1, \ldots, i_\ell\} \in K. \tag{1}$$

For convenience, a characteristic function λ is frequently represented by an $(n \times m)$ $\mathbb{F}_2$-matrix $\Lambda = (\lambda(1) \ \ldots b \ \lambda(m))$, called a characteristic matrix. Define a map $\theta \colon [m] \to \mathbb{F}_2^m$ so that $\theta(i)$ is the ith coordinate vector of $\mathbb{F}_2^m$. Then the homomorphism Λ (viewed as a matrix multiplication) satisfies $\Lambda \circ \theta = \lambda$. We will see in Lemma 3.1 that Condition (1) ensures that the group $\ker \Lambda \cong \mathbb{F}_2^{m-n}$ acts freely on $\mathbb{R}\mathcal{Z}_K$. We denote by M_λ the associated real toric space, which is defined to be $\mathbb{R}\mathcal{Z}_K / \ker \Lambda$. If K is a polytopal $(n-1)$-sphere then M_λ is known as a small cover [8] and if K is a star-shaped $(n-1)$-sphere then M_λ is known as a real topological toric manifold [10].

In [1, Theorem 2.21] it is shown that there is a homotopy equivalence

$$\Sigma\mathbb{R}\mathcal{Z}_K \simeq \Sigma \bigvee_{I \notin K} \Sigma|K_I|, \tag{2}$$

where K_I is the full subcomplex of K on the vertex set I and $|K_I|$ is its geometric realization. In this short note, we give an analogous odd primary decomposition of the suspension of M_λ.

Theorem 1.1 *Let M_λ be a real toric space. Localized at an odd prime p or the rationals (denoted by $p = 0$) there is a homotopy equivalence*

$$\Sigma(M_\lambda) \simeq_p \Sigma \bigvee_{I \in \mathrm{Row}(\lambda)} \Sigma|K_I|,$$

where $\mathrm{Row}(\lambda)$ *is the space of m-dimensional $\mathbb{F}_2$-vectors spanned by the rows of Λ associated to λ.*

The restriction to odd primes arises because the free action of $\ker \Lambda$ on $\mathbb{R}\mathcal{Z}_K$ implies that when $|\ker \Lambda|$ is inverted in a coefficient ring R then the quotient map

$\mathbb{R}\mathcal{Z}_K \longrightarrow M_\lambda$ induces an injection in cohomology with image the invariant subring $H^*(\mathbb{R}\mathcal{Z}_K; R)^{\ker\Lambda}$. This will be used to help analyze the topology of $\mathbb{R}\mathcal{Z}_K$. As $|\ker\Lambda|$ has order a power of 2 we can take R to be $\mathbb{Z}_{(p)}$ or $\mathbb{Q}$. In fact, Theorem 1.1 fails when $p = 2$ in simple cases. For example, if K is the boundary of a triangle and $\lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, then $M_\lambda = \mathbb{R}P^2$ but each $\Sigma|K_I|$ is contractible.

Recent work of Yu [12] gave a different decomposition of the suspension of certain quotient spaces of $\mathbb{R}\mathcal{Z}_K$. He considers a homomorphism $\Lambda : \mathbb{F}_2^m \to \mathbb{F}_2^n$ which is associated to a partition on the vertices of K, and proves that $\Sigma\mathbb{R}\mathcal{Z}_K/\ker\Lambda$ decomposes analogously to the Bahri, Bendersky, Cohen and Gitler decomposition. Yu's decomposition has the advantage of working integrally and also for some non-free actions, but it has the disadvantage of working only for particular homomorphisms Λ. Our decomposition, by contrast, works only after localizing at an odd prime but holds for all characteristic maps derived from free actions.

