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ORIGINAL ARTICLE

Poisson structures on wrinkled fibrations

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Abstract We provide local formulae for Poisson bivectors and symplectic forms on the leaves of Poisson structures associated with wrinkled fibrations on smooth 4-manifolds. When such a fibration structure does not have 2-spheres in its fibres, the associated Poisson structure is integrable.

Keywords Poisson structures · Smooth 4-manifolds · Wrinkled fibrations

Mathematics Subject Classification 57R17 · 53D17

1 Introduction

The study of smooth manifolds of dimension four has led to various interesting types of fibrations. One of the origins of this research direction can be found in the seminal paper by Auroux et al. [1]. They described near-symplectic forms adapted to a type of fibration that has since been called *broken Lefschetz fibration*, this name was introduced by Perutz [9].

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Definition 1.1 On a smooth 4-manifold X, a *broken Lefschetz fibration* is a smooth map $f : X \to S^2$ that is a submersion outside the singularity set. Moreover, the allowed singularities are of the following type:

(i) *Lefschetz* singularities: finitely many points

$$\{p_1, \dots, p_k\} \subset X,$$

which are locally modelled by complex charts

$$\mathbf{C}^2 \to \mathbf{C}, \qquad (z_1, z_2) \mapsto z_1^2 + z_2^2,$$

(ii) *indefinite fold* singularities, also called *broken*, contained in the smooth embedded 1-dimensional submanifold $\Gamma \subset X \setminus \{p_1, \dots, p_k\}$, which are locally modelled by the real charts

$$\mathbf{R}^4 \to \mathbf{R}^2, \qquad (t, x_1, x_2, x_3) \mapsto \left(t, -x_1^2 + x_2^2 + x_3^2\right).$$

The term indefinite in (ii) refers to the fact that the quadratic form $-x_1^2 + x_2^2 + x_3^2$ is neither negative nor positive definite. In the language of singularity theory, these subsets are known as fold singularities of corank 1. Since X is closed, Γ is homeomorphic to a collection of disjoint circles. For this reason, throughout this work, we will often refer to Γ as *singular circles*. On the other hand, we can only assert that $f(\Gamma)$ is a union of immersed curves. In particular, the images of the components of Γ need not be disjoint, and the image of each component can self-intersect.

In [6] the first named author together with García-Naranjo and Vera exhibited a Poisson structure whose symplectic leaves coincide with the fibres of a broken Lefschetz fibration, and the singular sets of both structures coincide.

The notion of *wrinkled fibration* on a smooth 4-manifold was introduced by Lekili [8], who showed that these wrinkled fibrations exist in every closed oriented smooth 4-manifold. Broken Lefschetz fibrations are not stable, as maps. In contrast, wrinkled fibrations are stable. So if one is interested in perturbations of broken Lefschetz fibrations, one is led naturally to the study of wrinkled fibrations. Lekili described a set of moves that include, up to homotopy, all the possible one-parameter deformations of broken and wrinkled fibrations. These were then further studied by Williams [12]. Both [8] and [12] are motivated by the *Lagrangian matching invariants* defined by Perutz [9]. These moves preserve the diffeomorphism type of the underlying smooth 4-manifold.

Let X be a closed 4-manifold, and Σ be a 2-dimensional surface. A map $f : X \to \Sigma$ is said to have a cusp singularity at a point p in X, if around p, f is locally modelled in oriented charts by the map:

$$(t, x, y, z) \mapsto \left(t, x^3 - 3xt + y^2 - z^2\right).$$

The critical point set is a smooth arc, $\{x^2 = t, y = 0, z = 0\}$, the critical value set is a cusp given by $\{(t, s) : 4t^3 = s^2\}$ (see Fig. 1).

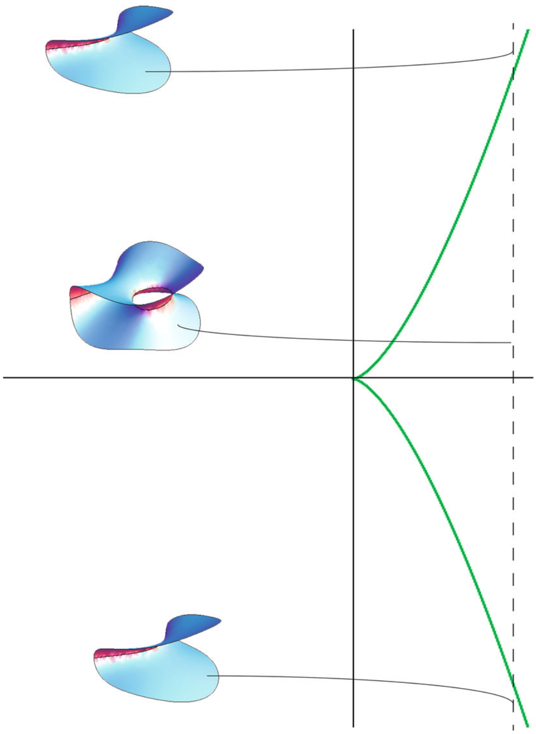

Fig. 1 A diagram depicting a fibration with a cusp singularity along the critical values depicted in *green*, which is a subset of the base of this fibration. The *black lines* indicate the points over which each of the fibres lie (color figure online)

Following Lekili we state:

Definition 1.2 A wrinkled fibration on a closed 4-manifold X is a smooth map f to a closed surface which is a broken fibration when restricted to $X \setminus C$, where C is a finite set such that around each point in C, f has cusp singularities. We say that a fibration is *purely wrinkled* if it has no isolated Lefschetz-type singularities.

Wrinkled fibrations may be obtained from broken Lefschetz fibrations by performing wrinkling moves. These eliminate a Lefschetz type singularity and introduce a wrinkled fibration structure. Conversely, it is possible to modify a wrinkled fibration locally by smoothing out the cusp singularity by introducing a Lefschetz type singularity, so obtaining a broken fibration (see [8,12]).

Next we will introduce the deformations of wrinkled fibrations that will appear in our Poisson structures. As Lekili, by a *deformation of wrinkled fibrations* we mean a one-parameter family of maps which is a wrinkled fibration for all but finitely many values. One of Lekili's major contributions in [8] was to show that any one-parameter family deformation of a purely wrinkled fibration is homotopic (relative endpoints) to one which realises a sequence of births, merges, flips, their inverses, and isotopies staying within the class of purely wrinkled fibrations. Moreover, these moves do not change the diffeomorphism type of the 4-manifold X on which they take place.

Let us briefly describe these moves, readers may consult both [8,12] for the corresponding descriptions in terms of how the fibres change and how these moves can

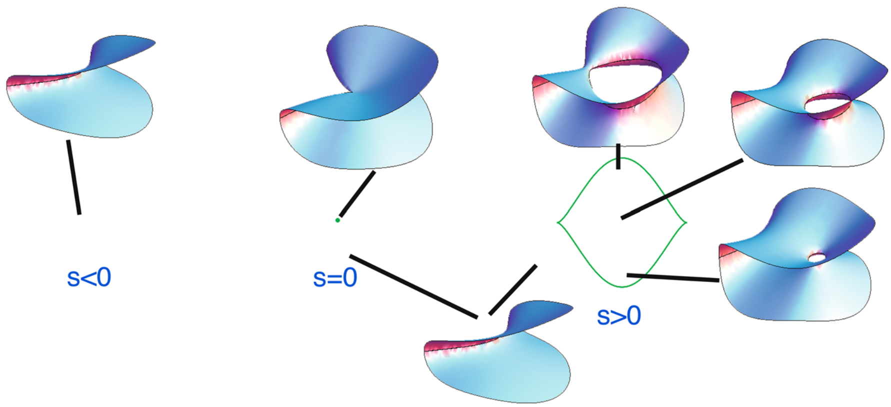

Fig. 2 **Birth move:** $b_s(x, y, z, t) = (t, x^3 - 3x(t^2 - s) + y^2 - z^2)$. For $s < 0$ the critical set is empty. Then when $s = 0$ the fibre above the critical point is shown to develop a singularity. As s becomes positive the wrinkled critical set appears, here depicted by the *green line*, which is a subset of the base of this fibration. The *black lines* indicate the points over which each of the fibres lies (color figure online)

be described using handlebodies. The moves we are interested in are to be considered as maps $\mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}$, given by the following equations, each depending on a real parameter s:

Move 1 (Birth, Fig. 2)

$$b_s(x, y, z, t) = (t, x^3 - 3x(t^2 - s) + y^2 - z^2).$$

Move 2 (Merging, Fig. 3)

$$m_s(x, y, z, t) = (t, x^3 - 3x(s - t^2) + y^2 - z^2).$$

Move 3 (Flipping, Fig. 4)

$$f_s(x, y, z, t) = (t, x^4 - x^2 s + xt + y^2 - z^2).$$

Move 4 (Wrinkling)

$$w_s(x, y, z, t) = (t^2 - x^2 + y^2 - z^2 + st, 2tx + 2yz).$$

The proof of the existence of a Poisson structure presented in [6] also implies that there exists a Poisson structure associated with wrinkled fibrations, where the fibres are leaves of the symplectic foliation and both structures share the same singularities.

The constructions presented in this paper use a general procedure to construct Poisson structures with prescribed Casimir functions. These ideas, in a general context, can be traced back to Damianou's thesis [3], to work of Damianou–Petalidou [4], and have also been attributed to Flaschka–Ratiu. However, the synthetic nature of the proof in [6], while complete, does not provide any information about the local forms of the Poisson bivector for wrinkled fibrations. The purpose of this note is to provide