2 Polyhedral product and its stable decomposition

Let us first recall Bahri, Bendersky, Cohen and Gitler's argument in [1]. To make it more clear, we present it in its full polyhedral product form. Let K be a simplicial complex on the vertex set $[m]$ and for $1 \leq i \leq m$ let (X_i, A_i) be pairs of pointed CW-complexes. If σ is a face of K let

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^{m} Y_i \qquad \text{where} \qquad Y_i = \begin{cases} X_i & \text{if } \ i \in K \\ A_i & \text{if } \ i \notin K. \end{cases}$$

The polyhedral product is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma\in K} (\underline{X}, \underline{A})^\sigma.$$

Notice that $(\underline{X}, \underline{A})^K$ is a subspace of the product $\prod_{i=1}^m X_i$. There is a canonical quotient map from the product to the smash product, $\prod_{i=1}^m X_i \longrightarrow \bigwedge_{i=1}^m X_i$. The smash polyhedral product $\widehat{(\underline{X}, \underline{A})}^K$ is the image of the composite $(\underline{X}, \underline{A})^K \longrightarrow \prod_{i=1}^m X_i \longrightarrow \bigwedge_{i=1}^m X_i$. In particular, mapping onto the image gives a map $(\underline{X}, \underline{A})^K \longrightarrow \widehat{(\underline{X}, \underline{A})}^K$.

Let $I \subset [m]$. As in [9, 2.2.3(i)], projecting $\prod_{i=1}^m X_i$ onto $\prod_{i\in I} X_i$ induces a map of polyhedral products $(\underline{X}, \underline{A})^K \longrightarrow (\underline{X}, \underline{A})^{K_I}$. We then obtain a composition into a smash polyhedral product:

$$p_I : (\underline{X}, \underline{A})^K \longrightarrow (\underline{X}, \underline{A})^{K_I} \longrightarrow \widehat{(\underline{X}, \underline{A})}^{K_I}.$$

Suspending, we can add every such composition over all full subcomplexes of K, giving a composition

$$\overline{H}\colon \Sigma(\underline{X},\underline{A})^K \xrightarrow{\text{comul}} \bigvee_{I\subset[m]} \Sigma(\underline{X},\underline{A})^{K_I} \xrightarrow{\vee\Sigma p_I} \bigvee_{I\subset[m]} \Sigma\widehat{(\underline{X},\underline{A})}^{K_I}.$$

Bahri, Bendersky, Cohen and Gitler [1, Theorem 2.10] show that $\overline{H}$ is a homotopy equivalence.

Further, in the special case when each X_i is contractible, they show that there is a homotopy equivalence $\widehat{(\underline{X},\underline{A})}^{K_I} \simeq \Sigma(|K_I| \wedge \widehat{A}^I)$ [1, Theoerem 2.19], where $\widehat{A}^I = \bigwedge_{j=1}^k A_{i_j}$ for $I = (i_1,\ldots,i_k)$. Consequently, when each X_i is contractible the map $\overline{H}$ specializes to a homotopy equivalence

$$H\colon \Sigma(\underline{X},\underline{A})^K \longrightarrow \bigvee_{I\subset[m]} \Sigma(\underline{X},\underline{A})^{K_I} \longrightarrow \bigvee_{I\subset[m]} \Sigma\widehat{(\underline{X},\underline{A})}^{K_I} \xrightarrow{\simeq} \bigvee_{I\subset[m]} \Sigma^2(|K_I|\wedge\widehat{A}^I).$$

In our case, each pair (X_i, A_i) equals (D^1, S^0) and D^1 is contractible. As there is a homotopy equivalence $S^0 \wedge S^0 \simeq S^0$, each $\widehat{A}^I$ is homotopy equivalent to S^0. Therefore there are homotopy equivalences

$$\widehat{\mathbb{R}\mathcal{Z}}_{K_I} := \widehat{(\underline{D^1},\underline{S^0})}^{K_I} \xrightarrow{\simeq} \Sigma|K_I| \wedge \widehat{A}^I \simeq \Sigma|K_I| \wedge S^0 \simeq \Sigma|K_I|. \tag{3}$$