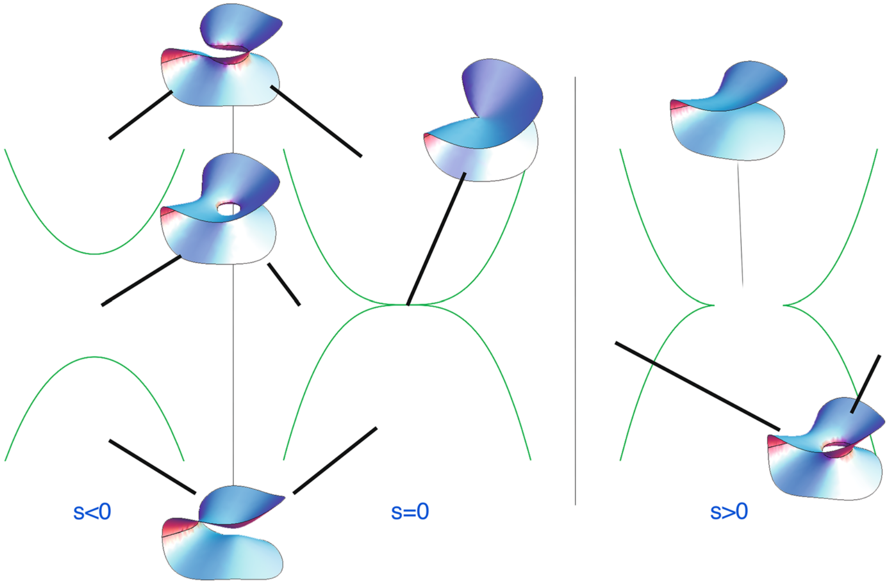

Fig. 3 **Merging move:** $m_s(x, y, z, t) = (t, x^3 - 3x(s - t^2) + y^2 - z^2)$. For $s < 0$ the critical set shows two connected components. Then when $s = 0$ these two components touch, and the fibre above the critical point is shown to develop a singularity. As s becomes positive the critical set (shown in *green*) separates again. The *black lines* indicate the points over which each of the fibres lies (color figure online)

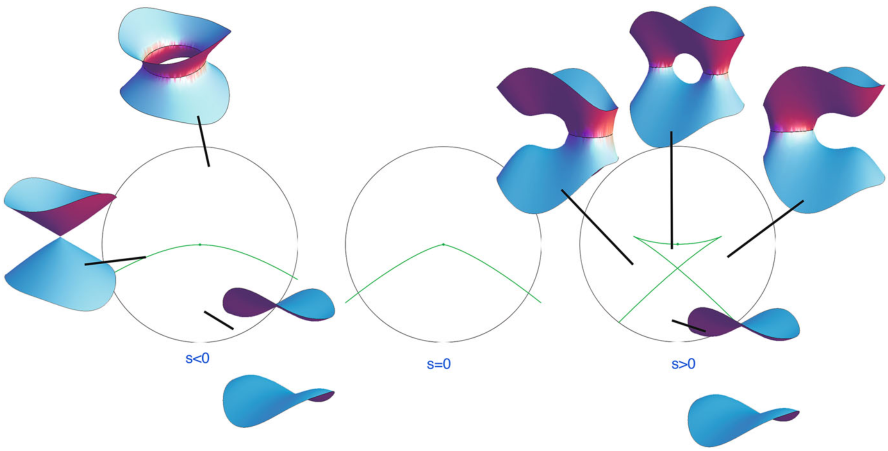

Fig. 4 **Flipping move:** $f_s(x, y, z, t) = (t, x^4 - x^2 s + xt + y^2 - z^2)$. For $s < 0$ the critical set corresponds to that of a broken Lefschetz fibration. As s becomes positive the critical set (shown in *green*) crosses itself. The *black lines* indicate the points over which each of the fibres lies (color figure online)

these details. In this paper, we continue to exploit the techniques of [6] in order to present local formulae for both the Poisson bivector and for the symplectic forms on the leaves of Poisson structures whose symplectic foliation and singularities are given by wrinkled fibrations and their deformations. Let us summarise the contributions of this paper in the following:

Theorem 1.3 *Let X be a closed, orientable, smooth* 4*-manifold equipped with a wrinkled fibration, or for a fixed s a fibration given by one of the birth, merging, flipping, or wrinkiling moves described above. Then there exists a complete Poisson structure whose symplectic leaves correspond to the fibres of the given fibration structure, and the singularities of both the fibration and the Poisson structures coincide. Moreover:*

(i) *In a neighbourhood of a cusp singularity the Poisson bivector is given by Eq.* (3.2)*, and the symplectic form on the leaves by Eq.* (3.7)*.*
(ii) *In a neighbourhood of a birth move the Poisson bivector is given by Eq.* (3.3)*, and the symplectic form on the leaves by Eq.* (3.8)*.*
(iii) *In a neighbourhood of a merging move the Poisson bivector is given by Eq.* (3.4)*, and the symplectic form on the leaves by Eq.* (3.9)*.*
(iv) *In a neighbourhood of a flipping move the Poisson bivector is given by Eq.* (3.5)*, and the symplectic form on the leaves by Eq.* (3.10)*.*
(v) *In a neighbourhood of a wrinkling move the Poisson bivector is given by Eq.* (3.6)*, and the symplectic form on the leaves by Eq.* (3.11)*.*

The existence of a Poisson structure with the stated properties follows from Theorem 2.4, originally shown in [6].

Given a Lie algebra $\mathfrak{g}$, the integrability problem for $\mathfrak{g}$ was solved by Lie's third theorem. It gives the existence of a Lie group G such that $Lie(G) \cong \mathfrak{g}$. Crainic and Fernandes found obstructions for the integrability of Lie algebroids. Their Corollary 5.13 in [2] establishes that any Poisson manifold whose symplectic leaves have trivial second homotopy groups is integrable.

Therefore we deduce:

Lemma 1.4 *A Poisson structure associated with a wrinkled fibration structure or its deformation moves as in Theorem* 1.3*, or to a broken Lefschetz fibration as in [6], none of whose symplectic leaves are, or contain,* 2*-spheres, is integrable.*

Definitions and useful related results can be found in Sect. 2. The computations of the bivectors and symplectic forms are carried out in Sect. 3.

2 Definitions

2.1 Poisson manifolds

In this section we include basic facts about Poisson geometry that we will use throughout the paper. Interested readers are invited to consult [5,7,11].

Definition 2.1 A *Poisson bracket* (or a *Poisson structure*) on a smooth manifold M is a bilinear operation $\{\cdot,\cdot\}$ on the set $C^\infty(M)$ of real valued smooth functions on M that satisfies

(i) $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra.

(ii) $\{gh, k\} = g\{h, k\} + h\{g, k\}$ for any $g, h, k \in C^\infty(M)$.

A manifold M with such a Poisson bracket is called a *Poisson manifold*. Symplectic manifolds (M, ω) provide examples of Poisson manifolds. In the symplectic case the bracket of M is defined by

$$\{g, h\} = \omega(X_g, X_h).$$

Hamiltonian vector fields X_h in symplectic manifolds are defined by $\mathbf{i}_{X_h}\omega = dh$.

Property (ii) in Definition 2.1 allows us to define Hamiltonian vector fields for Poisson manifolds. For $h \in C^\infty(M)$ we assign it the *Hamiltonian vector field* X_h, defined via

$$X_h(\cdot) = \{\cdot, h\}.$$

It follows from (ii) that a Poisson bracket $\{g, h\}$ depends solely on the first derivatives of g and h. Hence we may think of the bracket as defining a bivector field π determined by

$$\{g, h\} = \pi(dg, dh). \tag{2.1}$$

A Poisson bivector π can be described locally, for coordinates $(x^1, \ldots, x^n)$, by

$$\pi(x) = \frac{1}{2}\sum_{i,j=1}^{n} \pi^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

Here $\pi^{ij}(x) = \{x^i, x^j\} = -\{x^j, x^i\}$.

Poisson brackets satisfy the Jacobi identity, which translates into a P.D.E. for the components of the Poisson bivector [7].

Given a bivector π on M, a point $q \in M$, and $\alpha_q \in T_q^*M$ it is possible to define:

$$\mathcal{B} : T^*M \to TM\,;\ \mathcal{B}_q(\alpha_q)(\cdot) = \pi_q(\cdot, \alpha_q).$$

When π is Poisson, we have that $X_h = \mathcal{B}(dh)$.

We then define the *rank* of π at $q \in M$ to be equal to the rank of $\mathcal{B}_q : T_q^*M \to T_qM$. This is also the rank of the matrix $\pi^{ij}(x)$.

The distribution defined by $\mathcal{B}_q$ on T_qM is called the *characteristic distribution* of π. What is known as the *Symplectic Stratification Theorem* is the celebrated statement that this characteristic distribution of a Poisson tensor π gives rise to a (possibly singular) foliation by symplectic leaves. This foliation is integrable in the sense of Stefan–Sussman [5].

Call Σ_q the symplectic leaf of M through the point q. As a set Σ_q is also the collection of points that may be joined via piecewise smooth integral curves of Hamiltonian vector fields. Write ω_{Σ_q} for the symplectic form on Σ_q. Observe that $T_q\Sigma_q$ is exactly the characteristic distribution of π through p. Therefore, given $u_q, v_q \in T_q\Sigma_q$ there exist $\alpha_q, \beta_q \in T_q^*M$ that under $\mathcal{B}_q$ go to u_q and v_q. Using this we can describe ω_{Σ_q}:

$$\omega_{\Sigma_q}(q)(u_q, v_q) = \pi_q(\alpha_q, \beta_q) = \langle \alpha_q, v_q\rangle = -\langle \beta_q, u_q\rangle. \tag{2.2}$$

As the rank varies, so do the dimensions of the symplectic leaves of the foliation.

In the special case that the rank of the characteristic distribution of a bivector is less than or equal to two, the following were shown to hold in [6]:

Proposition 2.2 (i) *If π is a bivector field on M whose characteristic distribution is integrable and has rank less than or equal to two at each point, then π is Poisson.*
(ii) *Let π be a Poisson structure on M whose rank at each point is less than or equal to two. Then $\pi_1 := k\pi$ is also a Poisson structure where $k \in C^\infty(M)$ is an arbitrary non-vanishing function.*

In order to describe the bivectors locally we will use certain Casimir functions.

Definition 2.3 Let M be a Poisson manifold. A function $h \in C^\infty(M)$ is called a *Casimir* if $\{h, g\} = 0$ for every $g \in C^\infty(M)$.

Equivalently $\mathcal{B}(dh) = 0$.

The next result was shown in [6]:.

Theorem 2.4 *Let M be an orientable n-manifold, N an orientable $n-2$ manifold, and $f : M \to N$ a smooth map. Let μ and Ω be orientations of M and N, respectively. The bracket on M defined by*

$$\{g, h\}\mu = k\, dg \wedge dh \wedge f^*\Omega, \tag{2.3}$$

where k is any non-vanishing function on M is Poisson. Moreover, its symplectic leaves are

(i) *the 2-dimensional leaves $f^{-1}(s)$ where $s \in N$ is a regular value of f,*
(ii) *the 2-dimensional leaves $f^{-1}(s) \setminus \{$Critical Points of $f\}$ where $s \in N$ is a singular value of f.*
(iii) *the 0-dimensional leaves corresponding to each critical point.*

Formula (2.3) appeared in [3] (attributed to H. Flaschka and T. Ratiu).