Thus the map H becomes a homotopy equivalence

$$H\colon \Sigma\mathbb{R}\mathcal{Z}_K \longrightarrow \bigvee_{I\subset[m]} \Sigma\mathbb{R}\mathcal{Z}_{K_I} \longrightarrow \bigvee_{I\subset[m]} \Sigma\widehat{\mathbb{R}\mathcal{Z}}_{K_I} \xrightarrow{\simeq} \bigvee_{I\subset[m]} \Sigma^2|K_I|.$$

It is in this form that we will use the Bahri, Bendersky, Cohen and Gitler decomposition because, as we will see shortly, it corresponds to a module decomposition of a differential graded algebra R_K whose cohomology equals $H^*(\mathbb{R}\mathcal{Z}_K)$. But it is worth pointing out that in [1, Theorem 2.21] it was shown that when each X_i is contractible then $\widehat{(\underline{X},\underline{A})}^{K_I}$ is contractible if $I \in K$. So the usual Bahri, Bendersky, Cohen and Gitler decomposition is of the form

$$\Sigma(\underline{X},\underline{A})^K \simeq \bigvee_{I\notin K} \Sigma^2(|K_I| \wedge \widehat{A}^I),$$

giving the special case

$$\Sigma\mathbb{R}\mathcal{Z}_K \simeq \Sigma \bigvee_{I\notin K} \Sigma|K_I|,$$

which is the statement in (2).

3 Proof of the main theorem

First, recall that M_λ is the quotient of $\mathbb{R}\mathcal{Z}_K$ by $\ker\Lambda$.

Lemma 3.1 *Under Condition* (1)*,* $\ker \Lambda$ *acts on* $\mathbb{R}\mathcal{Z}_K$ *freely.*

Proof Let $\bar{g} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}\mathcal{Z}_K = (\underline{D^1}, \underline{S^0})^K$ be the fixed point of an element $g = (g_1, g_2, \ldots, g_m) \in \ker \Lambda \subset \mathbb{F}_2^m$. This means either $g_i = 0$ or $x_i \in (D^1)^{\mathbb{F}_2} = \{1/2\}$ for all $i \in [m]$. Let $\sigma \in K$ be the maximal simplex such that $x \in (\underline{D^1}, \underline{S^0})^\sigma$ and Λ_σ be the sub-matrix of Λ consisting of columns corresponding to σ. Let g_σ be the sub-vector of g corresponding to σ. Since $g \in \ker \Lambda$, we have

$$\Lambda g = \Lambda_\sigma g_\sigma + \Lambda_{[m]\setminus\sigma} g_{[m]\setminus\sigma} = 0.$$

Since $\mathbb{F}_2$ acts on S^0 freely, we have $g_i = 0$ for $i \notin \sigma$. Then, by the previous equation we have $\Lambda_\sigma g_\sigma = 0$. Therefore Condition (1) implies $g_\sigma = 0$ and we have $g = 0$. □

Next, consider the following diagram

$$\begin{array}{ccccc} \Sigma\mathbb{R}\mathcal{Z}_K & \xrightarrow{\bar{H}} & \Sigma \bigvee_{I\subset[m]} \widehat{\mathbb{R}\mathcal{Z}}_{K_I} & \xrightarrow{\simeq} & \Sigma \bigvee_{I\subset[m]} \Sigma|K_I| \\ \downarrow{\scriptstyle \Sigma q} & & \uparrow{\scriptstyle \Sigma incl} & & \\ \Sigma M_\lambda & \xleftarrow{\phi} & \Sigma \bigvee_{I\in \mathrm{Row}(\lambda)} \widehat{\mathbb{R}\mathcal{Z}}_{K_I} & \xrightarrow{\simeq} & \Sigma \bigvee_{I\in\mathrm{Row}(\lambda)} \Sigma|K_I| \end{array} \tag{4}$$

where, by definition, $\phi = \Sigma q \circ \bar{H}^{-1} \circ \Sigma incl$.

To prove Theorem 1.1 we will show that ϕ^* induces an isomorphism on cohomology with $\mathbb{Z}_{(p)}$-coefficients. From now on, assume that coefficients in cohomology are $\mathbb{Q}$ or $\mathbb{Z}_{(p)}$, where p is an odd prime.