Definition 2.5 A Poisson manifold M is said to be complete if every Hamiltonian vector field on M is complete.

Notice that M is complete if and only if every symplectic leaf is bounded in the sense that its closure is compact.

3 Local expressions for the Poisson structures

3.1 Local formulae for the Poisson bivectors

We will now construct explicit expressions for the Poisson structure and the corresponding symplectic forms in a neighbourhood of cusp singularities of wrinkled fibrations $X \to \Sigma$, as well as for all the possible moves described above. All of the expressions that we will give depend upon a choice of a non-vanishing function $k \in C^\infty(X)$ (see [6]).

Before proceeding we will describe the general strategy employed to find the local bivectors.

Step 1: Consider the coordinate functions C_1, C_2 that describe each fibration as Casimir functions for the Poisson structure that we want to find.

Step 2: Calculate the differentials dC_1, dC_2 of the Casimirs C_1, C_2.

Step 3: We use formula 2.3 to compute the skew-symmetric matrix with entries:

$$\pi^{ij} = \{x^i, x^j\}\mu = \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge dC_1 \wedge dC_2.$$

This matrix will then annihilate dC_1, dC_2, as this matrix is to be the endomorphism $\mathcal{B}$ associated with a Poisson structure with dC_1 and dC_2 as Casimirs. The components of the bivector field will be given by:

$$\{x^i, x^j\} = \det\left(\epsilon^i, \epsilon^j, dC_1, dC_2\right).$$

Here ϵ^i is the 4×1 canonical basis column vector, whose i-th component is 1 and all others are zero.

For the cusp singularity and the birth, merging, and flipping moves, t is a Casimir, so we only have to compute

$$\{x_, y\}, \quad \{x, z\} \quad \text{and} \quad \{y, z\}.$$

In fact, for these four cases if we denote by dC_2^i the components of the column vector dC_2 we obtain

$$\begin{aligned} \{x, y\} &= -dC_2^3, \\ \{x, z\} &= \ \ dC_2^2, \\ \{y, z\} &= -dC_2^1. \end{aligned}$$

Step 4: According to Proposition 2.2 (ii), we write the Poisson bivector using the skew-symmetric matrix entries.

Near a wrinkling move the Poisson bivector will be obtained using formula 2.3. For the other cases the bivector admits a general expression given by:

$$\pi = k(x, y, z, t)\left[-dC_2^3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + dC_2^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - dC_2^1 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right] \tag{3.1}$$

for any non-vanishing smooth function k.

The corresponding results for neighbourhoods of Lefschetz singularities and broken singular circles were obtained in [6]. We proceed to describe the results this general strategy yields for cusp singularities and the moves described above.

Attentive readers might wonder why we are not using Eq. (2.3) directly. The computations needed to be carried out to take the volume form from the left hand side to the right hand side of the Eq. (2.3) are cumbersome and lengthy. The method we present below yields equivalent results, and may be easily verified.

3.1.1 Local expressions near a cusp singularity

The local coordinate model around a cusp singularity is given by:

$$(x, y, z, t) \mapsto (C_1(x, y, z, t), C_2(x, y, z, t)) = (t, x^3 - 3xt + y^2 - z^2).$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{aligned} dC_1 &= (0 \quad 0 \quad 0 \quad 1), \\ dC_2 &= (3x^2 - 3t \quad 2y \quad -2z \quad -3x). \end{aligned}$$

The following matrix annihilates dC_1 and dC_2 and its entries satisfy the Jacobi identity :

$$\begin{pmatrix} 0 & 2kz & 2ky & 0 \\ -2kz & 0 & k(3t - 3x^2) & 0 \\ -2ky & k(3x^2 - 3t) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which means that the Poisson bivector in the local coordinates of a cusp singularity is described by:

$$\pi = k(x, y, z, t)\left[2z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + (3t - 3x^2)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right]. \tag{3.2}$$

3.1.2 Local expressions near a birth move

The local coordinate model around a birth move is given by:

$$b_s(x, y, z, t) = (C_1(x, y, z, t), C_2(x, y, z, t)) = (t, x^3 - 3x(t^2 - s) + y^2 - z^2).$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{aligned} dC_1 &= (0 \quad 0 \quad 0 \quad 1), \\ dC_2 &= (3x^2 - 3(t^2 - s) \quad 2y \quad -2z \quad -6xt). \end{aligned}$$

From which we can obtain the following matrix:

$$\begin{pmatrix} 0 & 2kz & 2ky & 0 \\ -2kz & 0 & k\left(3\left(t^2 - s\right) - 3x^2\right) & 0 \\ -2ky & k\left(3x^2 - 3\left(t^2 - s\right)\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence the Poisson bivector near a birth move has the form:

$$\pi_s = k(x, y, z, t)\left[2z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - 3(s - t^2 + x^2)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right]. \tag{3.3}$$