First, by [4, Theorem 5.1] the cohomology ring of $\mathbb{R}\mathcal{Z}_K$ is given as follows. Let $\mathbb{Z}_{(p)}\langle u_1, \ldots, u_m, t_1, \ldots, t_m\rangle$ be the free associative algebra over the indeterminants of $\deg u_i = 1$, $\deg t_i = 0$ $(i = 1, \ldots, m)$. Define a differential graded algebra R_K by

$$R_K = \frac{\mathbb{Z}_{(p)}\langle u_1, \ldots, u_m, t_1, \ldots, t_m\rangle}{(u_\sigma \mid \sigma \notin K, u_i^2, u_iu_j + u_ju_i, u_it_i - u_i, t_iu_i, t_iu_j - u_jt_i, t_i^2 - t_i, t_it_j - t_jt_i)}$$

where $i \neq j$ and $d(t_i) = u_i$ for each $i = 1, \ldots, m$. Then $H^*(\mathbb{R}\mathcal{Z}_K) = H^*(R_K)$. We shall use the notation u_σ (respectively, t_σ) for the monomial $u_{i_1} \ldots u_{i_k}$ (respectively, $t_{i_1} \ldots t_{i_k}$) where $\sigma = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$, is a subset of $[m]$. For $I \subset [m]$, denote by R_{K_I} the differential graded sub-module of R_K spanned by the monomials $\{u_\sigma t_{I\setminus\sigma} \mid \sigma \in K_I\}$. Observe from the definitions of R_K and R_{K_I} that there is an additive isomorphism $R_K = \bigoplus_{I\subset[m]} R_{K_I}$.

Lemma 3.2 *There is an additive isomorphism*

$$H^*(R_{K_I}) \simeq \tilde{H}^*(\widehat{\mathbb{R}\mathcal{Z}}_{K_I})$$

and the projection $p_I\colon \mathbb{R}\mathcal{Z}_K \to \widehat{\mathbb{R}\mathcal{Z}}_{K_I}$ *induces the inclusion* $p_I^* : H^*(R_{K_I}) \hookrightarrow H^*(R_K)$.

Proof The first assertion follows from $\widehat{\mathbb{R}\mathcal{Z}}_{K_I} \simeq \Sigma|K_I|$ [see (3)] and the isomorphism $H^*(R_{K_I}) \simeq \tilde{H}^{*-1}(|K_I|)$ given by

$$\begin{aligned} R_{K_I} &\to C^*(K_I) \\ u_\sigma t_{I\setminus\sigma} &\mapsto \sigma^*, \end{aligned}$$

where $C^*(K_I)$ is the simplicial cochain complex of K_I ([4, Proposition 3.3]).

To show the second assertion, we look more closely at the isomorphism $H^*(R_K) \simeq H^*(\mathbb{R}\mathcal{Z}_K)$. From [4, §3.2], the monomials $u_\sigma t_{I\setminus\sigma}$ are mapped into the image of $p_I^* : C_e^*(\mathbb{R}\mathcal{Z}_{K_I}) \to C_e^*(\mathbb{R}\mathcal{Z}_K)$, where C_e^* denotes the cellular cochain complex. By combining this with the first assertion, we deduce the second assertion. □

Now we investigate the maps appearing in (4). Since the action of $\ker\Lambda$ on $\mathbb{R}\mathcal{Z}_K$ is free and $|\ker\Lambda|$ is a unit in the coefficient ring $\mathbb{Z}_{(p)}$, the map q^* is injective with image $H^*(\mathbb{R}\mathcal{Z}_K)^{\ker\Lambda}$. Notice that in cohomology incl induces the projection $\text{incl}^* : \bigoplus_{I\subset[m]} H^*(R_{K_I}) \to \bigoplus_{I\in\text{Row}(\lambda)} H^*(R_{K_I})$. Recall that $\bar{H} = \Sigma\bigvee_{I\subset[m]} p_I \circ \text{comul}$ and $\phi = \Sigma q \circ \bar{H}^{-1} \circ \Sigma\text{incl}$. So ϕ^* is the composite