3.1.3 Local expressions near a merging move

The local coordinate model around a merging move is given by:

$$m_s(x, y, z, t) = (C_1(x, y, z, t), C_2(x, y, z, t)) = (t, x^3 - 3x(s - t^2) + y^2 - z^2).$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{array}{rcccc} dC_1 = & (0 & 0 & 0 & 1), \\ dC_2 = (3x^2 - 3(s - t^2) & 2y & -2z & 6xt). \end{array}$$

The associated matrix is then:

$$\begin{pmatrix} 0 & 2kz & 2ky & 0 \\ -2kz & 0 & k\left(3\left(s - t^2\right) - 3x^2\right) & 0 \\ -2ky & k\left(3x^2 - 3\left(s - t^2\right)\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the Poisson bivector in a neighbourhood of a merging move is described as:

$$\pi_s = k(x, y, z, t)\left[2z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - 3(s - t^2 - x^2)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right]. \quad (3.4)$$

3.1.4 Local expressions near a flipping move

The local coordinate model around a flipping move is given by:

$$f_s(x, y, z, t) = (C_1(x, y, z, t), C_2(x, y, z, t)) = (t, x^4 - x^2 s + xt + y^2 - z^2).$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{array}{rcccc} dC_1 = & (0 & 0 & 0 & 1), \\ dC_2 = & (4x^3 - 2xs + t & 2y & -2z & x). \end{array}$$

The corresponding matrix is:

$$\begin{pmatrix} 0 & 2kz & 2ky & 0 \\ -2kz & 0 & k\left(-4x^3 + 2sx - t\right) & 0 \\ -2ky & k\left(4x^3 - 2sx + t\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Poisson bivector in a neighbourhood of a flipping move can then be written in the following way:

$$\pi_s = k(x, y, z, t)\left[2z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - (t - 2sx + 4x^3)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right]. \quad (3.5)$$

3.1.5 Local expressions near a wrinkling move

The local coordinate model around a wrinkling move is given by:

$$w_s(x,y,z,t) = (C_1(x,y,z,t), C_2(x,y,z,t)) = (t^2 - x^2 + y^2 - z^2 + st, 2tx + 2yz).$$

The differentials dC_1 and dC_2 are therefore:

$$\begin{aligned} dC_1 &= (-2x \;\; 2y \;\; -2z \;\; 2t+s), \\ dC_2 &= \;\;(2t \;\;\; 2z \;\;\; 2y \;\;\;\; 2x). \end{aligned}$$

The matrix we are interested in is given by:

$$\begin{pmatrix} 0 & k(sy+2ty+2xz) & k(2xy-(s+2t)z) & -2k\left(y^2+z^2\right) \\ -k(sy+2ty+2xz) & 0 & k\left(st+2\left(t^2+x^2\right)\right) & k(2tz-2xy) \\ k((s+2t)z-2xy) & -k\left(st+2\left(t^2+x^2\right)\right) & 0 & 2k(ty+xz) \\ 2k\left(y^2+z^2\right) & 2k(xy-tz) & -2k(ty+xz) & 0 \end{pmatrix}.$$

The expression for the Poisson bivector in a neighbourhood of a wrinkling move is then:

$$\begin{aligned} \pi_s = k(x,y,z,t)[&(-2sy-4ty-4xz)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + (-4xy+2sz+4tz)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} \\ &+(4y^2+4z^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} - (2st+4t^2+4x^2)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} \\ &+4(xy-tz)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t} - 4(ty+xz)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}]. \end{aligned} \tag{3.6}$$

3.1.6 Linearization

We follow chapters 3 and 4 of [5], where more details and examples may be found. Let $\mathfrak{l}$ be a finite-dimensional Lie algebra. Denote by $\mathfrak{r}$ the radical of $\mathfrak{l}$, i.e., the maximal solvable ideal of $\mathfrak{l}$. Then $\mathfrak{g} = \mathfrak{l}/\mathfrak{r}$ is a semi-simple Lie algebra. The Levi–Malcev theorem states that $\mathfrak{l}$ can be decomposed as a semi-direct product:

$$\mathfrak{l} = \mathfrak{g} \ltimes \mathfrak{r}.$$

In analogy with this Levi–Malcev decomposition, we have a Levi decomposition for Poisson structures. Let π be a Poisson structure and denote by π_0 its linear part. By definition we obtain that π_0 generates a Lie algebra $\mathfrak{l}$. We take the Levi–Malcev decomposition of $\mathfrak{l}$, with the previous notation. Let $\{x_1,\dots,x_m,y_1,\dots,y_m\}$ be a basis for $\mathfrak{l}$, such that $\{x_1,\dots,x_m\}$ span $\mathfrak{g}$, and $\{y_1,\dots,y_m\}$ spans a complement $\mathfrak{r}$ of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{g}$ on $\mathfrak{l}$.

A Levi decomposition for π at 0, with $\pi(0) = 0$, provides local coordinates such that;

$$\{x_i, x_j\} \in \mathfrak{g} \quad \text{and} \quad \{x_i, y_j\} \in \mathfrak{r}.$$

Any analytic Poisson structure π, which vanishes at 0, admits a Levi decomposition in a neighbourhood of 0.