$$\phi^* : H^*(\Sigma M_\lambda) \simeq H^*(\Sigma\mathbb{R}\mathcal{Z}_K)^{\ker\Lambda} \hookrightarrow H^*(\Sigma\mathbb{R}\mathcal{Z}_K) \xrightarrow{\simeq} \bigoplus_{I\subset[m]} H^*(\Sigma R_{K_I})$$
$$\to \bigoplus_{I\in\text{Row}(\lambda)} H^*(\Sigma R_{K_I}),$$

where Σ for graded modules means the degree shift in the positive degree parts.

We aim to show that ϕ^* is an isomorphism. To see this, first observe that $H^*(\mathbb{R}\mathcal{Z}_K)^{\ker\Lambda} \simeq H^*(R_K^{\ker\Lambda})$. We need two lemmas.

Lemma 3.3 ([7, Section 4]) *The Reynolds operator*

$$N(x) := \frac{1}{|\ker\Lambda|}\sum_{g\in\ker\Lambda} gx$$

induces an additive isomorphism $\bigoplus_{I\in\text{Row}(\lambda)} R_{K_I} \xrightarrow{\simeq} R_K^{\ker\Lambda}$, *where* $R_K^{\ker\Lambda}$ *is the* $\ker\Lambda$*-invariant ring of* R_K. *Furthermore, for a monomial* $x = u_\sigma t_{I\setminus\sigma}$, $N(x)$ *has the unique maximal term* x, *where the order is given by the containment of the index set.* □

Lemma 3.4 *The composite*

$$\Phi : R_K^{\ker\Lambda} \hookrightarrow R_K \simeq \bigoplus_{I\subset[m]} R_{K_I} \xrightarrow{\pi} \bigoplus_{I\in\text{Row}(\lambda)} R_{K_I}$$

is an isomorphism, where π *is the projection.*

Proof We first show this is surjective. Take an element $x \in \bigoplus_{I\in\text{Row}(\lambda)} R_{K_I}$. We induct on the size of the index set of x. By Lemma 3.3, the terms in $\pi(N(x) - x) \in \bigoplus_{I\in\text{Row}(\lambda)} R_{K_I}$ has an index set strictly smaller than that for x. By induction hypothesis, there is an element $y \in R_K^{\ker\Lambda}$ such that $\Phi(y) = \pi(N(x) - x)$. Put $z = N(x) - y \in R_K^{\ker\Lambda}$ and we have $\Phi(z) = \pi(N(x)) - \Phi(y) = \pi(x) = x$.

On the other hand, suppose $\Phi(y) = 0$ for some $y \in R_K^{\ker \Lambda}$. By Lemma 3.3, there is $x \in \bigoplus_{I \in \mathrm{Row}(\lambda)} R_{K_I}$ such that $y = N(x)$ and y must contain the maximal terms in x. Thus, $\Phi(y) = 0$ implies $x = 0$ and $y = N(x) = 0$. □

Proof of Theorem 1.1 Since $H^*(\mathbb{R}\mathcal{Z}_K)^{\ker \Lambda} \simeq H^*(R_K^{\ker \Lambda})$, the definitions of ϕ and Φ imply that $\Phi^* = \phi^*$. Therefore, by Lemma 3.4, ϕ^* is an isomorphism. □

Remark 3.5 In [7], they only consider the case when the coefficients ring is either $\mathbb{Q}$ or $\mathbb{Z}/q\mathbb{Z}$ for $q > 2$. However, since $H^*(M_\lambda) = H^*(\mathbb{R}Z_K/\ker \Lambda) \simeq H^*(\mathbb{R}Z_K)^{\ker \Lambda}$ holds for any coefficients ring such that 2 is invertible, their theorem is valid also for $\mathbb{Z}_{(p)}$ for an odd prime p.

4 Stable rigidity of real toric spaces

In this section, we give an application of Theorem 1.1 to a stable rigidity property of real toric spaces.

Corollary 4.1 *Let M_λ be a real toric space over K. When K_I for any $I \in \mathrm{Row}(\lambda)$ suspends to a wedge of spheres after localization at an odd prime p, ΣM_λ is homotopy equivalent to a wedge of spheres after localization at p. Let N_μ be another real toric spaces over K', where K'_I for any $I \in \mathrm{Row}(\mu)$ suspends to a wedge of spheres after localization at p. Then, if $H^*(M_\lambda; \mathbb{F}_p) \simeq H^*(N_\mu; \mathbb{F}_p)$ as modules, we have $\Sigma M_\lambda \simeq_p \Sigma N_\mu$.* □

Real toric spaces associated to graphs

Given a connected simple graph G with $n+1$ nodes $[n+1]$, the *graph associahedron* P_G ([5]) of dimension n is a convex polytope whose facets correspond to the connected subgraphs of G. Let K be the boundary complex of P_G. We can describe K directly from G: the vertex set of K consists proper subsets $T \subsetneq [n+1]$ such that $G|_T$ are connected and the simplices are the tubings of G. We define a mod 2 characteristic map λ_G on K as follows:

$$\lambda_G(T) = \begin{cases} \sum_{t \in T} \mathbf{e}_t, & \text{if} \quad n+1 \notin T; \\ \sum_{t \notin T} \mathbf{e}_t, & \text{if} \quad n+1 \in T, \end{cases}$$

where $\mathbf{e}_t$ is the tth coordinate vector of $\mathbb{F}_2^n$. Then we have a real toric manifold $M(G) := M_{\lambda_G}$ associated to G.

The signed a-number $\mathrm{sa}(G)$ of G is defined recursively by

$$\mathrm{sa}(G) = \begin{cases} 1, & \text{if} \quad G = \varnothing; \\ 0, & \text{if} \quad \text{G has a connected component with odd number of nodes;} \\ -\sum_{T \subsetneq [n+1]} \mathrm{sa}(G|_T), & \text{otherwise,} \end{cases}$$

and the a-number $a(G)$ of G is the absolute value of $sa(G)$. As shown in [6], there is a bijection φ from $\mathrm{Row}(\lambda_G)$ to the set of subgraphs of G having an even number nodes and $|K_I|$ for $I \in \mathrm{Row}(\lambda_G)$ is homotopy equivalent to $\bigvee^{a(\varphi(I))} S^{|\varphi(I)|/2-1}$ where $|\varphi(I)|$ is the number of nodes of $\varphi(I)$. By Theorem 1.1 we obtain the following.

Corollary 4.2 *We have a homotopy equivalence*

$$\Sigma M(G)_{(p)} \simeq_p \bigvee_{I \in \mathrm{Row}(\lambda_G)} \bigvee^{a(\varphi(I))} S^{|\varphi(I)|/2+1} \quad \text{for any odd prime } p.$$

□

Now, we define the a_i-number $a_i(G)$ of G by

$$a_i(G) = \sum_{\substack{T \subseteq [n+1] \\ |T| = 2i}} a(G|_T).$$

Then, $a_i(G)$ coincides the ith Betti number $\beta^i(M(G); \mathbb{F}_p)$ of $M(G)$. It should be noted that, by Corollary 4.2, if two graphs G_1 and G_2 have the same a_i-numbers for all i's, then $\Sigma M(G_1) \simeq_p \Sigma M(G_2)$ for any odd prime p.