Now we will focus on the expressions for the bivectors obtained in Eqs. (3.2), (3.3), (3.4), (3.5), and (3.6). We fix $k \equiv 1$. It can be seen that in the case of cusp singularities, the linear part of the corresponding Poisson structure (3.2) defines a Lie algebra through the commutation relations:

$$[x, z] = 2y, \qquad [x, y] = 2y, \qquad [y, z] = 3t.$$

For the Birth, Merge and Flipping moves, corresponding to the bivectors (3.3), (3.4), and (3.5), respectively, their linear part in all these cases is generated by:

$$[x, z] = 2y, \qquad [x, y] = 2y.$$

Notice that this Lie algebra contains a nonzero Abelian ideal; hence, it is not semi-simple.

So in all these cases the linear part of the Poisson structure admits a decomposition of the form $\mathbb{R} \times L_3$, where L_3 is a semi-direct product of Lie algebras:

$$L_3 = \mathbb{R} \ltimes_A \mathbb{R}^2.$$

Here $\mathbb{R}$ acts on $\mathbb{R}^2$ linearly by the matrix:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

For the case of wrinkled fibrations, corresponding to (3.6), we see that all the commutation relations are trivial. Therefore the corresponding linear part of its Poisson structure spans an Abelian Lie algebra, which is not semi-simple.

Conn's theorem asserts that, provided the linear part of an analytic Poisson structure π that vanishes at 0, corresponds to a semi-simple Lie algebra, π admits a local analytic linearizaton at 0. Hence, in the spirit of Conn's theorem, the linearization of all these Poisson structures is not guaranteed. Moreover, the semi-direct product L_3 is degenerate (formally, analytically, and smoothly).

Finally, in the general case when k is a non vanishing smooth function, we obtain other Poisson structures.

Question 3.1 *Does there exist a function k such that an expression given by one of the bivectors* (3.2), (3.3), (3.4), (3.5) *or* (3.6) *is linearizable?*

3.2 Equations for the symplectic forms on the leaves near singularities

Proposition 3.2 *Let $q \in B^4 \backslash \{0\}$ and let π be the Poisson structure near a cusp singularity. Then the symplectic form induced by π on the symplectic leaf Σ_q through $q = (x, y, z, t)$ at the point q is given by*

$$\omega_{\Sigma_q} = \frac{1}{k(x, y, z, t)3\left(x^2 - t\right)}\omega_{\text{Area}}(q). \tag{3.7}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^4.

Proof If u_q, v_q are tangent vectors to the leaves there exist co-vectors $\alpha_q, \beta_q \in T_q^*M$ such that $\mathcal{B}_q(\alpha_q) = u_q$ and $\mathcal{B}_q(\beta_q) = v_q$, where $\mathcal{B}_q$ is given by the rule:

$$\mathcal{B}_q(\alpha)(\cdot) = \pi_q(\cdot, \alpha).$$

Finding two tangent vectors to the symplectic leaves is equivalent to detecting vectors annihilated simultaneously by the differential of two Casimir functions for the corresponding Poisson structure. Note that the characteristic distribution has rank 2.

In this case we find that the vectors:

$$u_q = -\frac{1}{3\left(t - x^2\right)}\left(2z\frac{\partial}{\partial x} + 3\left(t - x^2\right)\frac{\partial}{\partial z}\right),$$

$$v_q = \frac{1}{3(t - x^2)}\left(2y\frac{\partial}{\partial x} + 3(t - x^2)\frac{\partial}{\partial y}\right)$$

are tangent to Σ_q at q. Using the local expression of the Poisson structure for a cusp singularity given by Eq. (3.2), one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = -\frac{\mathrm{d}y}{k(x, y, z, t)3(t - x^2)}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{1}{k(x, y, z, t)}\left(-\frac{1}{2z}\mathrm{d}x + \frac{y}{3(t - x^2)z}\mathrm{d}y\right).$$

A direct calculation now implies that the symplectic form is:

$$\omega_{\Sigma_q}(q)(u_q, v_q) = \langle \alpha_q, v_q \rangle = \frac{1}{k(x, y, z, t)3(x^2 - t)}\omega_{\text{Area}}(q).$$

□

Proposition 3.3 *Let $q \in B^4 \backslash \{0\}$ and π_s be the Poisson structure near a birth move. The symplectic form induced by π_s on the symplectic leaf Σ_q through $q = (x, y, z, t)$ at the point q is given by*

$$\omega_{\Sigma_q} = \frac{1}{k(x, y, z, t)3(s - t^2 + x^2)} \omega_{\text{Area}}(q). \tag{3.8}$$

here ω_{Area} is the area form on Σ_q induced by the Euclidean metric on B^4.

Proof Making use of the corresponding Casimir functions for the Poisson structure associated with a birth move we obtain that the vectors

$$u_q = \frac{1}{3\left(s - t^2 + x^2\right)} \left(2z \frac{\partial}{\partial x} + 3\left(s - t^2 + x^2\right) \frac{\partial}{\partial z}\right),$$

$$v_q = -\frac{1}{3(s - t^2 + x^2)} \left(2y \frac{\partial}{\partial x} + 3(s - t^2 + x^2) \frac{\partial}{\partial y}\right)$$

are tangent to Σ_q at q. Using the local expression (3.3) of the Poisson structure one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{\mathrm{d}y}{k(x, y, z, t)3(s - t^2 + x^2)}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{1}{k(x, y, z, t)} \left(-\frac{1}{2z} \mathrm{d}x - \frac{y}{3(s - t^2 + x^2)z} \mathrm{d}y\right).$$

The expression for the symplectic form follows from:

$$\omega_{\Sigma_q}(q)(u_q, v_q) = \langle \alpha_q, v_q \rangle = -\langle \beta_q, u_q \rangle.$$

□

Proposition 3.4 *Let $q \in B^4 \backslash \{0\}$ and let π_s be the Poisson structure near a merging move. The symplectic form induced by π_s on the symplectic leaf Σ_q through $q = (x, y, z, t)$ at the point q is given by*

$$\omega_{\Sigma_q} = \frac{1}{k(x, y, z, t)3(t^2 - s + x^2)} \omega_{\text{Area}}(q), \tag{3.9}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^4.