Example 4.3 Let P_4 be a path graph of length 3, and $K_{1,3}$ a tree with one internal node and 3 leaves (known as a claw). One can compute $a_i(G) := \sum_{\substack{T \subseteq [n+1] \\ |T| = 2i}} a(G|_T)$ as follows:

$$\begin{aligned} a_0(P_4) &= a_0(K_{1,3}) = 1, \\ a_1(P_4) &= a_1(K_{1,3}) = 3, \\ a_2(P_4) &= a_2(K_{1,3}) = 2, \\ a_i(P_4) &= a_i(K_{1,3}) = 0 \quad \text{for } i > 2. \end{aligned}$$

Hence, by Corollary 4.2, $\Sigma M(P_4) \simeq_p \Sigma M(K_{1,3})$ for any odd prime p although $\Sigma M(P_4)$ and $\Sigma M(K_{1,3})$ are not homotopy equivalent since they have different mod-2 cohomology.

Real toric spaces over fillable complexes

There is a wide class of simplicial complexes on which every real toric space satisfies the assumption in Corollary 4.1.

Definition 4.4 ([11, Definition 4.8]) Let K be a simplicial complex. Let $K_1, \ldots, K_s$ be the connected components of K, and let $\hat{K}_i$ be a simplicial complex obtained from K_i by adding all of its minimal non-faces. Then K is said to be $\mathbb{F}_p$-homology fillable if (1) for each i there are minimal non-faces $M_1^i, \ldots, M_r^i$ of K such that $K_i \cup M_1^i \cup \cdots \cup M_r^i$ is acyclic over $\mathbb{F}_p$, and (2) $\hat{K}_i$ is simply connected for each i.

Moreover, we say that K is totally $\mathbb{F}_p$-homology fillable when K_I is $\mathbb{F}_p$-homology fillable for any $\emptyset \neq I \subset [m]$.

Proposition 4.5 ([11, Proposition 4.15]) *If K is $\mathbb{F}_p$-homology fillable, then $\Sigma|K|_{(p)}$ is a wedge of p-local spheres.* □

There is a large class of simplicial complexes which are totally homology fillable.

Proposition 4.6 ([11, Propositions 5.18 and 5.19]) *If the Alexander dual of K is sequentially Cohen–Macaulay over $\mathbb{F}_p$ ([3]), then K is totally $\mathbb{F}_p$-homology fillable.*

Note that the Alexander duals of shifted and shellable simplicial complexes are sequentially Cohen–Macaulay over $\mathbb{F}_p$.

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References

1. Bahri, A., Bendersky, M., Cohen, F.R., Gitler, S.: The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces. Adv. Math. **225**(3), 1634–1668 (2010)
2. Buchstaber, V.M., Panov, T.E.: Torus Actions and Their Applications in Topology and Combinatorics, University Lecture Series, vol. 24. American Mathematical Society, Providence (2002)
3. Björner, A., Wachs, M., Welker, V.: On sequentially Cohen–Macaulay complexes and posets. Israel J. Math. **169**, 295–316 (2009)
4. Cai, L.: On products in a real moment-angle manifold. J. Math. Soc. Japan. arXiv:1301.1518 (to appear)
5. Carr, M., Devadoss, S.: Coxeter complexes and graph-associahedra. Topol. Appl. **153**, 2155–2168 (2006)
6. Choi, S., Park, H.: A new graph invariant arises in toric topology. J. Math. Soc. Japan **67**(2), 699–720 (2015)
7. Choi, S., Park, H.: On the Cohomology and Their Torsion of Real Toric Objects. arXiv:1311.7056
8. Davis, M.W., Januszkiewicz, T.: Convex polytopes, coxeter orbifolds and torus actions. Duke Math. J. **62**(2), 417–451 (1991)
9. Denham, G., Suciu, A.: Moment-angle complexes, monomial ideals and Massey products. Pure Appl. Math. Q. **3**, 25–60 (2007)
10. Ishida, H., Fukukawa, Y., Masuda, M.: Topological toric manifolds. Moscow Math. J. **13**(1), 57–98 (2013)
11. Iriye, K., Kishimoto, D.: The Fat Wedge Filtration and a Homotopy Decomposition of a Polyhedral Product. arXiv:1412.4866
12. Yu, L.: On a Class of Quotient Spaces of Moment-Angle Complexes. arXiv:1406.7392