Proof We proceed similarly to the previous cases above. We find that the vectors

$$u_q = \frac{1}{3\left(s - t^2 - x^2\right)}\left(-2z\frac{\partial}{\partial x} + 3\left(s - t^2 - x^2\right)\frac{\partial}{\partial z}\right),$$
$$v_q = \frac{1}{3(s - t^2 - x^2)}\left(2y\frac{\partial}{\partial x} + 3(s - t^2 - x^2)\frac{\partial}{\partial y}\right)$$

are tangent to Σ_q at q. Using the local expression (3.4) of the corresponding Poisson structure one can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = -\frac{\mathrm{d}y}{k(x, y, z, t)3(s - t^2 - x^2)}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{1}{k(x, y, z, t)}\left(-\frac{1}{2z}\mathrm{d}x + \frac{y}{3(s - t^2 - x^2)z}\mathrm{d}y\right).$$

As before the symplectic form is obtained by computing:

$$\omega_{\Sigma_q}(q)(u_q, v_q) = \langle \alpha_q, v_q\rangle = -\langle \beta_q, u_q\rangle.$$

□

Proposition 3.5 *Let $q \in B^4\backslash\{0\}$ and let π_s be the Poisson structure near a flipping move. The symplectic form induced by π_s on the symplectic leaf Σ_q through $q = (x, y, z, t)$ at the point q is given by*

$$\omega_{\Sigma_q} = \frac{1}{k(x, y, z, t)(t - 2sx + 4x^3)}\omega_{\mathrm{Area}}(q), \tag{3.10}$$

here ω_{Area} is the area form on Σ_q induced by the euclidean metric on B^4.

Proof In this case, the following vectors:

$$u_q = \frac{1}{t - 2sx + 4x^3}\left(2z\frac{\partial}{\partial x} + (t - 2sx + 4x^3)\frac{\partial}{\partial z}\right),$$

$$v_q = \frac{1}{t - 2sx + 4x^3}\left(-2y\frac{\partial}{\partial x} + (t - 2sx + 4x^3)\frac{\partial}{\partial y}\right)$$

are tangent to Σ_q at q. Using Eq. (3.5) we can check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = -\frac{\mathrm{d}y}{k(x, y, z, t)(-t + 2sx - 4x^3)}.$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{1}{k(x, y, z, t)} \left(-\frac{1}{2z}\mathrm{d}x - \frac{y}{(t - 2sx + 4x^3)z}\mathrm{d}y \right).$$

A straightforward calculation shows that:

$$\omega_{\Sigma_q} = \frac{1}{k(x, y, z, t)(t - 2sx + 4x^3)z}\omega_{\text{Area}}(q).$$

□

Proposition 3.6 *Let $q \in B^4 \backslash \{0\}$ and let π_s be the Poisson structure near a wrinkling move. The symplectic form induced by π_s on the symplectic leaf Σ_q through $q = (x, y, z, t)$ at the point q is given by*

$$\omega_{\Sigma_q} = -\frac{1}{k(x, y, z, t)2(ty + xz)}\omega_{\text{Area}}(q). \tag{3.11}$$

here ω_{Area} is the area form on Σ_q induced by the Euclidean metric on B^4.

Proof Using the corresponding Poisson structure for a wrinkling move given in Eq. (3.6) we obtain:

$$u_q = -\frac{1}{2(ty + xz)} \left((2xy - sz - 2tz)\frac{\partial}{\partial x} + (st + 2t^2 + 2x^2)\frac{\partial}{\partial y} - 2(ty + xz)\frac{\partial}{\partial t} \right).$$

$$v_q = -\frac{1}{(ty + xz)} \left((y^2 + z^2)\frac{\partial}{\partial x} + (xy - tz)\frac{\partial}{\partial y} - (ty + xz)\frac{\partial}{\partial z} \right)$$

and makes v_q an unitary vector. These vectors are tangent to Σ_q at q. Using the local expression of the Poisson structure in (3.6) we check that $\mathcal{B}_q(\alpha_q) = u_q$, for

$$\alpha_q = \frac{1}{k(x, y, z, t)} (\frac{1}{2(y^2 + z^2)}\mathrm{d}x - \frac{-2xy + sz + 2tz}{4(ty + xz)(y^2 + z^2)}\mathrm{d}t).$$

Similarly, $\mathcal{B}_q(\beta_q) = v_q$, for

$$\beta_q = \frac{\mathrm{d}t}{2(ty + xz)}.$$

The proposition is shown as in the previous cases by calculating the symplectic form explicitly using the above equations. □

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